On a piecewise linear link isotopy group

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction

Throughout this paper we shall only be concerned with the piecewise linear category of polyhedra and piecewise linear maps (for the PL-category, see [12]). The purpose of the paper makes a study of a group of order preserving ambient isotopy classes of $(n, k, \mu)$-links $L=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ in the metastable range.

In § 2 we shall define a group $SI(n, k, \mu)$ of $(n, k, \mu)$-link isotopy classes in the strong sense under $n-k\geq 3$ (Cor. for Th. 1) and we will introduce a concept of cobordism between $(n, k, \mu)$-link isotopy classes. Their cobordism classes form an abelian group $\mathcal{L}(n, k, \mu)$ provided $n-k\geq 3$ (Th. 1). In § 3 we will introduce a homotopy on the group $SI(n, k, \mu)$. Then their homotopy classes form an abelian group $\mathcal{A}^\prime(n, k, \mu)$ (Th. 2). In § 4 we introduce a group $LM(n, k, \mu)$ as follows, called the $(n, k, \mu)$-link matrix group. Let $LM(n, k, \mu)$ be a set of $(\mu \times \mu)$-matrixes of the form

$$\begin{bmatrix}
e & \lambda_2^1 & \cdots & \lambda_\mu^1 \\
\lambda_1^2 & e & \cdots & \cdots \\
\lambda_1^\mu & \lambda_2^\mu & \cdots & \lambda_{\mu-1}^\mu & e \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda_1^n & \lambda_2^n & \cdots & \lambda_{\mu-1}^n & e
\end{bmatrix}$$

satisfying that

1) $\lambda_j^i$ is an element of $\pi_k(S^{n-k-1})$
2) $e$ is the identity element of $\pi_k(S^{n-k-1})$ and
3) $\lambda_j^i=(-1)^{n-k} \lambda_i^j$

Then $LM(n, k, \mu)$ becomes an abelian group under the usual sum operation of matrices. We will study relations between $SI(n, k, \mu)$, $\mathcal{L}(n, k, \mu)$, $\mathcal{A}(n, k, \mu)$ and $LM(n, k, \mu)$ under some conditions.

$SI(n, k, \mu)\cong \mathcal{L}(n, k, \mu)$ provided $n-k\geq 3$ (Th. 4): $\mathcal{L}(n, k, \mu)\cong \mathcal{A}(n, k, \mu)$ provided $2n\geq 3k+4$ (Th. 5 and 6). Hence we shall obtain the following main result of the paper.

Theorem 7. $SI(n, k, \mu)$ is isomorphic to $LM(n, k, \mu)$ provided $2n\geq 3k+4$. 
§2. The strong link isotopy group and the link cobordism group.

In the paper a manifold, say $M$, is an orientable combinatorial manifold. Then $\partial M$ and $\text{Int}M$ stand for its boundary and its interior.

Let $S^n, D^n$ be standard $n$-dimensional combinatorial sphere and disk and $\bigcup_{j=1}^\mu S_j^k, \bigcup_{j=1}^\mu D_j^k$ be disjoint unions of $\mu$-spheres and disks of dimension $k$. Let $I= [-1, 1], I_0= [0, 1]$ and $\Delta_k$ be a $k$-simplex. We shall always assign an orientation to a manifold and we shall consider a manifold pair $V=(M, N)$ (where the inclusion map $N \subset M$ is proper in the sense of Zeeman). Homeomorphisms will be orientation preserving unless otherwise stated. Set theoretic notations such as $\subset, =, \sigma, \text{Int}$ when applied to oriented objects respect orientations.

**DEFINITION 1.** For the sake of convenience, a locally flat $(n, k)$-sphere pair $L= (S^n, \bigcup_{j=1}^\mu S_j^k)$ is called an $(n, k, \mu)$-link and a locally flat $(n, k)$-disk pair $B= (D^n, \bigcup_{j=1}^\mu D_j^k)$ is called an $(n, k, \mu)$-braid for $k \geq 0$. As we consider links and braids, it is assumed that $n-k \geq 2$ throughout this paper without mentions. If $B= (D^{n+1}, \bigcup_{j=1}^\mu D_j^{k+1})$ is an $(n+1, k+1, \mu)$-braid, then it is obvious that $\partial B = \partial (D^{n+1}, \bigcup_{j=1}^\mu D_j^{k+1}) = (\partial D^{n+1}, \bigcup_{j=1}^\mu \partial D_j^{k+1})$ is an $(n, k, \mu)$-link. When $\partial B = L$, we always assume $\partial D_j^k = S_j^k, j=1,2,\cdots, \mu$.

**DEFINITION 2.** Let $L_i=(S^n, \bigcup_{j=1} S_{ij}^k)$ be an $(n, k, \mu)$-link, $i=1, 2$. We say $L_1$ isotopic to $L_2$ if there is a level preserving homeomorphism, that is, an isotopy $H$ of $S^n \times I_0$ onto $S^n \times I_0$ satisfying

1) \[ H(x, 0) = (x, 0), x \in S^n \] and

2) \[ H(\bigcup_{j=1}^\mu S_{ij}^k, 1) = (\bigcup_{j=1}^\mu S_{ij}^k, 1), \] and written $L_1 \approx L_2$.

The set of $(n, k, \mu)$-links is classified by this equivalence relation into classes which are called $(n, k, \mu)$-link isotopy classes. We denote the set of these classes $\text{Iso}(n, k, \mu)$.

We can define the same thing for braids and denote the set of $(n, k, \mu)$-braid isotopy classes $\text{Biso}(n, k, \mu)$.

**DEFINITION 3.** Let $L_1 \approx L_2$. We say $L_1$ strong isotopic to $L_2$, written $L_1 \approx L_2$, if the isotopy $H$ of Definition 2 satisfies the following condition 2)' instead of the above 2);

2)' \[ H(S_{ij}^k, 1) = (S_{ij}^k, 1), j=1, 2,\cdots, \mu. \]
The set of $(n, k, \mu)$-links is classified by this equivalence relation into classes which are called $(n, k, \mu)$-link isotopy classes in the strong sense. We denote the set of these classes $SI(n, k, \mu)$. We can define the same thing for braids and denote the set of $(n, k, \mu)$-braid isotopy classes in the strong sense $BSI(n, k, \mu)$.

**Definition 4.** Let $L_i = (S^n, \bigcup_{j=1}^{\mu} S_{ij}^k)$ an $(n, k, \mu)$-link, $i = 1, 2, \ldots$.

We say that $L_1$ is equivalent to $L_2$ if there is an orientation preserving homeomorphism $G$ of $S^n$ onto itself satisfying the conditions:

1) $G(\bigcup_{j=1}^{\mu} S_{1j}^k) = \bigcup_{j=1}^{\mu} S_{2j}^k$

2) $G|\bigcup_{j=1}^{\mu} S_{1j}^k$ is orientation preserving.

Throughout this paper we will always assume that $G|\bigcup_{j=1}^{\mu} S_{1j}^k$ is orientation preserving (or reversing) provided that $G$ is orientation preserving (or reversing).

We can define the same thing for braids. The sets of $(n, k, \mu)$-links and braids are classified by this equivalence relation into classes which are called $(n, k, \mu)$-link types and braid types respectively.

**Remark 1.** Two links $L_1$ and $L_2$ belong to the same link type if and only if they belong to the same isotopy class by [2].

**Definition 5.** If two links (or braids) say $L_1$ and $L_2$, belong to the same link type and if the homeomorphism $G$ of Definition 4 satisfies the following condition $1)'$ instead of 1);

$1)' \quad G(S_{1j}^k) = S_{2j}^k, j = 1, 2, \ldots, \mu,$

we say $L_1$ to be equivalent to $L_2$ in the strong sense. The sets of $(n, k, \mu)$-links and braids are classified by this equivalence relation into classes which are called strong $(n, k, \mu)$-link types and strong braid types respectively.

**Remark 2.** Two links belong to the same strong link type if and only if they belong to the same strong isotopy class by [2].

**Definition 6.** The standard $(n, k, \mu)$-pair is the pair $\square = (D^n, \bigcup_{j=1}^{\mu} D_{1j}^k)$ such that there is a commutative diagram

\[ \begin{array}{ccc} \bigcup_{j=1}^{\mu} D_{1j}^k & \xrightarrow{\beta} & D^n \times I \\ \downarrow & & \downarrow \times 0 \\ \bigcup_{j=1}^{\mu} D_{1j}^k & \xrightarrow{s} & (D^n \times I) \times I^{n-k-1} \end{array} \]
where $\beta$ is the proper embedding satisfying $\beta(D_j^k) = \Delta^k \times \left\{-1 + \frac{2j}{\mu + 1}\right\}$, $j=1, 2, \cdots, \mu$ and $s$ is a canonical homeomorphism.

An $(n, k, \mu)$-link $L$ is trivial if $L$ is equivalent (or equivalently equivalent in the strong sense) to the boundary of the standard $(n+1, k+1, \mu)$-pair and an $(n, k, \mu)$-braid $B$ is trivial if $B$ is equivalent (or equivalent in the strong sense) to the standard $(n, k, \mu)$-pair. We shall denote both the trivial $(n, k, \mu)$-link type and braid type by $0$. Let $L=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ be an $(n, k, \mu)$-link and $B=(D^n, U \bigcup_{j=1}^{\mu} D_j^k)$ be an $(n, k, \mu)$-braid. If $D_j^k \subset S_j^k$, $j=1, 2, \cdots, \mu$, $D^n \subset S^n$ and $D^n \cap S_j^k = D_j^k$, $j=1, 2, \cdots, \mu$, then we call $B$ is contained in $L$, denoted by $B \supset L$.

