On higher order non-singular immersions of RP^n in CP^m

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Haruo SUZUKI

§ 0. Introduction

We denote by RP^n the real projective space of dimension n and by CP^n the complex projective space of complex dimension n. We know many informations on higher order non-sigular immersions of RP^n in RP^m or those of CP^n in CP^m. They can be seen in H. Suzuki [11, 12] and H. Ôike [8]. Results on higher order non-singular immersions of RP^n in euclidean spaces by H. Suzuki [10] and T. Kobayashi [7] are translated easily into those on higher order non-singular immersions of RP^n in RP^m or CP^n which are homotopically trivial. As for higher order non-sigular immersions of RP^n in $RP^m (n<m)$, some results are anounced in W.-L. Ting [13].

Let m, n be integers such that $1<n\leqq 2m$. The homotopy set $[RP^n, CP^m]$ consists of just two elements. We consider in this article, higher order non-singular immersions of RP^n in CP^m which are not homotopically trivial, and also consider relations between the homotopy classes of higher order non-singular immersions of RP^n in CP^m and its order, under some additional conditions on dimensions n and m. Our main tools are Stiefel-Whitney classes of pth order normal (conormal) bundles. Mod 2 S-relations of stunted real projective spaces constructed from some independent cross-sections of a multiple of the canonical line bundle over RP^n can be used in quite restricted cases.

We state results in Section 1. We prove, in Section 2, the key lemma on the homotopically non-trivial maps of RP^n to CP^m and prove theorems of section 1 in section 3. Finally we give supplementary results on the higher order non-singular immersions which are homotopically trival and results from the mod 2 S-relations of stunted projective spaces.

§ 1. Statement of results

Let $\tau_p(RP^n)$ and $\tau_p(CP^m)$ denote pth order tangent bundles of RP^n and CP^m respectively. The dimensions of their fibres are $\nu(n,p)=\binom{n+p}{p}-1$ and

$\nu(2m, p)=\binom{2m+p}{p}-1$. Let $f: RP^n \to CP^m$ be a C^p-differentiable map and $\tau_p(f): \tau_p(RP^n) \to \tau_p(CP^m)$ be the homomorphism of pth order tangent bundles induced by f. If we take a sequence of qth order dissections $\{D^{(q)}\}$, $q=1, 2, \cdots, p-1$ on CP^m, we have a pth order osculating map of f with respect to $\{D^{(q)}\}$, which is a vector bundle homomorphism

$$D_p \cdot \tau_p(f) = D^{(1)} \cdots D^{(p-1)} \cdot \tau_p(f) : \tau_p(RP^n) \to \tau(CP^m)$$

covering f. If f is an immersion and $D_p \cdot \tau_p(f)$ is of maximal rank on each fibre, f is called a *pth order non-singular immersion* with respect to the dissections. If $p \geqq 2$, $2m \geqq \nu(n, p)$ and $D_p \cdot \tau_p(f)|_x (x \in RP^n)$ is not of maximal rank for a C^p-immersion f, one can say that x is an *inflexion point* with respect to the dissections, of order $\leqq p-1$. (See W. F. Pohl [9] and E. A. Feldman [2, 3, 4, 5].)

The homotopy set $[RP^n, CP^m]$ $(1<n \leqq 2m)$ are $1-1$ correspondence with the cohomology group $H^2(RP^n; Z) \cong Z_2$. For a continuous map $f: RP^n \to CP^m$ $(1<n \leqq 2m)$, we call the homotopy class $\{f\} \in [RP^n, CP^m] \cong Z_2$ a *degree* of f and denote it by $\deg(f)$. This is expressed by an integer d mod 2. By computations of the Stiefel-Whitney classes of pth order normal (conormal) bundle of the pth order non-sigular immersion, we obtain the following results.

