On necessary and sufficient conditions for 
$L^2$-well-posedness of mixed problems for hyperbolic equations II

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction

We consider hyperbolic mixed problems \((P, B_j)\) in a quadrant \(R_+ \times R^n_+:\n\[Pu = f \quad \text{in } t > 0, \, x_n > 0, \, x' \in R^{n-1},\]
\[B_j u = 0 \quad (j = 1, \ldots, l) \quad \text{in } t > 0, \, x_n = 0, \, x' \in R^{n-1},\]
\[\partial_t^j u = u_k \quad (k = 0, 1, \ldots, m - 1) \quad \text{in } t = 0, \, x_n > 0, \, x' \in R^{n-1} \]

where \(R^n_+ = \{x = (x', x_n) \in R^n ; x_n > 0\}, \partial_t = \frac{\partial}{\partial t}, D_{x_j} = (-i)\frac{\partial}{\partial x_j}, P = P(t, x; \partial_t, D_x)\)
is a strictly \(t\)-hyperbolic operator of order \(m\) and \(B_j = B_j(t, x'; \partial_t, D_x)\) are boundary operators of order \(m_j < m\).
Furthermore \(P\) is assumed non-characteristic with respect to the hyperplane \(x_n = 0\).

This paper consists of two parts. The first part (§ 2 and § 3) is concerned with constant coefficient problems for homogeneous operators \(P\) and \(B_j\). In previous paper [2] we assume that \(\{B_j\}\) is normal; that is, \(m_j \neq m_k\) if \(j \neq k\) and the hyperplane \(x_n = 0\) is non-characteristic for \(B_j\). However, when \(\{B_j\}\) is not always normal, it will be suitable for our purpose to use the following

**Definition.** The mixed problem \((P, B_j)\) with homogeneous initial condition is \(L^2\)-well-posed with decreasing order \(\nu\) \((\nu \geq 0, \text{ an integer})\) if and only if there exist positive constants \(a\) and \(C\) such that for any \(f\) with \(e^{-at} f \in H_0^{\nu+1}(R^n_+ \times R^n_+)\) the problem has a unique solution \(u\) with \(e^{-at} u \in H^m(R_+ \times R^n_+)\) satisfying

\[
\sum_{j+|a| \leq m-1} \int_0^\infty \int_{R^n_+} e^{-2at} |(\partial_t^j D_{x}^a u)(t, x)|^2 \, dt \, dx
\]
\[
\leq C \sum_{j+|a| \leq m-1} \int_0^\infty \int_{R^n_+} e^{-2at} |(\partial_t^j D_{x}^a f)(t, x)|^2 \, dt \, dx.
\]

In § 2 a necessary and sufficient condition for \(L^2\)-well-posedness with decreasing order \(\nu\) is given by the terms of compensating function (cf.
Theorem 4.1 in [2]). In §3 we show that under the $L^2$-well-posedness with decreasing order $\nu$ Lopatinskii's determinant $R(\tau_0, \sigma_0) \neq 0$ for $Re \tau_0 > 0$ and $\sigma_0 \in R^{n-1}$ is equivalent to the fact that $B_j(\tau_0, \sigma_0, \lambda)$ are linearly independent as polynomials in $\lambda$.

The second part is concerned with variable coefficient problems. In this part we always assume that $\{B_j\}$ is normal. For initial data $U(t_0) = (u_0, u_1, \ldots, u_{m-1})$ we set up a convenient function space $\mathcal{K}(t_0)$; that is, $U(t_0) \in \mathcal{K}(t_0)$ if and only if $u_k \in H^{m-k}(\mathbb{R}_+^n)$ and satisfy the compatibility condition

$$\sum_{k=0}^{m_j} b_{j,k}(t, x') D_x u_k = 0 \quad \text{on} \quad t = t_0, x_n = 0$$

where

$$B_j(t, x' ; \partial_t, D_x) = \sum_{k=0}^{m_j} b_{j,k}(t, x' ; D_x) \partial_t^k.$$

**DEFINITION.** The mixed problem $(P, B_j)$ is strongly $L^2$-well-posed if and only if there exist positive constants $T$ and $C$ such that for an arbitrarily fixed time $t_0 \in [0, T)$ the problem $(P, B_j)$ with $f \in H_0^1((t_0, T) \times \mathbb{R}_+^n)$ and $U(t_0) \in \mathcal{K}(t_0)$ has a unique solution $u \in \mathcal{E}^0((t_0, T), H^m(\mathbb{R}_+^n)) \cap \cdots \cap \mathcal{E}^m((t_0, T), H^0(\mathbb{R}_+^n))$ which satisfies energy inequalities

(1.2) \[ \|u(t, \cdot)\|_{m-1}^2 \leq C(\|U(t_0)\|_{m-1}^2 + \int_{t_0}^t \|f(s, \cdot)\|_0^2 ds), \]

(1.3) \[ \|u(t, \cdot)\|_m^2 \leq C(\|U(t_0)\|_m^2 + \int_{t_0}^t \|f(s, \cdot)\|_1^2 ds) \]

for any $t \in [t_0, T]$, where

$$\|u(t, \cdot)\|_k^2 = \sum_{j=0}^k \|\partial_t^j u(t, \cdot)\|_{k-j}^2,$$

$$\|U(t_0)\|_k^2 = \sum_{j=0}^k \|u_j(\cdot)\|_{k-j}^2,$$

$$\|u(\cdot)\|_k^2 = \sum_{|\alpha| \leq k} \int_{\mathbb{R}_+^n} |D_x^\alpha u(x)|^2 dx.$$

In §4 we show the following: If a variable coefficient problem $(P, B_j)$ is strongly $L^2$-well-posed, then each constant coefficient problem arising from freezing coefficients of their principal parts at a boundary point is $L^2$-well-posed (with decreasing order $\nu = 0$), provided that the corresponding Lopatin- skii's determinant $R(t, x' ; 1, 0) \neq 0$ on the boundary. Combining this and results in [1] we obtain a certain characterization of strongly $L^2$-well-posed problems with real boundary condition for the case of second order.

