On necessary and sufficient conditions for L^2-well-posedness of mixed problems for hyperbolic equations II

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction

We consider hyperbolic mixed problems (P, B_j) in a quadrant $\boldsymbol{R}_+ \times \boldsymbol{R}^n_+$:

$$\begin{array}{lll} Pu = f & & \text{in } t>0, x_n>0, x' \in \boldsymbol{R}^{n-1}, \\ B_j u = 0 & (j=1, \cdots, l) & \text{in } t>0, x_n=0, x' \in \boldsymbol{R}^{n-1}, \\ \partial_t^k u = u_k & (k=0, 1, \cdots, m-1) & \text{in } t=0, x_n>0, x' \in \boldsymbol{R}^{n-1} \end{array}$$

where $\boldsymbol{R}^n_+ = \{x=(x', x_n) \in \boldsymbol{R}^n ; x_n>0\}$, $\partial_t = \frac{\partial}{\partial t}$, $D_{x_j} = (-i)\frac{\partial}{\partial x_j}$, $P=P(t, x ; \partial_t, D_x)$ is a strictly t-hyperbolic operator of order m and $B_j = B_j(t, x'; \partial_t, D_x)$ are boundary operators of order $m_j < m$. Furthermore P is assumed non-characteristic with respect to the hyperplane $x_n = 0$.

This paper consists of two parts. The first part (§ 2 and § 3) is concerned with constant coefficient problems for homogeneous operators P and B_j. In previous paper [2] we assume that $\{B_j\}$ is normal; that is, $m_j \neq m_k$ if $j \neq k$ and the hyperplane $x_n = 0$ is non-characteristic for B_j. However, when $\{B_j\}$ is not always normal, it will be suitable for our purpose to use the following

DEFINITION. *The mixed problem (P, B_j) with homogeneous initial condition is L^2-well-posed with decreasing order ν ($\nu \geq 0$, an integer) if and only if there exist positive constants a and C such that for any f with $e^{-at} f \in H_0^{\nu+1}(\boldsymbol{R}_+ \times \boldsymbol{R}^n_+)$ the problem has a unique solution u with $e^{-at} u \in H^m(\boldsymbol{R}_+ \times \boldsymbol{R}^n_+)$ satisfying*

$$\sum_{j+|\alpha| \leq m-1} \int_0^\infty \int_{\boldsymbol{R}^n_+} e^{-2at} \left|(\partial_t^j D_x^\alpha u)(t, x)\right|^2 dt\, dx \leq C \sum_{j+|\alpha| \leq \nu} \int_0^\infty \int_{\boldsymbol{R}^n_+} e^{-2at} \left|(\partial_t^j D_x^\alpha f)(t, x)\right|^2 dt\, dx . \tag{1.1}$$

In § 2 a necessary and sufficient condition for L^2-well-posedness with decreasing order ν is given by the terms of compensating function (cf.

Theorem 4. 1 in [2]). In § 3 we show that under the L^2-well-posedness with decreasing order ν Lopatinskii's determinant $R(\tau_0, \sigma_0)\neq 0$ for $Re\ \tau_0>0$ and $\sigma_0\in \boldsymbol{R}^{n-1}$ is equivalent to the fact that $B_j(\tau_0, \sigma_0, \lambda)$ are linearly independent as polynomials in λ.

The second part is concerned with variable coefficient problems. In this part we always assume that $\{B_j\}$ is normal. For initial data $U(t_0)=(u_0, u_1, \cdots, u_{m-1})$ we set up a convenient function space $\mathscr{H}(t_0)$; that is, $U(t_0) \in\mathscr{H}(t_0)$ if and only if $u_k\in H^{m-k}(\boldsymbol{R}^n_+)$ and satisfy the compatibility condition

$$\sum_{k=0}^{m_j} b_{j,k}(t, x'; D_x)u_k=0 \quad \text{on} \quad t=t_0, x_n=0$$

where

$$B_j(t, x'; \partial_t, D_x)=\sum_{k=0}^{m_j} b_{j,k}(t, x'; D_x)\partial_t^k.$$

DEFINITION. *The mixed problem (P, B_j) is strongly L^2-well-posed if and only if there exist positive constants T and C such that for an arbitrarily fixed time $t_0\in[0, T)$ the problem (P, B_j) with $f\in H^1_0((t_0, T)\times R^n_+)$ and $U(t_0)\in\mathscr{H}(t_0)$ has a unique solution $u\in \mathcal{E}^0((t_0, T), H^m(R^n_+))\cap\cdots\cap\mathcal{E}^m((t_0, T), H^0(R^n_+))$ which satisfies energy inequalities*

$$|||u(t, \cdot)|||^2_{m-1}\leqq C(\|U(t_0)\|^2_{m-1}+\int_{t_0}^t \|f(s, \cdot)\|^2_0\, ds), \tag{1. 2}$$

$$|||u(t, \cdot)|||^2_m\leqq C(\|U(t_0)\|^2_m+\int_{t_0}^t \|f(s, \cdot)\|^2_1\, ds) \tag{1. 3}$$

for any $t\in[t_0, T]$, where

$$|||u(t, \cdot)|||^2_k=\sum_{j=0}^k\|(\partial_t^k u)(t, \cdot)\|^2_{k-j},$$

$$\|U(t_0)\|^2_k=\sum_{j=0}^k\|u_j(\cdot)\|_{k-j},$$

$$\|u(\cdot)\|^2_k=\sum_{|\alpha|\leqq k}\int_{R^n_+}|D^\alpha_x u)(x)|^2\, dx.$$

In § 4 we show the following: If a variable coefficient problem (P, B_j) is strongly L^2-well-posed, then each constant coefficient problem arising from freezing coefficients of their principal parts at a boundary point is L^2-well-posed (with decreasing order $\nu=0$), provided that the corresponding Lopatinskii's determinant $R(t, x'; 1, 0)\neq 0$ on the boundary. Combining this and results in [1] we obtain a certain characterization of strongly L^2-well-posed problems with real boundary condition for the case of second order.

This note is the supplement of our previous papers [1] and [2].

§ 2. Necessary and sufficient condition

In this section and the following we consider constant coefficient problems (P, B_j) with homogeneous initial condition. Here P and B_j are homogeneous operators.

