On Riemannian manifolds satisfying a certain condition on the curvature tensors

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 0. Introduction

If a Riemannian manifold M is locally symmetric, then its curvature tensor R satisfies

$$(*) \qquad R(X, Y)R = 0 \qquad \text{for any tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on R as a derivation of tensor algebra at each point of M. As a converse problem, there is a following conjecture by K. Momizu ([1]).

CONJECTURE. *Let M be a complete, irreducible Riemannian manifold with dim-$M \geqq 3$. If M satisfies the condition $(*)$, then M is a locally symmetric space.*

For this conjecture, K. Nomizu gave an affirmative answer in case that M is a complete hypersurface in a Euclidean space ([2]). P. J. Ryan gave an affirmative answer in case that M is a complete hypersurface in a space of constant curvature ([3]). Furthermore there are some results in this direction.

In this paper we shall prove the following theorems

THEOREM 1. *Let $\bar{M}$ be an $n+1$-dimensional Riemannian manifold satisfying the condition $\bar{R}(X, Y)\bar{R}=0$, and M(dim-$M \geqq 5$) be a hypersurface of $\bar{M}$ whose curvature tensor satisfies the condition $(*)$ If M satisfies $\lambda_i=\lambda(=const \neq 0)$, $1 \leqq i \leqq n$, then M is a space of constant curvature, where $\bar{R}$ and R are curvature tensors of $\bar{M}$ and M respectively, and λ_i is a eigenvalue of A, A is a field of symmetric endomorphism which corresponds to the second fundamental form h, that is, $h(X, Y)=g(AX, Y)$ for tangent vectors X and Y.*

THEOREM 2. *Let M be 3-dimensional Riemannian manifold whose curvature tensor satisfies the condition $(*)$. If the scalar curvature K of M is constant, then M is a locally symmetric space.*

I should like to express my hearty thanks to Prof. Y. Katsurada for her kind suggestions and many valuable criticisms.

§ 1. Proof of Theorem 1.

The following is a purely local argument. Let U be a neighborhood of a point $x \in M$ on which we choose a unit vector field ξ normal to M. For any vector field X and Y tangent to M, we have the formulas of Gauss and Weingarten

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi,$$
$$\bar{\nabla}_X \xi = -AX,$$

where $\bar{\nabla}$ and ∇ denote covariant differentiation for $\bar{M}$ and M respectively. Using these formulas we have the following equation

$$\bar{R}(X, Y)Z = R(X, Y)Z - (AX \wedge AY)Z + \left\{(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z)\right\}\xi,$$

where, in general, $X \wedge Y$ denotes the endomorphism which maps Z upon $g(Z, Y)X - g(Z, X)Y$, g being the Riemannian metric of M.

At a point $x \in M$, let $\{e_1, e_2, \cdots e_n\}$ be an orthonormal basis of the tangent space $T_x(M)$ such that $Ae_i = \lambda_i e_i$, $1 \leqq i \leqq n$.

LEMMA: If $\lambda_i = \lambda = \text{const}$, then $(\nabla_X h)(Y, Z) = 0$.

PROOF.
$$\begin{aligned}(\nabla_X h)(Y, Z) &= X \cdot h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \\ &= X \cdot g(AY, Z) - g(A\nabla_X Y, Z) - g(AY, \nabla_X Z) \\ &= X \cdot g(AY, Z) - g(\nabla_X(AY) - (\nabla_X A)Y, Z) - g(AY, \nabla_X Z) \\ &= g((\nabla_X A)Y, Z).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(\nabla_X A)Y &= \nabla_X(AY) - A(\nabla_X Y) \\ &= \nabla_X(\lambda Y) - \lambda\nabla_X Y \\ &= X\lambda \cdot Y + \lambda\nabla_X Y - \lambda\nabla_X Y \\ &= 0.\end{aligned}$$

Hence it follws that

$$(\nabla_X h)(Y, Z) = 0.$$

In the following computation, assuming that any two indices in i, j, k, l, have not the same value, we have

$$\begin{aligned}\left(\bar{R}(e_i, e_j)\bar{R}\right)(e_i, e_l)e_j &= \left[\bar{R}(e_i, e_j), \bar{R}(e_i, e_l)\right]e_j - \bar{R}\left(\bar{R}(e_i, e_j)e_i, e_l\right)e_j \\ &\quad - \bar{R}\left(e_i, \bar{R}(e_i, e_j)e_l\right)e_j \\ &= \bar{R}(e_i, e_j)\left(R(e_i, e_l)e_j - \lambda^2(\delta_{lj}e_i - \delta_{ij}e_l)\right)\end{aligned}$$

