On necessary and sufficient conditions for $L^2$-well-posedness of mixed problems for hyperbolic equations II

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Rentaro Agemi and Taira Shirota

§ 1. Introduction

We consider hyperbolic mixed problems $(P, B_j)$ in a quadrant $\mathbb{R}^n_+ \times \mathbb{R}^n_+$:

$$
Pu = f \quad \text{in } t > 0, x_n > 0, x' \in \mathbb{R}^{n-1},
$$

$$
B_j u = 0 \quad (j = 1, \ldots, l) \quad \text{in } t > 0, x_n = 0, x' \in \mathbb{R}^{n-1},
$$

$$
\partial_t^j u = u_k \quad (k = 0, 1, \ldots, m-1) \quad \text{in } t = 0, x_n > 0, x' \in \mathbb{R}^{n-1}
$$

where $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n; x_n > 0\}$, $\partial_t = \frac{\partial}{\partial t}$, $D_{x_j} = (-i)\frac{\partial}{\partial x_j}$, $P = P(t, x; \partial_t, D_x)$ is a strictly $t$-hyperbolic operator of order $m$ and $B_j = B_j(t, x'; \partial_t, D_x)$ are boundary operators of order $m_j < m$. Furthermore $P$ is assumed non-characteristic with respect to the hyperplane $x_n = 0$.

This paper consists of two parts. The first part (§ 2 and § 3) is concerned with constant coefficient problems for homogeneous operators $P$ and $B_j$. In previous paper [2] we assume that $\{B_j\}$ is normal; that is, $m_j \neq m_k$ if $j \neq k$ and the hyperplane $x_n = 0$ is non-characteristic for $B_j$. However, when $\{B_j\}$ is not always normal, it will be suitable for our purpose to use the following

**Definition.** The mixed problem $(P, B_j)$ with homogeneous initial condition is $L^2$-well-posed with decreasing order $\nu$ ($\nu \geq 0$, an integer) if and only if there exist positive constants $a$ and $C$ such that for any $f$ with $e^{-at} f \in H_{0}^{\nu+1}(\mathbb{R}^+_+ \times \mathbb{R}^+_+)$ the problem has a unique solution $u$ with $e^{-at} u \in H^m(\mathbb{R}^+ \times \mathbb{R}^+)$ satisfying

$$
\sum_{j+|\alpha| \leq m-1} \int_0^\infty \int_{\mathbb{R}^n_+} e^{-2at} \left| (\partial_t^j D_x^\alpha u)(t, x) \right|^2 dt \, dx \\
\leq C \sum_{j+|\alpha| \leq m-1} \int_0^\infty \int_{\mathbb{R}^n_+} e^{-2at} \left| (\partial_t^j D_x^\alpha f)(t, x) \right|^2 dt \, dx.
$$

(1.1)

In § 2 a necessary and sufficient condition for $L^2$-well-posedness with decreasing order $\nu$ is given by the terms of compensating function (cf.
Theorem 4.1 in [2]). In §3 we show that under the $L^2$-well-posedness with decreasing order $\nu$ Lopatinskii's determinant $R(\tau_0, \sigma_0) \neq 0$ for $Re \tau_0 > 0$ and $\sigma_0 \in R^{n-1}$ is equivalent to the fact that $B_j(\tau_0, \sigma_0, \lambda)$ are linearly independent as polynomials in $\lambda$.

The second part is concerned with variable coefficient problems. In this part we always assume that $\{B_j\}$ is normal. For initial data $U(t_0) = (u_0, u_1, \ldots, u_{m-1})$ we set up a convenient function space $\mathscr{K}(t_0)$; that is, $U(t_0) \in \mathscr{K}(t_0)$ if and only if $u_k \in H^{m-k}(\mathbb{R}^n_+)$ and satisfy the compatibility condition

$$
\sum_{k=0}^{m_j} b_{j,k}(t, x'; D_x) u_k = 0 \quad \text{on} \quad t = t_0, \ x_n = 0
$$

where

$$B_j(t, x'; \partial_t, D_x) = \sum_{k=0}^{m_j} b_{j,k}(t, x'; D_x) \partial_t^k.$$

**Definition.** The mixed problem $(P, B_j)$ is strongly $L^2$-well-posed if and only if there exist positive constants $T$ and $C$ such that for an arbitrarily fixed time $t_0 \in [0, T]$ the problem $(P, B_j)$ with $f \in H_0^1((t_0, T) \times \mathbb{R}^n_+)$ and $U(t_0) \in \mathscr{K}(t_0)$ has a unique solution $u \in \mathcal{E}^0((t_0, T), H^m(\mathbb{R}^n_+)) \cap \cdots \cap \mathcal{E}^m((t_0, T), H^0(\mathbb{R}^n_+))$ which satisfies energy inequalities

(1. 2)\[ \|u(t, \cdot)\|_{m-1}^2 \leq C(\|U(t_0)\|_{m-1}^2 + \int_{t_0}^{t} \|f(s, \cdot)\|_0^2 ds), \]

(1. 3)\[ \|u(t, \cdot)\|_{m}^2 \leq C(\|U(t_0)\|_{m}^2 + \int_{t_0}^{t} \|f(s, \cdot)\|_1^2 ds) \]

for any $t \in [t_0, T]$, where

\[
\|u(t, \cdot)\|_k^2 = \sum_{j=0}^{k} \|\partial_t^j u(t, \cdot)\|_{k-j}^2,
\]

\[
\|U(t_0)\|_k^2 = \sum_{j=0}^{k} \|u_j(\cdot)\|_{k-j}^2,
\]

\[
\|u(\cdot)\|_k^2 = \sum_{|\alpha|=k} \int_{\mathbb{R}^n} |D_x^\alpha u(x)|^2 dx.
\]

In §4 we show the following: If a variable coefficient problem $(P, B_j)$ is strongly $L^2$-well-posed, then each constant coefficient problem arising from freezing coefficients of their principal parts at a boundary point is $L^2$-well-posed (with decreasing order $\nu = 0$), provided that the corresponding Lopatinskii's determinant $R(t, x'; 1, 0) \neq 0$ on the boundary. Combining this and results in [1] we obtain a certain characterization of strongly $L^2$-well-posed problems with real boundary condition for the case of second order.

