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<td>Author(s)</td>
<td>Tsuzuku, Tosiro</td>
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<tr>
<td>Citation</td>
<td>Journal of the Faculty of Science Hokkaido University, Ser. 1 Mathematics = 北海道大学理学部紀要, 22(3-4): 104-107</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1972</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/54693">http://hdl.handle.net/2115/54693</a></td>
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<td>Type</td>
<td>bulletin (article)</td>
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<td>File Information</td>
<td>JFSHIU_22_N3-4_104-107.pdf</td>
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On $\text{LF}_3(3)$

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Tosiro TSUZUKU

Let $\Omega$ be a set of $p=4q+1$ symbols where $p$ and $q$ are prime numbers. Let $G$ be an unsolvable transitive permutation group on $\Omega$. Appel and Parker ([1], theorem 2) showed that there is no such a permutation group $(G, \Omega)$ for $q\leq 79$ except for the group $(\text{LF}_3(3), P)$ with $q=3$, where $\text{LF}_3(3)$ denotes the linear fractional group over the field of three elements and $P$ denotes the projective plane over the field of three elements. In this note we prove the following theorem.

**Theorem.** Let $H$ be the stabilizer of a symbol, say 0, of $\Omega$. If $H$ is imprimitive on $\Omega-\{0\}$, then $(G, \Omega)$ is isomorphic to $(\text{LF}_3(3), P)$.

**Proof.** By the result of Appel-Parker, we can assume that $q>79$, and we only need to prove non existence of such a permutation group $(G, \Omega)$ for $q>79$.

Let $P$ be a Sylow $p$-subgroup of $G$ and let $N(P)$ be the normalizer of $P$ in $G$. Let $N$ be a minimal normal subgroup of $G$. Then $N$ is a simple group and transitive on $\Omega$, and $G$ is a product of $N$ and $N(P)$. Hence $G/N$ is cyclic and so $N$ is an unsolvable transitive group on $\Omega$. Hence hereafter we can assume that $G$ is simple.

Let $K$ be the stabilizer of two symbols 0 and 1 of $\Omega$. Then, by a theorem of Witt, $N(K)$ is doubly transitive on $F(K)$, where

$$F(K) = \{a \in \Omega | a^a = a \text{ for any } a \in K\},$$

and hence $4qp$ is divisible by $f(f-1)$ where $f=|F(K)|$. From our assumption we have that $f=2$. Since $(H, \Omega-\{0\})$ is imprimitive, $K$ is not a maximal subgroup of $H$. Let $L$ be a maximal subgroup of $H$ containing $K$. Since $[H:K]=4q$ and $f=2$, three cases arise: (i) $[H:L]=2$ and $[L:K]=2q$, (ii) $[H:L]=4$ and $[L:K]=q$, and (iii) $[H:L]=q$, and $[L:K]=4$.

Let $\Omega=\{\Omega_1, \Omega_2, \cdots, \Omega_t\}$, $t[L:K]=4q$, be a complete system of sets of imprimitivity of $(H, \Omega-\{0\})$ corresponding to $L$, namely, $\Omega-\{0\} = \bigcup \Omega_i$, $|\Omega_i| = [L:K]$, $\Omega_i \cap \Omega_j = \phi$ for $i \neq j$, every element of $H$ induces a permutation on the set $\tilde{\Omega}$ and $L$ is the stabilizer of one symbol of $\tilde{\Omega}$, say $\Omega_1$. Then we may assume that $\Omega_1 \ni 1$. 

Case (i) cannot occur. Assume that \([L:K]=2q\). Then \(L\) is a normal subgroup of \(H\) of index 2. Let \(1_{L}\) be the principal character of \(L\). Then \(\nu_{1_{L}}\), the character of \(H\) induced by \(1_{L}\), is the sum of the principal character \(1_{H}\) of \(H\) and a non trivial linear character \(\varepsilon\) of \(H\). Since \((G, \Omega)\) is doubly transitive, \(\eta_{1_{L}}=1_{\theta}+\chi\) where \(\chi\) is an irreducible character of degree \(p-1\). Since \(\eta_{1_{L}}=1_{\theta}+\eta\) by the reciprocity theorem of Frobenius, \(\eta\) is a non principal linear character of \(G\). This is a contradiction.

Dase (ii) cannot occur. Assume that \([L:H]=q\). If \(\nu_{1_{L}}\) contains a non principal linear character, then we have a contradiction in the similar way as the case (i). Hence we have that

\[ \nu_{1_{L}}=1_{H}+\phi \]

where \(\phi\) is an irreducible character of \(H\) of degree 3. Then by the reciprocity theorem of Frobenius we have that

\[ \eta_{1_{L}}=1_{\theta}+2\chi_{0}+\phi \]

where \(\chi_{0}\) is an irreducible character of \(G\) of degree \(p-1\) and \(\phi\) is a character of \(G\) of degree \(2p+1\). By a result of Tuan ([4], Theorem 3), we conclude that \(\phi\) is irreducible or decomposed into two irreducible characters. In the latter case, by a theorem of Brauer [2], the degrees of irreducible characters are \(p\) and \(p+1\). Let us consider \(G\) as a permutation group on \(G/L\), the set of left cosets by \(L\). Then it is known ([5], Proposition 29.2) that the number of domains of transitivity of \((L, G/L)\) equals to the norm of \(\eta_{1_{L}}\). Assume that \(\phi\) is irreducible. Then the norm of \(\eta_{1_{L}}\) is six, and hence the number of domains of transitivity of \((L, G/L)\) is six, and it is easy to see that the lengths of those domains are 1, 3, \(g\), 3\(g\), 3\(g\) and 9\(g\). Then, by a theorem of Frame ([3], Theorem B), we have that