THEOREM 1. 1. *Let p be an even positive integer, and let n, m be integers such that $1<n$ and $\nu(n, p)-n<2m<\nu(n, p)+n$. Suppose there exists a pth order non-singular immersion $f: RP^n \to CP^m$ with respect to dissections $\{D^{(q)}\}$ on CP^m such that* $\deg(f) \neq 0$. (a) *If $k=2m-\nu(n, p) \geqq 0$ then we have* $\binom{\binom{n+p}{p}+k+1}{k+i}=0$ mod 2 *and* $\binom{\binom{n+p}{p}+k+i}{k+i}=0$ mod 2 *for* $1 \leqq i \leqq n-k$. (b) *If $k=\nu(n, p)-2m \geqq 0$ then we have* $\binom{\binom{n+p}{p}+1}{k+i}=0$ mod 2 *and* $\binom{\binom{n+p}{p}+i}{k+i}=0$ mod 2 *for* $1 \leqq i \leqq n-k$.

COROLLARY 1. 2. *Let n, m be positive integers, $1<n$ and let p be an even positive integer. We put* $s(n, p)=\max\{j | 0<j \leqq n, \binom{\binom{n+p}{p}+j}{j} \neq 0 \text{ mod } 2\}$ *and* $d(n, p)=\max\{j | 0<j \leqq n, \binom{\binom{n+p}{p}+1}{j} \neq 0 \text{ mod } 2\}$. *If* $\nu(n, p)-d(n, p)<2m<\nu(n, p)+s(n, p)$ *then there is no homotopically non-trivial pth order non-singular immersion of RP^n in CP^m with respect to dissections on CP^m.*

We can also show easily the following theorem by Theorem 1. 1.

THEOREM 1. 3. *Assume* $p=2^r(2q+1)$ *where* q, r *are integers and* $r\geqq 1$, *and assume* $n=2^l-1$ *for an integer* $l>r$ *and* $0\leqq 2m-\nu(n,p)<2^{l-r}-1$. *If* RP^n *can be immersed in* CP^m *by a homotopically non-trivial* C^p*-map* f, *then* f *has at least one inflexion point of order* $\leqq p-1$ *with respect to dissections* $\{D^{(q)}\}$ *on* CP^m.

As for relations between $\deg(f)$ and the order p, we obtain some theorems. First we show an analogy of Theorem 1. 1 of H. Suzuki [12] in a certain sence.

THEOREM 1. 4. *Suppose* n, m *and* p *are integers such that* $n>1, p\geqq 1$, $\binom{n+p}{p}\not\equiv n \bmod 2$ *and suppose* $\nu(n,p)\leqq 2m\leqq \nu(n,p)+n-2$ *and* $\binom{\binom{n+p}{p}+n-1}{n}\not\equiv 0 \bmod 2$. *If* $f: RP^n\to CP^m$ *is a pth order non-singular immersion with respect to dissections* $\{D^{(q)}\}$ *on* CP^m *then we have* $\deg(f)=p \bmod 2=0$.

We can find the values of n, m and p satisfying conditions of Theorem 1. 4. For examples, we have $n=2, p=5$ and $m=10$; $n=2, p=8$ and $m=22$. Under these assumtions for n, m and p, a C^p-immersion $f: RP^n\to CP^m$ has at least one inflexion point of order $\leqq p-1$ with respect to dissections $\{D^{(q)}\}$ on CP^m if $\deg(f)\neq 0$ or $p\not\equiv 0 \bmod 2$.

In the above theorem, we assume $\binom{n+p}{p}\not\equiv n \bmod 2$ and $\binom{\binom{n+p}{p}+n-1}{n}\not\equiv 0 \bmod 2$. Without these assumptions, we can prove a result.

THEOREM 1. 5. *Suppose* n, m *and* p *are integers such that* $n>1, p\geqq 1$ *and* $\nu(n,p)\leqq 2m\leqq \nu(n,p)+n-1$. *If there exists a pth order non-singular immersion* $f: RP^n\to CP^m$ *with respect to dissections* $\{D^{(q)}\}$ *on* CP^m, *then one of the following cases takes place*: $\mathrm{Deg}(f)\neq 0$ *and* $p\equiv 0 \bmod 2$; $\deg(f)=0$ *and* $p\not\equiv 0 \bmod 2$; $\deg(f)=0$ *and* $p\equiv 0 \bmod 2$.