This note is the supplement of our previous papers [1] and [2].
§ 2. Necessary and sufficient condition

In this section and the following we consider constant coefficient problems $(P, B_j)$ with homogeneous initial condition. Here $P$ and $B_j$ are homogeneous operators.

We take Laplace transform in $t$ and Fourier transform in $x$ and $\hat{u}(\tau, \sigma, x_n)$ denote the Fourier-Laplace image of $u(t, x', x_n)$ with respect to $(t, x', x_n)$ and $(t, x')$ respectively. By the assumption on $P$ the number $l(m-l)$ of roots $\lambda_{j}^{+}(\tau, \sigma)(\lambda_{k}^{-}(\tau, \sigma))$ of $P(\tau, \sigma, \lambda)=0$ in $\lambda$, which have positive (negative) imaginary part, is independent of $(\tau, \sigma)\in C_{+}\times R^{n-1}$, where $C_{+} = \{\tau\in C; Re\tau>0\}$.

Taking now Fourier-Laplace transform the problem $(P, B_j)$ becomes formally to the boundary value problem of ordinary differential equations depending on parameters $(\tau, \sigma)\in C_{+}\times R^{n-1}$:

\begin{equation}
\begin{align*}
P(\tau, \sigma, D_{x_n}) \hat{u}(\tau, \sigma, x_n) &= \hat{f}(\tau, \sigma, x_n) \quad \text{in } x_n>0, \\
B_j(\tau, \sigma, D_{x_n}) \hat{u}(\tau, \sigma, x_n) &= 0 \quad (j=1, \cdots, l) \quad \text{on } x_n=0.
\end{align*}
\end{equation}

Let $R(\tau, \sigma)$ be Lopatinskii’s determinant; that is,

\[
R(\tau, \sigma) = \det \left(B_j(\tau, \sigma, \lambda_{j}^{+}(\tau, \sigma))\right) \prod_{j>k}(\lambda_{j}^{+}(\tau, \sigma)-\lambda_{k}^{+}(\tau, \sigma))
\]

and $R_j(\tau, \sigma, x_n)$ be the determinant replacing $j$-column in $R(\tau, \sigma)$ by the transposed vector $(\exp(ix_n \lambda_{1}^{+}(\tau, \sigma)), \ldots, \exp(ix_n \lambda_{l}^{+}(\tau, \sigma)))$. Then $R(\tau, \sigma)$ and $R_j(\tau, \sigma, x_n)$ ($j=1, \cdots, l$) are analytic in $(\tau, \sigma)\in C_{+}\times R^{n-1}$. If $R(\tau, \sigma)\neq 0$ for some $(\tau, \sigma)\in C_{+}\times R^{n-1}$, then it is well known that for any $f\in C_{r}^{0}(R_{+})$ the problem (2.1) has a unique bounded solution $u\in C^{\infty}(R_{+})$ which is written by the form

\begin{equation}
\hat{u}(\tau, \sigma, x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix_n \hat{f}(\lambda)}}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \int_{0}^{\infty} G(x_n, s, \tau, \sigma) f(s) ds
\end{equation}

where

\[
G(x_n, s, \tau, \sigma) = -\sum_{j=1}^{l} \frac{R_j(\tau, \sigma, x_n)}{R(\tau, \sigma)} \int_{\Gamma} \frac{B_j(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} e^{-is\lambda} d\lambda
\]

and $\Gamma = \Gamma(\tau, \sigma)$ denotes a simple closed curve in the lower half $\lambda$-plane enclosing all the roots $\lambda_{j}(\tau, \sigma)$.

Let $\Sigma_{+}$ be the set $\{(\tau', \sigma')\in C_{+}\times R^{n-1}; |\tau'|^2 + |\sigma'|^2 = 1\}$ and $\Sigma_{+}$ be its closure. Furthermore let $V$ be the zeros of $R(\tau, \sigma)$ in $C_{+}\times R^{n-1}$ and $V'$ be $V\cap \Sigma_{+}$. $V^{\text{re}}$ and $V^{\text{v}}$ denote the complement of $V'$ and $V$ in $\Sigma_{+}$ and $C_{+}\times R^{n-1}$ respectively. Then we obtain the following
THEOREM 2.1. Suppose that \( R(\tau, \sigma) \) is identically not zero. Then the mixed problem \((P, B_j)\) is \(L^2\)-well-posed with decreasing order \( \nu \) if and only if the following condition is satisfied:

For every \((\tau'_0, \sigma'_0) \in (\Sigma_+ - \Sigma_+) \cup V'\) there exist a constant \( C(\tau'_0, \sigma'_0) \) and a neighborhood \( U(\tau'_0, \sigma'_0) \) such that

\[
\|(D_{x_n}^k G)(x_n, s, \tau', \sigma')\|_{L^2(x_n > 0)} \leq C(\tau'_0, \sigma'_0)(Re \tau')^{-\nu-1}
\]

for any \((\tau', \sigma') \in U(\tau'_0, \sigma'_0) \cap \Sigma_+ \cap V'\) and \( k=0, 1, \ldots, m-1 \),

Since the proof of the theorem is accomplished by the almost same considerations as those in Theorem 4.1 [2], we show only different points.

(1) SUFFICIENCY (EXISTENCE OF SOLUTIONS). Let \( S \) be the set \{\( \sigma \in \mathbb{R}^{n-1} ; R(\tau, \sigma) \) is identically zero in \( \tau \)\}. Then \( S \) is a null set with respect to Lebesgue measure in \( \mathbb{R}^{n-1} \) because \( V \) is so in \( C_+ \times \mathbb{R}^{n-1} \). In what follows we assume \( \sigma \in S \) and for \((\tau, \sigma) \in C_+ \times \mathbb{R}^{n-1} \) \((\tau', \sigma')\) denotes \( (\rho^{-1} \tau, \rho^{-1} \sigma) \) where \( \rho = (|\tau|^2 + |\sigma|^2)^{\frac{1}{2}} \).