Similarly if $\tilde{B}=(\tilde{D}^n, U \bigcup_{j=1}^{\mu} \tilde{D}_j^k)$ is another $(n, k, \mu)$-braid, then we say $B$ is contained in $\tilde{B}$ ($B \subset \tilde{B}$) if $D_j^k \subset \text{Int} \tilde{D}_j^k$, $j=1, 2, \cdots, \mu$, $D^n \subset \text{Int} \tilde{D}^n$ and $D^n \cap \tilde{D}_j^k = D_j^k$, $j=1, 2, \cdots, \mu$. Let $L_i=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ be $(n, k, \mu)$-links, $i=1, 2$ and $B_j=(D^n, U \bigcup_{j=1}^{\mu} D_j^k)$ be an $(n, k, \mu)$-braid. We will denote $B_j=L_i \cap L_j$ if $D^n \subset S^n$, $S_j^k \cap S_j^k = D_j^k$, $j=1, 2, \cdots, \mu$, $S_i^k \cap S_j^k = \phi$ ($j \neq i$), $D^n \cap S_j^k = D^n \cap S_j^k = D_j^k$ and $\bigcup_{j=1}^{\mu} S_j^k$ induce orientations on $\bigcup_{j=1}^{\mu} D_j^k$ which are opposite to one induced from $\bigcup_{j=1}^{\mu} S_j^k$.

Remark 3. If $B=(D^n, \bigcup_{j=1}^{\mu} D_j^k)$ is a trivial $(n, k, \mu)$-braid, then there exist an orientation preserving homeomorphism $H$ of $D^n$ and a commutative diagram same as Definition 6:

\[
\begin{array}{ccc}
\bigcup_{j=1}^{\mu} D_j^k & \xrightarrow{H} & \bigcup_{j=1}^{\mu} \tilde{D}_j^k \\
\downarrow \subset & & \downarrow \subset \\
D^n & \xrightarrow{H} & \tilde{D}^n
\end{array}
\]

\[
\begin{array}{ccc}
\beta & \xrightarrow{\Delta^k \times I} & 0 \\
\downarrow \subset & & \downarrow \subset \\
\Delta^k \times I \times I^n-k-1 & \xrightarrow{H^{-1} \times I} & X \times 0
\end{array}
\]

where $(D^n, \bigcup_{j=1}^{\mu} \tilde{D}_j^k)$ is a standard pair. Joining $\beta(\bigcup_{j=1}^{\mu} \tilde{D}_j^k)$ each other by simple arc $\Delta^k \times \left[-1 + \frac{2j}{\mu + 1}, 1 - \frac{2j}{\mu + 1}\right]$ in $\Delta^k \times I$ where $\Delta^k$ is the barycenter of $\Delta^k$ and let the resultant complex be $X$. Then $\Delta^k \times I \setminus X$ and hence $(\Delta^k \times I) \times I^{n-k-1} \setminus X \times 0$. Hence $D^n$ is a regular neighborhood of $H^{-1} \times I$.

Lemma 0. If $L=(S^n, \bigcup_{j=1}^{\mu} S_j^k)$ and $B=(D^n, \bigcup_{j=1}^{\mu} D_j^k)$ are both trivial $(n, k, \mu)$-link and braid respectively such that $B \subset L$, then the complementary braid
On a piecewise linear link isotopy group

$L-B=(S^n-\text{Int }D^n, \bigcup_{j=1}^{\mu} (S_j^k-\text{Int }D_j^k))$ is also a trivial $(n, k, \mu)$-braid.

**Proof.** Let \(\square=(\overline{D}^{n+1}, \bigcup_{j=1}^{\mu} \tilde{D}_j^{k+1})\) be a standard \((n+1, k+1, \mu)\)-pair such that

\[
\begin{array}{ccc}
\bigcup_{j=1}^{\mu} \tilde{D}_j^{k+1} & \xrightarrow{\beta} & \Delta^{k+1} \times I \\
\subset & & \downarrow \times 0 \\
\overline{D}^{n+1} & \xrightarrow{\tau} & (\Delta^{k+1} \times I) \times I^{n-k-1}
\end{array}
\]

is commutative. Since \(L\) is a trivial link, there exists an orientation preserving homeomorphism \(H\) of \(S^n\) onto \(\partial\overline{D}^{n+1}\) such that \(H(S_j^k)= \partial \overline{D}_j^{k+1}\), \(j=1, 2, \ldots, \mu\).

Furthermore we may suppose that \(\beta H|\bigcup_{j=1}^{\mu} D_j^k\) is properly embedded in \(\Delta^k \times I\) where \(\Delta^k\) is a face of \(\partial \Delta^{k+1}\). We construct a complex \(X\) in \(\Delta^k \times I\) same as Remark 3. Since \(B\) is a trivial braid, using Remark 3 \(D^n\) and \(H^{-1} s((\Delta^k \times I) \times I^{n-k-1})\) are regular neighborhoods of \(H^{-1} s(X \times 0)\) mod \(\bigcup_{j=1}^{\mu} (S_j^k-\text{Int }D_j^k)\) in \(S^n\). Then there is an ambient isotopy of \(S^n\) moving \(D^n\) onto \(H^{-1} s((\Delta^k \times I) \times I^{n-k-1})\) keeping \(H^{-1} s(X \times 0)\) fixed [6, Th. 1]. The end of isotopy throws \((S^n-\text{Int }D^n, \bigcup_{j=1}^{\mu} (S_j^k-\text{Int }D_j^k))\) onto \(H^{-1} s(\beta(\bigcup_{j=1}^{\mu} \partial \tilde{D}_j^{k+1}) \times I^{n-k-1}, \beta(\bigcup_{j=1}^{\mu} \partial \tilde{D}_j^{k+1}) \times 0)\), which is trivial, where \(\overline{\partial D}_j^{k+1}=\partial \overline{D}_j^{k+1}\).

Hence \(L-B\) is trivial.

**Lemma 1.** Let \(B_i=(D_i^n, \bigcup_{j=1}^{\mu} D_{ij}^k), i=1, 2\) be trivial \((n, k, \mu)\)-braids. If

\(B_3= B_1 \cap B_2 = \partial B_1 \cap \partial B_2=(D_3^{n-1}, \bigcup_{j=1}^{\mu} \hat{D}_j^{k-1})\) is a trivial \((n-1, k-1, \mu)\)-braid such that

\(D_i^n \cap D_3^{n-1} = \partial D_i^n \cap \partial D_3^{n-1} = D_3^{k-1}, D_{i,j}^k \cap D_{3,j}^{k-1} = \partial D_{i,j}^k \cap \partial D_{3,j}^{k-1} = D_{3,j}^{k-1}, j=1, 2, \ldots, \mu,\)

then \(B_1 \cup B_2\) is also a trivial \((n, k, \mu)\)-braid.

**Proof.** Since \(B_i, i=1, 2, 3\) are trivial braids, there are homeomorphisms \(H_i, i=1, 2, 3\) of \(D_i\) and commutative diagrams

\[
\begin{array}{ccc}
\bigcup_{j=1}^{\mu} D_{ij}^k & \xrightarrow{H_i} & \bigcup_{j=1}^{\mu} \tilde{D}_j^k \\
\subset & \xrightarrow{H_i} & \subset \\
D_i^k & \xrightarrow{H_i} & \tilde{D}_i^k
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\bigcup_{j=1}^{\mu} D_{3j}^{k-1} & \xrightarrow{H_3} & \bigcup_{j=1}^{\mu} \hat{D}_j^{k-1} \\
\subset & \xrightarrow{H_3} & \subset \\
D_3^{k-1} & \xrightarrow{H_3} & \hat{D}_3^{k-1}
\end{array}
\]

where \((\tilde{D}_i^k, \bigcup_{j=1}^{\mu} \tilde{D}_j^k)\) \(i=1, 2\) and \((\hat{D}_3^{k-1}, \bigcup_{j=1}^{\mu} \hat{D}_j^{k-1})\) are standard pairs.
We may suppose that $\tilde{D}_{3}^{n-1} = \partial\tilde{D}_{1}^{n} \cap \partial\tilde{D}_{2}^{n}$ and $D_{3j}^{k-1} = \partial\tilde{D}_{1j}^{k} \cap \partial\tilde{D}_{2j}^{k}, \ j = 1, 2, \cdots, \mu$ and $H_{3} = H_{1} \cap H_{2}$. Hence we obtain the following commutative diagram,

\[
\begin{array}{c}
\bigcup_{J=1}^{\mu}(D_{1j}^{k} \cup_{\subset} D_{2j}^{k}) \rightarrow \bigcup_{j=1, D_{3j}^{k-1}}^{\mu}(\tilde{D}_{1j}^{k} \cup_{\subset \downarrow \downarrow} \tilde{D}_{2j}^{k}) \rightarrow H_{1} \cup H_{2} \rightarrow \bigcup_{j=1}^{\mu}(\tilde{D}_{1j}^{k} \cup_{\subset} \tilde{D}_{2j}^{k}) \\
(D_{1}^{k} \times I) \cup (D_{2}^{k} \times I) \rightarrow (\Delta_{1}^{k} \times I) \cup (\Delta_{2}^{k} \times I) \rightarrow (\Delta_{1}^{k} \times I) \times I^{n-k-1} \cup (\Delta_{2}^{k} \times I) \times I^{n-k-1}.
\end{array}
\]

If $A_{i}^{k} \cap A_{j}^{k}$ is a common $(k-1)$-face of the boundaries, there is a homeomorphism $A_{i}^{k} \cup A_{j}^{k} \rightarrow A^{k}$, [11, Lemma 1]. So there is a commutative diagram

\[
\begin{array}{c}
(A_{i}^{k} \times I) \cup (A_{j}^{k} \times I) \rightarrow \bigcup_{\lambda=1}^{\mu}(A_{i}^{k} \times I) \cup (A_{j}^{k} \times I) \\
\rightarrow (A_{i}^{k} \times I) \times I^{n-k-1} \cup (A_{j}^{k} \times I) \times I^{n-k-1}.
\end{array}
\]

where $a : A_{i}^{k} \cup A_{j}^{k} \rightarrow A^{k}$ is a natural homeomorphism and where 1 is the identity homeomorphism. From (1) and (2), $B_{1} \cup B_{2}$ is a trivial $(n, k, \mu)$-braid, complete the proof.