§ 2. Homotopically non-trivial map of RP^n in CP^m.

Let f : $RP^n\to CP^m$ $(1<n\leqq 2m)$ be the homotopically non-trivial map. Let η be the canonical complex line bundle over CP^m. We denote by r the realification of a complex vector bundle. We shall compute the induced bundle $f^! r\eta$ over RP^n. Let ξ be the canonical real line bundle over RP^n.

LEMMA 2. 1. *Under the above notations, we have*

$$f^! r\eta=\xi\oplus\xi.$$

PROOF. Complex line bundles over RP^n are classified by the homotopy set $[RP^n, CP^m]$ since we assume $n \leqq 2m$. The correspondence is given by taking the induced bundle. On the other hand, the set $[RP^n, CP^m]$ are $1-1$ correspondence with the cohomology group $H^2(RP^n; Z)$. This correspondence is given by $c_1(f^! \eta) = f^*(c_1 \eta) \in H^2(RP^n; Z)$ where c_1 is the first Chern class. By the assumption $1 < n \leqq 2m$, $H^2(RP^n; Z)$ is isomorphic to Z_2 and hence the set of isomorphism classes of complex line bundles over RP^n consists of just two elements which are the class of the trivial bundle and that of the non-trivial bundle. The induced complex line bundle $f^! \eta$ is not trivial because f is not homotopically trivial.

Let w denote the total Stiefel-Whitney class, let a be the generator of $H^1(RP^n; Z)$ and let c be the complexification of a real vector bundle. Then we have

$$rc\xi = \xi \oplus \xi,$$

and

$$w(rc\xi) = 1 + a^2.$$

This shows that the complex line bundle $c\xi$ is not trivial, and it follows that

$$f^! \eta = c\xi.$$

Therefore, we get the relation

$$f^!(r\eta) = rf^! \eta = rc\xi = \xi \oplus \xi,$$

which completes the proof. (Cf. Lemma 7. 1 of J. F. Adams [1].)

§ 3. Even order non-singular immersions of RP^n in CP^m.

Let $f: RP^n \to CP^m (1 < n \leqq 2m)$ be a pth order non-singular immersion with respect to dissections $\{D^{(q)}\}$ on CP^m and assume that $\nu(n, p) - n < 2m < \nu(n, p) + n$, $p > 1$. Let τ and τ_p denote the tangent bundle and the pth order tangent bundle. Let ξ be the canonical line bundle over RP^n and η be the canonical complex line bundle over CP^m. Let c and r denote respectively the complexification and realification operator as in Section 2.

PROOF OF THEOREM 1. 1. Let p be even and f be homotopically non-trivial. We prove first (a) If $2m \geqq \nu(n, p)$, then we denote the pth order normal bundle (the cokernel of the pth order osculating map of f with respect to dissections $\{D^{(q)}\}$) by $\mu_p(f)$. It follows that

$$\mu_p(f) \oplus \tau_p(RP^n) = f^! \tau(CP^m).$$

By taking the Whitney sum with the trivial 2-vector bundle and by making use the relation of H. Suzuki [10],

$$\tau_p(RP^n)\oplus 1=\binom{n+p}{p}\xi^p\,,$$

we have

$$\mu_p(f)\oplus\left(\binom{n+p}{p}+1\right)=f^!(m+1)r\eta\,.$$

We put $k=2m-\nu(n,p)$. From Lemma 2. 1, it follows that

$$\mu_p(f)\oplus\left(\binom{n+p}{p}+1\right)=\left(\binom{n+p}{p}+k+1\right)\xi\,. \tag{1}$$