LEMMA A. There exists an analytic extension \( \tilde{G}(\tau, \sigma) \) in \( \tau \in C_+ \) of \( G(x_n, s, \tau, \sigma) \) as an operator from \( H_0^{\nu}(s>0) \) to \( L^2(x_n>0) \) such that \( \tilde{G}(\tau, \sigma)f \in H^{m-1}(x_n>0) \) for \( f \in H_0^{\nu}(s>0) \) and

\[
\|(D_{x_n}^k \tilde{G}(\tau, \sigma)f)\|_{L^2(x_n>0)} \leq C(\tau'_0, \sigma'_0)(Re \tau')^{-\nu-1} \rho^{k-m+1}(\sum_{\mu=0}^{\nu} \rho^{-2\mu} \int_0^s |(D_{x_n}^\mu f)(s)|^2 ds)^{\frac{1}{2}}
\]

for any \((\tau_0, \sigma_0)\) with \( R(\tau_0, \sigma_0) = 0 \), \((\tau, \sigma) \in U(\tau'_0, \sigma'_0) \cap (C_+ \times \mathbb{R}^{n-1}) \) and \( k=0, 1, \ldots, m-1 \). Here \( C(\tau'_0, \sigma'_0) \) and \( U(\tau'_0, \sigma'_0) \) are the same ones in Theorem 2.1.

PROOF. If \( \sigma_0 \in S \) and \( R(\tau_0, \sigma_0) = 0 \), then \( \tau_0 \) is an isolated point in \( C_+ \). In virtue of (2.3) and the relations \( (D_{x_n}^k G)(x_n, s, \tau, \sigma) = \rho^{k-m+1}(D_{x_n}^k \tilde{G}(\tau, \sigma)\rho x_n, \rho s, \tau', \sigma') \) \((k=0, 1, \ldots, m-1)\), we obtain for \( f \in H_0^{\nu}(s>0) \) and \( g \in L^2(x_n>0) \)

\[
\left| \left( \int_0^s g(x_n)dx_n \int_0^\infty \rho^{k-m+1}(D_{x_n}^k \tilde{G}(\tau, \sigma)\rho x_n, \rho s, \tau', \sigma')f(s)ds \right) \right| \leq \rho^{k-m+1}\rho^{2\nu}(\sum_{\mu=0}^\nu \rho^{-2\mu} \int_0^s |(D_{x_n}^\mu f)(s)|^2 ds)^{\frac{1}{2}}
\]

(2.5) \( \leq \rho^{k-m+1}\rho^{2\nu}(\sum_{\mu=0}^\nu \rho^{-2\mu} \int_0^s |(D_{x_n}^\mu f)(s)|^2 ds)^{\frac{1}{2}} \)
necessary and sufficient conditions for $L^2$-well-posedness of mixed problems

$$= C(\tau_0', \sigma_0')(Re\tau)^{-\nu-1}\rho^{k-m+1}\Vert g\Vert_{L^2(x_n>0)}\left(\sum_{\mu=0}^{\nu}\rho^{2\nu-2\mu}\int_0^\infty|(D^{\mu}f)(s)|^2ds\right)^{1/2}$$

where $(\tau, \sigma)$ with $(\tau', \sigma')\in(\tau_0', \sigma_0')\cap V'^c\cap\Sigma_+$. In particular, it follows from above that, with some $C(\tau_0, \sigma_0)$,

$$(2.6)$$

$$|\left(\int_0^\infty(D^{k}G)(x_n, s, \tau, \sigma_0)f(s)ds, g(x_n)\right)_{L^2(x_n>0)}| \leq C(\tau_0, \sigma_0)\Vert f\Vert_{H_0^\nu(s>0)}\Vert g\Vert_{L^2(x_n>0)}$$

in a small neighborhood of $\tau_0$. Hence $\left(\int_0^\infty(D^{k}G)(x_n, s, \tau, \sigma_0)f(s)ds, g(x_n)\right)_{L^2(x_n>0)}$ has an analytic extension in $C_+$. By Riesz theorem and (2.6) there exist operators $\tilde{G}_k(\tau, \sigma_0)(k=0,1, \cdots, m-1)$ from $H_0^\nu(s>0)$ to $L^2(x_n>0)$ such that

$$(\tilde{G}_k(\tau, \sigma_0)f, g)_{L^2(x_n>0)}$$

is analytic in $C_+$ and

for $\tau \neq \tau_0$. In virtue of Banach-Steinhaus theorem there exist uniform derivatives

$$(\frac{d}{d\tau}\tilde{G}_k)(\tau, \sigma_0)\frac{1}{h}\lim_{h\rightarrow 0}(\tilde{G}_k(\tau+h, \sigma_0)-\tilde{G}_k(\tau, \sigma_0))$$

such that

$$(\frac{d}{d\tau}\tilde{G}_k(\tau, \sigma_0)f, g)_{L^2(x_n>0)}=(\tilde{G}_k(\tau, \sigma_0)f, g)_{L^2(x_n>0)}.$$

This shows that $\tilde{G}_k(\tau, \sigma_0)$ is an analytic extension in $C_+$ of $(D^{k}G)(x_n, s, \tau, \sigma_0)$ as an operator from $H_0^\nu(s>0)$ to $L^2(x_n>0)$ and $\|\tilde{G}_k(\tau, \sigma)f\|_{L^2(x_n>0)}$ is the same bound as (2.4). Since

$$(2.7)$$

$$(\int_0^\infty(D^{k}G)(x_n, s, \tau_0, \sigma_0)f(s)ds, (D^{k}G)(x_n)\|_{L^2(x_n>0)}$$

$$(\int_0^\infty(D^{k}G)(x_n, s, \tau, \sigma_0)f(s)ds, g(x_n)\|_{L^2(x_n>0)}$$

$$(\tilde{G}_k(\tau_0, \sigma_0)f, g)_{L^2(x_n>0)}$$

where $\tilde{G}(\tau, \sigma_0)=\tilde{G}_0(\tau, \sigma_0)$ and $g\in C_0^\infty(\mathbb{R}_+). D^{k}_x(\tilde{G}(\tau_0, \sigma_0)f)$ is in $L^2(x_n>0)$ and equal to $\tilde{G}_k(\tau_0, \sigma_0)f$. This finishes the proof.