We take Laplace transform in t and Fourier transform in x and $\hat{u}(\tau, \sigma, \lambda)$ and $\hat{u}(\tau, \sigma, x_n)$ denote the Fourier-Laplace image of $u(t, x', x_n)$ with respect to (t, x', x_n) and (t, x') respectively. By the assumption on P the number $l(m-l)$ of roots $\lambda_j^+(\tau, \sigma)(\lambda_k^-(\tau, \sigma))$ of $P(\tau, \sigma, \lambda)=0$ in λ, which have positive (negative) imaginary part, is independent of $(\tau, \sigma)\in \boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$, where $\boldsymbol{C}_+ = \{\tau \in \boldsymbol{C}\,;\ Re\ \tau > 0\}$.

Taking now Fourier-Laplace transform the problem (P, B_j) becomes formally to the boundary value problem of ordinary differential equations depending on parameters $(\tau, \sigma)\in \boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$:

$$\text{(2. 1)} \qquad \begin{aligned} &P(\tau, \sigma, D_{x_n})\,\hat{u}(\tau, \sigma, x_n) = \hat{f}(\tau, \sigma, x_n) && \text{in } x_n > 0\,, \\ &B_j(\tau, \sigma, D_{x_n})\,\hat{u}(\tau, \sigma, x_n) = 0 \quad (j=1, \cdots, l) && \text{on } x_n = 0\,. \end{aligned}$$

Let $R(\tau, \sigma)$ be Lopatinskii's determinant; that is,

$$R(\tau, \sigma) = \det\big(B_j(\tau, \sigma, \lambda_k^+(\tau, \sigma))\big)\Big/ \prod_{j>k} \big(\lambda_j^+(\tau, \sigma) - \lambda_k^+(\tau, \sigma)\big)$$

and $R_j(\tau, \sigma, x_n)$ be the determinant replacing j-column in $R(\tau, \sigma)$ by the transposed vector $(\exp(ix_n\lambda_1^+(\tau, \sigma)), \cdots, \exp(ix_n\lambda_l^+(\tau, \sigma)))$. Then $R(\tau, \sigma)$ and $R_j(\tau, \sigma, x_n)\,(j=1, \cdots, l)$ are analytic in $(\tau, \sigma)\in \boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$. If $R(\tau, \sigma)\neq 0$ for some $(\tau, \sigma)\in \boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$, then it is well known that for any $f\in C_0^\infty(\boldsymbol{R}_+)$ the problem (2. 1) has a unique bounded solution $u\in C^\infty(\boldsymbol{R}_+)$ which is written by the form

$$\text{(2. 2)} \qquad \hat{u}(\tau, \sigma, x_n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\lambda x_n}\hat{f}(\lambda)}{P(\tau, \sigma, \lambda)}\,d\lambda + \frac{1}{2\pi}\int_0^\infty G(x_n, s, \tau, \sigma) f(s)\,ds$$

where

$$G(x_n, s, \tau, \sigma) = -\sum_{j=1}^{l} \frac{R_j(\tau, \sigma, x_n)}{R(\tau, \sigma)} \int_\Gamma \frac{B_j(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)}\, e^{-is\lambda}\,d\lambda$$

and $\Gamma = \Gamma(\tau, \sigma)$ denotes a simple closed curve in the lower half λ-plane enclosing all the roots $\lambda_j^-(\tau, \sigma)$.

Let Σ_+ be the set $\{(\tau', \sigma')\in \boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}\,;\ |\tau'|^2 + |\sigma'|^2 = 1\}$ and $\bar{\Sigma}_+$ be its closure. Furthermore let V be the zeros of $R(\tau, \sigma)$ in $\boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$ and V' be $V\cap \Sigma_+$. V'^c and V^c denote the complement of V' and V in Σ_+ and $\boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$ respectively. Then we obtain the following

THEOREM 2.1. *Suppose that $R(\tau, \sigma)$ is identically not zero. Then the mixed problem (P, B_j) is L^2-well-posed with decreasing order ν if and only if the following condition is satisfied*:

For every $(\tau'_0, \sigma'_0)\in(\bar{\Sigma}_+-\Sigma_+)\cup V'$ there exist a constant $C(\tau'_0, \sigma'_0)$ and a neighborhood $U(\tau'_0, \sigma'_0)$ such that

$$\|(D^k_{x_n}G)(x_n, s, \tau', \sigma')\|_{\mathscr{L}(H^\nu_0(s>0), L^2(x_n>0))}\leq C(\tau'_0, \sigma'_0)(Re\,\tau')^{-\nu-1} \tag{2.3}$$

for any $(\tau', \sigma')\in U(\tau'_0, \sigma'_0)\cap\Sigma_+\cap V'^c$ and $k=0, 1, \cdots, m-1$, where $\|\cdot\|_{\mathscr{L}(H^\nu_0(s>0), L^2(x_n>0))}$ denotes the operator norm from $H^\nu_0(s>0)$ to $L^2(x_n>0)$.

Since the proof of the theorem is accomplished by the almost same cosiderations as those in Theorem 4.1 [2], we show only different points.

(I) SUFFICIENCY (EXISTENCE OF SOLUTIONS). Let S be the set $\{\sigma\in\boldsymbol{R}^{n-1}; \{R(\tau, \sigma)$ is identically zero in $\tau\}$. Then S is a null set with respect to Lebesgue measure in $\boldsymbol{R}^{n-1}$ because V is so in $\boldsymbol{C}_+\times\boldsymbol{R}^{n-1}$. In what follows we assume $\sigma\notin S$ and for $(\tau, \sigma)\in\boldsymbol{C}_+\times\boldsymbol{R}_{n-1}$ (τ', σ') denotes $(\rho^{-1}\tau, \rho^{-1}\sigma)$ where $\rho=(|\tau|^2+|\sigma|^2)^{\frac{1}{2}}$.