The Stiefel-Whitney classes of $\mu_p(f)$ are

$$\begin{aligned} w_j(\mu_p(f)) &= w_j\left(\left(\binom{n+p}{p}+k+1\right)\xi\right)\\ &=\binom{\binom{n+p}{p}+k+1}{j}a^j\,, \end{aligned}$$

where $a\in H^1(RP^n\,;Z_2)$ is the canonical generator, and they must be zero in dimensions $j\geqq k+1$. Since we $a^j\neq 0$, $1\leqq j\leqq n$, it follows that

$$\binom{\binom{n+p}{p}+k+1}{k+i}\equiv 0 \bmod 2,\quad i=1,\cdots,n-k\,.$$

We multiply the both sides of (1) by ξ and we get

$$\mu_p(f)\otimes\xi\oplus\left(\binom{n+p}{p}+1\right)\xi=\binom{n+p}{p}+k+1 \tag{1'}$$

and in $K0$-theory we have

$$\mu_p(f)\otimes\xi=-\left(\binom{n+p}{p}+1\right)\xi+\binom{n+p}{p}+k+1\,.$$

The Stiefel-Whitney classes of $\mu_p(f)\otimes\xi$ are

$$w_j(\mu_p(f)\otimes\xi)=\binom{\binom{n+p}{p}+j}{j}a^j$$

and they must be zero in dimensions $j\geqq k+1$. Since we have $a^j\neq 0$, $1\leqq j\leqq n$, it follows that

$$\binom{\binom{n+p}{p}+k+i}{k+i}\equiv 0 \bmod 2,\quad i=1,\cdots,n-k\,.$$

Thus the proof of (a) is completed.

The proof of (b) is similar to that of (a) and is briefly indicated in the following. Let p be again even and assume that $k=\nu(n,p)-2m\geqq 0$. Let f be homotopically non-trivial. We denote the pth order conormal bundle (the kernel of the pth order osculating map of f with respect to dissections $\{D^{(q)}\}$) by $\mu'_p(f)$ and we have

$$\mu'_p(f)\oplus f^!\tau(CP^m)=\tau_p(RP^n),$$

and hence by Lemma 2. 1,

$$\mu'_p(f)\oplus\left(\binom{n+p}{p}-k+1\right)\xi=\binom{n+p}{p}+1. \tag{2}$$

By multiplying the both sides of (2) by ξ, we get

$$\mu'_p(f)\otimes\xi\oplus\left(\binom{n+p}{p}-k+1\right)=\left(\binom{n+p}{p}+1\right)\xi. \tag{2'}$$

From the vanishing of the Stiefel-Whitney classes $w_j(\mu'_p(f)\otimes\xi)$ for $j=k+1, \cdots, n$, it follows that

$$\binom{\binom{n+p}{p}+1}{k+i}\equiv 0 \bmod 2,\quad i=1,\cdots,n-k.$$

In $K0$-theory, we obtain from (2), the relation,

$$\mu'_p(f)=-\left(\binom{n+p}{p}-k+1\right)\xi+\binom{n+p}{p}+1.$$

From the vanishing the Stiefel-Whitney classes $w_j(\mu'_p(f))$ for $j=k+1,\cdots,n$, it follows that

$$\binom{\binom{n+p}{p}+i}{k+i}\equiv 0 \bmod 2,\quad i=1,\cdots,n-k.$$

Thus the proof of (b) is completed.

PROOF OF THEOREM 1. 3. Under the assumption of Theorem 1. 3, $\binom{n+p}{p}$ is divisible by 2^{l-r}. If we put $k=2m-\nu(n,p)$, then we can see easily that

$$\binom{\binom{n+p}{p}+k+1}{k+1}\not\equiv 0 \bmod 2.$$

The required result follows immediately from Theorem 1. 1.

§4. Relations between deg(f) and order p

Now we consider relations between the homotopy class of the pth order non-singular immersion $f: RP^n \to CP^m$ with respect to dissections $\{D^{(q)}\}$ on CP^m and the order p, and prove Theorems 1.4, 1.5. We use the same notations as in section 3.