Let us set
\[ \hat{u}(\tau, \sigma, x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\imath \sigma \lambda} \hat{f}(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \tilde{G}(\tau, \sigma) \hat{f}(\tau, \sigma, s) \]

where \( f \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}_+^n) \).

Then by Lemma A and the method of the proof of Theorem 4.1 in [2] we have the following

**Lemma B.** \( \hat{u}(\tau, \sigma, x_n) (\sigma \in \mathcal{S}) \) is a solution in \( H^m(x_n > 0) \) of (2.1) and satisfies

\[ \|\hat{u}(\tau, \cdot, \cdot)\|_{L^2(D)} \leq C(Re\tau)^{-2(v+1)}\|f(\tau, \cdot, \cdot)\|_{L^2(D)} \]

where

\[ \|\hat{u}(\tau, \cdot, \cdot)\|^2 \leq C(Re\tau)^{-2(v+1)}\|f(\tau, \cdot, \cdot)\|_{L^2(D)} \]

Let us set

\[ u(t, x) = \frac{1}{(2\pi)^n} \int_{a-i\infty}^{a+i\infty} \int_{\mathcal{S}} \hat{u}(\tau, \sigma, x_n) d\sigma d\tau \]

where \( f \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}_+^n) \) and \( \hat{u}(\tau, \sigma, x_n) \) is defined by (2.7). Then by Lemma B \( u(t, x) \) is a solution of the problem \((P, B_j)\) which satisfies \( e^{-at}u \in H^m(\mathbb{R}_+ \times \mathbb{R}_+^n) \), (1.1) and

\[ a^{2(v+1)} \sum_{j+|\alpha| \leq m} \int e^{-2at} |(D^\alpha_x \partial_t^j u)(t, x)|^2 dt dx \]

\[ \leq C \sum_{j+|\alpha| \leq m} \int e^{-2at} |(D^\alpha_x f)(t, x)|^2 dt dx \]

\( C_0^\infty(\mathbb{R}_+ \times \mathbb{R}_+^n) \) is dense in \( H_0^{v+1}(\mathbb{R}_+ \times \mathbb{R}_+^n) \). By (1.1), (2.9) and the limit process we obtain a solution \( u(t, x) \) for \( f \in H_0^{v+1}(\mathbb{R}_+ \times \mathbb{R}_+^n) \).

**II. Necessity and Uniqueness of Solutions.**

**Lemma C.** Suppose that for fixed \( \tau_0 \in C_+ \{B_j(\tau_0, \sigma, D_{x_n})\} \) is normal in a bounded open set \( D \subset \mathbb{R}^{n-1} \) and \( R(\tau_0, \sigma) \neq 0 \) for \( \sigma \in D \). Let \( \hat{u}(\sigma, x_n) \) be a function whose distribution derivatives in \( x_n \) up to \( m \) belong to \( L^2(D \times \mathbb{R}_+) \). Then the problem (2.1) has the following uniqueness property; that is, if for any \( \hat{\varphi}(\sigma, x_n) \in C_0^\infty(D \times \mathbb{R}_+) \) whose support in \( \sigma \) is contained in \( D \)

\[ \left( P(\tau_0, \sigma, D_{x_n}) \hat{u}, \hat{\varphi} \right)_{L^2(D \times \mathbb{R}_+)} = 0 , \]

\[ \left( B_j(\tau_0, \sigma, D_{x_n}) \hat{u} \right)_{x_n=0} = 0, \quad \hat{\varphi} \mid_{x_n=0} = 0 \quad (j=1, \cdots, l) , \]

then \( \hat{u} = 0 \) in \( L^2(D \times \mathbb{R}_+) \).

To prove Lemma C we may use the dual problem.
**Lemma D.** Suppose that \( \{B_j(\tau_0, \sigma_0, D_{x_n})\} \) is not normal and \( R(\tau_0, \sigma_0) \neq 0 \). Then there exist a point \((\tilde{\tau}_0, \tilde{\sigma}_0)\) sufficiently close to \((\tau_0, \sigma_0)\) and a normal set \( \{B_j(\tilde{\tau}_0, \sigma, D_{x_n})\} \) in a sufficiently small neighborhood \( D(\tilde{\sigma}_0) \) such that Lopatinskii's determinant for \( (P(\tilde{\tau}_0, \sigma, D_{x_n}), B_j(\tilde{\tau}_0, \sigma, D_{x_n})) \) does not vanish in \( D(\tilde{\sigma}_0) \) and \( (P(\tilde{\tau}_0, \sigma, D_{x_n}), B_j(\tilde{\tau}_0, \sigma, D_{x_n})) \) and \( (P(\tilde{\tau}_0, \sigma, D_{x_n}), B_j(\tilde{\tau}_0, \sigma, D_{x_n})) (\sigma \in D(\tilde{\sigma}_0)) \) have the same solutions.