**Lemma 2.** Let $B_{i} = (D_{i}^{n}, \bigcup_{j=1}^{\mu}D_{ij}^{k})$ be $(n, k, \mu)$-braids such that $B_{1} \cap B_{2} = \partial B_{1} \cap \partial B_{2} = (D_{3}^{n-1}, \bigcup_{j=1}^{\mu}D_{3j}^{k-1})$ is a trivial $(n-1, k-1, \mu)$-braid. If $B_{1}$ is trivial, then $B_{1} \cup B_{2}$ is equivalent to $B_{2}$.

**Proof.** Using [11] we may suppose that the homeomorphism $a$ of Lemma 1 is satisfying the condition

1) $a : A_{i}^{k} \cup A_{j}^{k} \rightarrow A_{i}^{k}$

2) $a|\partial A_{i}^{k} - \text{Int}(A_{i}^{k} \cap A_{j}^{k}) = \text{identity}$. Then by Lemma 1 we obtain the commutative diagram,
On a piecewise linear link isotopy group

Hence we may define $H = H^{-1}s_{2}(a \times 1 \times 1)(s_{1}Us_{2})^{-1}(H_{1}UH_{2})$.

Let $B = (D^{n}, \bigcup_{j=1}^{\mu} D_{j}^{k})$ be an $(n, k, \mu)$-braid and $\bar{B} = (D^{n-1}, \bigcup_{j=1}^{\mu} D_{j}^{k-1})$ be a trivial $(n-1, k-1, \mu)$-braid such that $\bar{B} \subset \partial B$. Since $\bar{B}$ is a trivial braid there exist a homeomorphism $H$ of $D^{n-1}$ onto $D^{n-1}$ and a commutative diagram

\[
\begin{array}{ccc}
\bigcup_{j=1}^{\mu} D_{j}^{k-1} & \overset{H}{\longrightarrow} & \bigcup_{j=1}^{\mu} \bar{D}_{j}^{k-1} \\
D^{n-1} & \overset{\subset}{\longrightarrow} & \hat{D}^{n-1} \bigcup_{j=1}^{\mu} \bar{D}_{j}^{k-1} & \overset{\bar{\beta}}{\longrightarrow} & \Delta^{k-1} \times I \\
\downarrow \subset & \downarrow \subset & \downarrow \hat{s} & \downarrow \times 0 & (\Delta^{k-1} \times I) \times I^{n-k-1}
\end{array}
\]

We construct a complex $X$ in $\Delta^{k-1} \times I$ defined as Remark 3.

Let $N$ be the second derived neighborhood of $H^{-1}s(X \times 0) \bigcup_{j=1}^{\mu} D_{j}^{k}$ in $D^{n}$. Since the complex $H^{-1}s(X \times 0) \bigcup_{j=1}^{\mu} D_{j}^{k}$ is collapsible, $N$ is an $n$-ball [12. chap. 3]. And the inclusion $\bigcup_{j=1}^{\mu} D_{j}^{k} \subset N$ is proper.

**Lemma 3.** The braid $C = (N, \bigcup_{j=1}^{\mu} D_{j}^{k})$ is trivial.

**Proof.** Let $N_{j}$ be the second derived neighborhood of $D_{j}^{k}$ in $D^{n}$, $j = 1, 2, \ldots, \mu$. Then $N_{j} \subset N$ and $N_{j} \bigcap \bigcup_{j=1}^{\mu} D_{j}^{k} = D_{j}^{k}$. If $D^{n} \setminus D_{0}^{n}$ and $(D_{0}^{n}, D^{k})$ is an unknotted ball pair for any $n$-balls $D^{n}, D_{0}^{n}$ and $k$-ball $D^{k}$, then $(D^{n}, D^{k})$ is also an unknotted ball pair (see [5. Cor. 10]).

Since $N \setminus \bigcup_{j=1}^{\mu} N_{j}$ and $(N_{j}, D_{j}^{k})$ is an unknotted ball pair, $j = 1, 2, \ldots, \mu$ [5, Cor. 10], $(N_{j}, D_{j}^{k})$ is an unknotted ball pair for $j = 1, 2, \ldots, \mu$.

And since it is obviously that $\bigcup_{j=1}^{\mu} D_{j}^{k}$ is split each other in $N$, $C = (N, \bigcup_{j=1}^{\mu} D_{j}^{k})$ is a trivial braid.

**Lemma 4.** Let $B_{i} = (D_{i}^{n}, \bigcup_{j=1}^{\mu} D_{ij}^{k})$ be $(n, k, \mu)$-braids, $i = 1, 2$, and $B_{3} = (D_{3}^{n-1}, \bigcup_{j=1}^{\mu} D_{3j}^{k-1}) = B_{1} \cap B_{2} = \partial B_{1} \cap \partial B_{2}$ be a trivial $(n-1, k-1, \mu)$-braid. If $B_{i}$ is trivial, then $B_{1} \cup B_{2}$ is equivalent to $B_{2}$.

**Proof.** Let $B = (\bar{D}_{n}^{n-1}, \bigcup_{j=1}^{\mu} \bar{D}_{j}^{k-1})$ be a standard $(n-1, k-1, \mu)$-pair. Since $B_{3}$ is trivial there exist a homeomorphism $H : D_{3}^{n-1} \longrightarrow \bar{D}_{n}^{n-1}$ and a commutative diagram,

\[
\begin{array}{ccc}
\bigcup_{j=1}^{\mu} D_{j}^{k-1} & \overset{H}{\longrightarrow} & \bigcup_{j=1}^{\mu} \bar{D}_{j}^{k-1} \\
D_{3}^{n-1} & \overset{\subset}{\longrightarrow} & \bar{D}_{n}^{n-1} \bigcup_{j=1}^{\mu} \bar{D}_{j}^{k-1} & \overset{\beta}{\longrightarrow} & \Delta^{k-1} \times I \\
\downarrow \subset & \downarrow \subset & \downarrow \hat{s} & \downarrow \times 0 & (\Delta^{k-1} \times I) \times I^{n-k-1}
\end{array}
\]
Let $X$ be a complex in $\Delta^{k-1} \times I$ defined as Remark 3.

Then $(\Delta^{k-1} \times I) \times I = \Delta^{k-1} \times I \searrow X$. Hence $D_{m+1}^{n} \cup \bigcup_{j=1}^{\mu} D_{m+1}^{k} \searrow \Delta^{k-1} \times I \searrow X$. Hence $D_{3}^{n-1} \cup \bigcup_{j=1}^{\mu} D_{2j}^{k} \searrow \Delta^{k-1} \times I \searrow X$.

Then $N$ collapses to $X \cup \bigcup_{j=1}^{\mu} D_{m+1}^{k}$. Hence $B=(N, \cup \bigcup_{j=1}^{\mu} D_{m+1}^{k})$ is a trivial $(n, k, \mu)$-braid by Lemma 3.

**LEMMA 5.** Let $L=(S^{n}, \cup S_{j}^{k})$ be an $(n, k, \mu)$-link and $B_{i}=(D_{i}^{n}, \cup D_{ij}^{k})$, $i=1, 2, \cdots, m$ be trivial $(n, k, \mu)$-braids. If each $B_{i}$ is contained in $L$ and disjoint from each other (that is $D_{ij}^{k} \cap D_{ip}^{k}=\phi$, $D_{ij}^{n} \cap D_{ip}^{n}=\phi$ for $i \neq p$) then there is a trivial $(n, k, \mu)$-braid $B=(\bigcup_{j=1}^{\mu} D_{j}^{k})$ such that $B_{i} \subset B \subset L$ for each $B_{i}$.

**PROOF.** Since $B_{1}, \cdots, B_{m}$ are disjoint from each other, there is a trivial $(n, k, \mu)$-braid $B_{0} \subset L$ such that $B_{0}, B_{1}, \cdots, B_{m}$ are disjoint from each other. Let $B_{m+1}=L-\text{Int} B_{0}$, then $B_{m+1}$ is a locally flat pair. Since $B_{0}$ is the trivial braid, $\partial B_{0}=\partial B_{m+1}$ is a trivial $(n-1, k-1, \mu)$-link. We take a trivial $(n-1, k-1, \mu)$-braid $\tilde{B}=(\bigcup_{j=1}^{\mu} D_{j}^{k-1})$ in $\partial B_{0}$ such that $D_{ij}^{k-1} \subset \partial D_{ij}^{k}$, $j=1, 2, \cdots, \mu$, $D^{n-1} \subset \partial D_{0}^{n}$. Then there are a homeomorphism $H$ of $D^{n}$ onto $\tilde{D}^{n}$ and a commutative diagram,

\[
\begin{array}{ccc}
\bigcup_{j=1}^{\mu} D_{j}^{k-1} & \xrightarrow{H} & \bigcup_{j=1}^{\mu} \tilde{D}_{j}^{k-1} \\
D^{n-1} & \xrightarrow{H} & D^{n-1}
\end{array}
\]

We construct a complex $X$ in $\Delta^{k-1} \times I$ as Remark 3 and let $N$ be a regular neighborhood of $H^{-1}s(X \times 0) \cup \bigcup_{j=1}^{\mu} D_{m+1j}^{k}$ in $D_{m+1}^{n}$.

Then the braid $B=(N, \bigcup_{j=1}^{\mu} D_{m+1j}^{k})$ is a trivial $(n, k, \mu)$-braid by Lemma 3 which is the required braid.

