On LF_3 (3)

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Tosiro TSUZUKU

Let Ω be a set of $p=4q+1$ symbols where p and q are prime numbers. Let G be an unsolvable transitive permutation group on Ω. Appel and Parker ([1], theorem 2) showed that there is no such a permutation group (G, Ω) for $q \leqq 79$ except for the group $(LF_3(3), \boldsymbol{P})$ with $q=3$, where $LF_3(3)$ denotes the linear fractional group over the field of three elements and $\boldsymbol{P}$ denotes the projective plane over the field of three elements. In this note we prove the following theorem.

THEOREM. *Let H be the stabilizer of a symbol, say* 0, *of Ω. If H is imprimitive on $\Omega-\{0\}$, then (G, Ω) is isomorphic to $(LF_3(3), \boldsymbol{P})$.*

PROOF. By the result of Appel-Parker, we can assume that $q>79$, and we only need to prove non existence of such a permutation group (G, Ω) for $q>79$.

Let P be a Sylow p-subgroup of G and let $N(P)$ be the normalizer of P in G. Let N be a minimal normal subgroup of G. Then N is a simple group and transitive on Ω, and G is a product of N and $N(P)$. Hence G/N is cyclic and so N is an unsolvable transitive troup sn Ω. Hence hereafter we can assume that G is simple.

Let K be the stabilizer of two symbols 0 and 1 of Ω. Then, by a theorem of Witt, $N(K)$ is doubly transitive on $F(K)$, where

$$F(K)=\{\alpha \in \Omega \mid \alpha^a=\alpha \text{ for any } a \in K\},$$

and hence $4qp$ is divisible by $f(f-1)$ where $f=|F(K)|$. From our assumption we have that $f=2$. Since $(H, \Omega-\{0\})$ is imprimitive, K is not a maximal subgroup of H. Let L be a maximal subgroup of H containing K. Since $[H:K]=4q$ and $f=2$ three cases arise: (*i*) $[H:L]=2$ and $[L:K]=2q$, (*ii*) $[H:L]=4$ and $[L:K]=q$ and (*iii*) $[H:L]=q$, and $[L:K]=4$. Let $\tilde{\Omega}=\{\Omega_1, \Omega_2, \cdots, \Omega_t\}$, $t[L:K]=4q$, be a complete system of sets of imprimitivity of $(H, \Omega-\{0\})$ corresponding to L, namely, $\Omega-\{0\}=\cup \Omega_i$, $|\Omega_i|=[L:K]$, $\Omega_i \cap \Omega_j=\phi$ for $i \neq j$, every element of H induces a permutation on the set $\tilde{\Omega}$ and L is the stabilizer of one symbol of $\tilde{\Omega}$, say Ω_1. Then we may assume that $\Omega_1 \ni 1$.

Case (*i*) *cannot occur.* Assume that $[L:K]=2q$. Then L is a normal subgroup of H of index 2. Let 1_L be the principal character of L. Then ${}^H 1_L$, the character of H induced by 1_L, is the sum of the principal character 1_H of H and a non trivial linear character ε of H. Since (G, Ω) is doubly transitive, ${}^G 1_H = 1_G + \chi$ where χ is an irreclucible character of degree $p-1$. Since ${}^H 1_K = \chi_{1H}$ (the restriction of χ to H), we have that ${}^G\varepsilon = \chi + \eta$ by the reciprocity theorem of Frobenius, where η is a non principal linear character of G. This is a contradiction.

Dase (*ii*) *cannot occur.* Assume that $[L:H]=q$. If ${}^H 1_L$ contoins a non principal linear character, then we have a contradiction in the similar way as the case (*i*). Hence we have that

$${}^H 1_L = 1_H + \psi$$

where ψ is a irreducible character of H of degree 3. Then by the reciprocity theorem of Frobenius we have that

$${}^G 1_L = 1_G + 2\chi_0 + \phi$$

where χ_0 is a irreducible character of G of degree $p-1$ and ϕ is a character of G of degree $2p+1$. By a result of Tuan ([4], Theorem 3), we conclude that ϕ is irreducible or decomposed into two irreducible characters. In the latter cese, by a theorem of Brauer [2], the degrees of irreducible characters are p and $p+1$. Let us consider G as a permutation group on G/L, the set of left cosets by L. Then it is known ([5], Proposition 29. 2) that the number of domains of transitivity of $(L, G/L)$ equals to the norm of ${}^G 1_L$. Assume that ϕ is irreducible. Then the norm of ${}^G 1_L$ is six, and hence the number of domains of transitivity of $(L, G/L)$ is six, and it is easy to see that the lengths of those domains are $1, 3, q, 3q, 3q$ and $9q$. Then, by a theorem of Frame ([3], Theorem B), we have that

$$\frac{(4q)^4 \cdot 3 \cdot q \cdot 3q \cdot 3q \cdot 9q}{(p-1)^4 (2p+1)}$$

is a ratronal integer. This is a contradiction. Hence ϕ is reducible. Then the number of domains of transitivity of $(L, G/L)$ is seven and the lengths of those domains are $1, 3, q, 3q, 3q, 3q$ and $6q$. Then, by a theorem of Frame,

$$\frac{(4p)^5 \cdot 3 \cdot q \cdot 3q \cdot 3q \cdot 3q \cdot 6q}{(p-1)^4 \cdot p(p+1)}$$

is a rational integer. This is also a contradiction.