LEMMA A. *There exists an analytic extension $\widetilde{G}(\tau, \sigma)$ in $\tau\in\boldsymbol{C}_+$ of $G(x_n, s, \tau, \sigma)$ as an operator from $H^\nu_0(s>0)$ to $L^2(x_n>0)$ such that $\widetilde{G}(\tau, \sigma)f\in H^{m-1}(x_n>0)$ for $f\in H^\nu_0(s>0)$ and*

$$\begin{aligned}&\|D^k_{x_n}(\widetilde{G}(\tau, \sigma)f)\|_{L^2(x_n>0)}\\ &\leq C(\tau'_0, \sigma'_0)(Re\,\tau)^{-\nu-1}\rho^{k-m+1}\Big(\sum_{\mu=0}^{\nu}\rho^{2(\nu-\mu)}\int_0^\infty|(D^\mu_s f)(s)|^2\,ds\Big)^{\frac{1}{2}}\end{aligned} \tag{2.4}$$

for any (τ_0, σ_0) with $R(\tau_0, \sigma_0)=0$, (τ, σ) with $(\tau', \sigma')\in U(\tau'_0, \sigma'_0)\cap(\boldsymbol{C}_+\times\boldsymbol{R}^{n-1})$ and $k=0, 1, \cdots, m-1$. Here $C(\tau'_0, \sigma'_0)$ and $U(\tau'_0, \sigma'_0)$ are the same ones in Theorem 2.1.

PROOF. If $\sigma_0\notin S$ and $R(\tau_0, \sigma_0)=0$, then τ_0 is an isolated point in $\boldsymbol{C}_+$. In virtue of (2.3) and the relations $(D^k_{x_n}G)(x_n, s, \tau, \sigma)=\rho^{k-m+1}(D^k_{x_n}G)(\rho x_n, \rho s, \tau', \sigma')$ $(k=0, 1, \cdots, m-1)$, we obtain for $f\in H^\nu_0(s>0)$ and $g\in L^2(x_n>0)$

$$\begin{aligned}&\Big|\Big(\int_0^\infty(D^k_{x_n}G)(x_n, s, \tau, \sigma)f(s)\,ds, g(x_n)\Big)_{L^2(x_n>0)}\Big|\\ &=\Big|\int_0^\infty\overline{g(x_n)}dx_n\int_0^\infty\rho^{k-m+1}(D^k_{x_n}G)(\rho x_n, \rho s, \tau', \sigma')f(s)\,ds\Big|\\ &=\Big|\rho^{k-m-1}\int_0^\infty\overline{g(\rho^{-1}x_n)}dx_n\int_0^\infty(D^k_{x_n}G)(x_n, s, \tau', \sigma')f(\rho^{-1}s)\,ds\Big|\\ &\leq\rho^{k-m-1}\|g(\rho^{-1}x_n)\|_{L^2(x_n>0)}\Big\|\int_0^\infty(D^k_{x_n}G)(x_n, s, \tau', \sigma')f(\rho^{-1}s)ds\Big\|_{L^2(x_n>0)} \qquad (2.5)\\ &\leq\rho^{k-m-\frac{1}{2}}C(\tau'_0, \sigma'_0)(Re\,\tau')^{-\nu-1}\|g\|_{L^2(x_n>0)}\Big(\sum_{\mu=0}^{\nu}\rho^{-2\mu}\int_0^\infty|(D^\mu_s f)(\rho^{-1}s)|^2\,ds\Big)^{\frac{1}{2}}\end{aligned}$$

$$= C(\tau_0', \sigma_0')(Re\tau)^{-\nu-1}\rho^{\nu-m+k+\frac{1}{2}}\|g\|_{L^2(x_n>0)}\Big(\sum_{\mu=0}^{\nu}\rho^{-2\mu}\int_0^\infty |(D_s^\mu f)(\rho^{-1}s)|^2\,ds\Big)^{\frac{1}{2}}$$

$$= C(\tau_0', \sigma_0')(Re\tau)^{-\nu-1}\rho^{k-m+1}\|g\|_{L^2(x_n>0)}\Big(\sum_{\mu=0}^{\nu}\rho^{2\nu-2\mu}\int_0^\infty |(D_s^\mu f)(s)|^2\,ds\Big)^{\frac{1}{2}}$$

where (τ, σ) with $(\tau', \sigma')\in(\tau_0', \sigma_0')\cap V'^c\cap\Sigma_+$. In particular, it follows from above that, with some $C(\tau_0, \sigma_0)$,

$$\Big|\Big(\int_0^\infty (D_{x_n}^k G)(x_n, s, \tau, \sigma_0)f(s)\,ds,\ g(x_n)\Big)_{L^2(x_n>0)}\Big| \leqq C(\tau_0, \sigma_0)\|f\|_{H_0^\nu(s>0)}\|g\|_{L^2(x_n>0)} \tag{2. 6}$$

in a small neighborhood of τ_0. Hence $\Big(\int_0^\infty (D_{x_n} G)(x_n, s, \tau, \sigma_0)f(s)\,ds,\ g(x_n)\Big)_{L^2(x_n>0)}$ has an analytic extension in $\boldsymbol{C}_+$. By Riesz theorem and (2. 6) there exist operators $\tilde{G}_k(\tau, \sigma_0)\,(k=0, 1, \cdots, m-1)$ from $H_0^\nu(s>0)$ to $L^2(x_n>0)$ such that $\tilde{G}(_k(\tau, \sigma_0)f, g)_{L^2(x_n>0)}$ is analytic in $\boldsymbol{C}_+$ and

$$\Big(\tilde{G}_k(\tau, \sigma_0)f,\ g\Big)_{L^2(x_n>0)} = \Big(\int_0^\infty (D_{x_n}^k G)(x_n, s, \tau, \sigma_0)f(s)\,ds,\ g(x_n)\Big)_{L^2(x_n>0)}$$

for $\tau \neq \tau_0$. In virtue of Banach-Steinhaus theorem there exist uniform derivatives

$$\Big(\frac{d}{d\tau}\tilde{G}_k\Big)(\tau, \sigma_0) = \lim_{h\to 0}\frac{1}{h}\Big(\tilde{G}_k(\tau+h, \sigma_0) - \tilde{G}_k(\tau, \sigma_0)\Big) \quad \text{in} \quad \mathscr{L}\Big(H_0^\nu(s>0), L^2(x_n>0)\Big)$$

such that

$$\frac{d}{d\tau}\Big(\tilde{G}_k(\tau, \sigma_0)f,\ g\Big)_{L^2(x_n>0)} = \Big(\Big(\frac{d}{d\tau}\tilde{G}_k\Big)(\tau, \sigma_0)f,\ g\Big)_{L^2(x_n>0)}.$$