This note is the supplement of our previous papers [1] and [2].
§ 2. Necessary and sufficient condition

In this section and the following we consider constant coefficient problems \((P, B_j)\) with homogeneous initial condition. Here \(P\) and \(B_j\) are homogeneous operators.

We take Laplace transform in \(t\) and Fourier transform in \(x\) and \(\hat{u}(\tau, \sigma, \lambda)\) and \(\hat{u}(\tau, \sigma, x_n)\) denote the Fourier-Laplace image of \(u(t, x', x_n)\) with respect to \((t, x', x_n)\) and \((t, x')\) respectively. By the assumption on \(P\) the number \(l(m-l)\) of roots \(\lambda_{j}^{+}(\tau, \sigma)(\lambda_{k}^{-}(\tau, \sigma))\) of \(P(\tau, \sigma, \lambda)=0\) in \(\lambda\), which have positive (negative) imaginary part, is independent of \((\tau, \sigma)\in C_+ \times R^{n-1}\), where \(C_+=\{\tau \in C; \text{Re}\,\tau>0\}\).

Taking now Fourier-Laplace transform the problem \((P, B_j)\) becomes formally to the boundary value problem of ordinary differential equations depending on parameters \((\tau, \sigma)\in C_+ \times R^{n-1}\):

\[
P(\tau, \sigma, D_{x_{n}}) \hat{u}(\tau, \sigma, x_{n}) = \hat{f}(\tau, \sigma, x_{n}) \quad \text{in } x_{n}>0,\]
\[
B_{j}(\tau, \sigma, D_{x_{n}}) \hat{u}(\tau, \sigma, x_{n}) = 0 \quad (j=1, \cdots, l) \quad \text{on } x_{n}=0.
\]

Let \(R(\tau, \sigma)\) be Lopatinskii’s determinant; that is,

\[
R(\tau, \sigma) = \det \left( B_j(\tau, \sigma, \lambda_{j}^{+}(\tau, \sigma))/\prod_{j>k}(\lambda_{j}^{+}(\tau, \sigma)-\lambda_{k}^{+}(\tau, \sigma)) \right)
\]

and \(R_{j}(\tau, \sigma, x_{n})\) be the determinant replacing \(j\)-column in \(R(\tau, \sigma)\) by the transposed vector \((\exp(ix_{n}\lambda_{1}^{+}(\tau, \sigma)), \cdots, \exp(ix_{n}\lambda_{l}^{+}(\tau, \sigma)))\). Then \(R(\tau, \sigma)\) and \(R_{j}(\tau, \sigma, x_{n})(j=1, \cdots, l)\) are analytic in \((\tau, \sigma)\in C_+ \times R^{n-1}\). If \(R(\tau, \sigma)\neq 0\) for some \((\tau, \sigma)\in C_+ \times R^{n-1}\), then it is well known that for any \(f\in C_{0}^{\infty}(R_{+})\) the problem (2.1) has a unique bounded solution \(u\in C^{\infty}(R_{+})\) which is written by the form

\[
\hat{u}(\tau, \sigma, x_{n}) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix_{n}\lambda} \hat{f}(\lambda)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \int_{0}^{\infty} G(x_{n}, s, \tau, \sigma)f(s) ds
\]

where

\[
G(x_{n}, s, \tau, \sigma) = -\sum_{j=1}^{l} \frac{R_{j}(\tau, \sigma, x_{n})}{R(\tau, \sigma)} \int_{\Gamma} \frac{B_{j}(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} e^{-is\lambda} d\lambda,
\]

and \(\Gamma = \Gamma(\tau, \sigma)\) denotes a simple closed curve in the lower half \(\lambda\)-plane enclosing all the roots \(\lambda_{j}(\tau, \sigma)\).

Let \(\Sigma_+\) be the set \(\{(\tau', \sigma')\in C_+ \times R^{n-1}; \ |\tau'|^2 + |\sigma'|^2 = 1\}\) and \(\Sigma_+\) be its closure. Furthermore let \(V\) be the zeros of \(R(\tau, \sigma)\) in \(C_+ \times R^{n-1}\) and \(V'\) be \(V\cap \Sigma_+\). \(V^\circ\) and \(V'^\circ\) denote the complement of \(V'\) and \(V\) in \(\Sigma_+\) and \(C_+ \times R^{n-1}\) respectively. Then we obtain the following
THEOREM 2.1. Suppose that $R(\tau, \sigma)$ is identically not zero. Then the mixed problem $(P, B_{j})$ is $L^{2}$-well-posed with decreasing order $\nu$ if and only if the following condition is satisfied:

For every $(\tau_{0}', \sigma_{0}') \in (\Sigma_{+} - \Sigma_{+}) \cup V'$ there exist a constant $C(\tau_{0}', \sigma_{0}')$ and a neighborhood $U(\tau_{0}', \sigma_{0}')$ such that

\[
\|D_{x_{n}}^{k}G(x_{n}, s, \tau', \sigma')\|_{L^{2}(x_{n} > 0)} \leq C(\tau_{0}', \sigma_{0}') (Re \tau')^{-\nu-1}
\]

for any $(\tau', \sigma') \in U(\tau_{0}', \sigma_{0}') \cap C_{+} \times R^{n-1}$ and $k=0, 1, \ldots, m-1$, where $\|\cdot\|_{L^{2}(x_{n} > 0)}$ denotes the operator norm from $L^{2}(x_{n} > 0)$ to $L^{2}(x_{n} > 0)$.

Since the proof of the theorem is accomplished by the almost same considerations as those in Theorem 4.1 [2], we show only different points.

(1) **SUFFICIENCY (EXISTENCE OF SOLUTIONS).** Let $S$ be the set $\{\sigma \in R^{n-1}; R(\tau, \sigma)$ is identically zero in $\tau\}$. Then $S$ is a null set with respect to Lebesgue measure in $R^{n-1}$ because $V$ is so in $C_{+} \times R^{n-1}$. In what follows we assume $\sigma \in S$ and for $(\tau, \sigma) \in C_{+} \times R^{n-1}$ (\tau', \sigma') denotes $(\rho^{-1}\tau, \rho^{-1}\sigma)$ where $\rho =(|\tau|^{2} + |\sigma|^{2})^{\frac{1}{2}}$.