**LEMMA 6.** Let $B_{i}$ be trivial $(n, k, \mu)$-braids, $i=0, 1, 2$, such that $B_{0}, B_{2} \subset B_{1}$, $B_{i}=(D_{i}^{n}, \cup D_{ij}^{k})$. Then there is an automorphism $G$ of $D_{0}^{n}$ such that $G(D_{i}^{n})=D_{0}^{n}$, $G(D_{0j}^{k})$ and $G(D_{ij}^{k})=D_{ij}^{k}$. Furthermore $G$ is isotopic to the identity
with an isotopy $G_t, t \in [0,1]$ which keep $\partial D^n_j, \bigcup_{j=1}^{\mu} \partial D^n_{ij}$ fixed.

**Proof.** Since $B_0$ is trivial, there is a commutative diagram,

$$
\begin{array}{ccc}
\bigcup_{j=1}^{\mu} D^n_{0j} & \xrightarrow{H} & \bigcup_{j=1}^{\mu} \bar{D}^n_{ij} \\
\downarrow \subset & & \downarrow \subset \\
D^n_0 & \xrightarrow{H} & \bar{D}^n
\end{array}
\xrightarrow{\beta} \Delta^k \times I \xrightarrow{\times 0} (\Delta^k \times I) \times I^{n-k-1}.
$$

Let $\Delta^{j+1}_j(\epsilon) = \Delta^k \times \left[ -1 + \frac{2j}{\mu+1} - \epsilon, -1 + \frac{2j}{\mu+1} + \epsilon \right] \subset \Delta^k \times I, \ j = 1, 2, \cdots, \mu, \ 0 < \epsilon \leq \frac{2}{3(\mu+1)}$. Since $\beta H(D^n_{ij}) \subset \beta (\text{Int} \ \bar{D}^n_j) \subset \text{Int} \ \Delta^{j+1}_j(\epsilon), \ i = 1, 2$, there is an automorphism $k_j$ of $\beta(D^n_{ij})$ such that $k_j \beta H(D^n_{ij}) = \beta H(D^n_{ij})$ and $k_j$ is isotopic to the identity which keeps $\beta(\partial \bar{D}^n_j)$ fixed, by [2. (I. Th. 3).

Then $k_j$ can obviously be extended to an automorphism $K_j$ over $\Delta^{j+1}_j(\epsilon)$ so that $K_j$ is isotopic to the identity by an isotopy which keeps $\partial \Delta^{j+1}_j(\epsilon)$ fixed. Hence there exists an automorphism $\overline{K}$ over $\Delta^k \times I$ such that $\overline{K}$ is isotopic to the identity by an isotopy which keeps $(\Delta^k \times I - \bigcup_{j=1}^{\mu} \Delta^{j+1}_j(\epsilon))$ fixed and $\overline{K}(\text{Int} \ \Delta^{j+1}_j(\epsilon)) = K_j$.

We can extend $\overline{K}$ to an automorphism $\tilde{G}$ over $(\Delta^k \times I) \times I^{n-k-1}$ by $\tilde{G}(x, t) = (K(x), t)$ for $x \in \Delta^k \times I, t \in I^{n-k-1}$ which is isotopic to the identity and keeps $(\Delta^k \times I - \bigcup_{j=1}^{\mu} \text{Int} \ \Delta^{j+1}_j(\epsilon)) \times I^{n-k-1}$ fixed. Let $G = H^{-1} s \tilde{G}$. Then $G$ is the required automorphism which is isopopic to the identity.

**Lemma 7.** Let $L = (S^n, \bigcup_{j=1}^{\mu} S^n_j)$ be an $(n, k, \mu)$-link and $B_1, B_2$ be trivial $(n, k, \mu)$-braids such that $B_1, B_2 \subset L, B_i = (D^n_i, \bigcup_{j=1}^{\mu} D^n_{ij}), \ i = 1, 2$. Then there is an automorphism $G : S^n \rightarrow S^n$ such that $G(D^n_i) = D^n_2$ and $G(D^n_{ij}) = D^n_{2j}, \ j = 1, 2, \cdots, \mu$. Furthermore $G$ is isotopic to the identity by an isotopy $G_t$.

**Proof.** We may assume that $B_1$ and $B_2$ are disjoint, otherwise, using Lemma 5 and 6, we make them disjoint. Then by Lemma 5 there is a trivial $(n, k, \mu)$-braid $B_0 = (D^n_0, \bigcup_{j=1}^{\mu} D^n_{ij}) \subset L$ such that $B_1, B_2 \subset B_0$. By Lemma 6 there are an automorphism $G$ of $D^n_0$ and an isotopy $G_t$, which can be extended to the whole of $S^n$ which is identity outside of $D^n_0$.

The extension is denoted by also $G, G_t$ respectively. Then by the choice of $G$ and Lemma 6 it can be easily checked that they satisfy the conditions of Lemma.

**Lemma 8.** Let $B_i = (D^n_i, \bigcup_{j=1}^{\mu} D^n_{ij}), \ i = 1, 2$, be $(n, k, \mu)$-braids and $B_0 = (D^n_0,$
\[ \bigcup_{j=1}^{\mu} D_{3j}^{k-1} = \partial B \cap \partial B_{2} \text{ be } (n-1, k-1, \mu)\text{-braid.} \]

Let \( D_{j}^{k} = D_{ij}^{k} \cup D_{ij}^{k} \) and \( D^{n} = D_{i}^{n} \cup D_{2}^{n} \). Then \( (D^{n}, \bigcup_{j=1}^{\mu} D_{j}^{k}) \) is an \( (n, k, \mu)\text{-braid.} \)

**Proof.** Let \( U \) be a triangulation of \( D^{n} \) such that pairs of subcomplexes \( U_{i} \) of \( U \) cover \( D_{i}^{n} \), \( i=1, 2, 3 \). Let \( x \) be a point of \( \bigcup_{j=1}^{\mu} D_{j}^{k} \). If \( x \in D_{j}^{k} - D_{3j}^{k-1} \), the pair \( (St(x, D^{n}), St(x, D_{j}^{k})) \) is flat. If \( x \in D_{3j}^{k-1} \), then \( (St(x, D^{n}), St(x, D_{j}^{k})) = (St(x, D_{i}^{n}), St(x, D_{i}^{k})) \cup (St(x, D_{2j}^{k}), St(x, D_{2j}^{k})) \), where \( (St(x, D_{i}^{n}), St(x, D_{i}^{k})) \cap (St(x, D_{2j}^{k}), St(x, D_{2j}^{k})) = (St(x, D_{3j}^{k}), St(x, D_{3j}^{k})) \), which is flat because \( B_{3} \) is locally flat.

Then by Lemma 2 \( (St(x, D^{n}), St(x, D_{j}^{k})) = (St(x, D_{i}^{n} \cup D_{2j}^{k}), St(x, D_{i}^{k} \cup D_{2j}^{k})) \) is flat. Hence \( B_{1} \cup B_{2} \) is an \( (n, k, \mu)\text{-braid.} \)

Similarly the following is proved.

**LEMMA 9.** Let \( B_{i} = (D_{i}^{n}, \bigcup_{j=1}^{\mu} D_{ij}^{k}) \) be \( (n, k, \mu)\text{-braids such that } D_{ij}^{k} \cap D_{2j}^{k} = \partial D_{ij}^{k}, \text{ and } D_{i}^{n} \cap D_{2j}^{k} = \partial D_{ij}^{k} = -\partial D_{2j}^{k} \).

Let \( S_{j}^{k} = (D_{ij}^{k} \cup D_{ij}^{k}) \), \( j=1, 2, \ldots, \mu \), \( S^{n} = D_{i}^{n} \cup D_{2j}^{k} \) be \( k-, n\text{-spheres. Then } (S^{n}, \bigcup_{j=1}^{\mu} S_{j}^{k}) \) is an \( (n, k, \mu)\text{-link.} \)

**LEMMA 10.** Let \( B = (D^{n}, \bigcup_{j=1}^{\mu} D_{j}^{k}) \) be an \( (n, k, \mu)\text{-braid and } L_{i} = (S_{i}^{n}, \bigcup_{j=1}^{\mu} S_{ij}^{k}) \), \( i=1, 2 \) be \( (n, k, \mu)\text{-links such that } L_{1} \cap L_{2} = B. \)

Let \( S_{j}^{k} = (S_{ij}^{k}, S_{ij}^{k}) - \text{Int } D_{j}^{k}, \) \( j=1, 2, \ldots, \mu. \) Then \( (S^{n}, \bigcup_{j=1}^{\mu} S_{j}^{k}) \) is an \( (n, k, \mu)\text{-link.} \)

**Proof.** Let \( x \in S_{j}^{k}. \) If \( x \in S_{3j}^{k} - D_{j}^{k}, \) \( i=1, 2, \) then \( (St(x, S^{n}), St(x, S_{ij}^{k})) \) is flat. Suppose that \( x \in \partial D_{j}^{k}. \) Since \( L_{i} \) and \( B \) are locally flat pairs, the pairs \( (St(x, S^{n}), St(x, S_{ij}^{k})) \) and \( (St(x, D^{n}), St(x, D_{j}^{k})) \) are flat. Since \( (St(x, S^{n}), St(x, S_{ij}^{k})) = (St(x, S^{n} \cup \text{Int } D_{j}^{k}), St(x, S_{ij}^{k} - \text{Int } D_{j}^{k}) \cup (St(x, D^{n}), St(x, D_{j}^{k})) \), the intersection of the summand is \( (St(x, \partial D^{n}), St(x, \partial D_{j}^{k})) \) which is flat. \( (St(x, S^{n} \cup \text{Int } D_{j}^{k}), St(x, S_{ij}^{k} - \text{Int } D_{j}^{k})) \) is flat by Lemma 2, \( i=1, 2. \) Since the pair \( (St(x, S^{n}), St(x, S_{ij}^{k})) = (St(x, S_{ij}^{k} - \text{Int } D^{n}), St(x, S_{ij}^{k} - \text{Int } D_{j}^{k})) \) \( \cup (St(x, S^{n} \cup \text{Int } D_{j}^{k}), St(x, S_{ij}^{k} - \text{Int } D_{j}^{k})) \) and since the intersection of the summands is \( (St(x, \partial D^{n}), St(x, \partial D_{j}^{k})) \), \( (St(x, S^{n}), St(x, S_{ij}^{k})) \) is flat by Lemma 1. Hence \( L \) is an \( (n, k, \mu)\text{-link.} \)

**LEMMA 11.** Let \( B = (D^{n}, \bigcup_{j=1}^{\mu} D_{j}^{k}) \) be a trivial \( (n, k, \mu)\text{-braid and } H \) is an orientation preserving automorphism of \( D^{n} \) such that \( H(D_{j}^{k}) = D_{j}^{k}, j=1, 2, \ldots, \mu. \) Then \( H \) is isotopic to the identity.