This shows that $\tilde{G}_k(\tau, \sigma_0)$ is an analytic extension in $\boldsymbol{C}_+$ of $(D_{x_n}^k G)$ (x_n, s, τ, σ_0) as an operator from $H_0^\nu(s>0)$ to $L^2(x_n>0)$ and $\|\tilde{G}_k(\tau, \sigma)f\|_{L^2(x_n>0)}$ is the same bound as (2. 4). Since

$$\begin{aligned}
&(-1)^k\Big(\tilde{G}(\tau_0, \sigma_0)f,\ D_{x_n}^k g\Big)_{L^2(x_n>0)}\\
&= \lim_{\tau\to\tau_0}(-1)^k\Big(\int_0^\infty G(x_n, s, \tau, \sigma_0)f(s)\,ds,\ (D_{x_n}^k g)(x_n)\Big)_{L^2(x_n>0)}\\
&= \lim_{\tau\to\tau_0}\Big(\int_0^\infty (D_{x_n}^k G)(x_n, s, \tau, \sigma_0)f(s)\,ds,\ g(x_u)\Big)_{L^2(x_n>0)}\\
&= \Big(\tilde{G}_k(\tau_0, \sigma_0)f,\ g\Big)_{L^2(x_n>0)}
\end{aligned}$$

where $\tilde{G}(\tau, \sigma_0) = \tilde{G}_0(\tau, \sigma_0)$ and $g\in C_0^\infty(\boldsymbol{R}_+)$. $D_{x_n}^k(\tilde{G}(\tau_0, \sigma_0)f)$ is in $L^2(x_n>0)$ and equal to $\tilde{G}_k(\tau_0, \sigma_0)f$. This finishes the proof.

Let us set

(2. 7) $$\hat{u}(\tau, \sigma, x_n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ix_n\lambda}\hat{f}(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi}\tilde{G}(\tau, \sigma)\hat{f}(\tau, \sigma, s)$$

where $f \in C_0^\infty(\boldsymbol{R}_+ \times \boldsymbol{R}_+^n)$. Then by Lemma A and the method of the proof of Theorem 4. 1 in [2] we have the following

LEMMA B. *$\hat{u}(\tau, \sigma, x_n)(\sigma \notin S)$ is a solution in $H^m(x_n>0)$* of (2. 1) and satisfies

(2. 8) $$\begin{aligned} |||\hat{u}(\tau, \cdot, \cdot)|||_{m-1}^2 &\leq C(Re\tau)^{-2(\nu+1)}|||f(\tau, \cdot, \cdot)|||_\nu^2, \\ |||\hat{u}(\tau, \cdot, \cdot)|||_m^2 &\leq C(Re\tau)^{-2(\nu+1)}|||f(\tau, \cdot, \cdot)|||_{\nu+1}^2 \end{aligned} \qquad (Re\tau>0)$$

where

$$|||\hat{u}(\tau, \cdot, \cdot)|||_k^2 = \sum_{j=0}^k \int_{R^{n-1}} \rho^{2(k-j)} d\sigma \int_0^\infty |(D_{x_n}^j \hat{u})(\tau, \sigma, x_n)|^2 dx_n .$$

Let us set

$$u(t, x) = \frac{1}{(2\pi)^n} \int_{a-i\infty}^{a+i\infty}\int_{R^{n-1}} \hat{u}(\tau, \sigma, x_n)\, d\sigma\, d\tau \qquad (a>0)$$

where $f \in C_0^\infty(\boldsymbol{R}_+ \times \boldsymbol{R}_+^n)$ and $\hat{u}(\tau, \sigma, x_n)$ is defined by (2. 7). Then by Lemma B $u(t, x)$ is a solution of the problem (P, B_j) which satisfies $e^{-at}u \in H^m(\boldsymbol{R}_+ \times \boldsymbol{R}_+^n)$, (1. 1) and

(2. 9) $$\begin{aligned} & a^{2(\nu+1)} \sum_{j+|\alpha|\leqq m} \int_{R_+ \times R_+^n} e^{-2at} |(D_x^\alpha \partial_t^j u)(t, x)|^2 dt\, dx \\ & \leqq C \sum_{j+|\alpha|\leqq \nu+1} \int_{R_+ \times R_+^n} e^{-2at} |(\partial_t^j D_x^\alpha f)(t, x)|^2 dt\, dx . \end{aligned}$$

$C_0^\infty(\boldsymbol{R}_+ \times \boldsymbol{R}_+^n)$ is dense in $H_0^{\nu+1}(\boldsymbol{R}_+ \times \boldsymbol{R}_+^n)$. By (1. 1), (2. 9) and the limit process we obtain a solution $u(t, x)$ for $f \in H_0^{\nu+1}(\boldsymbol{R}_+ \times \boldsymbol{R}_+^n)$.

(II) NECESSITY AND UNIQENESS OF SOLUTIONS.

LEMMA C. *Suppose that for fixed $\tau_0 \in \boldsymbol{C}_+$ $\{B_j(\tau_0, \sigma, D_{x_n})\}$ is normal in a bounded open set $D \subset R^{n-1}$ and $R(\tau_0, \sigma) \neq 0$ for $\sigma \in D$. Let $\hat{u}(\sigma, x_n)$ be a function whose distribution derivatives in x_n up to m belong to $L^2(D \times \boldsymbol{R}_+)$. Then the problem* (2. 1) *has the following uniqueness property*; *that is, if for any $\hat{\varphi}(\sigma, x_n) \in C_0^\infty(D \times \boldsymbol{R}_+)$ whose support in σ is contained in D*

$$\begin{aligned} & \left(P(\tau_0, \sigma, D_{x_n})\hat{u}, \hat{\varphi}\right)_{L^2(D \times R_+)} = 0 , \\ & \left(B_j(\tau_0, \sigma, D_{x_n})\hat{u}\Big|_{x_n=0}, \hat{\varphi}\Big|_{x_n=0}\right)_{L^2(D)} = 0 \qquad (j=1, \cdots, l), \end{aligned}$$

then $\hat{u}=0$ in $L^2(D \times \boldsymbol{R}_+)$.