\[ \frac{(4q)^{5} \cdot 3 \cdot q \cdot 3q \cdot 3q \cdot 3q \cdot 9q}{(p-1)^{4} \cdot (2p+1)} \]

is a rational integer. This is a contradiction. Hence \(\phi\) is reducible. Then the number of domains of transitivity of \((L, G/L)\) is seven and the lengths of those domains are 1, 3, \(g\), 3\(g\), 3\(g\), 3\(g\) and \(6g\). Then, by a theorem of Frame,

\[ \frac{(4p)^{6} \cdot 3 \cdot q \cdot 3q \cdot 3q \cdot 3q \cdot 3q \cdot 6q}{(p-1)^{4} \cdot p(p+1)} \]

is a rational integer. This is also a contradiction.

Case (iii) cannot occur. Assume that \([L:K]=4\). We can assume that
there is no subgroup $T$ of $H$ such that $H > T > K$ and $[T : K] \neq 4$. Let $M$ be the kernel of $H$ as a permutation group on $\tilde{\Omega}$, namely,

$$M = \{a \in H | \Omega^a_i = \Omega_i \text{ for } i = 1, \ldots, q\}.$$ 

If $M \leq K$, then $|M| = 1$ and then a Sylow $q$ subgroup $Q$ of $H$ is a normal subgroup of $H$ because $H$ is considered as a Frobenius group of degree $q$. Then $KQ$ is a subgroup of $H$ and $[KQ : K] = q$. This is a contradiction.

If $M$ is not transitive on $\Omega_i$, then we have that $KM$ is a subgroup of $H$ and $[KM : K] = 2$. This is also a contradiction. Hence $M$ is transitive on $\Omega_i$, also on $\Omega_i$ for any index $i$. Now we assume that $H$ is unsolvable. Then $(H/M, \tilde{\Omega})$ is an unsolvable permutation group of degree $q$ since $M$ is solvable, and hence $L$ is transitive on $\Omega_2 \cup \Omega_3 \cup \cdots \cup \Omega_q$ because $L$ is transitive on $\tilde{\Omega} - \{\Omega_1\}$ and $M$ is transitive on $M$, for any $i$. Let $N$ be the kernel of $L$ as a permutation group on $\Omega_i$. Then $N$ is half-transitive on $\Omega_2 \cup \cdots \cup \Omega_q$ and so we have that

$$F(N)\left(= \{\alpha \in \Omega | \alpha^a = \alpha \text{ for any } a \in \Omega\}\right) = \{0\} \cup \Omega_1 \text{ or } \Omega.$$

If $F(N) = \Omega$, then we have that $q - 1 \leq 6$ which is in contradiction with our assumption. Hence $F(N) = \{0\} \cup \Omega_1$. Then, by a theorem of Witt, $[G : N] \equiv 0 \pmod{5}$. This is a contradiction. Hence $H$ is solvable. Since $|FK| = 2$, $\Omega_1 - \{1\}$ is an orbit of length 3 of $(K, Q)$. If $\Omega_1 - \{1\}$ is the unique orbit of length 3 of $(K, Q)$, then $\{0\} \cup \Omega_1$ is uniquely determined by a (unordered) set $\{0, 1\}$, and then it is easily seen that $\{0\} \cup \Omega_1$ is also uniquely determined by any two symbols of $\{0\} \cup \Omega_1$. Hence we conclude that $p(p - 1)/5 \cdot 4$ is a rational integer which is in contradiction with $p(p - 1) = 4qp$. Therefore the permutation group $(K, \Omega)$ has at least two orbits of length 3. For a symbol $\alpha$ of $\Omega - \{0\}$, we set

$$\bar{\alpha} = \{\Omega_\alpha | \Omega_\alpha \cap F(M_\alpha) \neq \phi\}$$

where $M_\alpha = \{a \in M | a^\alpha = \alpha\}$ and $F(M_\alpha) = \{\beta \in Q | \beta^\alpha = \beta\}$ for any $x \in M_\alpha$. Then we have that $\bar{\alpha} = \bar{\beta}$ for $\alpha, \beta \in \Omega_\alpha$ and, if $\bar{\alpha} \neq \bar{\beta}$, then $\bar{\alpha} \cap \bar{\beta} = \phi$ and $|\bar{\alpha}| = |\bar{\beta}|$. Hence if $|\bar{\alpha}| \geq 1$ then $\bar{\alpha} = \tilde{\Omega}$. Then $M$ is faithful on $\Omega_1$ and we conclude that $|M| \leq 24$. Since $(H/M, \tilde{\Omega})$ is a Frobenius group, $H > MQ$ where $Q$ is a Sylow $q$-subgroup of $H$. Since $q > 24$, $MQ = M \times Q$, namely $H \triangleright Q$. Hence we conclude that $KQ$ is a subgroup of $H$ and $[KQ : K] = q$. This is in contradiction with our assumption. Hence $|\bar{\alpha}| = 1$ for any $\alpha \in \Omega - \{0\}$. Then the set of orbits of $(M_\alpha, \Omega)$ consists of the following orbits; two orbits of length 1, one orbit of length 3 and other orbits of even length. Therefore this
conclude that $(K, \Omega)$ has only one orbit of length 3. This is a contradiction. Thus we have completed the proof of the Theorem.

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Reference


(Received August 17, 1971)