**Proof.** Let us set

\[
B_j(\tau, \sigma, D_{x_n}) = \sum_{k=0}^{m_j} b_{j,k}(\tau, \sigma) D_{x_n}^k.
\]

Here we may assume that \( m_j \leq m_j' \) and \( b_{j,m_j'}(\tau, \sigma) \) is identically not zero in a neighborhood of \((\tau_0, \sigma_0)\). Furthermore we may assume that \( R(\tau, \sigma) \neq 0 \) in a neighborhood of \((\tau_0, \sigma_0)\). We carry out the following two processes in this neighborhood: First if \( b_{j,m_j'}(\tau_0, \sigma_0) \neq 0 \) then there is a point \((\tilde{\tau}_0, \tilde{\sigma}_0)\) closed to \((\tau_0, \sigma_0)\) such that \( b_{j,m_j'}(\tilde{\tau}_0, \sigma_0) \neq 0 \) in a neighborhood of \( \tilde{\sigma}_0 \). Thus we replace \( B_j(\tau_0, \sigma_0, D_{x_n}) \) by \( b_{j,m_j'}(\tilde{\tau}_0, \sigma_0)^{-1} B_j(\tilde{\tau}_0, \sigma, D_{x_n}) \). Second, after the first process, if \( m_j' = m_j'' \) then we replace \( B_j \) by \( B_k \) by \( B_j - B_k \). Remark that in these processes it is invariant that Lopatinskii's determinant does not vanish. Hence, after each process, it does not occur the case that \( B_j \) and \( B_k (j \neq k) \) are monomials in \( D_{x_n} \) of same degree. Therefore Lemma D is obtained by carrying out successively these processes from \( B_j \) of highest order in \( D_{x_n} \).

To prove the uniqueness let \( \hat{u}(\tau, \sigma, x_n)(Re\tau \geq a) \) be the Fourier-Laplace transform of a solution \( u \) of \( (P, B_j)(f = 0, e^{-s'} u \in H^n(\mathbb{R}^n \times \mathbb{R}_+^n)) \). Then for an arbitrarily fixed point \( \tau_0 \) with \( Re\tau_0 \geq a \) the two equations in Lemma C are valid for any \( \tilde{\tau}_0, \tilde{\sigma}_0 \in \mathbb{C}^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n) \). From the proof of Lemma D we see that the set \( Q \) of all the points \((\tilde{\tau}_0, \tilde{\sigma}_0)\) with \( Re\tilde{\tau}_0 \geq a \) satisfying the conclusion in Lemma D is almost everywhere equal to \((\tau; Re\tau \geq a) \times \mathbb{R}^{n-1} \). Therefore it follows from Lemma C that \( \hat{u} = 0 \) in \( L^2(Q \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{n-1}) \), which implies that for some \( a'(a' \geq a) \) \( \hat{u}(a' + i\eta, \sigma, x_n) = 0 \) almost everywhere in \((\eta, \sigma, x_n)\).

Now we prove the necessity of Theorem 2.1. First, in the proof of Theorem 4.1 in [2] pp. 142–144, a sequence \( \{(\tau'_p, \sigma'_p)\} \) may be replaced by \( \{ (\tilde{\tau}_p', \tilde{\sigma}_p') \} \) where the conclusion of Lemma D is satisfied for each point \((\tilde{\tau}_p', \tilde{\sigma}_p') \). Here it may be assumed that if \( p \to \infty \),

\[ (Re\tilde{\tau}_p')^{a+1} \left\| (D_{x_n}^k G)(x_n, s, \tilde{\tau}_p', \tilde{\sigma}_p') \right\|_{L^2(\mathbb{R}_0^+ \times \mathbb{R}^+ \times \mathbb{R}_0^+)} \to \infty \]

because \( \left\| (D_{x_n}^k G)(x_n, s, \tau, \sigma) \right\|_{L^2(\mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+)} \) is continuous in \((\tau, \sigma)\).

Second we use Lemma C in order to show the inequality in lines 7–10 in [2] p. 143. Thus we may prove our assertion as we have done in [2].
§ 3. Lopatinskii's determinant

In this section we prove the following

**Theorem 3.1.** Suppose that a constant coefficient problem \((P, B_j)\) is \(L^2\)-well-posed with decreasing order \(\nu\) and \(R(\tau, \sigma)\) is identically not zero. Then \(R(\tau_0, \sigma_0)\neq 0\) for \((\tau_0, \sigma_0)\in C_+\times \mathbb{R}_{n-1}\) is equivalent to the fact that \(B_j(\tau_0, \sigma_0, \lambda)\) \((j=1, \cdots, l)\) are linearly independent as polynomials in \(\lambda\).

From Theorem 3.1 we obtain immediately

**Corollary 3.2.** Under the assumptions of Theorem 3.1 and the normality of \(\{B_j\}\), \(R(\tau, \sigma)\neq 0\) for any \((\tau, \sigma)\in C_+\times \mathbb{R}^{n-1}\).

**Proof of Theorem 3.1.** \(R(\tau_0, \sigma_0)\neq 0\) for \((\tau_0, \sigma_0)\in C_+\times \mathbb{R}^{n-1}\) is equivalent to the fact that \(B_j(\tau_0, \sigma_0, \lambda)\) are linearly independene modulo \(\prod_{j=1}^{l}(\lambda-\lambda_j^+)(\tau_0, \sigma_0)\) as polynomials in \(\lambda\). Hence the necessity is obvious.

Let us set \(B_j(\sigma, \tau, \lambda) = \sum_k b_{j,k}(\tau, \sigma)\lambda^k\). Since \(B_j(\tau_0, \sigma_0, \lambda)\) are linearly independent, the matrix \((b_{j,k}(\tau, \sigma))\) has rank \(l\); that is, there exist \((k_1, \cdots, k_l)\) and a neighborhood \(U(\tau_0, \sigma_0)\) in \(C_+\times \mathbb{R}^{n-1}\) such that

\[
\text{det} (b_{j,k_h}(\tau, \sigma);_{h\rightarrow}^{j\downarrow}1, \cdots, l) \neq 0 \quad \text{in} \quad U(\tau_0, \sigma_0).\tag{3.1}
\]

Using (3.1) we can construct a function \(v\) satisfying

\[
B_j(\tau, \sigma, D_{x_n})v|_{x_n=0} = g_j \quad (j=1, \cdots, l)\tag{3.2}
\]

for any \(g_j\in \mathcal{C}\) and \((\tau, \sigma)\in U(\tau_0, \sigma_0)\). In fact, if \(v(\tau, \sigma, x_n) = \sum_{h=1}^{l} v_{k_h}(\tau, \sigma) x_n^{k_h} (k_h!)^{-1}\) then (3.2) becomes to

\[
\sum_{h=1}^{l} b_{j,k_h}(\tau, \sigma) v_{k_h}(\tau, \sigma) = g_j \quad (j=1, \cdots, l).
\]

In the rest of this section \((\tau, \sigma)\) are considered as parameters and belong to \(U(\tau_0, \sigma_0)\cap V^c\), where \(U(\tau_0, \sigma_0)\) is assumed, if necessary, sufficiently small.