To prove Lemma C we may use the dual problem.

LEMMA D. *Suppose that $\{B_j(\tau_0, \sigma_0, D_{x_n})\}$ is not normal and $R(\tau_0, \sigma_0)\neq 0$. Then there exist a point $(\tilde{\tau}_0, \tilde{\sigma}_0)$ sufficiently closed to (τ_0, σ_0) and a normal set $\{B'_j(\tilde{\tau}_0, \sigma, D_{x_n})\}$ in a sufficiently small neighborhood $D(\tilde{\sigma}_0)$ such that Lopatinskii's determinant for $(P(\tilde{\tau}_0, \sigma, D_{x_n}), B'_j(\tilde{\tau}_0, \sigma, D_{x_n}))$ does not vanish in $D(\tilde{\sigma}_0)$ and $(P(\tilde{\tau}_0, \sigma, D_{x_n}), B_j(\tilde{\tau}_0, \sigma, D_{x_n}))$ and $(P(\tilde{\tau}_0, \sigma, D_{x_n}), B'_j(\tilde{\tau}_0, \sigma, D_{x_n}))$ $(\sigma\in D(\tilde{\sigma}_0))$ have the same solutions.*

PROOF. Let us set

$$B_j(\tau, \sigma, D_{x_n}) = \sum_{k=0}^{m'_j} b_{j,k}(\tau, \sigma) D^k_{x_n}.$$

Here we may assume that $m'_j\leq m_j$ and $b_{j,m'_j}(\tau, \sigma)$ is identically not zero in a neighborhood of (τ_0, σ_0). Furthermore we may assume that $R(\tau, \sigma)\neq 0$ in a neighborhood of (τ_0, σ_0). We carry out the following two processes in this neighborhood: First if $b_{j,m'_j}(\tau_0, \sigma_0)=0$ then there is a point $(\tilde{\tau}_0, \tilde{\sigma}_0)$ closed to (τ_0, σ_0) such that $b_{j,m'_j}(\tilde{\tau}_0, \sigma)\neq 0$ in a neighborhood of $\tilde{\sigma}_0$. Thus we replace $B_j(\tau_0, \sigma_0, D_{x_n})$ by $b_{j,m'_j}(\tilde{\tau}_0, \sigma)^{-1} B_j(\tilde{\tau}_0, \sigma, D_{x_n})$. Second, after the first process, if $m'_j=m'_k$ then we replace B_j or B_k by B_j-B_k. Remark that in these processes it is invariant that Lopatinskii's determinant does not vanish. Hence, after each process, it does not occur the case that B_j and B_k $(j\neq k)$ are monomials in D_{x_n} of same degree. Therefore Lemma D is obtained by carrying out successively these processes from B_j of highest order in D_{x_n}.

To prove the uniqueness let $\hat{u}(\tau, \sigma, x_n)$ $(Re\tau\geq a)$ be the Fourier-Laplace transform of a solution u of (P, B_j) $(f=0, e^{-at}u\in H^m(\boldsymbol{R}_+\times \boldsymbol{R}^n_+))$. Then for an arbitrarily fixed point τ_0 with $Re\tau_0\geq a$ the two equations in Lemma C are valid for any $\hat{\varphi}(\sigma, x_n)\in C_0^\infty(\boldsymbol{R}^{n-1}\times \boldsymbol{R}_+)$. From the proof of Lemma D we see that the set Q of all the points $(\tilde{\tau}_0, \tilde{\sigma}_0)$ with $Re\tilde{\tau}_0\geq a$ satisfying the conclusion in Lemma D is almost everywhere equal to $\{\tau\,;\, Re\tau\geq a\}\times \boldsymbol{R}^{n-1}$. Therefore it follows from Lemma C that $\hat{u}=0$ in $L^2(Q\times \boldsymbol{R}_+)$, which implies that for some $a'(a'\geq a)$ $\hat{u}(a'+i\eta, \sigma, x_n)=0$ almost everywhere in (η, σ, x_n).

Now we prove the necessity of Theorem 2.1. First, in the proof of theorem 4.1 in [2] pp. 142–144, a sequence $\{(\tau'_p, \sigma'_p)\}$ may be replaced by $\{(\tilde{\tau}'_p, \tilde{\sigma}'_p)\}$ where the conclusion of Lemma D is satisfied for each point $(\tilde{\tau}'_p, \tilde{\sigma}'_p)$. Here it may be assumed that if $p\to\infty$,

$$(Re\,\tilde{\tau}'_p)^{\nu+1}\left\|(D^{k_0}_{x_n}G)(x_n, s, \tilde{\tau}'_p, \tilde{\sigma}'_p)\right\|_{\mathscr{L}(H_0^\nu(s>0),\, L^2(x_n>0))}\to\infty$$

because $\|(D^k_{x_n}G)(x_n, s, \tau, \sigma)\|_{\mathscr{L}(H_0^\nu(s>0),\,(L^2(x_n>0))}$ is continuous in (τ, σ).

Second we use Lemma C in order to show the inequality in lines 7–10 in [2] p. 143. Thus we may prove our assertion as we have done in [2].

§ 3. Lopatinskii's determinant

In this section we prove the following

THEOREM 3.1. *Suppose that a constant coefficient problem* (P, B_j) *is* L^2*-well-posed with decreasing order* ν *and* $R(\tau, \sigma)$ *is identically not zero. Then* $R(\tau_0, \sigma_0)\neq 0$ *for* $(\tau_0, \sigma_0)\in \boldsymbol{C}_+\times \boldsymbol{R}_{n-1}$ *is equivalent to the fact that* $B_j(\tau_0, \sigma_0, \lambda)$ $(j=1, \cdots, l)$ *are linearly independent as polynomials in* λ.

From Theorem 3.1 we obtain immediately

COROLLARY 3.2. *Under the assumptions of Theorem* 3.1. *and the normality of* $\{B_j\}$, $R(\tau, \sigma)\neq 0$ *for any* $(\tau, \sigma)\in \boldsymbol{C}_+\times \boldsymbol{R}^{n-1}$.