**LEMMA A.** There exists an analytic extension $\tilde{G}(\tau, \sigma)$ in $\tau \in C_{+}$ of $G(x_{n}, s, \tau, \sigma)$ as an operator from $H_{0}^{y}(s > 0)$ to $L^{2}(x_{n} > 0)$ such that $G \sim (\tau, \sigma)f \in H^{m-1}(x_{n} > 0)$ for $f \in H_{0}^{\nu}(s > 0)$ and

\[
\|D_{x_{n}}^{k}(\tilde{G}(\tau, \sigma)f)\|_{L^{2}(x_{n} > 0)} \leq C(\tau_{0}', \sigma_{0}') (Re \tau)^{-\nu-1} \rho^{k-m+1}(\sum_{\mu=0}^{\nu}\rho^{-2\mu}\int_{0}^{\infty}|(D_{s}^{\mu}f)(s)|^{2}ds)^{\frac{1}{2}}
\]

for any $(\tau_{0}', \sigma_{0}')$ with $R(\tau_{0}', \sigma_{0}')=0$, $(\tau, \sigma)$ with $(\tau', \sigma') \in U(\tau_{0}', \sigma_{0}') \cap (C_{+} \times R^{n-1})$ and $k=0, 1, \ldots, m-1$. Here $C(\tau_{0}', \sigma_{0}')$ and $U(\tau_{0}', \sigma_{0}')$ are the same ones in Theorem 2.1.

**Proof.** If $\sigma_{0} \in S$ and $R(\tau_{0}, \sigma_{0})=0$, then $\tau_{0}$ is an isolated point in $C_{+}$. In virtue of (2.3) and the relations $(D_{x_{n}}^{k}G)(x_{n}, s, \tau, \sigma)=\rho^{k-m+1}(D_{x_{n}}^{k}G)(\rho x_{n}, \rho s, \tau', \sigma')(k=0, 1, \ldots, m-1)$, we obtain for $f \in H_{0}^{y}(s > 0)$ and $g \in L^{2}(x_{n} > 0)$

\[
\int_{0}^{\infty}(D_{x_{n}}^{k}G)(x_{n}, s, \tau, \sigma)f(s)ds, g(x_{n})\int_{0}^{\infty}(D_{x_{n}}^{k}G)(x_{n}, s, \tau', \sigma')f'(\rho^{-1}s)ds
\]

\[
\leq \rho^{k-m-1}||g(\rho^{-1}x_{n})||_{L^{2}(x_{n} > 0)}|| \int_{0}^{\infty}(D_{x_{n}}^{k}G)(x_{n}, s, \tau', \sigma')f(\rho^{-1}s)ds||_{L^{2}(x_{n} > 0)}
\]

\[
\leq \rho^{k-m-\frac{1}{2}}C(\tau_{0}', \sigma_{0}')(Re \tau)^{-\nu-1}||g||_{L^{2}(x_{n} > 0)}(\sum_{\mu=0}^{\nu}\rho^{-2\mu} \int_{0}^{\infty}|(D_{x_{n}}^{k}f)(\rho^{-1}s)|^{2}ds)^{\frac{1}{2}}
\]
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\[
\begin{align*}
C(\tau_0, \sigma_0)(Re\tau)^{-\nu-1}\rho^{k-m+1}\|g\|_{L^2(\sigma_0^2>0)}(\sum_{\mu=0}^{\nu}\rho^{-2\mu}\int_{0}^{\infty}|(D_{x_{n}}^{k}G)(\rho^{-1}s)|^{2}ds)^{\frac{1}{2}}
\end{align*}
\]

where $(\tau, \sigma)$ with $(\tau', \sigma') \in (\tau_0, \sigma_0) \cap V^\infty \cap \Sigma_+$. In particular, it follows from above that, with some $C(\tau_0, \sigma_0)$,

\[
\begin{align*}
|\left(\int_{0}^{\infty}(D_{x_{n}}^{k}G)(x_n, s, \tau, \sigma_0) f(s) ds, g(x_n)\right)_{L^2(x_n>0)}|
\leqq C(\tau_0, \sigma_0)\|f\|_{H_0^\nu(x_n>0)}\|g\|_{L^2(x_n>0)}
\end{align*}
\]

in a small neighborhood of $\tau_0$. Hence $\left(\int_{0}^{\infty}(D_{x_{n}}^{k}G)(x_n, s, \tau, \sigma_0) f(s) ds, g(x_n)\right)_{L^2(x_n>0)}$ has an analytic extension in $C_+$. By Riesz theorem and (2.6) there exist operators $\tilde{G}_k(\tau, \sigma_0)$ $(k=0, 1, \cdots, m-1)$ from $H_0^\nu(x_n>0)$ to $L^2(x_n>0)$ such that $\tilde{G}(\tau, \sigma_0)f, g)_{L^2(x_n>0)}$ is analytic in $C_+$ and

\[
\begin{align*}
(\tilde{G}_k(\tau, \sigma_0)f, g)_{L^2(x_n>0)} = \left(\int_{0}^{\infty}(D_{x_{n}}^{k}G)(x_n, s, \tau, \sigma_0) f(s) ds, g(x_n)\right)_{L^2(x_n>0)}
\end{align*}
\]

for $\tau \neq \tau_0$. In virtue of Banach-Steinhaus theorem there exist uniform derivatives

\[
\begin{align*}
\left(\frac{d}{d\tau}\tilde{G}_k(\tau, \sigma_0)f, g\right)_{L^2(x_n>0)} = \left(\frac{d}{d\tau}\tilde{G}_k(\tau+h, \sigma_0) - \tilde{G}_k(\tau, \sigma_0)\right)_{L^2(x_n>0)}
\end{align*}
\]

such that

\[
\begin{align*}
\frac{d}{d\tau}\tilde{G}_k(\tau, \sigma_0)f, g\right)_{L^2(x_n>0)} = \left(\frac{d}{d\tau}\tilde{G}_k(\tau, \sigma_0)f, g\right)_{L^2(x_n>0)}.
\end{align*}
\]

