On a certain property of a Riemannian space admitting a special concircular scalar field

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 0. Introduction.

The purpose of the present paper is to investigate the property of a Riemannian space which admits a scalar field $\Phi$ characterised by the property

\[(0.1) \quad \Phi_{k;l} = \rho \Phi g_{kl}, \quad \rho = \text{non-zero constant},\]

(such a scalar field $\Phi$ is called the special concircular scalar field in this paper) where $\Phi_{k} = \Phi_{,k}$ and $g_{kl}$ means the metric tensor of the space. In § 1, we consider a Riemannian space with certain special curvature tensor, and prove the property that the space is of constant curvature. Next, in § 2, we give some corollaries of it.

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§ 1. Riemannian space with certain special curvature tensor.

We suppose an $n$-dimensional Riemannian space $M \ (n \geq 3)$ of class $C^r \ (r \geq 3)$ which has local coordinates $x^i$ and admits the special concircular scalar field $\Phi$ defined by the equation (0.1). First, substituting the relation obtained from (0.1) into the Ricci identity

$$2\Phi_{[j;k]} = -R^a_{ij,k} \Phi_a,$$

we have

\[(1.1) \quad \rho(\Phi_{k} g_{ij} - \Phi_{j} g_{ik}) = -R^a_{ij,k} \Phi_a,\]

from which, by covariant differentiation with respect to $x^i$,

$$\rho(\Phi_{k;i} g_{ij} - \Phi_{j;i} g_{ik}) = -R^a_{ij,k;i} \Phi_a - R^a_{ij,k} \Phi_{a;i},$$

and consequently, inserting the relation (0.1), we obtain

\[(1.2) \quad \rho \Phi \left\{ \rho (g_{kl} g_{ij} - g_{jl} g_{ik}) + R_{lijk} \right\} = -R^a_{ij,k;l} \Phi_a.\]

Moreover, on differentiating (1.2) covariantly with respect to $x^k$, we have
On a certain property of a Riemannian space admitting a special concircular

\[
\rho \Phi_h \left\{ \rho \left( g_{kl} g_{ij} - g_{jl} g_{ik} \right) + R_{lijk} \right\} + \rho \Phi R_{lijk;h} = -R^a_{ijkl;h} \Phi_a - R^a_{ijkl} \Phi_{a;h},
\]

and inserting the values of \( \Phi_{ij} \) given by (0.1) we obtain

\[
(1.3)
\]

\[
\rho \Phi_h \left\{ \rho \left( g_{kl} g_{ij} - g_{jl} g_{ik} \right) + R_{lijk} \right\} = -R^a_{ijkl;h} \Phi_a - \rho \Phi \left( R_{lijk;h} + R_{lijk} \right),
\]

so that multiplying the expression (1.3) by \( \Phi^h \) and summing with respect to \( h \) we get

\[
(1.4)
\]

\[
\rho \Phi_h \left\{ \rho \left( g_{kl} g_{ij} - g_{jl} g_{ik} \right) + R_{lijk} \right\} = -R^a_{ijkl;h} \Phi_a \Phi^h - \rho \Phi \left( R_{lijk;h} + R_{lijk} \right) \Phi^h.
\]

Also, multiplying both sides of (1.3) by \( \Phi^l \) and summing with respect to \( l \) we have

\[
\rho \Phi_h \left\{ \rho \left( \Phi_k g_{ij} - \Phi_j g_{ik} \right) + R_{lijk} \right\} = -R^a_{ijkl;h} \Phi_a \Phi^l - \rho \Phi \left( R_{lijk;h} + R_{lijk} \right) \Phi^l.
\]

From (1.1), it is evident that the left hand side of these equations are equal to zero, and consequently we get

\[
-R^a_{ijkl;h} \Phi_a \Phi^l - \rho \Phi \left( R_{lijk;h} + R_{lijk} \right) \Phi^l = 0,
\]

from which, by interchanging the indices \( h \) and \( l \), we obtain

\[
(1.5)
\]

\[-R^a_{ijkl;h} \Phi_a \Phi^h - \rho \Phi \left( R_{lijk;h} + R_{lijk} \right) \Phi^h = 0.
\]

And subtracting from (1.4) the equation (1.5), we find

\[
(1.6)
\]

\[
\rho \Phi_h \left\{ \rho \left( g_{kl} g_{ij} - g_{jl} g_{ik} \right) + R_{lijk} \right\} = \left( R^a_{ijkl;h} - R^a_{ijkl;h} \right) \Phi_a \Phi^h.
\]