PROOF OF THEOREM 1.4. Suppose $\binom{n+p}{p} \not\equiv n \bmod 2$. Since we have $m \leqq \frac{1}{2}\left(\binom{n+p}{p}+n-3\right)=m'$ by the assumption of our theorem, we can take the standard inclusion $i: CP^m \to CP^{m'}$ which induces the injection of tangent bundles $\tau(i): \tau(CP^m) \to \tau(CP^{m'})$. We denote the composition homomorphism of vector bundles,

$$\tau_p(RP^n) \xrightarrow{D_p \cdot \tau_p(f)} \tau(CP^m) \xrightarrow{\tau(i)} \tau(CP^{m'})$$

by H. $D_p \cdot \tau_p(f)$ is an injection by the assumption of our theorem and hence H is also an injection. Obviously, the map H covers the C^p-map $h = i \cdot f: RP^n \to CP^m \to CP^{m'}$. We denote the cokernel of H by $\mu_p(H)$ which is a real vector bundle over RP^n. By the relation $\tau_p(RP^n)+1=\binom{n+p}{p}\xi^p$ of H. Suzuki [10], we have, in $KO(RP^n)$,

$$\begin{aligned}\mu_p(H) &= h^! \tau(CP^{m'}) - \tau_p(RP^n) \\ &= h^!(m'+1) r\eta - \binom{n+p}{p}\xi^p - 1 .\end{aligned}$$

Let d be an integer such that $d \bmod 2 = \deg(f)$. From Lemma 2.1, it follows that

$$\begin{aligned}\mu_p(H) &= 2(m'+1)\xi^d - \binom{n+p}{p}\xi^p - 1 \\ &= \left(\binom{n+p}{p}+n-1\right)\xi^d - \binom{n+p}{p}\xi^p - 1 .\end{aligned}$$

The total Stiefel-Whitney class of $\mu_p(H)$ is given by

$$(3) \qquad w(\mu_p(H)) = (1+da)^{\left(\binom{n+p}{p}+n-1\right)}(1+pa)^{-\binom{n+p}{p}} \bmod 2 .$$

Since the dimension of $\mu_p(H)$ is $n-2$, it follows, therefore, that

$$w_n(\mu_p(H)) = w_{n-1}(\mu_p(H)) = 0 .$$

By the formula (3), we get

$$w_n(\mu_p(H)) = \binom{\binom{n+p}{p}+n-1}{n}(d-p)^n a^n = 0 \bmod 2 .$$

Since we have $\binom{\binom{n+p}{p}+n-1}{n} \neq 0 \bmod 2$ by the assumption of our theorem and $a^n \neq 0$, we obtain

$$d \equiv p \bmod 2\,,$$

that is

$$\deg(f) = p \bmod 2\,.$$

By this last relation and the formula (3), we have

$$w(\mu_p(H)) = (1+da)^{n-1} \bmod 2\,,$$

and hence

$$w_{n-1}(\mu_p(H)) = d^{n-1}a^{n-1} \bmod 2\,.$$

Since we have shown in the above that $w_{n-1}(\mu_p(H))=0$ and have $a^{n-1} \neq 0$, it follows that

$$d \equiv 0 \bmod 2\,,$$

which completes the proof of theorem 1. 4.

PROOF OF THEOREM 1. 5. Let $f: RP^n \to CP^m$ be a pth order non-singular immersion with respect to dissections $\{D^{(q)}\}$ on CP^m, where $n>1$, $p \geqq 1$ and $\nu(n, p) \leqq 2m \leqq \nu(n, p)+n-1$. Contrary to the consequence of our theorem, we suppose that $\deg(f) \neq 0$ and $p \neq 0 \bmod 2$. Let $\mu_p(f)$ be the pth order normal bundle of f and put $k=2m-\nu(n, p) \geqq 0$. By the formula $\tau_p(RP^n)+1=\binom{n+p}{p}\xi^p$ and Lemma 2. 1, we have, in $KO(RP^n)$,

$$\begin{aligned}\mu_p(f) &= 2(m+1)\xi-\binom{n+p}{p}\xi-1 \\ &= (k+1)\xi-1\,.\end{aligned}$$

It follows that

$$w_{k+1}(\mu_p(f)) = a^{k+1}\,.$$

Since we have $k+1 \leqq n$ by the assumption of our theorem, we obtain $w_{k+1}(\mu_p(f))=a^{k+1} \neq 0$. But the dimension of $\mu_p(f)$ is k. This contradicts to $w_{k+1}(\mu_p(f)) \neq 0$ and completes the proof of Theorem 1. 5.