This shows that $\tilde{G}_k(\tau, \sigma_0)$ is an analytic extension in $C_+$ of $(D_{x_{n}}^{k}G)(x_n, s, \tau, \sigma_0)$ as an operator from $H_0^\nu(x_n>0)$ to $L^2(x_n>0)$ and $\|\tilde{G}_k(\tau, \sigma)f\|_{L^2(x_n>0)}$ is the same bound as (2.4). Since

\[
\begin{align*}
(-1)^k(\tilde{G}(\tau_0, \sigma_0)f, D_{x_{n}}^{k}g)_{L^2(x_n>0)}
= \lim_{\tau \rightarrow \tau_0}(-1)^k\left(\int_{0}^{\infty}G(x_n, s, \tau, \sigma_0) f(s) ds, (D_{x_{n}}^{k}g)(x_n)\right)_{L^2(x_n>0)}
\end{align*}
\]

where $\tilde{G}(\tau, \sigma_0) = \tilde{G}_0(\tau, \sigma_0)$ and $g \in C_0^\nu(\mathbb{R}_+)$. $D_{x_{n}}^{k}(\tilde{G}(\tau_0, \sigma_0)f)$ is in $L^2(x_n>0)$ and equal to $\tilde{G}_k(\tau_0, \sigma_0)f$. This finishes the proof.

Let us set
\[ \hat{u}(\tau, \sigma, x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix_n^\lambda} \hat{f}(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \hat{G}(\tau, \sigma) \hat{f}(\tau, \sigma, s) \]

where \( f \in C^\infty_0(R_+ \times R^n_+) \). Then by Lemma A and the method of the proof of Theorem 4.1 in [2] we have the following

**Lemma B.** \( \hat{u}(\tau, \sigma, x_n) (\sigma \in \mathcal{S}) \) is a solution in \( H^m(x_n > 0) \) of (2.1) and satisfies

\[ \|\hat{u}(\tau, \cdot, \cdot)\|_{m-1}^2 \leq C(Re\tau)^{-2(\nu+1)} \|f(\tau, \cdot, \cdot)\|_{\nu}^2 \]

(Re\tau > 0)

\[ \|\hat{u}(\tau, \cdot, \cdot)\|_{m}^2 \leq C(Re\tau)^{-2(\nu+1)} \|f(\tau, \cdot, \cdot)\|_{\nu+1}^2 \]

(2.8)

where

\[ \|\hat{u}(\tau, \cdot, \cdot)\|_k^2 = \sum_{j=0}^{k} \int_{R^{n-1}} \rho^{2(k-j)} d\sigma \int_{0}^{\infty} |(D^{i}x_{n})\hat{u}(\tau, \sigma, x_n)|^2 dx_n. \]

Let us set

\[ u(t, x) = \frac{1}{(2\pi)^n} \int_{a- i\infty}^{a+i\infty} \int_{R^{n-1}} \hat{u}(\tau, \sigma, x_n) d\sigma d\tau \]

(\( a > 0 \))

where \( f \in C^\infty_0(R_+ \times R^n_+) \) and \( \hat{u}(\tau, \sigma, x_n) \) is defined by (2.7). Then by Lemma B \( u(t, x) \) is a solution of the problem \((P, B_j)\) which satisfies \( e^{-at}u \in H^m(R_+ \times R^n_+), (1.1)\) and

\[ a^{2(\nu+1)} \sum_{j+|\alpha| \leq m} \int e^{-2at} |(D^\alpha x_{\frac{\partial}{\partial t}})u)(t, x)|^2 dtdx \]

(2.9)

\[ \leq C \sum_{j+|\alpha| \leq \nu+1} \int_{R_+ \times R^n_+} e^{-2at} |(D^\alpha f)(t, x)|^2 dtdx. \]

\( C^\infty_0(R_+ \times R^n_+) \) is dense in \( H_{0}^{\nu+1}(R_+ \times R^n_+) \). By (1.1), (2.9) and the limit process we obtain a solution \( u(t, x) \) for \( f \in H_{0}^{\nu+1}(R_+ \times R^n_+) \).

(II) **Necessity and uniqueness of solutions.**

**Lemma C.** Suppose that for fixed \( \tau_0 \in C_+ \) \( \{B_j(\tau_0, \sigma, D_{x_n})\} \) is normal in a bounded open set \( D \subset R^{n-1} \) and \( R(\tau_0, \sigma) \neq 0 \) for \( \sigma \in D \). Let \( \hat{u}(\sigma, x_n) \) be a function whose distribution derivatives in \( x_n \) up to \( m \) belong to \( L^2(D \times R_+) \). Then the problem (2.1) has the following uniqueness property; that is, if for any \( \hat{\varphi}(\sigma, x_n) \in C^\infty_0(D \times R_+) \) whose support in \( \sigma \) is contained in \( D \)

\[ (P(\tau_0, \sigma, D_{x_n})\hat{u}, \hat{\varphi})_{L^2(D \times R_+)} = 0, \]

\[ (B_j(\tau_0, \sigma, D_{x_n})\hat{u})_{|_{x_n=0}, \hat{\varphi}_{|_{x_n=0}}} = 0 \quad (j=1, \cdots, l) \]

then \( \hat{u} = 0 \) in \( L^2(D \times R_+) \).

To prove Lemma C we may use the dual problem.
Lemma D. Suppose that \( \{B_j(\tau_0, \sigma_0, D_{x_n})\} \) is not normal and \( R(\tau_0, \sigma_0) \equiv 0 \). Then there exist a point \((\tilde{\tau}_0, \tilde{\sigma}_0)\) sufficiently close to \((\tau_0, \sigma_0)\) and a normal set \( \{B_j(\tilde{\tau}_0, \sigma, D_{x_n})\} \) in a sufficiently small neighborhood \( D(\tilde{\sigma}_0) \) such that Lopatinskii’s determinant for \( (P(\tilde{\tau}_0, \sigma, D_{x_n}), B_j(\tilde{\tau}_0, \sigma, D_{x_n})) \) does not vanish in \( D(\tilde{\sigma}_0) \) and \( (P(\tilde{\tau}_0, \sigma, D_{x_n}), B_j(\tilde{\tau}_0, \sigma, D_{x_n})) \) and \( (P(\tilde{\tau}_0, \sigma, D_{x_n}), B_j(\tilde{\tau}_0, \sigma, D_{x_n})) \) \( \sigma \in D(\tilde{\sigma}_0) \) have the same solutions.