Suppose that our space has the curvature tensor satisfying \( R^a_{ijkl;h} = 0 \) ([2], p. 222). Then, \( \rho \) defined by (0.1) being different from zero, the equation (1.6) can be written as follows:

\[
(1.7)
\]

\[
\Phi_h \Phi^h \left\{ \rho \left( g_{kl} g_{ij} - g_{jl} g_{ik} \right) + R_{lijk} \right\} = 0.
\]

We assume, moreover, that there exists no open set \( U \) such that \( \Phi = \text{constant} \) at any point of it. And then it follows that there exists no open set \( V \) such that \( \Phi = 0 \) at any point of it. Under these assumptions, from (1.7), we obtain the following relation:

\[
R_{lijk} = \rho \left( g_{ij} g_{ek} - g_{ik} g_{ej} \right),
\]
that is, our space is of constant curvature. Hence we have the following

**Theorem.** Let $M$ be a Riemannian space of dimension $n$ which has the curvature tensor satisfying

\[ R_{ijkl;n=m} = 0 \]

and admits the special concircular scalar field $\Phi$ defined by (0.1). Then $M$ is of constant curvature.

§ 2. Some corollaries.

Suppose that a Riemannian space $M$ is symmetric. Then it is evident that the condition (1.8) is satisfied. Therefore we have

**Corollary 1.** Let $M$ be a symmetric Riemannian space which admits the special concircular scalar field $\Phi$. Then $M$ is of constant curvature. ([1])

Next we consider an $n$-dimensional Einstein space $M(n > 2)$ which has the scalar curvature $R \neq 0$ and admits a proper conformal Killing vector field $\xi^i$, that is, $\xi^i$ satisfies an equations:

\[ \mathcal{L}_\xi g_{ij} = \xi_{i;j} + \xi_{j;i} = 2\phi g_{ij} \quad ([2], \text{p. 32}), \]

where $\mathcal{L}_\xi g_{ij}$ means the Lie derivative of the metric tensor $g_{ij}$ with respect to $\xi^i$. Then the Lie derivative of the curvature tensor $R^h_{\,ijkl}$ with respect to the conformal Killing vector field $\xi^i$ is given by

\[ \mathcal{L}_\xi R^h_{\,ijkl} = \delta^h_{[i} \phi_{nj]k} - \delta^h_{[j} \phi_{ni]k} + g_{i\ell} \phi_{nk}^\ell - g_{nj} \phi^h_{nk}, \quad ([2], \text{p. 160}) \]

where $\phi^\ell_{\,i} \equiv \phi_i^\ell, \phi^i \equiv g^{ij} \phi_j$ and $\delta^h_{[i}$ is the Kronecker delta. Since $M$ is an Einstein space, we have

\[ R_{ij} = \frac{R}{n} g_{ij} \quad (R = \text{constant}), \]

where $R_{ij}$ is the Ricci tensor and $R$ the scalar curvature. On making use of (2.2) and (2.3), after some calculations we obtain the following result:

\[ \phi_{ij} = - \frac{R}{n(n-1)} \phi g_{ij} \]

and, remembering the round brackets of (2.3) and the assumption $R \neq 0$, we have $- \frac{R}{n(n-1)} = \text{non-zero constant.}$

From (2.4) we can see that the Einstein space $M$ admitting the proper conformal Killing vector field $\xi^i$ must always admit the scalar field $\phi$, which
is the special concircular scalar field. Hence we have the following corollary:

**Corollary 2.** Let $M$ be an $n$-dimensional Einstein space $(n > 2)$ which has the scalar curvature $R \neq 0$, the curvature tensor such that $R_{abcdef;[g;h]} = 0$ and admits a proper conformal Killing vector field $\xi^i$. Then $M$ is of constant curvature.

On the other hand, multiplying both sides of (1.6) by $g^{kl}$ and summing with respect to $i$ and $j$, we get

$$\rho(x^a x^b) \{ \rho (n-1) g_{kl} + R_{kl} \} = (R_{ak;hl} - R_{ak;hl}) \Phi^a \Phi^h.$$  

When we think of the vector field $\Phi^a$ as being provided with the assumption with respect to $\Phi$ in the manner described in §1, we have the following

**Proposition.** Let $M$ be a Riemannian space of dimension $n$ which has the Ricci tensor such that

(2.5)  
$$R_{ak;[hl]} = 0$$

and admits the special concircular scalar field $\Phi$. Then $M$ is an Einstein space.

Now, suppose that a Riemannian space $M$ is Ricci symmetric (defined by $R_{ijkl} = 0$). Then it is evident that the condition (2.5) is satisfied, and so that we obtain

**Corollary.** Let $M$ be a Ricci symmetric space of dimension $n$ admitting a special concircular scalar field $\Phi$. Then $M$ is an Einstein space.

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References


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