PROOF OF THEOREM 3.1. $R(\tau_0, \sigma_0)\neq 0$ for $(\tau_0, \sigma_0)\in \boldsymbol{C}_+\times \boldsymbol{R}^{n-1}$ is equivalent to the fact that $B_j(\tau_0, \sigma_0, \lambda)$ are linearly independene modulo $\prod_{j=1}^{l}(\lambda-\lambda_j^+(\tau_0, \sigma_0))$ as polynomials in λ. Hence the necessity is obvious.

Let us set $B_j(\sigma, \tau, \lambda)=\sum_k b_{j,k}(\tau, \sigma)\lambda^k$. Since $B_j(\tau_0, \sigma_0, \lambda)$ are linearly independent, the matrix $(b_{j,k}(\tau, \sigma))$ has rank l; that is, there exist $(k_1, \cdots, k_l)$ and a neighborhood $U(\tau_0, \sigma_0)$ in $\boldsymbol{C}_+\times \boldsymbol{R}^{n-1}$ such that

$$\det\left(b_{j,k_h}(\tau, \sigma)\,;\ \overset{j\,\downarrow}{\underset{h\rightarrow}{}}\ 1, \cdots, l\right)\neq 0 \ \text{ in } U(\tau_0, \sigma_0). \tag{3.1}$$

Using (3.1) we can construct a function v satisfying

$$B_j(\tau, \sigma, D_{x_n})v|_{x_n=0}=g_j \qquad (j=1, \cdots, l) \tag{3.2}$$

for any $g_j\in \boldsymbol{C}$ and $(\tau, \sigma)\in U(\tau_0, \sigma_0)$. In fact, if $v(\tau, \sigma, x_n)=\sum_{h=1}^{l}v_{k_h}(\tau, \sigma)x_n^{k_h}(k_h!)^{-1}$ then (3.2) becomes to

$$\sum_{h=1}^{l}b_{j,k_h}(\tau, \sigma)v_{k_h}(\tau, \sigma)=g_j \qquad (j=1, \cdots, l).$$

In the rest of this section (τ, σ) are considered as parameters and belong to $U(\tau_0, \sigma_0)\cap V^c$, where $U(\tau_0, \sigma_0)$ is assumed, if necessary, sufficiently small.

Let u_1 be *a* solution of the Caucly problem:

$$\begin{aligned} &P(\tau, \sigma, D_{x_n})u_1=P(\tau, \sigma, D_{x_n})(\varphi v) && x_n>0, \\ &D_{x_n}^k u_1=0 \qquad (k=0, \cdots, m-1) && x_n=0 \end{aligned}$$

where $\varphi\in C_0^\infty(\overline{\boldsymbol{R}}_+)$ with $\varphi=1$ $(0\leq x_n\leq 2^{-1})$ and $\varphi=0$ $(x_n\geq 1)$. We consider the problem:

$$\begin{aligned} &P(\tau, \sigma, D_{x_n})u=P(\tau, \sigma, D_{x_n})(\varphi v-\varphi u_1) && x_n>0, \\ &B_j(\tau, \sigma, D_{x_n})u=0 \qquad (j=1, \cdots, l) && x_n=0. \end{aligned}$$

Since $f\equiv P(\tau, \sigma, D_{x_n})(\varphi(v-u_1))\in C_0^\infty(x_n>0)$, the problem has a unique solution:

$$u(\tau, \sigma, x_n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ix_n\lambda}\hat{f}(\tau, \sigma, \lambda)}{P(\tau, \sigma, \lambda)} d\lambda + \frac{1}{2\pi}\int_0^{\infty} G(x_n, s, \tau, \sigma) f(\tau, \sigma, s)\, ds$$
$$= \tilde{u}_1(\tau, \sigma, x_n) + \tilde{u}_2(\tau, \sigma, x_n).$$

Since $|P(\tau, \sigma, \lambda)|^2 \geq C(Re\,\tau)^2 (|\tau|^2 + |\sigma|^2 + |\lambda|^2)^{m-1}$, we obtain for some constant $C(\tau_0, \sigma_0) > 0$

(3. 3) $$\|\tilde{u}_1(\tau, \sigma, x_n)\|_{H^{m-1}(x_n>0)} \leq C(\tau_0, \sigma_0)\| f(\tau, \sigma, x_n)\|_{L^2(x_n>0)}\,.$$

Furthermore it follows from the assumption and Theorem 2. 1 that for some constant $C(\tau_0, \sigma_0) > 0$

(3. 4) $$\|\tilde{u}_2(\tau, \sigma, x_n)\|_{L^2(x_n>0)} \leq C(\tau_0, \sigma_0)\| f(\tau, \sigma, x_n)\|_{H_0^{\nu}(x_n>0)}\,.$$

From (3. 3) and (3. 4) we have

(3. 5) $$\|u(\tau, \sigma, x_n)\|_{L^2(x_n>0)} \leq C(\tau_0, \sigma_0)\| f(\tau, \sigma, x_n)\|_{H_0^{\nu}(x_n>0)}\,.$$

If we put $w = \varphi v - \varphi u_1 - u$, w is an L^2-solution of

(3. 6) $$\begin{aligned} &P(\tau, \sigma, D_{x_n}) w = 0 && x_n > 0\,, \\ &B_j(\tau, \sigma, D_{x_n}) w = g_j \qquad (j=1, \cdots, l) && x_n = 0\,. \end{aligned}$$

Furthermore, by (3. 5) and the construction of v and u_1, we see that for some $C(\tau_0, \sigma_0) > 0$

(3. 7) $$\|w(\tau, \sigma, x_n)\|^2_{L^2(x_n>0)} \leq C(\tau_0, \sigma_0) \sum_{j=1}^{l} |g_j|^2.$$

On the other hand, if $R(\tau, \sigma) \neq 0$ then the problem (3. 6) has a unique solution in $L^2(x_n > 0)$ which is written by the form

(3. 8) $$w(\tau, \sigma, x_n) = \sum_{j=1}^{l} \frac{R_j(\tau, \sigma, x_n)}{R(\tau, \sigma)} g_j\,.$$