Proof. Let us set

\[
B_j(\tau, \sigma, D_{x_n}) = \sum_{k=0}^{m_j} b_{j,k}(\tau, \sigma) D_{x_n}^k.
\]

Here we may assume that \( m_j < m_j \) and \( B_{j,m_j}(\tau, \sigma) \) is identically not zero in a neighborhood of \((\tau_0, \sigma_0)\). Furthermore we may assume that \( R(\tau, \sigma) \equiv 0 \) in a neighborhood of \((\tau_0, \sigma_0)\). We carry out the following two processes in this neighborhood: First if \( b_{j,m_j}(\tau_0, \sigma_0) = 0 \) then there is a point \((\tilde{\tau}_0, \tilde{\sigma}_0)\) closed to \((\tau_0, \sigma_0)\) such that \( b_{j,m_j}(\tilde{\tau}_0, \tilde{\sigma}_0) \neq 0 \) in a neighborhood of \( \tilde{\sigma}_0 \). Thus we replace \( B_j(\tau_0, \sigma_0, D_{x_n}) \) by \( b_{j,m_j}(\tilde{\tau}_0, \tilde{\sigma}_0)^{-1} B_j(\tilde{\tau}_0, \sigma, D_{x_n}) \). Second, after the first process, if \( m_j = m_k \) then we replace \( B_j \) or \( B_k \) by \( B_j - B_k \). Remark that in these processes it is invariant that Lopatinskii’s determinant does not vanish. Hence, after each process, it does not occur the case that \( B_j \) and \( B_k \) \( j \neq k \) are monomials in \( D_{x_n} \) of same degree. Therefore Lemma D is obtained by carrying out successively these processes from \( B_j \) of highest order in \( D_{x_n} \).

To prove the uniqueness let \( \hat{u}(\tau, \sigma, x_n)(Re\tau \geq a) \) be the Fourier-Laplace transform of a solution \( u \) of \( (P, B_j)(f = 0, e^{-at} \hat{u}(\tau, \sigma, x_n)) \in H^m(R_+ \times R^n) \). Then for an arbitrarily fixed point \( \tau_0 \) with \( Re\tau_0 \geq a \) the two equations in Lemma C are valid for any \( \hat{\varphi}(\sigma, x_n) \in C^0(\hat{R}^{n-1} \times \hat{R}_+) \). From the proof of Lemma D we see that the set \( Q \) of all the points \((\tilde{\tau}_0, \tilde{\sigma}_0)\) with \( Re\tilde{\tau}_0 \geq a \) satisfying the conclusion in Lemma D is almost everywhere equal to \( \{\tau; Re\tau \geq a\} \times \hat{R}^{n-1} \). Therefore it follows from Lemma C that \( \hat{u} = 0 \) in \( L^2(Q \times \hat{R}_+) \), which implies that for some \( a' \geq a \) \( \hat{u}(a' + i\eta, \sigma, x_n) = 0 \) almost everywhere in \( (\eta, \sigma, x_n) \).

Now we prove the necessity of Theorem 2.1. First, in the proof of theorem 4.1 in [2] pp. 142–144, a sequence \( \{\hat{\varphi}(\tau', \sigma_p)\} \) may be replaced by \( \{(\tilde{\tau}', \tilde{\sigma}')\} \) where the conclusion of Lemma D is satisfied for each point \((\tilde{\tau}', \tilde{\sigma}')\). Here it may be assumed that if \( p \rightarrow \infty \),

\[
(Re\tilde{\tau}')^{p+1} \langle D_{x_n}^k G(x_n, s, \tilde{\tau}', \tilde{\sigma}') \rangle_{H^0(\sigma > 0), L^2(x_n > 0)} \rightarrow \infty
\]

because \( \langle D_{x_n}^k G(x_n, s, \tau, \sigma) \rangle_{H^0(\sigma > 0), L^2(x_n > 0)} \) is continuous in \( (\tau, \sigma) \).

Second we use Lemma C in order to show the inequality in lines 7–10 in [2] p. 143. Thus we may prove our assertion as we have done in [2].
§ 3. Lopatinskii's determinant

In this section we prove the following

**Theorem 3.1.** Suppose that a constant coefficient problem \((P, B_j)\) is \(L^2\)-well-posed with decreasing order \(\nu\) and \(R(\tau, \sigma)\) is identically not zero. Then \(R(\tau_0, \sigma_0) \neq 0\) for \((\tau_0, \sigma_0) \in C_+ \times \mathbb{R}^{n-1}\) is equivalent to the fact that \(B_j(\tau_0, \sigma_0, \lambda) (j=1, \cdots, l)\) are linearly independent as polynomials in \(\lambda\).

From Theorem 3.1 we obtain immediately

**Corollary 3.2.** Under the assumptions of Theorem 3.1 and the normality of \(\{B_j\}\), \(R(\tau, \sigma) \neq 0\) for any \((\tau, \sigma) \in C_+ \times \mathbb{R}^{n-1}\).

**Proof of Theorem 3.1.** \(R(\tau_0, \sigma_0) \neq 0\) for \((\tau_0, \sigma_0) \in C_+ \times \mathbb{R}^{n-1}\) is equivalent to the fact that \(B_j(\tau_0, \sigma_0, \lambda)\) are linearly independent modulo \(\prod_{j=1}^{l} (\lambda - \lambda^j_\tau(\tau_0, \sigma_0))\) as polynomials in \(\lambda\). Hence the necessity is obvious.