**Proof.** Let us call the lemma **proposition** \( B(n, k, \mu). \) By **proposition**
$L(n, k, \mu)$ we shall denote the similar proposition concerning with a trivial $(n, k, \mu)$-link $L$. The proposition $B(n-k, 0, \mu)$ is trivial. In the first place let us prove that the proposition $L(n-1, k-1, \mu)$ implies the proposition $B(n, k, \mu)$. Let $H$ be an automorphism of $D^n$ such that $H(D_j^k) = D_j^k$, $j = 1, 2, \ldots, \mu$.

Then $\partial H$ is also an automorphism of $\partial D^n$ such that $\partial H(\partial D_j^k) = \partial D_j^k$, $j = 1, 2, \ldots, \mu$. Then by $L(n-1, k-1, \mu)$ $\partial H$ is isotopic to the identity, that is, there is a level preserving homeomorphism $G: \partial D^n \times I_0 \rightarrow \partial D^n \times I_0$ such that $G(x, 0) = (\partial H(x), 0)$, $G(\partial D_j^k, t) = (\partial D_j^k, t)$, $j = 1, 2, \ldots, \mu$ and $G(x, 1) = (x, 1)$. Let us define a homeomorphism $a: \partial (D^n \times I_0) \rightarrow \partial (D^n \times I_0)$ be taking $a|D^n \times 0 = H \times 0$, $a|\partial D^n \times t = G|\partial D^n \times t$, $t \epsilon I_0$ and $a|D^n \times 1 = \text{identity}$.

Since $B$ is a trivial braid, $B \times I_0 = (D^n \times I_0, \bigcup_{j=1}^{\mu} D_j^k \times I_0)$ is also trivial. Hence there is a commutative diagram

$$
\begin{array}{c}
\bigcup_{j=1}^{\mu} D_j^k \times I_0 & \overset{\mu}{\rightarrow} & \bigcup_{j=1}^{\mu} D_j^k \times I_0 \\
\downarrow C \times 1 & & \downarrow C \times 1 \\
D^n \times I_0 & \overset{H}{\rightarrow} & D^n \times I_0 \\
\end{array}
$$

where $1 : I_0 \rightarrow I_0$ is the identity map. Let $D_j^k = y_j * \partial D_j^k$, $j = 1, 2, \ldots, \mu$ where $y_j$ is an inner point of $D_j^k$. Then we may assume that $(y, \frac{1}{2}) = \rho\left((\hat{A}^k, -1 + \frac{2j}{\mu+1}), \frac{1}{2}\right)$ where $\rho = H^{-1}(s \times 1) \times 0 \times 1$ and $\hat{A}^k$ is the barycenter of $A^k$. Let $p: (A^k \times I) \times I^{n-k-1} \times I_0 \rightarrow (A^k \times I) \times I^{n-k-1} \times I_0$ be an automorphism defined by $p((\hat{A}^k, t_1), t_2, \frac{1}{2}) = ((\hat{A}^k, t_1), t_2, \frac{1}{2})$ for any $t_1 \epsilon I$, $t_2 \epsilon I^{n-k-1}$, $\frac{1}{2} \epsilon I_0$, $p((x, t_1), t_2, t_3) = (s \times 1)^{-1} H a H^{-1}((x, t_1), t_2, t_3)$ for any $((x, t_1), t_2, t_3) \epsilon \partial((A^k \times I) \times I^{n-k-1} \times I_0)$ and $p((y, t_1), t_2, t_3) = \lambda((\hat{A}^k, t_1), t_2, \frac{1}{2}) + (1-\lambda)(s \times 1)^{-1} H a H^{-1}((s \times 1))((x, t_1), t_2, t_3)$ provided $((y, t_1), t_2, t_3) = \lambda((\hat{A}^k, t_1), t_2, \frac{1}{2}) + (1-\lambda)((x, t_1), t_2, t_3)$. We define a homeomorphism $K: D^n \times I_0 \rightarrow D^n \times I_0$ by $K = H^{-1}(s \times 1)p(s \times 1)^{-1}H$.

Then $K$ is a required isotopy. Next, let us prove that the proposition $B(n, k, \mu)$ implies the proposition $L(n, k, \mu)$.

Let $H$ be a homeomorphism of $S^n$ such that $H(S_j^k) = S_j^k$, $j = 1, 2, \ldots, \mu$.

By virtue of Lemma 5 and 6, it is assumed without loss of generality, that there is a trivial $(n, k, \mu)$-braid $B(D^n, \bigcup_{j=1}^{\mu} D_j^k) \subset L$ such that $B$ and $H(B)$ are disjoint. Then again by Lemma 5 and 6, $H$ is isotopic to an automorphism $G$ of $S^n$ such that $G|D^n = \text{id}$. Since $L$ and $B$ are trivial, $L$—
Int $B$ is a trivial $(n, k, \mu)$-braid by lemma 0.

Then by $B(n, k, \mu)$ we may construct an isotopy $K$ between $G[S^n - \text{Int } D^n]$ and identity on $S^n - \text{Int } D^n$ which keep $\partial(S^n - \text{Int } D^n)$ fixed. Then $G$ is isotopic to the identity on $S^n$ and so $H$.

**Definition 7.** Let $(S^n, S^{n-1})$ be a standard sphere pair, (i.e., an unknotted sphere pair) and let $S^n - S^{n-1} = D^+ \cup D^-$. Choose thin cylinders $D^+_j \times D^+_j$ embedded in $S^n$, $j = 1, 2, \ldots, \mu$ satisfying

1. $(D^+_j \times D^+_j) \cap (\bigcup_{j=1}^{\mu} (S^+_j \cup S^+_j)) = S^+_j \cup S^+_j$, $j = 1, 2, \ldots, \mu$
2. $(D^+_j \times D^+_j) \cap S^+_j = 0$, $(D^+_j \times D^+_j) \cap S^+_j = 1 \times D^+_j$ where $D^+_j \cong [0, 1]$ and $\mu$
3. $S^+_j$ induces an orientation on $0 \times D^+_j$ which is opposite to one induced from $D^+_j \times D^+_j$ and $S^+_j$ induces an orientation on $1 \times D^+_j$ which is opposite to one induced from $D^+_j \times D^+_j$.

Since $S^+_j \cup \partial(D^+_j \times D^+_j) \cup S^+_j - 0 \times \text{Int } D^+_j \cup I \times \text{Int } D^+_j$ is again $k$-sphere $S^+_j$, $(n, k)$-sphere pair $L = (S^n, \bigcup_{j=1}^{\mu} S^+_j)$ is also an $(n, k, \mu)$-link.

Let $l$ be strong $(n, k, \mu)$-link type containing $L$. We define the sum to be $l = l_1 + l_2$.

**Lemma 12.** The sum defined as above is well defined under $n - k \geq 3$, so that the set of strong $(n, k, \mu)$-link types $SI(n, k, \mu)$ from a commutative semigroup with 0 (the trivial type) provided $n - k \geq 3$.

**Proof.** Since $n - k \geq 3$, $\partial(D^+_j \times D^+_j)$, $j = 1, 2, \ldots, \mu$ are embedded locally flatly in $S^n$. Then by Lemma 10 $L = (S^n, \bigcup_{j=1}^{\mu} S^+_j)$ is an $(n, k, \mu)$-link. We must prove the sum is independent of a choice of (1) the embedding of $D^+_j \times D^+_j$; (2) representatives $L_\nu$.

(1) Let $\tilde{D}^+_j \times \tilde{D}^+_j$ be another cylinders is $S^n$, $j = 1, 2, \ldots, \mu$ satisfying

1. $(\tilde{D}^+_j \times \tilde{D}^+_j) \cap (\bigcup_{j=1}^{\mu} (S^+_j \cup S^+_j)) = S^+_j \cup S^+_j$, $j = 1, 2, \ldots, \mu$
2. $(\tilde{D}^+_j \times \tilde{D}^+_j) \cap S^+_j = 0 \times \tilde{D}^+_j$ and
3. $(\tilde{D}^+_j \times \tilde{D}^+_j) \cap S^+_j = 1 \times \tilde{D}^+_j$ where $\tilde{D}^+_j = [0, 1]$.

Since there is an automorphism $f_{ij}$ of $S^+_j$ $i = 1, 2$, $j = 1, 2, \ldots, \mu$ such that $f_{ij}(\tilde{D}^+_j) = D^+_j$ and since $f_{ij}$ is isotopic to the identity by [2 (I)], we may suppose that

2. $(\tilde{D}^+_j \times \tilde{D}^+_j) \cap S^+_j = 0 \times D^+_j$
3. $(\tilde{D}^+_j \times \tilde{D}^+_j) \cap S^+_j = 1 \times D^+_j$

by an isotopy extended on the small neighborhood of $\bigcup_{j=1}^{\mu} (S^+_j \cup S^+_j)$.

