レムマンマンフォールドについての条件を満たすリーマン球面における応用についての論文。

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On Riemannian manifolds satisfying a certain condition on the curvature tensors

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Kazunari Yamauchi

§ 0. Introduction

If a Riemannian manifold $M$ is locally symmetric, then its curvature tensor $R$ satisfies

\[ (*) \quad R(X, Y) R = 0 \]

for any tangent vectors $X$ and $Y$, where the endomorphism $R(X, Y)$ operates on $R$ as a derivation of tensor algebra at each point of $M$. As a converse problem, there is a following conjecture by K. Nomizu ([1]).

Conjecture. Let $M$ be a complete, irreducible Riemannian manifold with \( \text{dim} \cdot M \geq 3 \). If $M$ satisfies the condition \((*)\), then $M$ is a locally symmetric space.

For this conjecture, K. Nomizu gave an affirmative answer in case that $M$ is a complete hypersurface in a Euclidean space ([2]). P. J. Ryan gave an affirmative answer in case that $M$ is a complete hypersurface in a space of constant curvature ([3]). Furthermore there are some results in this direction.

In this paper we shall prove the following theorems

Theorem 1. Let $\overline{M}$ be an $n+1$-dimensional Riemannian manifold satisfying the condition $\overline{R}(X, Y) \overline{R} = 0$, and $M(\text{dim} \cdot M \geq 5)$ be a hypersurface of $\overline{M}$ whose curvature tensor satisfies the condition \((*)\). If $M$ satisfies $\lambda_i = \lambda (= \text{const} \neq 0), 1 \leq i \leq n$, then $M$ is a space of constant curvature, where $\overline{R}$ and $R$ are curvature tensors of $\overline{M}$ and $M$ respectively, and $\lambda_i$ is an eigenvalue of $A$, $A$ is a field of symmetric endomorphism which corresponds to the second fundamental form $h$, that is, $h(X, Y) = g(AX, Y)$ for tangent vectors $X$ and $Y$.

Theorem 2. Let $M$ be 3-dimensional Riemannian manifold whose curvature tensor satisfies the condition \((*)\). If the scalar curvature $K$ of $M$ is constant, then $M$ is a locally symmetric space.

I should like to express my hearty thanks to Prof. Y. Katsurada for her kind suggestions and many valuable criticisms.
§ 1. Proof of Theorem 1.

The following is a purely local argument. Let $U$ be a neighborhood of a point $x \in M$ on which we choose a unit vector field $\xi$ normal to $M$. For any vector field $X$ and $Y$ tangent to $M$, we have the formulas of Gauss and Weingarten

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi,$$
$$\overline{\nabla}_\xi = -AX,$$

where $\overline{\nabla}$ and $\nabla$ denote covariant differentiation for $\overline{M}$ and $M$ respectively. Using these formulas we have the following equation

$$\overline{R}(X, Y)Z = R(X, Y)Z - (AX \wedge AY)Z + \left((\overline{\nabla}_X h)(Y, Z) - (\overline{\nabla}_Y h)(X, Z)\right)\xi,$$

where, in general, $X \wedge Y$ denotes the endomorphism which maps $Z$ upon $g(Z, Y)X - g(Z, X)Y$, $g$ being the Riemannian metric of $M$.

At a point $x \in M$, let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of the tangent space $T_x(M)$ such that $\lambda e_i = \lambda_i e_i$, $1 \leq i \leq n$.

**Lemma:** If $\lambda_i = \lambda = \text{const}$, then $(\overline{\nabla}_X h)(Y, Z) = 0$.

**Proof.** $(\overline{\nabla}_X h)(Y, Z) = X \cdot h(Y, Z) - h(\overline{\nabla}_X Y, Z) - h(Y, \overline{\nabla}_X Z)$

$$= X \cdot g(AY, Z) - g(A\overline{\nabla}_X Y, Z) - g(AY, A\overline{\nabla}_X Z)$$
$$= X \cdot g(AY, Z) - g(\overline{\nabla}_X (AY) - (\overline{\nabla}_X A)Y, Z) - g(AY, \overline{\nabla}_X Z)$$
$$= g((\overline{\nabla}_X A)Y, Z).$$

On the other hand, we have

$$(\overline{\nabla}_X A)Y = \overline{\nabla}_X (AY) - A(\overline{\nabla}_X Y)$$
$$= \overline{\nabla}_X (\lambda Y) - \lambda \overline{\nabla}_X Y$$
$$= X\lambda \cdot Y + \lambda \overline{\nabla}_X Y - \lambda \overline{\nabla}_X Y$$
$$= 0.$$

Hence it follows that

$$(\overline{\nabla}_X h)(Y, Z) = 0.$$

In the following computation, assuming that any two indices in $i, j, k, l$, have not the same value, we have

$$\left(\overline{R}(e_\alpha, e_\beta)\overline{R}(e_\alpha, e_\gamma)e_\beta - \overline{R}(e_\alpha, e_\gamma)e_\beta\overline{R}(e_\alpha, e_\beta)e_\gamma\right)e_j$$
$$= \overline{R}(e_\alpha, e_\beta)\overline{R}(e_\alpha, e_\gamma)e_\beta - \overline{R}(e_\alpha, e_\gamma)e_\beta\overline{R}(e_\alpha, e_\beta)e_\gamma$$
$$= \overline{R}(e_\alpha, e_\beta)\overline{R}(e_\alpha, e_\gamma)e_\beta - \overline{R}(e_\alpha, e_\gamma)e_\beta\overline{R}(e_\alpha, e_\beta)e_\gamma.$$
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\[-\overline{R}(e_i, e_j)(R(e_i, e_j)e_j - \lambda^2(\delta_{ij}e_i - \delta_{i}e_j))\]
\[-\overline{R}(R(e_i, e_j)e_i - \lambda^2(\delta_{ij}e_i - \delta_{i}e_j), e_j)e_j\]
\[-\overline{R}(e_i, e_i)(R(e_i, e_i)e_j - \lambda^2(\delta_{ij}e_i - \delta_{i}e_j))e_j\]