Let us set \(B_j(\sigma, \tau, \lambda) = \sum_k b_{j,k}(\tau, \sigma) \lambda^k\). Since \(B_j(\tau_0, \sigma_0, \lambda)\) are linearly independent, the matrix \(\{b_{j,k}(\tau, \sigma)\}\) has rank \(l\); that is, there exist \((k_1, \cdots, k_l)\) and a neighborhood \(U(\tau_0, \sigma_0)\) in \(C_+ \times \mathbb{R}^{n-1}\) such that

\[
\det \left( b_{j,k_h}(\tau, \sigma) ;_{h=1}^{j=1} \cdots , l \right) \neq 0 \text{ in } U(\tau_0, \sigma_0).
\]

Using (3.1) we can construct a function \(v\) satisfying

\[
B_j(\tau, \sigma, D_{x_n}) v|_{x_n=0} = g_j \quad (j=1, \cdots, l)
\]

for any \(g_j \in C\) and \((\tau, \sigma) \in U(\tau_0, \sigma_0)\). In fact, if \(v(\tau, \sigma, x_n) = \sum_{h=1}^{l} v_{k_h}(\tau, \sigma) x_n^{k_h} (k_h !)^{-1}\)

then (3.2) becomes

\[
\sum_{h=1}^{l} b_{j,k_h}(\tau, \sigma) v_{k_h}(\tau, \sigma) = g_j \quad (j=1, \cdots, l).
\]

In the rest of this section \((\tau, \sigma)\) are considered as parameters and belong to \(U(\tau_0, \sigma_0) \cap V^c\), where \(U(\tau_0, \sigma_0)\) is assumed, if necessary, sufficiently small.

Let \(u_1\) be a solution of the Cauchy problem:

\[
P(\tau, \sigma, D_{x_n}) u_1 = P(\tau, \sigma, D_{x_n})(\varphi v) \quad x_n > 0 ,
\]

\[
D_{x_n}^k u_1 = 0 \quad (k=0, \cdots, m-1) \quad x_n = 0
\]

where \(\varphi \in C_0^\infty(\overline{R}_+\) with \(\varphi = 1 \quad (0 \leq x_n \leq 2^{-1})\) and \(\varphi = 0 \quad (x_n \geq 1)\). We consider the problem:

\[
P(\tau, \sigma, D_{x_n}) u = P(\tau, \sigma, D_{x_n})(\varphi v - \varphi u_1) \quad x_n > 0 ,
\]

\[
B_j(\tau, \sigma, D_{x_n}) u = 0 \quad (j=1, \cdots, l) \quad x_n = 0.
\]

Since \(f = P(\tau, \sigma, D_{x_n})(\varphi (v - u_1)) \in C_0^\infty(x_n > 0)\), the problem has a unique solution:
On necessary and sufficient conditions for \(L^2\)-well-posedness of mixed problems

\[
\begin{align*}
\mathcal{L}(\tau, \sigma, x_n) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix_n \lambda}}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \int_0^\infty G(x_n, s, \tau, \sigma) f(\tau, \sigma, s) d\sigma
\end{align*}
\]

where

\[
\begin{align*}
\tilde{u}_1(\tau, \sigma, x_n) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix_n \lambda}}{P(\tau, \sigma, \lambda)} d\lambda, \\
\tilde{u}_2(\tau, \sigma, x_n) &= \frac{1}{2\pi} \int_0^\infty G(x_n, s, \tau, \sigma) f(\tau, \sigma, s) d\sigma.
\end{align*}
\]

Since \(|P(\tau, \sigma, \lambda)|^2 \geq C(Re\tau)^2 (|\tau|^2 + |\sigma|^2 + |\lambda|^2)^{m-1}\), we obtain for some constant \(C(\tau_0, \sigma_0) > 0\)

\[||\tilde{u}_1(\tau, \sigma, x_n)||_{B^{m-1}(x_n>0)} \leq C(\tau_0, \sigma_0) ||f(\tau, \sigma, x_n)||_{L^2(x_n>0)}.\]

Furthermore it follows from the assumption and Theorem 2.1 that for some constant \(C(\tau_0, \sigma_0) > 0\)

\[||\tilde{u}_2(\tau, \sigma, x_n)||_{L^2(x_n>0)} \leq C(\tau_0, \sigma_0) ||f(\tau, \sigma, x_n)||_{L^2(x_n>0)}.\]

From (3.3) and (3.4) we have

\[||u(\tau, \sigma, x_n)||_{L^2(x_n>0)} \leq C(\tau_0, \sigma_0) ||f(\tau, \sigma, x_n)||_{L^2(x_n>0)}.\]

If we put \(w = \varphi v - \varphi u_1 - u\), \(w\) is an \(L^2\)-solution of

\[
P(\tau, \sigma, D_{x_n})w = 0 \quad x_n > 0,
\]

\[
B_j(\tau, \sigma, D_{x_n})w = g_j \quad (j = 1, \cdots, l) \quad x_n = 0.
\]

Furthermore, by (3.5) and the construction of \(v\) and \(u_1\), we see that for some \(C(\tau_0, \sigma_0) > 0\)

\[||w(\tau, \sigma, x_n)||_{L^2(x_n>0)} \leq C(\tau_0, \sigma_0) \sum_{j=1}^{l} |g_j|^2.\]

On the other hand, if \(R(\tau, \sigma) \neq 0\) then the problem (3.6) has a unique solution in \(L^2(x_n>0)\) which is written by the form

\[
w(\tau, \sigma, x_n) = \sum_{j=1}^{l} \frac{R_j(\tau, \sigma, x_n)}{R(\tau, \sigma)} g_j.
\]

Now we arrange the roots \(\lambda_j^+(\tau, \sigma)\) into \(q\)-groups \(\{\lambda_{k,h}^+(\tau, \sigma)\}_{h = 1, 2, \cdots, k'}\) \((k = 1, \cdots, q)\) in a sufficiently small neighborhood \(U(\tau_0, \sigma_0)\) such that \(\lambda_{k,1}^+(\tau, \sigma) = \cdots = \lambda_{k,k'}^+(\tau_0, \sigma_0)\). Let us set

\[
\gamma_{k,1}(\tau, \sigma, x_n) = \exp(ix_n \lambda_{k,1}^+(\tau, \sigma)),
\]

\[
\gamma_{k,h}(\tau, \sigma, x_n) = (ix_n)^{h-1} \int_0^1 d\theta_1 \cdots d\theta_{h-2} \int_0^\theta \theta_{h-2} \cdots \theta_{h-1} \exp(ix_n g_{k,h}(\tau, \sigma, \theta)) d\theta_{h-1},
\]