§ 5. Supplementary results.

1. Let f be a pth order non-singular immersion of RP^n in CP^m with respect to dissections $\{D^{(q)}\}$ on CP^m where $1<n \leqq 2m$. We consider the isomorphism classes of the pth order normal bundle $\mu_p(f)$ or the pth order conormal bundle $\mu'_p(f)$. The induced bundle $f^! \tau(CP^m)$ is trivial if we have

$\deg(f)=0$, and we see that the Whitney sum relations,

$$\mu_p(f)\oplus\tau_p(RP^n)=f^!\tau(CP^m)\cong 2m$$

and

$$\tau_p(RP^n)=\mu'_p(f)\oplus f^!\tau(CP^m)\cong\mu'_p(f)\oplus 2m$$

are just the same as the corresponding formulas for the pth order non-singular immersions of RP^n in the euclidean space R^{2m} with the standard dissections on R^{2m}. In the latter case, we know many results and they are translated into the former case. We illustrate two theorems below.

THEOREM. 5. 1. *Let n, m and p be integers such that p is odd positive, $1<n\leqq m$ and $\nu(n,p)-n<2m<\nu(n,p)+n$. Suppose that there exists a pth order non-singular immersion $f: RP^n\to CP^m$ with respect to dissections $\{D^{(q)}\}$ such that* $\deg(f)=0$. (a) *If $0\leqq 2m-\nu(n,p)=k$, then we have* $\binom{\binom{n+p}{p}+k+i-1}{k+i}\equiv 0 \bmod 2$ *and* $\binom{\binom{n+p}{p}+k}{k+i}\equiv 0 \bmod 2$ *for* $1\leqq i\leqq n-k$. (b) *If $0\leqq\nu(n,p)-2m=k$, then we have* $\binom{\binom{n+p}{p}}{k+i}\equiv 0$ mod *and* $\binom{\binom{n+p}{p}+i-1}{k+i}$ $\equiv 0 \bmod 2$ *for* $1\leqq i\leqq n-k$.

The proof of Theorem 5. 1 is easy and omitted.

Let ξ be the canonical line bundle over RP^n and $n\geqq k>0$, $k'\geqq 0$ be integers such that $\binom{k+k'}{k}\not\equiv 0 \bmod 2$. According to § 4 of T. Kobayashi [7], if $(k+k')\xi$ has k' independent cross-sections, then stunted projective spaces RP^n/RP^{k-1} and $RP^{n+k'}/RP^{k-1+k'}$ are mod 2 S-related. According to I. M. James [6], RP^n/RP^{k-1} is S-reducible if and only if $(n+1)\equiv 0 \bmod 2^{\varphi(n-k)}$, where $\varphi(s)$ is the number of integers i such that $0<i\leqq s$ and $i\equiv 0, 1, 2$ and 4 mod 8. We put $n+1=(2b+1)2^{c+4d}$ $(0\leqq c\leqq 3)$ and $j(n)=2^c+8d$, where b, c and d are positive integers. Theorem 6. 1 of T. Kobayashi [7] is interpreted as follows.