Next since $\tilde{D}^+_j \times \tilde{D}^+_j$ is a regular neighborhood of the arc $\tilde{D}^+_j$ mod $\partial \tilde{D}^+_j$, there is an ambient isotopy $G$ of $S^n$ which keeps $\bigcup_{j=1}^{\mu} (S^+_j \cup S^+_j)$ and such that
On a piecewise linear link isotopy group

$G(\tilde{D}_j \times \tilde{D}_j) = D_j \times D_j^*$ provided $n-k \geq 3$. Hence $\tilde{L} = (S^n, \bigcup_{j=1}^\nu S^*_{j})$ is ambient isotopic to the link $L=(S^n, \bigcup_{j=1}^\nu S^*_{j})$ in the strong sense where $\tilde{S}^*_{j} = S^*_{j} \cup \partial(\tilde{D}_j \times \tilde{D}_j) \cup S^*_{2j} - 0 \times \operatorname{Int} \tilde{D}_j^* \cup 1 \times \operatorname{Int} \tilde{D}_j^*$.

(2) Let $L_i = (S^n, \bigcup_{j=1}^\nu S^*_{ij})$, $i = 1, 2$ be other representatives of $l_i$ satisfying $\bigcup_{j=1}^\nu S^*_{ij} \subset \operatorname{Int} D^*_j$ and $\bigcup_{j=1}^\nu S^*_{2j} \subset \operatorname{Int} D^n$. And let $D_j \times D_j^*$, $j = 1, 2, \ldots, \mu$ be cylinders in $S^n$ satisfying $1), 2), 3$) same as Definition 7.

Then $L = (S^n, \bigcup_{j=1}^\nu S^*_{j})$ is an $(n, k, \mu)$-link by Lemma 10 where $S^*_{j} = S^*_{ij}$ $\bigcup \partial(D_j \times D_j^*) \cup S^*_{2j} - 0 \times \operatorname{Int} D_j^* \cup 1 \times \operatorname{Int} D_j^*$. Since $L_i$ and $L_{j}$, $i = 1, 2$ are strong isotopic, there is an ambient isotopy $F$ of $S^n$ such that $F(S^*_{ij}) = S^*_{ij}$, $i = 1, 2$; $j = 1, 2, \ldots, \mu$. Let $F(D_j \times D_j^*) = D_j \times D_j^*$, then $L = (S^n, \bigcup_{j=1}^\nu F(S^*_{ij})) = (S^n, \bigcup_{j=1}^\nu S^*_{ij})$ is also an $(n, k, \mu)$-link strong isotopic to $L$ where $S^*_{ij} = S^*_{ij} \cup \partial(D_j \times D_j^*) \cup S^*_{2j} - 0 \times \operatorname{Int} D_j^* \cup 1 \times \operatorname{Int} D_j^*$.

Then $L$ is ambient isotopic to $L$ by [12] provided $n-k \geq 3$. Hence $L$ and $L$ belong to the same strong isotopy class $l$.

Consequently, the sum $l_1 + l_2$ is well defined. The commutative and associative laws follow from the definition of sum and Lemma 7.

And the trivial type is clear 0.

DEFINITION 8. Let $l$ be an $(n, k, \mu)$-link type with a representative $(n, k, \mu)$-link $L=(S^n, \bigcup_{j=1}^\nu S^*_{j})$. We say $l$ is link cobordant to zero, written $l \sim 0$, if there exists an $(n+1, k+1, \mu)$-braid $B=(D^{n+1}, \bigcup_{j=1}^\nu D_j^{n+1})$ such that $\partial B = L$, that is $\partial D_j^{n+1} = S^*_j$, $j = 1, 2, \ldots, \mu$ and $\partial D^{n+1} = S^n$.

Clearly the definition is independent of the representative $L$.

We define the link type $-l$ to be represented by $-L = (-S^n, \bigcup_{j=1}^\nu (-S^*_{j}))$ where $L=(S^n, \bigcup_{j=1}^\nu S^*_{j})$. Let $l_i$ be strong link types, $i = 1, 2$. We say $l_i$ is link cobordic to $l_i$ written $l_i \sim l_i$ if $l_i - l_i \sim 0$. The link cobordant is a symmetric relation provided $n-k \geq 3$.

LEMMA 13. Let $n-k \geq 3$ and let $l_i, l_2$ be strong $(n, k, \mu)$-link types such that $l_i \sim 0$. Then $l_i + l_2 \sim 0$ if and only $l_i \sim 0$.

PROOF. Let $L_i = (S^n, \bigcup_{j=1}^\nu S^*_{ij})$, $i = 1, 2$ be representatives of $l_i$.

Choose thin cylinders $D_j \times D_j^*$ embedded in $S^n$ same as Definition 7, then $L=(S^n, \bigcup_{j=1}^\nu S^*_{j})$ is a representative of $l_1 + l_2$ by Lemma 12 where $S^*_j = S^*_{ij}$.
\[ \partial(D_j^1 \times D_j^2) \cup S_{ij}^2 - 0 \times \text{Int } D_j^2 \cup 1 \times \text{Int } D_j^2. \]

In the first place, suppose \( l \sim 0 \). Same as Definition 7 we may suppose \( \bigcup_{j=1}^\mu S_{ij}^2 \subset D_j^2 \) and \( \bigcup_{j=1}^\mu S_{ij}^2 \subset D^n \). Since \( l \sim 0 \) and \( l \sim 0 \) there are \((n+1, k+1, \mu)\)-braids \( \tilde{B}_i = (D^{n+1}, \bigcup_{j=1}^\mu D_j^{i+1}) \) such that \( \partial \tilde{B}_i = L_i \).

And we may suppose \( \bigcup_{j=1}^\mu D_j^{i+1} \subset D_i^{n+1} \) and \( \bigcup_{j=1}^\mu D_j^{i+1} \subset D_i^{n+1} \) where \( D_i^{n+1} \cap S^n = D_i^n \) and \( D_i^{n+1} \cap S^n = D_i^n \). We push \( D_j \times \text{Int } D_j^1, j = 1, 2, \ldots, \mu \) in \( \text{Int } D_i^{n+1} \) slightly. Then \( \tilde{B} = (D^{n+1}, \bigcup_{j=1}^\mu D_j^{i+1}) \) is an \((n+1, k+1, \mu)\)-braid using Lemma 8 such that \( \partial \tilde{B} = L \) where \( D_j^{i+1} = D_j^{i+1} \cup D_j^1 \times D_j^1 \cup D_j^{i+1} \), and hence \( l_1 + l_2 \sim 0 \).

Next, suppose that \( l_1 + l_2 \sim 0 \) and \( l_2 \sim 0 \). Then there are \((n+1, k+1, \mu)\)-braids \( \tilde{B} = (D^{n+1}, \bigcup_{j=1}^\mu D_j^{i+1}) \) and \( \tilde{B} = (D^{n+1}, \bigcup_{j=1}^\mu D_j^{i+1}) \) such that \( \partial \tilde{B} = L = (S_n, \bigcup_{j=1}^\mu S_j^2) \) and \( \partial \tilde{L} = L_1 = (S_n, \bigcup_{j=1}^\mu S_j^{i+1}) \). We suppose \( D_i^{n+1} \cap D_i^{n+1} = S^n \) and \( D_j^{i+1} \cap D_j^{i+1} = S_j^{i+1} \), \( j = 1, \ldots, \mu \). Since \( \bigcup_{j=1}^\mu S_{ij}^2 \subset D_j^1 \) we may suppose \( \bigcup_{j=1}^\mu D_j^{i+1} \subset D_i^{n+1} \) where \( D_i^{n+1} = D_i^n \cup D_i^2 \cap S^n \). Then \( \tilde{L} = (\partial(D^{n+1}, D_j^{i+1}), \bigcup_{j=1}^\mu (S_j^2 \cup (\partial(D_j^1 \times D_j^2) - 1 \times \text{Int } D_j^{i+1}))) \) is an \((n, k, \mu)\)-link strong isotopic to \( L_2 = (S_n, \bigcup_{j=1}^\mu S_j^{i+1}) \). Clearly \( \partial(D^{n+1}, D_j^{i+1}), \bigcup_{j=1}^\mu (D_j^{i+1} \cup D_j^{i+1}) = \tilde{L} \).

Hence \( l_2 \sim 0 \).

**Lemma 14.** For any strong \((n, k, \mu)\)-link type \( l, l \sim 0 \) provided \( n-k \geq 3 \).

**Proof.** Let an \((n, k, \mu)\)-link \( L = (S_n, \bigcup_{j=1}^\mu S_j^2) \) be a representative of \( l \) and \( B \) a trivial \((n, k, \mu)\)-braid (see the first part of the proof of Lemma 5), \((L - \text{Int } B) \times I \) is an \((n+1, k+1, \mu)\)-braid.

Then \( \partial(L - \text{Int } B) \times I) = (L - \text{Int } B) \cup (\partial B \times I) \cup (-L - \text{Int } (-B)) \) where \((L - \text{Int } B) \cap (\partial B \times I) = \partial B, \) \((-L - \text{Int } (-B)) \cap (\partial B \times I) = -\partial B \). Since \( B \cup (\partial B \times I) \cup (-B) = \partial(B \times I) \) is a trivial \((n, k, \mu)\)-link, \( \partial(L - \text{Int } B) \times I \) is a representative of \( l + 0 - l \) by Lemma 12. Since \( (L - \text{Int } B) \times I \) is an \((n+1, k+1, \mu)\)-braid, we have \( l \sim 0 \).

**Theorem 1.** Link cobordism is an equivalence relation between strong \((n, k, \mu)\)-link types under \( n-k \geq 3 \). Link cobordism classes form an abelian group \( L(n, k, \mu) \) under the sum operation. The identity element 0 of \( L(n, k, \mu) \) is the class of the strong link type 0. For a link cobordism class \( L \) containing a strong link type \( l \), the inverse \(-L \) is the class containing \(-l \).
Proof. By Lemma 14 link cobordism is reflexive. It is clearly symmetric. To prove transitivity suppose that \( l_1 - l_2 + l_2 - l_3 = (l_2 - l_2) + l_1 - l_3 \sim 0 \). By Lemma 14, \( l_2 - l_2 \sim 0 \) hence Lemma 13 implies \( l_1 - l_3 \sim 0 \), proving transitivity and the link cobordism is an equivalence relation. Next suppose that \( l_1 \sim l_1 \) and \( l_2 \sim l_2 \). Since \( (l_1 + l_2) - (l_1 + \tilde{l}_2) = (l_1 - \overline{l}_1) + (l_2 - \overline{l}_2) \sim 0 \), \( l_1 + l_2 \sim \overline{l}_1 + l_2 \sim \) by Lemma 13. Hence the sum is well defined. Since the sum operation of link types is associative and commutative and since the inverse exists by Lemma 14, this complete the proof.

