On a certain property of a Riemannian space admitting a special concircular scalar field

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 0. Introduction.

The purpose of the present paper is to investigate the property of a Riemannian space which admits a scalar field Φ characterised by the property

$$\Phi_{k;l}=\rho\Phi g_{kl}, \qquad \rho=\text{non-zero constant}, \tag{0. 1}$$

(such a scalar field Φ is called the special concircular scalar field in this paper) where $\Phi_k \overset{\text{def}}{=} \Phi_{;k}$ and g_{kl} means the metric tensor of the space. In § 1, we consider a Riemannian space with certain special curvature tensor, and prove the property that the space is of constant curvature. Next, in § 2, we give some corollaries of it.

The author wishes to express to Prof. Y. Katsurada and Dr. T. Nagai his very sincere thanks for their kind guidance.

§ 1. Riemannian space with certain special curvature tensor.

We suppose an n-dimensional Riemannian space M ($n\geqq 3$) of class C^r ($r\geqq 3$) which has local coordinates x^i and admits the special concircular scalar field Φ defined by the equation (0. 1). First, substituting the relation obtained from (0. 1) into the Ricci identity

$$2\Phi_{i;[j;k]}=-R^a{}_{ijk}\Phi_a,$$

we have

$$\rho(\Phi_k g_{ij}-\Phi_j g_{ik})=-R^a{}_{ijk}\Phi_a, \tag{1. 1}$$

from which, by covariant differentiation with respect to x^l,

$$\rho(\Phi_{k;l} g_{ij}-\Phi_{j;l} g_{ik})=-R^a{}_{ijk;l}\Phi_a-R^a{}_{ijk}\Phi_{a;l},$$

and consequently, inserting the relation (0. 1), we obtain

$$\rho\Phi\left\{\rho(g_{kl}g_{ij}-g_{jl}g_{ik})+R_{lijk}\right\}=-R^a{}_{ijk;l}\Phi_a. \tag{1. 2}$$

Moreover, on differentiating (1. 2) covariantly with respect to x^h, we have

$$\rho\Phi_h\left\{\rho(g_{kl}g_{ij}-g_{jl}g_{ik})+R_{lijk}\right\}+\rho\Phi R_{lijk;h}$$
$$=-R^a{}_{ijk;l;h}\Phi_a-R^a{}_{ijk;l}\Phi_{a;h},$$

and inserting the values of $\Phi_{i;j}$ given by (0. 1) we obtain

$$\rho\Phi_h\left\{\rho(g_{kl}g_{ij}-g_{jl}g_{ik})+R_{lijk}\right\}$$
$$=-R^a{}_{ijk;l;h}\Phi_a-\rho\Phi(R_{lijk;h}+R_{hijk;l}), \tag{1. 3}$$

so that multiplying the expression (1. 3) by Φ^h and summing with respect to h we get

$$\rho(\Phi_h\Phi^h)\left\{\rho(g_{kl}g_{ij}-g_{jl}g_{ik})+R_{lijk}\right\}$$
$$=-R^a{}_{ijk;l;h}\Phi_a\Phi^h-\rho\Phi(R_{lijk;h}+R_{hijk;l})\Phi^h. \tag{1. 4}$$

Also, multiplying both sides of (1. 3) by Φ^l and summing with respect to l we have

$$\rho\Phi_h\left\{\rho(\Phi_k g_{ij}-\Phi_j g_{ik})+R_{lijk}\Phi^l\right\}$$
$$=-R^a{}_{ijk;l;h}\Phi_a\Phi^l-\rho\Phi(R_{lijk;h}+R_{hijk;l})\Phi^l.$$

From (1. 1), it is evident that the left hand side of these equations are equal to zero, and consequently we get

$$-R^a{}_{ijk;l;h}\Phi_a\Phi^l-\rho\Phi(R_{lijk;h}+R_{hijk;l})\Phi^l=0,$$

from which, by interchanging the indices h and l, we obtain

$$-R^a{}_{ijk;h;l}\Phi_a\Phi^h-\rho\Phi(R_{hijk;l}+R_{lijk;h})\Phi^h=0. \tag{1. 5}$$

And subtracting from (1. 4) the equation (1. 5), we find

$$\rho(\Phi_h\Phi^h)\left\{\rho(g_{kl}g_{ij}-g_{jl}g_{ik})+R_{lijk}\right\}$$
$$=(R^a{}_{ijk;h;l}-R^a{}_{ijk;l;h})\Phi_a\Phi^h. \tag{1. 6}$$