Let \(u_1\) be a solution of the Cauchy problem:

\[
P(\tau, \sigma, D_{x_n})u_1 = P(\tau, \sigma, D_{x_n})(\varphi v) \quad x_n > 0,\]
\[
D_{x_n}^k u_1 = 0 \quad (k=0, \cdots, m-1) \quad x_n = 0
\]

where \(\varphi \in C_0(\overline{\mathbb{R}_+})\) with \(\varphi = 1 \quad (0 \leq x_n \leq 2^{-1})\) and \(\varphi = 0 \quad (x_n \geq 1)\). We consider the problem:

\[
P(\tau, \sigma, D_{x_n})u = P(\tau, \sigma, D_{x_n})(\varphi v - \varphi u_1) \quad x_n > 0,\]
\[
B_j(\tau, \sigma, D_{x_n})u = 0 \quad (j=1, \cdots, l) \quad x_n = 0.
\]

Since \(f=P(\tau, \sigma, D_{x_n})(\varphi(v-u_1))\in C_0(\mathbb{R}_+\cap 0<x_n<\infty)\), the problem has a unique solution:
On necessary and sufficient conditions for $L^2$-well-posedness of mixed problems

\[ u(\tau, \sigma, x_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ix_n\lambda} \hat{f}(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi} \int_{0}^{\infty} G(x_n, s, \tau, \sigma) f(\tau, \sigma, s) ds \]

\[ = \bar{u}_1(\tau, \sigma, x_n) + \bar{u}_2(\tau, \sigma, x_n). \]

Since $|P(\tau, \sigma, \lambda)|^2 \geq C(Re\tau)^2(|\tau|^2 + |\sigma|^2 + |\lambda|^2)^{m-1}$, we obtain for some constant $C(\tau_0, \sigma_0) > 0$

(3.3) \[ \|\bar{u}_1(\tau, \sigma, x_n)\|_{H^{m-1}(x_n>0)} \leq C(\tau_0, \sigma_0) \|f(\tau, \sigma, x_n)\|_{L^2(x_n>0)}. \]

Furthermore it follows from the assumption and Theorem 2.1 that for some constant $C(\tau_0, \sigma_0) > 0$

(3.4) \[ \|\bar{u}_2(\tau, \sigma, x_n)\|_{L^2(x_n>0)} \leq C(\tau_0, \sigma_0) \|f(\tau, \sigma, x_n)\|_{B_0^0(x_n>0)}. \]

From (3.3) and (3.4) we have

(3.5) \[ \|u(\tau, \sigma, x_n)\|_{L^2(x_n>0)} \leq C(\tau_0, \sigma_0) \|f(\tau, \sigma, x_n)\|_{H_0^0(x_n>0)}. \]

If we put $w = \varphi v - \varphi u_1 - u$, $w$ is an $L^2$-solution of

(3.6) \[ P(\tau, \sigma, D_{x_n}) w = 0, \quad x_n > 0, \]

\[ B_j(\tau, \sigma, D_{x_n}) w = g_j \quad (j = 1, \cdots, l) \quad x_n = 0. \]

Furthermore, by (3.5) and the construction of $v$ and $u$, we see that for some $C(\tau_0, \sigma_0) > 0$

(3.7) \[ \|w(\tau, \sigma, x_n)\|_{L^2(x_n>0)} \leq C(\tau_0, \sigma_0) \sum_{j=1}^{l} |g_j|^2. \]

On the other hand, if $R(\tau, \sigma) \neq 0$ then the problem (3.6) has a unique solution in $L^2(x_n>0)$ which is written by the form

(3.8) \[ w(\tau, \sigma, x_n) = \sum_{j=1}^{l} \frac{R_j(\tau, \sigma, x_n)}{R(\tau, \sigma)} g_j. \]

Now we arrange the roots $\lambda_j^+(\tau, \sigma)$ into $q$-groups $\{\lambda_{k,h}^+(\tau, \sigma) h = 1, 2, \cdots, k'\}$ ($k=1, \cdots, q$) in a sufficiently small neighborhood $U(\tau_0, \sigma_0)$ such that $\lambda_{k,1}^+(\tau_0, \sigma_0) = \cdots = \lambda_{k,k'}^+(\tau_0, \sigma_0)$. Let us set

\[ \gamma_{k,1}(\tau, \sigma, x_n) = \exp(ix_n \lambda_{k,1}^+(\tau, \sigma)), \]

\[ \gamma_{k,h}(\tau, \sigma, x_n) = (ix_n)^{h-1} \int_{0}^{1} d\theta_1 \cdots d\theta_{h-2} \int_{0}^{1} \theta_1^{h-2} \cdots \theta_{h-2} \exp(ix_n g_{k,h}(\tau, \sigma, \theta)) d\theta_{h-1}, \]

\[ B_j^{k,h}(\tau, \sigma) = \int_{0}^{1} d\theta_1 \cdots d\theta_{h-2} \int_{0}^{1} \theta_1^{h-2} \cdots \theta_{h-2} (\partial_{\lambda}^{h-1} B_j)(\tau, \sigma, g_{k,h}(\tau, \sigma, \theta)) d\theta_{h-1}, \]

\[ g_{k,h}(\tau, \sigma, \theta) = \lambda_{k,1}^+(\tau, \sigma) + (\lambda_{k,2}^+(\tau, \sigma) - \lambda_{k,1}^+(\tau, \sigma)) \theta_1 + \cdots + (\lambda_{k,h}^+(\tau, \sigma) - \lambda_{k,h-1}^+(\tau, \sigma)) \theta_1 \cdots \theta_{h-1} \quad (h \geq 2). \]
Then we have