THEOREM 5. 2. *Let p and r be integers such that p is odd positive, $r>3$ and $2^r>p-1$. If $n=2^r-1$ and $2m=\nu(n,p)+n-j(n)$, then RP^n can not be immersed in CP^m by a homotopically trivial pth order non-singular map with respect to dissections $\{D^{(q)}\}$ on CP^m.*

2. In general, we denote by $\varphi(n,k)$ the number of integers i such that $k<i\leqq n$ and $i\equiv 0, 1, 2$ and 4 mod 8. By making use of the mod 2 S-relations of stunted projective spaces and Adams operations Ψ^3: $\widetilde{K}O(RP^{n+k}/RP^{k'-1+k})\longrightarrow\widetilde{K}O(RP^{n+k}/RP^{k'-1+k})$, $\widetilde{K}O(S^r(RP^{n+k}/RP^{k'-1+k}))\longrightarrow\widetilde{K}O(S^r(RP^{n+k}/$

$RP^{k'-1+k})$), we have following analogies of Theorems 5.6, 5.8 of §5 [7].

Assume that $\binom{\binom{n+p}{p}+k+1}{k} \not\equiv 0 \bmod 2$ and $\binom{n+p}{p}+1\equiv 0 \bmod 8$. If there exists a homotopially non-trivial pth order non-singular immersion $f: RP^n \to CP^m$ with respect to dissections $\{D^{(q)}\}$ on CP^m, where p is even positive and $k=2m-\nu(n,p)\geqq 0$ then we have $\binom{n+p}{p}+1\equiv 0 \bmod 2^{\varphi-1}$, where $\varphi=\varphi(n,k-1)$ for $k\not\equiv 0 \bmod 4$ and $\varphi=\varphi(n,k)$ for $k\equiv 0 \bmod 4$.

Assume that $\binom{\binom{n+p}{p}+1}{k} \not\equiv 0 \bmod 2$ and $\binom{n+p}{p}+1-k\equiv 0 \bmod 8$. If there exists a homotopically non-trivial pth order non-singular immersion $f: RP^n \to CP^m$ with respect to dissections $\{D^{(q)}\}$ on CP^m, where p is even positive and $k=\nu(n,p)-2m\geqq 0$ then we have $\binom{n+p}{p}+1-k\equiv 0 \bmod 2^{\varphi-1}$.

Department of Mathematics
Hokkaido University

References

[1] J. F. ADAMS: Vector fields on spheres, Ann. of Math. (2) 75 (1962), 603–632.

[2] E. A. FELDMAN: Geometry of immersions. I, Bull. Amer. Math. Soc. 69 (1963), 693–698.

[3] E. A. FELDMAN: Geometry of immersions. II, Bull. Amer. Math. Soc. 70 (1964), 600–607.

[4] E. A. FELDMAN: Geometry of immersions. I, Trans. Amer. Math. Soc. 120 (1965), 185–224.

[5] E. A. FELDMAN: Geometry of immersions. II, Trans. Amer. Math. Soc. 125 (1966), 181–215.

[6] I. M. JAMES: Spaces associated with Stiefel manifolds, Proc. London Math. Soc. (3) 9 (1959), 114–140.

[7] T. KOBAYASHI: On the odd order non-singular immersions of real projective spaces, J. Sci. Hiroshima Univ. Ser. A-I Math. 33 (1969), 197–207.

[8] H. ÔIKE: Higher order tangent bundles of projective spaces and lens spaces, Tôhoku Math. J. (2) 22 (1970), 200–207.

[9] W. F. POHL: Differential geometry of higher order, Topology 1 (1962), 169–211.

[10] H. SUZUKI: Bounds for dimensions of odd order non-singular immersions of RP^n, Trans. Amer. Math. Soc. 121 (1966), 269–275.

[11] H. SUZUKI: Characteristic classes of some higer order tangent bundles of complex projective spaces, J. Math. Soc. Japan 18 (1966), 386–393.

[12] H. SUZUKI: Higher order non-singular immersions in projective spaces, Quart. J. Math. Oxford Ser. (2) 20 (1969), 33–44.

[13] W.-L. TING, On odd order nondegenerate immersions of RP^n, 70T-G172, Notices Amer. Math. Soc. 124 (1970), 974.

(Received August 31, 1971)