\[
B_j^{k,h}(\tau, \sigma) = \int_0^1 d\theta_1 \cdots d\theta_{h-2} \int_0^\theta \theta_{h-2} \cdots \theta_{h-1} \left( \frac{\partial^{h-1} B_j}{\partial \lambda} \right)(\tau, \sigma, g_{k,h}(\tau, \sigma, \theta)) d\theta_{h-1},
\]

\[
g_{k,h}(\tau, \sigma, \theta) = \lambda_{k,1}^+(\tau, \sigma) + (\lambda_{k,2}^+(\tau, \sigma) - \lambda_{k,1}^+(\tau, \sigma)) \theta_1 + \cdots + (\lambda_{k,h}^+(\tau, \sigma) - \lambda_{k,h-1}^+(\tau, \sigma)) \theta_1 \cdots \theta_{h-1} \quad (h \geq 2).
\]
Then we have

\[
R(\tau, \sigma) = \begin{vmatrix}
\cdots & \cdots & \cdots \\
B^{k,1}_1(\tau, \sigma) & \cdots & B^{k,1}_l(\tau, \sigma) \\
B^{k,k'}_1(\tau, \sigma) & \cdots & \cdots & \cdots & B^{k,k'}_l(\tau, \sigma) \\
\cdots & \cdots & \cdots \\
\end{vmatrix} / \Delta(\tau, \sigma)
= R'(\tau, \sigma) / \Delta(\tau, \sigma),
\]

where \( \Delta(\tau, \sigma) \neq 0 \) in \( U(\tau_0, \sigma_0) \). If follows from (3.8) that

\[
\omega(\tau, \sigma, x_n) = \sum_{k,h} \left( \sum_{j=1}^{l} \frac{A^{k,h}_j(\tau, \sigma)}{R'(\tau, \sigma)} g_j \right) \gamma_{k,h}(\tau, \sigma, x_n)
\]

where \( A^{k,h}_j(\tau, \sigma) \) is a cofactor of \( R'(\tau, \sigma) \) with respect to \( B^{k,h}_j(\tau, \sigma) \). Since \( \gamma_{k,h}(\tau, \sigma, x_n) \) are linearly independent, it follows from (3.7) and the same method in [2] p. 146 that for some \( C(\tau_0, \sigma_0) > 0 \)

(3.9)

\[
|A^{k,h}_j(\tau, \sigma)| / R'(\tau, \sigma) < C(\tau_0, \sigma_0)
\]

\( (j=1, \ldots, l, k=1, \ldots, q \text{ and } h=1, \ldots, k') \).

By the definition of \( A^{k,h}_j(\tau, \sigma) \) we have

\[
R'(\tau, \sigma)^{-1} = \det \left( \frac{A^{k,h}_j(\tau, \sigma)}{R'(\tau, \sigma)} \right).
\]

Hence it follows from this and (3.9) that for some \( C(\tau_0, \sigma_0) > 0 \)

\[
R'(\tau, \sigma) > C(\tau_0, \sigma_0).
\]

In virtue of the continuity of \( R(\tau, \sigma) \) we conclude that \( R(\tau_0, \sigma_0) \neq 0 \).

\S 4. Necessary condition for \( L^2 \)-well-posedness

(The case of variable coefficients)

In this section we consider variable coefficient problems \( (P, B_j) \). Here coefficients are smooth and constant except a compact set in \( \mathbb{R}^{n+1} \).
Let \((P^0, B^0_j)_{(t,x')}\) be a constant coefficient problem arising from freezing coefficients of their principal parts at a boundary point \((t, x', 0)\) and \(R(t, x'; 1, 0) \neq 0\) for any boundary point \((t, x', 0)\). Then we have the following

**Theorem 4.1.** Suppose that a variable coefficient problem \((P, B_j)\) is strongly \(L^2\)-well-posed and \(R(t, x'; 1, 0) \neq 0\) for any boundary point \((t, x', 0)\). Then each constant coefficient problem \((P^0, B^0_j)_{(t,x')}\) is \(L^2\)-well-posed (with \(\nu=0\)).

**Proof.** First we shall show that for an arbitrarily fixed boundary point \((t_0, x_0', 0)(0 \leq t_0 < T)\) the problem \((P^0, B^0_j)_{(t_0,x')}\) with \(f=0\) and initial data \(U(t_1) = (u_0, u_1, \cdots, u_{m-1})(0 \leq t_1 < T)\) is strongly \(L^2\)-well-posed. Here \(u_k \in C_0^\infty(\mathbb{R}^n)\) for \(k=0, 1, \cdots, m-1\). By the assumption there exists a unique solution \(v \in \mathcal{E}^0((t_0, T), H^m(\mathbb{R}_+^n)) \cap \cdots \cap \mathcal{E}^m((t_0, T), H^0(\mathbb{R}_+^n))\) of the problem \((P, B_j)\) with \(f=0\) and initial data \(V_{(t_0)} = (v_0, v_1, \cdots, v_{m-1})\) which satisfy energy inequalities (1.2) and (1.3). Here \(v_k, (x) = \epsilon^{-k} u_k((x' - x_0') \epsilon^{-1}, x_n \epsilon^{-1})\). Let us set \(u_k, (s, y) = v_k, (t_0 + \epsilon (s-t_1), x_0' + \epsilon y', \epsilon y_n)\). Then \(u_k, (s, y)\) becomes a solution of the equations:

\[
P(t_0 + \epsilon (s-t_1), x_0' + \epsilon y', \epsilon y_n; \epsilon^{-1} \partial_s, \epsilon^{-1} D_y) u = 0 \quad \text{in} \quad y \in \mathbb{R}_+^n,
\]

\[
B_j(t_0 + \epsilon (s-t_1), x_0' + \epsilon y'; \epsilon^{-1} \partial_s, \epsilon^{-1} D_y) u = 0 \quad (j=1, \cdots, l)
\]

in \(y_n=0, y' \in \mathbb{R}^{n-1}\),

with initial data \((s=t_1)\)

\[
\partial^k_s u_k, (x) = \epsilon^{-k} u_k, (x_0' + \epsilon y', \epsilon y_n) = u_k, (y) \quad (k=0, 1, \cdots, m-1)
\]

in \(y \in \mathbb{R}_+^n\).