\[
R(\tau, \sigma) = \left| \begin{array}{ccc} \cdots & \cdots & \cdots \\
B_{1}^{k,1}(\tau, \sigma) & \cdots & B_{l}^{k,1}(\tau, \sigma) \\
\cdots & \cdots & \cdots \\
B_{1}^{k,k'}(\tau, \sigma) & \cdots & B_{l}^{k,k'}(\tau, \sigma) \\
\cdots & \cdots & \cdots \\
\end{array} \right| / \Delta(\tau, \sigma)\]

\[
= R'(\tau, \sigma) / \Delta(\tau, \sigma)
\]

\[
R_{j}(\tau, \sigma, x_{n}) = \left| \begin{array}{ccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
B_{1}^{k,1}(\tau, \sigma) & \cdots & \tau_{k,1}(\tau, \sigma, x_{n}) & \cdots & B_{l}^{k,1}(\tau, \sigma) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
B_{1}^{k,k'}(\tau, \sigma) & \cdots & \tau_{k,k'}(\tau, \sigma, x_{n}) & \cdots & B_{l}^{k,k'}(\tau, \sigma) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array} \right| / \Delta(\tau, \sigma)
\]

\[
= R'_{j}(\tau, \sigma, x_{n}) / \Delta(\tau, \sigma)
\]

where \(\Delta(\tau, \sigma) \neq 0\) in \(U(\tau_{0}, \sigma_{0})\). If follows from (3.8) that

\[
\omega(\tau, \sigma, x_{n}) = \sum_{k, h} \left( \sum_{j=1}^{l} \frac{A_{j}^{k,h}(\tau, \sigma)}{R'(\tau, \sigma)} g_{j} \right) \tau_{k,h}(\tau, \sigma, x_{n})
\]

where \(A_{j}^{k,h}\) is a cofactor of \(R'(\tau, \sigma)\) with respect to \(B_{j}^{k,h}(\tau, \sigma)\). Since \(\tau_{k,h}(\tau, \sigma, x_{n})\) are linearly independent, it follows from (3.7) and the same method in [2] p. 146 that for some \(C(\tau_{0}, \sigma_{0}) > 0\)

(3.9) \[
\left| \frac{A_{j}^{k,h}(\tau, \sigma)}{R'(\tau, \sigma)} \right| < C(\tau_{0}, \sigma_{0})
\]

\((j = 1, \ldots, l, k = 1, \ldots, q \text{ and } h = 1, \ldots, k')\).

By the definition of \(A_{j}^{k,h}(\tau, \sigma)\) we have

\[
R'(\tau, \sigma)^{-1} = \det \left( \frac{A_{j}^{k,h}(\tau, \sigma)}{R'(\tau, \sigma)} \right).
\]

Hence it follows from this and (3.9) that for some \(C(\tau_{0}, \sigma_{0}) > 0\)

\[
R'(\tau, \sigma) > C(\tau_{0}, \sigma_{0}).
\]

In virtue of the continuity of \(R(\tau, \sigma)\) we conclude that \(R(\tau_{0}, \sigma_{0}) \neq 0\).

\section*{§ 4. Necessary condition for \(L^{2}\)-well-posedness (The case of variable coefficients)}

In this section we consider variable coefficient problems \((P, B_{j})\). Here coefficients are smooth and constant except a compact set in \(R^{n+1}\).
Let \((P^0, B^0_{j})_{(t,x')}\) be a constant coefficient problem arising from freezing coefficients of their principal parts at a boundary point \((t, x', 0)\) and \(R(t, x'; \tau, \sigma)\) be Lopatinskii's determinant for the problem \((P^0, B^0_{j})_{(t,x')}\). Then we have the following

**Theorem 4.1.** Suppose that a variable coefficient problem \((P, B_{j})\) is strongly \(L^2\)-well-posed and \(R(t, x'; 1, 0)\neq 0\) for any boundary point \((t, x', 0)\). Then each constant coefficient problem \((P^{0}, B_{j}^{0})_{(t,x')}\) is \(L^2\)-well-posed (with \(\nu=0\)).

**Proof.** First we shall show that for an arbitrarily fixed boundary point \((t_0, x'_0, 0)(0\leq t_0<T)\) the problem \((P^0, B^0_{j})_{(t_0,x'_0)}\) with \(f=0\) and initial data \(U(t_1)=\sum_{k=0}^{m-1} u_k\) \((0\leq t_1<T)\) is strongly \(L^2\)-well-posed. Here \(u_k\in C_0^\infty(R^+_{n})\) for \(k=0, 1, \ldots, m-1\). By the assumption there exists a unique solution \(v_k\in C_0^\infty(t_0, T), H^m_0(R^n_+)\cap \cdots \cap C^m_0(t_0, T), H^0_0(R^n_+)\) of the problem \((P, B_{j})\) with \(f=0\) and initial data \(V(t_0)=(v_{0,1}, v_{1,1}, \ldots, v_{m-1,1})\) which satisfy energy inequalities \((1.2)\) and \((1.3)\). Here \(v_k(x)=\epsilon^{-k}\sum_{h=0}^{k} u_k((x'-x'_0)\epsilon^{-1}, x_n\epsilon^{-1})\). Let us set \(u_k\) \((s, y)=u_k(t_0+\epsilon(s-t_1), x'_0+\epsilon y', \epsilon y_n)\). Then \(u_k(s, y)\) becomes a solution of the equations:

\[
P(t_0+\epsilon(s-t_1), x'_0+\epsilon y', \epsilon y_n; \epsilon^{-1}D_y)u=0 \quad \text{in } y\in R^+_n,
\]

(4.1) \[
B_{j}(t_0+\epsilon(s-t_1), x'_0+\epsilon y'; \epsilon^{-1}D_y)u=0 \quad (j=1, \ldots, l)
\]

with initial data \((s=t_1)\)

\[
\partial_s^k u_k=\epsilon^k v_k(x'_0+\epsilon y', \epsilon y_n)
\]