Corollary. When \( n - k \geq 3 \), the commutative semigroup of strong \((n, k, \mu)\)-link types become a commutative group under sum operation.

(We also denote this group \( SI(n, k, \mu) \). For a strong link type \( l \) with a representative \( L \), the inverse \(-1\) is the type containing \(-L\).

Proof. It is sufficiently to show that \( l - l \) becomes the class containing a trivial link. By Lemma 14 \( l - l \sim 0 \) and using [10. Th. 7] \( l - l \) is the class containing a trivial link provided \( n - k \geq 3 \).

Complete the proof.

§ 3. The link homotopy group.

Definition 9. Let \( l \) be an \((n, k, \mu)\)-link type with a representative \((n, k, \mu)\)-link \( L = (S^n, \bigcup_{j=1}^{\mu} S_j^k) \) and 0 be a trivial \((n, k, \mu)\)-link type with a representative \( L_0 = (S^n, \bigcup_{j=1}^{\mu} \tilde{S}_j^k) \). We say \( l \) link homotopic to zero, written \( l \sim 0 \), if there is a map

\[
G : \bigcup_{j=1}^{\mu} S^k_{0j} \times I_0 \to S^n
\]

satisfying

1) \( G(S^k_{0j}, 0) = S^k_j, \ j = 1, 2, \ldots, \mu \)

2) \( G(S^k_{0j}, 1) = \tilde{S}^k_j, \ j = 1, 2, \ldots, \mu \) and

3) \( G(S^k_{0i}, t) \cap G(S^k_{0j}, t) = \phi \) for any \( t \in I_0 \) and \( i \neq j \).

We say that \( l_1 \) is link homotopic to \( l_2 \), written \( l_1 \sim l_2 \), if \( l_1 - l_2 \sim 0 \). Whenever \( n - k \geq 3 \), the link homotopy is obviously well defined and an equivalence relation between strong \((n, k, \mu)\)-link types and the set of strong \((n, k, \mu)\)-link types are classified by this equivalence relation into classes which are called \((n, k, \mu)\)-link homotopy class.

By the sum operation given by Definition 7, the set of link homotopy classes form a commutative semi-group with 0 provided \( n - k \geq 3 \) (link homotopy class containing the trivial type). We now prove the existence of inverses.

Lemma 15. Let \( l \) be an strong \((n, k, \mu)\)-link type. If \( l \) is link cobordant
to zero, \( l \) is link homotopie to zero.

**Proof.** Let \( L = (S^n, \bigcup_{j=1}^{\mu} S^k_j) \) be a representative of \( l \). Then by the assumption there exists an \((n+1, k+1, \mu)\)-braid \( B = (D^{n+1}, \bigcup_{j=1}^{\mu} D^k_j) \) such that \( \partial B = L \). We take a trivial \((n+1, k+1, \mu)\)-braid \( \tilde{B} = (\tilde{D}^{n+1}, \bigcup_{j=1}^{\mu} D^k_j) \) such that \( \tilde{B} \subset B \). Then \( D^k_j + 1 - \text{Int} D^k_j + 1 \cong \partial D^k_j + 1 \times I_0 = S^k_j \times I_0 \) and \( D^{n+1} - \text{Int} \tilde{D}^{n+1} \cong \partial D^{n+1} \times I_0 = S^n \times I_0 \) where \( \tilde{S}^n = \partial \tilde{D}^{n+1} \).

Although \( \bigcup_{j=1}^{\mu} S^k_j \times I_0 \subset \tilde{S}^n \times I_0 \) is not a level preserving, it is a proper embedding \( g \) such that \( g(S^k_j, 0) = \partial \tilde{D}^k_j + 1 \subset (\tilde{S}^n, 0) \) and \( g(S^k_j, 1) = S^k_j \subset (\tilde{S}^n, 1) \). And we may further suppose that \( g(S^k_i, t), g(S^k_2, t), \cdots, g(S^k_\mu, t) \) are contained in the same set \( (\tilde{S}^n, \tilde{t}) \) for some \( \tilde{t} \in I_0 \). Let \( p : \tilde{S}^n \times I_0 \longrightarrow \tilde{S}^n \) be a projection onto the first factor. We define a map \( G : \bigcup_{j=1}^{\mu} S^k_j \times I_0 \longrightarrow \tilde{S}^n \) by \( G = pg \). Then \( G \) is the required homotopy between \( l \) and 0.

Hence we define the inverse of the link homotopy class containing \( l \) by the class containing \(-l\). Therefore we obtain the following.

**Theorem 2.** The set of \((n, k, \mu)\)-link homotopy classes forms an abelian group by the sum operation provided \( n-k \geq 3 \). We call this group \((n, k, \mu)\)-link homotopy group and denotes \( H(n, k, \mu) \).

**§ 4. The structure of** \( L(n, k, \mu) \), \( H(n, k, \mu) \) and the related results.

It is obviously that \( L(n, k, 1) \) is equivalent to knot cobordism group defined by [11].

Hence the following is obvions.

**Theorem 3.**
\[
L(n, k, 1) \begin{cases} 
\cong \{0\} & \text{if } n-k \geq 3 \\
\cong \{0\} & \text{if } n-k = 2, \ n = \text{even} \\
= \text{infinitely generated} & \text{if } n-k = 2, \ n = 3 \\
\end{cases} 
\begin{cases} 
\text{if } n \geq 5, \ odd \end{cases} \quad [13, \ 8, \ 1].
\]

**Theorem 4.** The group \( SI(n, k, \mu) \) is isomorphic to the group \( L(n, k, \mu) \) provided \( n-k \geq 3 \).

**Proof.** If a strong \((n, k, \mu)\)-link type \( l \) is equivalent to the trivial type, it is obvious that \( l \) is link cobordic to zero.

Conversely \( l \) is link cobordic to zero, it is equivalent to the trivial type provided \( n-k \geq 3 \) [10, Th. 7]. And there is a natural homomorphism of \( SI(n, k, \mu) \) onto \( L(n, k, \mu) \). Complete the proof.
On a piecewise linear link isotopy group

Denote by $\bigvee_{j=1}^{\mu}S_{j}^{n-k-1}$ the wedge of spheres $S_{1}^{n-k-1} \vee \cdots \vee S_{\mu}^{n-k-1}$.

Given an $(n, k, \mu)$-link $L=(S^{n}, \bigcup_{j=1}^{\mu}S_{j}^{k})$, there exists a map $g$ of $\bigvee_{j=1}^{\mu}S_{j}^{n-k-1}$ in the closure of the complement $X$ of $\bigcup_{j=1}^{\mu}S_{j}^{k}$ in $S^{n}$ (i.e. $X=S^{n}-\bigcup_{j=1}^{\mu}S_{j}^{k}$) whose homotopy class is well defined by the condition; $g(S_{j}^{n-k-1})$ is homotopic in $X$ to an $(n-k-1)$-sphere $\partial D_{j}^{n-k}$ which is the boundary of the fiber of a tubular neighborhood of $S_{j}^{k}$ and with linking number $+1$ with $S_{j}^{k}$.

Lemma 16. If $n-k \geq 3$ we obtain the isomorphism $\pi_{i}(\bigvee_{j=1}^{\mu}S_{j}^{n-k-1}) \cong \pi_{i}(X)$ for $i \leq n-2$ induced by $g$.

Proof. Using Alexander's duality theorem (see [9]), we obtain

\[
\tilde{H}_{i}(X; Z) = \tilde{H}^{n-i-1}(\bigcup_{j=1}^{\mu}S_{j}^{k}; Z) = \begin{cases} \overline{Z+\cdots+Z} & i=n-k-1 \\ Z & i=0 \\ 0 & \text{otherwise.} \end{cases}
\]

And $H_{0}(X) \cong Z$ since $X$ is connected and $H_{n-1}(X) \cong G$ where $G$ is some abelian group, for example $G \cong \mathbb{Z}$ if $\mu=2$.

On the other hand

\[
H_{i}(\bigvee_{j=1}^{\mu}S_{j}^{n-k-1}) = \begin{cases} \overline{Z+\cdots+Z} & i=n-k-1 \\ Z & i=0 \\ 0 & \text{otherwise.} \end{cases}
\]

Hence $H_{i}(X) = H_{i}(\bigvee_{j=1}^{\mu}S_{j}^{n-k-1})$ for $i \leq n-2$. Since both $X$ and $\bigvee_{j=1}^{\mu}S_{j}^{n-k-1}$ are simply connected provided $n-k \geq 3$, $g^{*}: \pi_{i}(\bigvee_{j=1}^{\mu}S_{j}^{n-k-1}) \rightarrow \pi_{i}(X)$ is an isomorphism for $i \leq n-2$ by the theorem of J.H.C. Whitehead (see [9]).

Lemma 17. Let $L=(S^{n}, \bigcup_{j=1}^{\mu}S_{j}^{k})$ be a homotopically trivial link. Then $L$ belongs to a trivial link type provided $2n \geq 3k+4$.