Furthermore it follows from (1.2) and (1.3) that \(u_k\) satisfy

\[
\|u_k, (s, \cdot)\|_{m-1, \epsilon} \leq C \|U(t_1)\|_{m-1, \epsilon} \quad (t_1 \leq s \leq t_1 + \epsilon^{-1} (T-t_0))
\]

\[
\|u_k, (s, \cdot)\|_{m, \epsilon} \leq C \|U(t_1)\|_{m, \epsilon}
\]

where

\[
\|u, (s, \cdot)\|_{k, \epsilon} = \|u, (s, \cdot)\|_k + \sum_{h=0}^{k-1} \epsilon^{k-h} \|u, (s, \cdot)\|_h
\]

and \(\|u, (s, \cdot)\|_k\) denotes the norm obtained by replacing \(|\alpha| \leq k\) in definition of \(\|u, (s, \cdot)\|_k\) by \(|\alpha|=k\). Therefore, by (1.1) and (1.2), there exists a weak limit \(u(s, \cdot)\) of a subsequence of \(\{u_k, (s, \cdot)\}\) in \(H^m(\mathbb{R}_+^n)\) as \(\epsilon \to 0\) such that \(u(s, y)\) is a solution in \(\mathcal{E}^0((t_1, T), H^m(\mathbb{R}_+^n)) \cap \cdots \cap \mathcal{E}^m((t_1, T), H^0(\mathbb{R}_+^n))\) of the problem \((P^0, B^0_j)_{(t,x')}\) and satisfies the energy inequalities.
$|u(s, \cdot)|_{m-1} \leq C|U(t_1)|_{m-1}$,  
$|u(s, \cdot)|_m \leq C|U(t_1)|_m$.

Since $R(t_0, x'_0 ; 1, 0) \neq 0$ the problem $(P^0, B^0(t, x'_0))$ has a finite propagation speed (See [6]). Hence the problem $(P^0, B^0_j(t, x'_0))$ has a unique solution. Using Poincaré lemma and the finiteness of propagation speed it follows from (4.3) that for any $s$ ($t_1 \leq s \leq T$)

$|u(s, \cdot)|_{m-1} \leq C(s)|U(t_1)|_{m-1}$

where $C(s)$ depends continuously on $s$ and propagation speed. Thus we can define an operator $G(s, t_1)$ from initial data $U(t_1) \in C_0^\infty(\mathbb{R}_+^n)^m$ to the solution $u(s, y)$.

Next we shall show that the problem $(P, B_j)_{(t_0, x'_0)}$ with $f \in C_0^\infty((0, T) \times \mathbb{R}_+^n)$ and zero initial data is $L^2$-well-posed ($\nu=0$). Let us set

$u(t, x) = \int_0^t G(t, s) F(s) ds$

where $F(s) = (0, \cdots, 0, f(s, x))$. Then $u(t, x)$ becomes a solution of the problem $(P^0, B^0_j(t, x'_0))$ such that, by (4.4),

$\|u(t, \cdot)\|_{m-1} \leq C(t) \int_0^t \|f(t, \cdot)\|^2 dt$  \quad (0 \leq t \leq T)

By integrating (4.5) from 0 to $T$ we obtain for some $C(T) > 0$

$\int_0^T \|u(t, \cdot)\|^2_{m-1} dt \leq C(T) \int_0^T \|f(t, \cdot)\|^2 dt$.

Therefore the problem $(P^0, B^0_j(t, x'_0))$ with $f \in C_0^\infty((0, T) \times \mathbb{R}_+^n)$ and homogeneous initial-boundary conditions has a unique solution $u \in H^m((0, T) \times \mathbb{R}_+^n)$ satisfying (4.6). Only this fact is used in the proof of the necessity of Theorem 4.1 in [2]. Thus the proof is complete.

Finally we consider mixed problems of second order:

$P = \partial_t^2 - 2 \sum_{j=1}^n a_j(t, x) \partial_t \partial x_j - \sum_{j,k=1}^n a_{jk}(t, x) \partial x_j \partial x_k + \text{first order term}$,

$B = \partial_{x_n} - \sum_{j=1}^{n-1} b_j(t, x') \partial x_j - c(t, x') \partial_t + h(t, x')$

where $\sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k > 0$ for any non zero vector $\xi \in \mathbb{R}^n$ and all the coefficients are real valued.

Combining Theorem 4.1 with results in [1] we obtain

**Theorem 4.2.** Suppose that $R(t, x' ; 1, 0) \not\equiv 0$ on the boundary. Then
a variable coefficient problem $(P, B)$ is strongly $L^2$-well-posed if and only if each constant coefficient problem $(P^0, B^0)_{(t, x')}$ is $L^2$-well-posed (with $\nu=0$).

**Remark.** Let $(P^0, B^0)$ denotes $(P^0, B^0)_{(t, x')}$ for a fixed point $(t, x', 0)$. Then the following statements (I), (II) and (III) are equivalent:

(I) $(P^0, B^0)$ is $L^2$-well-posed.

(II) Lopatinskii’s determinant for $(P^0, B^0)$ does not vanish in $C_+ \times \mathbb{R}^{n-1}$ and $(P^0, B^0)$ has no supersonic speeds.

(III) $a_{nn}c + a_n \geq 0$ and the quadratic form $H(\sigma) = (a_{nn}c + a_n)^2 (a_{nn}c - b^2) - 2(a_{nn}c + a_n)(a_{nn}a - a_n b)(a_{nn}b + b) - (a_{nn} + a_n^2)(a_{nn}b + b)^2$ is positive semi-definite, where

$$
e = \sum_{j,k=1}^{n-1} a_{jk} \sigma_j \sigma_k, \quad b = \sum_{j=1}^{n-1} a_{nj} \sigma_j, \quad a = \sum_{j=1}^{n-1} a_j \sigma_j, \quad b = \sum_{j=1}^{n-1} b_j \sigma_j \quad (\sigma \in \mathbb{R}^{n-1}).$$

The equivalence (I) and (III) has been proved in [1] and other equivalences are proved by using results in § 2 of [1] and [6].

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**References**


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