(4.2) \[
\|u_k(s, \cdot)\|_{l_1, m-1, l} \leq C\|U(t_1)\|_{l_1, m-1, l} \quad (t_1 \leq s \leq t_1+\epsilon^{-1}(T-t_0))
\]

where

\[
\|u(s, \cdot)\|_{k, l} = |u(s, \cdot)|_k + \sum_{h=0}^{k-1} \epsilon^{k-h} |u(s, \cdot)|_h
\]

and \(|u(s, \cdot)|_k\) denotes the norm obtained by replacing \(|\alpha|\leq k\) in definition of \(|u(s, \cdot)|_k\) by \(|\alpha|=k\). Therefore, by \((4.1)\) and \((4.2)\), there exists a weak limit \(u(s, \cdot)\) of a subsequence of \(\{u_k(s, \cdot)\}\) in \(H^m_0(R^n_+)\) as \(\epsilon\to 0\) such that \(u(s, y)\) is a solution in \(C^0((t_1, T), H^m_0(R^n_+))\cap \cdots \cap C^m_0((t_1, T), H^m_0(R^n_+))\) of the problem \((P^0, B_{j})_{(t_0,x'_0)}\) and satisfies the energy inequalities...
\[ \begin{aligned}
|u(s, \cdot)|_{m-1} & \leq C|U(t_1)|_{m-1}, \\
|u(s, \cdot)|_m & \leq C|U(t_1)|_m.
\end{aligned} \]

Since \( R(t_0, x'_0 ; 1, 0) \neq 0 \) the problem \((P^0, B^0_j)_{(t_0, x'_0)}\) has a finite propagation speed (See [6]). Hence the problem \((P^0, B^0_j)_{(t_0, x'_0)}\) has a unique solution. Using Poincaré lemma and the finiteness of propagation speed it follows from (4.3) that for any \( s \) \( (t_1 \leq s \leq T) \)

\[ |u(s, \cdot)|_m \leq C(s)|U(t_1)|_m. \]

where \( C(s) \) depends continuously on \( s \) and propagation speed. Thus we can define an operator \( G(s, t_1) \) from initial data \( U(t_1) \in C_0^\infty(\mathbb{R}^n) \) to the solution \( u(s, y) \).

Next we shall show that the problem \((P, B_j)_{(t_0, x'_0)}\) with \( f \in C_0^\infty((0, T) \times \mathbb{R}_+^n) \) and zero initial data is \( L^2 \)-well-posed \( (\nu = 0) \).

Let us set

\[ u(t, x) = \int_0^t G(t, s)F(s)ds \]

where \( F(s) = (0, \cdots, 0, f(s, x)) \). Then \( u(t, x) \) becomes a solution of the problem \((P^0, B^0_j)_{(t_0, x'_0)}\) such that, by (4.4),

\[ \|u(t, \cdot)\|_{m-1}^2 \leq C(t)\int_0^T \|f(t, \cdot)\|^2 dt \quad (0 \leq t \leq T) \]

By integrating (4.5) from 0 to \( T \) we obtain for some \( C(T) > 0 \)

\[ \int_0^T \|u(t, \cdot)\|_{m-1}^2 dt \leq C(T)\int_0^T \|f(t, \cdot)\|^2 dt. \]

Therefore the problem \((P^0, B^0_j)_{(t_0, x'_0)}\) with \( f \in C_0^\infty((0, T) \times \mathbb{R}_+^n) \) and homogeneous initial-boundary conditions has a unique solution \( u \in H^m((0, T) \times \mathbb{R}_+^n) \) satisfying (4.6). Only this fact is used in the proof of the necessity of Theorem 4.1 in [2]. Thus the proof is complete.

Finally we consider mixed problems of second order:

\[ P = \partial_t^2 - 2 \sum_{j=1}^n a_{j}(t, x) \partial_t \partial_{x_j} - \sum_{j,k=1}^n a_{jk}(t, x) \partial_{x_j} \partial_{x_k} + \text{first order term}, \]

\[ B = \partial_{x_n} - \sum_{j=1}^{n-1} b_j(t, x') \partial_{x_j} - c(t, x') \partial_t + h(t, x') \]

where \( \sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k > 0 \) for any non zero vector \( \xi \in \mathbb{R}^n \) and all the coefficients are real valued.

Combining Theorem 4.1 with results in [1] we obtain

**Theorem 4.2.** Suppose that \( R(t, x' ; 1, 0) \neq 0 \) on the boundary. Then
a variable coefficient problem \((P, B)\) is strongly \(L^2\)-well-posed if and only if each constant coefficient problem \((P^0, B^0)_{(t, x')}\) is \(L^2\)-well-posed (with \(\nu = 0\)).

**Remark.** Let \((P^0, B^0)\) denotes \((P^0, B^0)_{(t, x')}\) for a fixed point \((t, x', 0)\). Then the following statements (I), (II) and (III) are equivalent:

(I) \((P^0, B^0)\) is \(L^2\)-well-posed.

(II) Lopatinskii's determinant for \((P^0, B^0)\) does not vanish in \(C_+ \times \mathbb{R}^{n-1}\) and \((P^0, B^0)\) has no supersonic speeds.

(III) \(a_{nn}c + a_n \geq 0\) and the quadratic form \(H(\sigma) = (a_{nn}c + a_n)^2 (a_{nn}e - b^2) - 2(a_{nn}c + a_n)(a_{nn}a - a_n b)(a_{nn}b + b) - (a_{nn} + a_n)(a_{nn}b + b)^2\) is positive semi-definite, where

\[
e = \sum_{j, k=1}^{n-1} a_{jk} \sigma_j \sigma_k, \quad b = \sum_{j=1}^{n-1} a_{nj} \sigma_j, \quad a = \sum_{j=1}^{n-1} a_j \sigma_j, \quad b = \sum_{j=1}^{n-1} b_j \sigma_j \quad (\sigma \in \mathbb{R}^{n-1}).
\]

The equivalence (I) and (III) has been proved in [1] and other equivalences are proved by using results in § 2 of [1] and [6].

Department of Mathematics
Hokkaido University

**References**


(Received August 31, 1971)