Suppose that our space has the curvature tensor satisfying $R^a{}_{ijk;[h;l]}=0$ ([2], p. 222). Then, ρ defined by (0. 1) being different from zero, the equation (1. 6) can be written as follows:

$$(\Phi_h\Phi^h)\left\{\rho(g_{kl}g_{ij}-g_{jl}g_{ik})+R_{lijk}\right\}=0. \tag{1. 7}$$

We assume, moreover, that there exists no open set U such that $\Phi=$constant at any point of it. And then it follows that there exists no open set V such that $\Phi_i=0$ at any point of it. Under these assumptions, from (1. 7), we obtain the following relation:

$$R_{lijk}=\rho(g_{lj}g_{ik}-g_{lk}g_{ij}),$$

that is, our space is of constant curvature. Hence we have the following

THEOREM. *Let M be a Riemannian space of dimension n which has the curvature tensor satisfying*

$$R_{hijk;[l;m]}=0 \tag{1.8}$$

and admits the special concircular scalar field Φ defined by (0.1). *Then M is of constant curvature.*

§ 2. Some corollaries.

Suppose that a Riemannian space M is symmetric. Then it is evident that the condition (1.8) is satisfied. Therefore we have

COROLLARY 1. *Let M be a symmetric Riemannian space which admits the special concircular scalar field Φ. Then M is of constant curvature.* ([1])

Next we consider an n-dimensional Einstein space $M(n>2)$ which has the scalar curvature $R\neq 0$ and admits a proper conformal Killing vector field ξ^i, that is, ξ^i satisfies an equations:

$$\underset{\xi}{\mathscr{L}}\, g_{ij} \equiv \xi_{i;j}+\xi_{j;i}=2\phi g_{ij} \qquad ([2], \text{p. } 32), \tag{2.1}$$

where $\underset{\xi}{\mathscr{L}}\, g_{ij}$ means the Lie derivative of the metric tensor g_{ij} with respect to ξ^i. Then the Lie derivative of the curvature tensor $R^h{}_{ijk}$ with respect to the conformal Killing vector field ξ^i is given by

$$\underset{\xi}{\mathscr{L}}\, R^h{}_{ijk}=\delta^h_j\phi_{i;k}-\delta^h_k\phi_{i;j}+g_{ik}\phi^h_{;j}-g_{ij}\phi^h_{;k}, \qquad ([2], \text{p. } 160) \tag{2.2}$$

where $\phi_i \overset{\text{def}}{=} \phi_{;i}$, $\phi^i \overset{\text{def}}{=} g^{ij}\phi_j$ and δ^h_j is the Kronecker delta. Since M is an Einstein space, we have

$$R_{ij}=\frac{R}{n}g_{ij} \qquad (R=\text{constant}), \tag{2.3}$$

where R_{ij} is the Ricci tensor and R the scalar curvature. On making use of (2.2) and (2.3), after some calculations we obtain the following result:

$$\phi_{i;j}=-\frac{R}{n(n-1)}\phi g_{ij} \tag{2.4}$$

and, remembering the round brackets of (2.3) and the assumption $R\neq 0$, we have $-\dfrac{R}{n(n-1)}$ = non-zero constant.

From (2.4) we can see that the Einstein space M admitting the proper conformal Killing vector field ξ^i must always admit the scalar field ϕ, which is the special concircular scalar field. Hence we have the following corollary:

COROLLARY 2. *Let M be an n-dimensional Einstein space* $(n>2)$ *which has the scalar curvature* $R\neq 0$, *the curvature tensor such that* $R_{hijk;[l;m]}=0$ *and admits a proper conformal Killing vector field* ξ^i. *Then M is of constant curvature.*

On the other hand, multiplying both sides of (1. 6) by g^{ij} and summing with respect to i and j, we get

$$\rho(\Phi_h\Phi^h)\left\{\rho(n-1)g_{kl}+R_{kl}\right\}=(R_{ak;h;l}-R_{ak;l;h})\Phi^a\Phi^h .$$

When we think of the vecter field Φ_h as being provided with the assumption with respect to Φ in the manner described in §1, we have the following

PROPOSITION. *Let M be a Riemannian space of dimension n which has the Ricci tensor such that*

$$R_{ak;[h;l]}=0 \tag{2. 5}$$

and admits the special concircular scalar field Φ. *Then M is an Einstein space.*

Now, suppose that a Riemannian space M is Ricci symmetric (definded by $R_{ij;k}=0$). Then it is evident that the condition (2. 5) is satisfied, and so that we obtain

COROLLARY. *Let M be a Ricci symmetric space of dimension n admitting a special concircular scalar field* Φ. *Then M is an Einstein space.*

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References

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(Received August 30, 1971)