Now we arrange the roots $\lambda_j^+(\tau, \sigma)$ into q-groups $\{\lambda_{k,h}^+(\tau, \sigma)\ h = 1, 2, \cdots, k'\}$ $(k = 1, \cdots, q)$ in a sufficiently small neighborhood $U(\tau_0, \sigma_0)$ such that $\lambda_{k,1}^+(\tau_0, \sigma_0) = \cdots = \lambda_{k,k'}^+(\tau_0, \sigma_0)$. Let us set

$$\gamma_{k,1}(\tau, \sigma, x_n) = \exp\left(ix_n \lambda_{k,1}^+(\tau, \sigma)\right),$$

$$\gamma_{k,h}(\tau, \sigma, x_n) = (ix_n)^{h-1} \int_0^1 d\theta_1 \cdots d\theta_{h-2} \int_0^1 \theta_1^{h-2} \cdots \theta_{h-2} \exp\left(ix_n g_{k,h}(\tau, \sigma, \theta)\right) d\theta_{h-1},$$

$$B_j^{k,h}(\tau, \sigma) = \int_0^1 d\theta_1 \cdots d\theta_{h-2} \int \theta_1^{h-2} \cdots \theta_{h-2} (\partial_\lambda^{h-1} B_j)\left(\tau, \sigma, g_{k,h}(\tau, \sigma, \theta)\right) d\theta_{h-1}\,,$$

$$g_{k,h}(\tau, \sigma, \theta) = \lambda_{k,1}^+(\tau, \sigma) + \left(\lambda_{k,2}^+(, \sigma) - \lambda_{k,1}^+(\tau, \sigma)\right)\theta_1 +$$
$$\cdots + \left(\lambda_{k,h}^+(\tau, \sigma) - \lambda_{k,h-1}^+(\tau, \sigma)\right)\theta_1 \cdots \theta_{h-1} \qquad (h \geq 2)\,.$$

Then we have

$$R(\tau,\sigma)=\begin{vmatrix} \cdot & \cdots\cdots & \cdot \\ B_1^{k,1}(\tau,\sigma) & \cdots\cdots & B_l^{k,1}(\tau,\sigma) \\ \cdot & \cdots\cdots & \cdot \\ B_1^{k,k'}(\tau,\sigma) & \cdots\cdots & B_l^{k,k'}(\tau,\sigma) \\ \cdot & \cdots\cdots & \cdot \end{vmatrix} \Big/ \varDelta(\tau,\sigma)$$

$$= R'(\tau,\sigma)/\varDelta(\tau,\sigma),$$

$$R_j(\tau,\sigma,x_n)=\begin{vmatrix} \cdot & \cdots\cdots & \cdot & \cdots\cdots & \cdot \\ B_1^{k,1}(\tau,\sigma) & \cdots\cdots & \gamma_{k,1}(\tau,\sigma,x_n) & \cdots\cdots & B_l^{k,1}(\tau,\sigma) \\ \cdot & \cdots\cdots & \cdot & \cdots\cdots & \cdot \\ B_1^{k,k'}(\tau,\sigma) & \cdots\cdots & \gamma_{k,k'}(\tau,\sigma,x_n) & \cdots\cdots & B_l^{k,k'}(\tau,\sigma) \\ \cdot & \cdots\cdots & \cdot & \cdots\cdots & \cdot \end{vmatrix} \Big/ \varDelta(\tau,\sigma)$$

$$= R'_j(\tau,\sigma,x_n)/\varDelta(\tau,\sigma)$$

where $\varDelta(\tau,\sigma)\neq 0$ in $U(\tau_0,\sigma_0)$. If follows from (3. 8) that

$$w(\tau,\sigma,x_n)=\sum_{k,h}\Big(\sum_{j=1}^{l}\frac{A_j^{k,h}(\tau,\sigma)}{R'(\tau,\sigma)}g_j\Big)\gamma_{k,h}(\tau,\sigma,x_n)$$

where $A_j^{k,h}$ is a cofactor of $R'(\tau,\sigma)$ with respect to $B_j^{k,h}(\tau,\sigma)$. Since $\gamma_{k,h}(\tau,\sigma,x_n)$ are linearly independent, it follows from (3. 7) and the same method in [2] p. 146 that for some $C(\tau_0,\sigma_0)>0$

$$\left|\frac{A_j^{k,h}(\tau,\sigma)}{R'(\tau,\sigma)}\right|<C(\tau_0,\sigma_0) \tag{3. 9}$$

$$(j=1,\cdots,l,\ k=1,\cdots,q \ \text{ and } h=1,\cdots,k').$$

By the definition of $A_j^{k,h}(\tau,\sigma)$ we have

$$R'(\tau,\sigma)^{-1}=\det\Big(\frac{A_j^{k,h}(\tau,\sigma)}{R'(\tau,\sigma)}\Big).$$

Hence it follows from this and (3. 9) that for some $C(\tau_0,\sigma_0)>0$

$$R'(\tau,\sigma)>C(\tau_0,\sigma_0).$$

In virtue of the continuity of $R(\tau,\sigma)$ we conclude that $R(\tau_0,\sigma_0)\neq 0$.

§ 4. Necessary condition for L^2-well-posedness (The case of variable coefficients)

In this section we consider variable coefficient problems (P, B_j). Here coefficients are smooth and constant except a compact set in $\boldsymbol{R}^{n+1}$.