\[= R(e_i, e_j)(R(e_i, e_j)e_j) - \lambda^2 R_{jkl}^{r}(\delta_{r}e_i - \delta_{i}e_r)\]
\[-R(e_i, e_i)(R(e_i, e_i)e_j) + \lambda^2 R_{ijl}^{r}(\delta_{l}e_i - \delta_{i}e_l)\]
\[-R(e_i, R(e_i, e_i)e_j)e_j + \lambda^2 R_{lij}^{r}(\delta_{r}e_i - \delta_{i}e_r)\]

Hence we obtain

\[\lambda^2 R_{ijjl} = 0,\]
\[\lambda^2(R_{iijl} - R_{jjll}) = 0,\]
\[\lambda^2(R_{lij}^{r} - R_{rjkl}) = 0\]

\[(r \neq i, j, l)\]

Under same computation, we have

\[\overline{R}(e_i, e_j)^2 = (R(e_i, e_j)R)(e_i, e_i)e_j - \lambda^2 R_{jkl}^{r}e_r\]

Hence we obtain

\[\lambda^2 R_{jkl}^{r} = 0,\]
\[\lambda^2 R_{jkl}^{r} = 0,\]
\[\lambda^2 R_{rjkl} = 0\]

\[(r \neq i, j, k, l)\]

From the above results we have

\[R_{ijkl} = 0,\]
\[R_{ijll} = 0,\]
\[R_{ijll} = R_{lll}^{p} = \mu.\]

The sectional curvature \(\rho(X, Y)\) for the plane spanned by \(X\) and \(Y\) is given by

\[\rho(X, Y) = \frac{g(R(X, Y)Y, X)}{g(X, X)g(Y, Y) - (g(X, Y))^2}.\]

Let \(X = a^{i}e_i, Y = b^{j}e_j\), then using \(R_{ijkl} = 0, R_{ijll} = 0\), we have
\[
\sum_{i=1}^{n}(a^i)^2 \cdot \sum_{j=1}^{n}(b^j)^2 \cdot \rho(X, Y) = g(R(a^i e_i, p^j e_j) b^k e_k, a^l e_l) \\
= \sum_{i=1}^{n}(a^i)^2 \cdot \sum_{j=1}^{n}(b^j)^2 \cdot \mu.
\]

Hence it follows that
\[\rho(X, Y) = \mu\]

Therefore \(M\) is a space of constant curvature since the sectional curvature \(\rho(X, Y)\) does not depend on the direction.

\section*{§ 2. Proof of Theorem 2.}

Now we can see that there is an orthonormal basis \(\{e_1, e_2, e_3\}\) at each tangent space of \(M\) such that
\[(2.1) \quad R_{1212} = a, \quad R_{1313} = b, \quad R_{2323} = c,\]
otherwise zero. From (2.1) we have
\[R(e_i, e_j) = R_{ijij} e_i \wedge e_j\]

From the following relation;
\[\left( R(e_i, e_j) R \right)(e_k, e_l) = \left[ R(e_i, e_j), R(e_k, e_l) \right] - R(R(e_i, e_j) e_k, e_l) - R(e_k, R(e_i, e_j) e_l)\]
we find that it is zero except in case that \(k = i, \) and \(l \neq i, j(i \neq j).\) For this case we have
\[\left( R(e_i, e_j) R \right)(e_k, e_l) = R_{ijij}(R_{jljl} - R_{ilil}) e_j \wedge e_l.\]
Thus we see that condition (*') is equivalent to
\[(2.2) \quad R_{ijij}(R_{jljl} - R_{ilil}) = 0\]

From (2.2) we have
\[a(c - b) = 0,\]
\[b(c - a) = 0,\]
\[c(b - a) = 0.\]

We consider the following two cases

I. \[a = b = c\]

II. \[a = b = 0, \quad c \neq 0\]

In case of I, clearly \(M\) is a space of constant curvature.
In case of II, we obtain
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\[ R_{12} = R_{13} = R_{23} = R_{11} = 0, \]
\[ R_{22} = R_{33} = R_{2323}. \]

From the Bianchi's identity, we have
\[ \nabla_1 R_{1k} + \nabla_2 R_{2k} + \nabla_3 R_{3k} = \frac{1}{2} \nabla_k K. \]

Let us put \( k = 3 \). Then we have \( \nabla_2 R_{33} = \nabla_3 R_{22} = 0 \).

Putting \( k = 2 \), we have \( \nabla_2 R_{22} = \nabla_3 R_{33} = 0 \).

From the Bianchi's identity we have
\[ \nabla_1 R_{1kji} + \nabla_2 R_{2kj\ell} + \nabla_3 R_{3kji} = \nabla_i R_{jk} - \nabla_j R_{\ell k}. \]

Put \( j = k = 2 \) and \( i = 1 \), then \( \nabla_1 R_{22} = \nabla_1 R_{33} = 0 \).

Hence it follows that
\[ \nabla_j R_{\ell k} = 0 \]

Therefore \( M \) is a locally symmetric space.

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Bibliography


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