Proof. By Lemma 16, $X$ is $(n-k-2)$-connected. Since $L$ is a homotopically trivial link, $S_{j}^{k}$ is homotopic to a point in $S^{n}-\bigcup_{j=1}^{\mu}S_{j}^{k}$ for all $j$. Since $(n-k-2)-(2k-n+2)=2n-3k-4 \geq 0$ and since $2n \geq 3k+4$ implies $n-k \geq 3$ for all positive integers $n, k$, using Zeeman's Unknotting Theorem [12, Chap. 8], the link $L$ is ambient isotopic to the trivial link for $2n \geq 3k+4$.

Theorem 5. There exists an isomorphism between $\mathcal{L}(n, k, \mu)$ and $\mathcal{A}(n, k, \mu)$ provided $2n \geq 3k+4$.

Proof. There is a natural mapping $\omega$ of $\mathcal{L}(n, k, \mu)$ onto $\mathcal{A}(n, k, \mu)$ such that it maps a link cobordism class containing a strong link type $l$ to a link homotopy class containing $l$. The mapping $\omega$ is a homomorphism.
because $\omega([l_1] + [l_2]) = \omega([l_1 + l_2]) = \{l_1 + l_2\}$ where $[\ ]$ and $\{\}$ mean cobordism class and homotopy class respectively. Next we will show $\omega$ is an monomorphism. Since $l_1 \sim l_2$ implies $l_1 \sim^h l_2$ by Lemma 15, it is sufficient to show that $l_1 \sim^h l_2$ implies $l_1 \sim l_2$. Since $l_1 - l_2 \sim 0$, the strong link type $l = l_1 - l_2$ is homotopically trivial. Hence $l$ is a trivial link by Lemma 17 under the condition $2n \geq 3k + 4$. Hence $l_1 \sim l_2$. Therefore $\omega$ is an isomorphism.

**Definition 10.** Let $LM(n, k, \mu)$ be set of $(\mu \times \mu)$-matrixes $A$ of the form

$$A = \begin{bmatrix} e & \lambda_2^1 & \lambda_3^1 & \cdots & \lambda_\mu^1 & \lambda_1^2 & \cdots & \cdots & \cdots & \cdots & \lambda_\mu^\mu & \cdots & \lambda_2^\mu & e \\ \lambda_1^1 & e & \cdots & \cdots & \cdots & \lambda_2^\mu & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & e \end{bmatrix}$$

satisfying that

1) $\lambda_j^i$ is element of $\pi_k(S^{n-k-1})$

2) $e$ is the identity element of $\pi_k(S^{n-k-1})$

3) $\lambda_j^i = (\lambda_i^j)^*$ where $e = (-1)^{n-k}$

We call this matrix $(n, k, \mu)$-link matrix. Then $LM(n, k, \mu)$ becomes a group under the operation $*$ such that

$$\begin{bmatrix} e & \lambda_2^1 & \lambda_3^1 & \cdots & \lambda_\mu^1 & \lambda_1^2 & \cdots & \cdots & \cdots & \cdots & \lambda_\mu^\mu & \cdots & \lambda_2^\mu & e \\ \lambda_1^1 & e & \cdots & \cdots & \cdots & \lambda_2^\mu & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & e \end{bmatrix} \ast \begin{bmatrix} e & K_2^1 & K_3^1 & \cdots & K_\mu^1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} e & \lambda_2^1 \cdot K_1^1 & \lambda_3^1 \cdot K_1^1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where $\cdot$ is the operation on $\pi_k(S^{n-k-1})$. Hence $LM(n, k, \mu)$ becomes an additive group under the usual sum operation of matrixes because $\pi_k(S^{n-k-1})$ is the abelian group under $n \geq k + 3$. We call it the group of the $(n, k, \mu)$-link matrix and denote also $LM(n, k, \mu)$.

**Theorem 6.** There is an isomorphism between $\mathcal{X}(n, k, \mu)$ and $LM(n, k, \mu)$ provided $2n \geq 3k + 4$ and there is an epimorphism of $\mathcal{X}(n, k, \mu)$ onto $LM(n \backslash k \backslash \varphi)$ provided $2n \geq 3k + 3$ and $n - k \geq 3$.

**Proof.** First we will show that there is a map of $\mathcal{X}(n, k, \mu)$ onto $LM(n, k, \mu)$. Let $L = (S^n, \bigcup_{j=1}^{\mu} S_j^k)$ be a representative of a link homotopy class $\{l\}$ containing a strong $(n, k, \mu)$-link type $l$, then $S_\xi^k \subset S^n$ defines an element $\lambda_j^i$ of $\pi_k(S^{n-k-1}) \cong \pi_k(S^n - S_\xi^k)$ under the condition $n - k \geq 3$ andLemma 16. Then $S_\xi^k \subset S^n$ defines an element $\lambda_j^i$ of $\pi_k(S^{n-k-1}) \cong \pi_k(S^n - S_\xi^k)$ under the same conditions and $\lambda_j^i = (-1)^{n-k} \lambda_j^i$ by [12. Chap. 8].

And we define $e$ for $i = j$ formally. Let $\tilde{L} = (S^n, \bigcup_{j=1}^{\mu} \tilde{S}_j^k)$ be another rep-
On a piecewise linear link isotopy group

representative of \{l\}. Then \(L \sim \tilde{L}\) and \(\tilde{S}^k_i\) defines the same element of \(\pi_k(S^{n-k-1})\) as \(\lambda_j^i\) corresponding to \(S^k_i\). Hence \(L\) and \(\tilde{L}\) have same \((n, k, \mu)\)-link matrix and this map is independent of the representative of \{l\}. Conversely let

\[
A = \begin{bmatrix}
    e & \lambda_2^1 & \cdots & \lambda_\mu^1 \\
    \lambda_1^2 & e & \cdots & \lambda_\mu^2 \\
    \vdots & \vdots & \ddots & \vdots \\
    \lambda_1^\mu & \lambda_2^\mu & \cdots & e
\end{bmatrix}
\]

be a given element of \(LM(n, k, \mu)\). We shall construct a \((n, k, \mu)\)-link \(L\) having a given \((n, k, \mu)\)-link matrix. Let \(L_i = (S^k_i \cup \bigcup_{j \neq i} S^k_j), i=1, 2, \cdots, \mu\) be trivial \((n, k, \mu-1)\)-links and \(g_i : S^k_i \rightarrow (S^k_i \cup \bigcup_{j \neq i} S^k_j)\), \(i=1, 2, \cdots, \mu\) be continuous maps such that \(g_i(S^k_i)\) represents the element \(\lambda_j^i\) of \(\pi_k(S^k_i - S^k_j)\) for \(i > j\) and the identity \(e\) for \(i < j\).

Since \(\pi_p(S^k_i - \bigcup_{j \neq i} S^k_j) \cong \sum_{i \neq j} \pi_p(S^k_j)\) for \(p \leq n-2\) provided \(n-k \geq 3\) by Lemma 16, there exist embeddings \(f_i : S^k_i \rightarrow (S^k_i \cup \bigcup_{j \neq i} S^k_j)\) homotopic to \(g_i, i=1, 2, \cdots, \mu\) provided \(2n \geq 3k+3\) and \(n-k \geq 3\) by Irwin's Embedding Theorem ([7], [12]). Hence \(L_i = (S^k_i \cup \bigcup_{j \neq i} f_i(S^k_j))\) is an \((n, k, \mu)\)-link such that \(f_i(S^k_i)\) represents the element \(\lambda_j^i\) of \(\pi_k(S^k_i - S^k_j)\) for \(i > j\) and \(e\) for \(i < j\). Let \(L\) be a strong \((n, k, \mu)\)-link type containing \(L_i\) constructed above. We construct a connected sum \(S^n = \#S^k_i, i=1, 2, \cdots, \mu\) satisfying \(S^k_i \cap S^k_{i+1} = (S^k_i - \bigcup_{j \neq i} S^k_j) \cap (S^k_{i+1} - \bigcup_{j \neq i+1} S^k_j) = D^k_i, 1, 2, \cdots, \mu-1\) and \(S^k_i \cap S^k_j = \emptyset\) if \(j \neq i+1\) or \(i-1\).

Let \(l = l_1 + l_2 + \cdots + l_\mu\). Then it is obviously that \(l\) is a strong \((n, k, \mu)\)-link type having a given \((n, k, \mu)\)-link matrix \(A\). Hence there is a map \(\mathcal{L}(n, k, \mu)\) onto \(LM(n, k, \mu)\) provided \(n-k \geq 3\) and \(2n \geq 3k+3\).

Furthermore it is easy to show that this map is a homomorphism.

Next we will show that the epimorphism defined above is an isomorphism provided \(2n \geq 3k+4\). Let \(E = \begin{bmatrix}
    e & e & \cdots & e \\
    e & e & \cdots & e \\
    \vdots & \vdots & \ddots & \vdots \\
    e & e & \cdots & e
\end{bmatrix}\) be the identity of \(LM(n, k, \mu)\) and \(L = (S^k_n \cup \bigcup_{j=1}^{\mu} S^k_j)\) be a representative of \(E\) constructed as above under the conditions \(n-k \geq 3\) and \(2n \geq 3k+3\). Since \(S^k_i \in \pi_k(S^k_{j-i} - S^k_j)\) for any \(j \neq i\) provided \(n-k \geq 3\) and since \(\sum_{j \neq i} \pi_k(S^k_j - S^k_i) = \pi_k(S^n - S^k_j)\) using Lemma 16 and [4] under the condition \(2n \geq 3k+4, S^k_i \in \pi_k(S^n - \bigcup_{j \neq i} S^k_j)\). Hence for any \(i, S^k_i\) is homotopically trivial in \(S^n - \bigcup_{j \neq i} S^k_j\) and hence \(L\) is homotopically trivial link. Therefore \(\mathcal{L}(n, k, \mu) \cong LM(n, k, \mu)\).
provided $2n \geq 3k + 4$.

From Theorem 4, 5 and 6 we obtain the main result.

**Theorem 7.** $SI(n, k, \mu)$ is isomorphic to $LM(n, k, \mu)$ provided $2n \geq 3k + 4$.

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**References**


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