Case (*iii*) *cannot occur.* Assume that $[L:K]=4$. We can assume that there is no subgroup T of H such that $H\gneqq T\gneqq K$ and $[T:K]\neq 4$. Let M be the kernel of H as a permutation group on $\tilde{\Omega}$, namely,

$$M=\{a\ni H|\Omega_i^a=\Omega_i \text{ for } i=1,\cdots,q\}.$$

If $M\leqq K$, then $|M|=1$ and then a Sylow q subgroup Q of H is a normal subgroup of H because H is considered as a Frobenius group of degree q. Then KQ is a subgroup of H and $[KQ:K]=q$. This is a contradiction. If M is not transitive on Ω_1, then we have that KM is a subgroup of H and $[KM:K]=2$. This is also a contradiction. Hence M is transitive on Ω_1, also on Ω_i for any index i. Now we assume that H is unsolvable. Then $(H/M, \tilde{\Omega})$ is a unsolvable permutation group of degree q since M is solvable, and hence L is transitive on $\Omega_2\cup\Omega_3\cup\cdots\cup\Omega_q$ because L is transitive on $\tilde{\Omega}-\{\Omega_1\}$ and M is transitive on M_i for any i. Let N be the kernel of L as a permutation group on Ω_1. Then N is half-transitive on $\Omega_2\cup\cdots\cup\Omega_q$ and so we have that

$$\begin{aligned}F(N)&\Big(=\{\alpha\in\Omega|\alpha^a=\alpha \text{ for any } a\in N\}\Big)\\ &=\{0\}\cup\Omega_1 \text{ or } \Omega.\end{aligned}$$

If $F(N)=\Omega$, then we have that $q-1\leqq 6$ which is in contradiction with our assumption. Hence $F(N)=\{0\}\cup\Omega_1$. Then, by a theorem of Witt, $[G:N]\equiv 0 \pmod 5$. This is a contradiction. Hence H is solvable. Since $|FK|=2$, $\Omega_1-\{1\}$ is an orbit of length 3 of (K, Q). If $\Omega_1-\{1\}$ is the unique orbit of length 3 of (K, Q), then $\{0\}\cup\Omega_1$ is uniquely determined by a (unordered) set $\{0, 1\}$, and then it is easily seen that $\{0\}\cup\Omega_1$ is also uniquely determined by any two symbols of $\{0\}\cup\Omega_1$. Hence we conclude that $p(p-1)/5\cdot 4$ is a rational integer which is in contradiction with $p(p-1)=4qp$. Therefore the permutation group (K, Ω) has at least two orbits of length 3. For a symbol α of $\Omega-\{0\}$, we set

$$\tilde{\alpha}=\{\Omega_i|\Omega_i\cap F(M_\alpha)\neq\phi\}$$

where $M_\alpha=\{a\in M|\alpha^a=\alpha\}$ and $F(M_\alpha)=\{\beta\in\Omega|\beta^x=\beta \text{ for any } x\in M_\alpha\}$. Then we have that $\tilde{\alpha}=\tilde{\beta}$ for α, $\beta\in\Omega_i$ and, if $\tilde{\alpha}\neq\tilde{\beta}$, then $\tilde{\alpha}\cap\tilde{\beta}=\phi$ and $|\tilde{\alpha}|=|\tilde{\beta}|$. Hence if $|\tilde{\alpha}|\gneqq 1$ then $\tilde{\alpha}=\tilde{\Omega}$. Then M is faithful on Ω_1 and we conclude that $|M|\leqq 24$. Since $(H/M, \tilde{\Omega})$ is a Frobenius group, $H\triangleright MQ$ where Q is a Sylow q-subgroup of H. Since $q>24$, $MQ=M\times Q$, namely $H\triangleright Q$. Hence we conclude that KQ is a subgroup of H and $[KQ:K]=q$. This is in contradiction with our assumption. Hence $|\hat{\alpha}|=1$ for any $\alpha\in\Omega-\{0\}$. Then the set of orbits of (M_α, Ω) consists of the following orbits; two orbits of length 1, one orbit of length 3 and other orbits of even length. Therefore this

conclude that (K, Ω) has only one orbit of length 3. This is a contradiction.
Thus we have completed the proof of the Theorem.

Department of Mathematics
Hokkaido University

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