$$
\begin{aligned}
&-\bar{R}(e_i, e_l)\Big(R(e_i, e_j)e_j-\lambda^2(\delta_{jj}e_i-\delta_{ij}e_j)\Big)\\
&-\bar{R}\Big(R(e_i, e_j)e_i-\lambda^2(\delta_{ij}e_i-\delta_{ii}e_j), e_l\Big)e_j\\
&-\bar{R}\Big(e_i, R(e_i, e_j)e_l-\lambda^2(\delta_{lj}e_i-\delta_{il}e_j)\Big)e_j\\
=&\ R(e_i, e_j)\Big(R(e_i, e_l)e_j\Big)-\lambda^2R^r_{jil}(\delta_{rj}e_i-\delta_{ir}e_j)\\
&-R(e_i, e_l)\Big(R(e_i, e_j)e_j\Big)+\lambda^2R^r_{jij}(\delta_{rl}e_i-\delta_{ir}e_l)\\
&+\lambda^2\Big(R(e_i, e_l)e_i-\lambda^2(\delta_{il}e_i-\delta_{ii}e_l)\Big)\\
&-R\Big(R(e_i, e_j)e_i, e_l\Big)e_j-\lambda^2R^r_{iij}(\delta_{rj}e_l-\delta_{lj}e_r)\\
&-\lambda^2\Big(R(e_j, e_l)e_j-\lambda^2(\delta_{lj}e_j-\delta_{jj}e_l)\Big)\\
&-R\Big(e_i, R(e_i, e_j)e_l\Big)e_j+\lambda^2R^r_{lij}(\delta_{rj}e_i-\delta_{ij}e_r)\\
=&\Big(R(e_i, e_j)R\Big)(e_i, e_l)e_j+\lambda^2R_{ijjl}e_j+\lambda^2(R^r_{iil}-R^r_{jjl})e_r.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
&\lambda^2R_{ijjl}=0,\\
&\lambda^2(R_{liil}-R_{ljjl})=0,\\
&\lambda^2(R_{riil}-R_{rjjl})=0 \qquad\qquad (r\neq i, j, l).
\end{aligned}
$$

Under same computation, we have

$$
\Big(\bar{R}(e_i, e_j)\bar{R}\Big)(e_k, e_l)e_i=\Big(R(e_i, e_j)R\Big)(e_k, e_l)e_i-\lambda^2R_{jikl}e_i-\lambda^2R^r_{jkl}e_r.
$$

Hence we obtain

$$
\begin{aligned}
&\lambda^2R_{kjkl}=0,\\
&\lambda^2R_{ljkl}=0,\\
&\lambda^2R_{rjkl}=0 \qquad\qquad (r\neq i, j, k, l).
\end{aligned}
$$

From the above results we have

$$
\begin{aligned}
&R_{ijkl}=0,\\
&R_{ijjl}=0,\\
&R_{ijij}=R_{llil}(\overset{\text{put}}{=}\mu).
\end{aligned}
$$

The sectional curvature $\rho(X, Y)$ for the plane spanned by X and Y is given by

$$
\rho(X, Y)=\frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y)-(g(X, Y))^2}.
$$

Let $X=a^ie_i$, $Y=b^je_j$, then using $R_{ijkl}=0$, $R_{ijjl}=0$, we have

$$\sum_{i=1}^{n}(a^i)^2\cdot\sum_{j=1}^{n}(b^j)^2\cdot\rho(X, Y) = g\left(R(a^ie_i, p^je_j)b^ke_k, a^le_l\right)$$
$$= \sum_{i=1}^{n}(a^i)^2\cdot\sum_{j=1}^{n}(b^j)^2\cdot\mu .$$

Hence it follows that

$$\rho(X, Y) = \mu$$

Therefore M is a space of constant curvature since the sectional curvature $\rho(X, Y)$ does not depend on the direction.

§ 2. Proof of Theorem 2.

Now we can see that there is an orthonormal basis $\{e_1, e_2, e_3\}$ at each tangent space of M such that

(2. 1) $$R_{1212} = a , \quad R_{1313} = b , \quad R_{2323} = c ,$$

otherwise zero. From (2. 1) we have

$$R(e_i, e_j) = R_{ijij} e_i \wedge e_j$$

From the following relation;

$$\left(R(e_i, e_j)R\right)(e_k, e_l) = \left[R(e_i, e_j), R(e_k, e_l)\right] - R\left(R(e_i, e_j)e_k, e_l\right)$$
$$-R\left(e_k, R(e_i, e_j)e_l\right)$$

we find that it is zero except in case that $k=i$, and $l \neq i, j (i \neq j)$. For this case we have

$$\left(R(e_i, e_j)R\right)(e_i, e_l) = R_{ijij}(R_{jljl} - R_{ilil})e_j \wedge e_l .$$

Thus we see that condition $(*)$ is equivalent to

(2. 2) $$R_{ijij}(R_{jljl} - R_{ilil}) = 0$$

From (2. 2) we have

$$a(c-b) = 0 ,$$
$$b(c-a) = 0 ,$$
$$c(b-a) = 0 .$$

We consider the following two cases

I. $a = b = c$

II. $a = b = 0 , \quad c \neq 0$

In case of I, clearly M is a space of constant curvature.
In case of II, we obtain

$$R_{12}=R_{13}=R_{23}=R_{11}=0\,,$$
$$R_{22}=R_{33}=R_{2323}\,.$$

From the Bianchi's identity, we have

$$\nabla_1 R_{1k}+\nabla_2 R_{2k}+\nabla_3 R_{3k}=\frac{1}{2}\nabla_k K.$$

Let us put $k=3$. Then we have $\nabla_3 R_{33}=\nabla_3 R_{22}=0$.
Putting $k=2$, we have $\nabla_2 R_{22}=\nabla_2 R_{33}=0$.
From the Bianchi's identity we have

$$\nabla_1 R_{1kji}+\nabla_2 R_{2kji}+\nabla_3 R_{3kji}=\nabla_i R_{jk}-\nabla_j R_{ik}\,.$$

Put $j=k=2$ and $i=1$, then $\nabla_1 R_{22}=\nabla_1 R_{33}=0$.
Hence it follows that

$$\nabla_j R_{ik}=0$$

Therefore M is a locally symmetric space.

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(Received August 16, 1971)