Let $(P^0, B_j^0)_{(t,x')}$ be a constant coefficient problem arising from freezing coefficients of their principal parts at a boundary point $(t, x', 0)$ and $R(t, x'; \tau, \sigma)$ be Lopatinskii's determinant for the problem $(P^0, B_j^0)_{(t,x')}$. Then we have the following

THEOREM 4.1. *Suppose that a variable coefficient problem (P, B_j) is strongly L^2-well-posed and $R(t, x'; 1, 0) \neq 0$ for any boundary point $(t, x', 0)$. Then each constant coefficient problem $(P^0, B_j^0)_{(t,x')}$ is L^2-well-posed (with $\nu=0$).*

PROOF. First we shall show that for an arbitrarily fixed boundary point $(t_0, x_0', 0)$ $(0 \leq t_0 < T)$ the problem $(P^0, B_j^0)_{(t_0,x_0')}$ with $f=0$ and initial data $U(t_1)=(u_0, u_1, \cdots, u_{m-1})$ $(0 \leq t_1 < T)$ is strongly L^2-well-posed. Here $u_k \in C_0^\infty(\boldsymbol{R}_+^n)$ for $k=0, 1, \cdots, m-1$. By the assumption there exists a unique solution $v_\varepsilon \in \mathcal{E}^0((t_0, T), H^m(\boldsymbol{R}_+^n)) \cap \cdots \cap \mathcal{E}^m((t_0, T), H^0(\boldsymbol{R}_+^n))$ of the problem (P, B_j) with $f=0$ and initial data $V_\varepsilon(t_0)=(v_{0,\varepsilon}, v_{1,\varepsilon}, \cdots, v_{m-1,\varepsilon})$ which satisfy energy inequalities (1.2) and (1.3). Here $v_{k,\varepsilon}(x)=\varepsilon^{-k} u_k((x'-x_0')\varepsilon^{-1}, x_n \varepsilon^{-1})$. Let us set $u_\varepsilon(s, y) = v_\varepsilon(t_0 + \varepsilon(s - t_1), x_0' + \varepsilon y', \varepsilon y_n)$. Then $u_\varepsilon(s, y)$ becomes a solution of the equations:

for $s \in [t_1, t_1 + \varepsilon^{-1}(T-t_0))$

$$
\begin{aligned}
&P(t_0+\varepsilon(s-t_1), x_0'+\varepsilon y', \varepsilon y_n ; \varepsilon^{-1}\partial_s, \varepsilon^{-1}D_y) u = 0 && \text{in } y \in \boldsymbol{R}_+^n, \\
&B_j(t_0+\varepsilon(s-t_1), x_0'+\varepsilon y' ; \varepsilon^{-1}\partial_s, \varepsilon^{-1}D_y) u = 0 && (j=1, \cdots, l) \\
& && \text{in } y_n=0,\ y' \in \boldsymbol{R}^{n-1},
\end{aligned}
\tag{4.1}
$$

with initial data $(s=t_1)$

$$
\begin{aligned}
\partial_s^k u_\varepsilon &= \varepsilon^k v_{k,\varepsilon}(x_0'+\varepsilon y', \varepsilon y_n) \\
&= u_k(y) \qquad (k=0, 1, \cdots, m-1) \qquad \text{in } y \in \boldsymbol{R}_+^n .
\end{aligned}
$$

Furthermore it follows from (1.2) and (1.3) that u_ε satisfy

$$
\begin{aligned}
&|||u_\varepsilon(s, \cdot)|||_{m-1,\varepsilon} \leq C \| U(t_1) \|_{m-1,\varepsilon} \\
&|||u_\varepsilon(s, \cdot)|||_{m,\varepsilon} \leq C \| U(t_1) \|_{m,\varepsilon}
\end{aligned}
\qquad (t_1 \leq s \leq t_1 + \varepsilon^{-1}(T-t_0))
\tag{4.2}
$$

where

$$
|||u(s, \cdot)|||_{k,\varepsilon} = |u(s, \cdot)|_k + \sum_{h=0}^{k-1} \varepsilon^{k-h} |u(s, \cdot)|_h
$$

and $|u(s, \cdot)|_k$ denotes the norm obtained by replacing $|\alpha| \leq k$ in definition of $|||u(s, \cdot)|||_k$ by $|\alpha|=k$. Therefore, by (4.1) and (4.2), there exists a weak limit $u(s, \cdot)$ of a subsequence of $\{u_\varepsilon(s, \cdot)\}$ in $H^m(\boldsymbol{R}_+^n)$ as $\varepsilon \to 0$ such that $u(s, y)$ is a solution in $\mathcal{E}^0((t_1, T), H^m(\boldsymbol{R}_+^n)) \cap \cdots \cap \mathcal{E}^m((t_1, T), H^0(\boldsymbol{R}_+^n))$ of the problem $(P^0, B_j^0)_{(t_0,x_0')}$ and satisfies the energy inequalities

$$(4.3)\qquad \begin{aligned} |u(s,\cdot)|_{m-1} &\leq C|U(t_1)|_{m-1},\\ |u(s,\cdot)|_m &\leq C|U(t_1)|_m. \end{aligned}$$

Since $R(t_0, x_0'; 1, 0) \neq 0$ the problem $(P^0, B_j^0)_{(t_0, x_0')}$ has a finite propagation speed (See [6]). Hence the problem $(P^0, B_j^0)_{(t_0, x_0')}$ has a unique solution. Using Poincaré lemma and the finiteness of propagation speed it follows from (4.3) that for any s $(t_1 \leq s \leq T)$

$$(4.4)\qquad |||u(s,\cdot)|||_{m-1} \leq C(s)\|U(t_1)\|_{m-1}$$

where $C(s)$ depends continuously on s and propagation speed. Thus we can define an operator $G(s, t_1)$ from initial data $U(t_1) \in C_0^\infty(\boldsymbol{R}_+^n)^m$ to the solution $u(s, y)$.

Next we shall show that the problem $(P, B_j)_{(t_0, x_0')}$ with $f \in C_0^\infty((0, T) \times \boldsymbol{R}_+^n)$ and zero initial data is L^2-well-posed $(\nu = 0)$. Let us set

$$u(t, x) = \int_0^t G(t, s) F(s)\, ds$$

where $F(s) = (0, \cdots, 0, f(s, x))$. Then $u(t, x)$ becomes a solution of the problem $(P^0, B_j^0)_{(t_0, x_0')}$ such that, by (4.4),

$$(4.5)\qquad |||u(t,\cdot)|||_{m-1}^2 \leq C(t)\int_0^T \|f(t,\cdot)\|_0^2\, dt \qquad (0 \leq t \leq T)$$

By integrating (4.5) from 0 to T we obtain for some $C(T) > 0$

$$(4.6)\qquad \int_0^T |||u(t,\cdot)|||_{m-1}^2\, dt \leq C(T)\int_0^T \|f(t,\cdot)\|_0^2\, dt.$$

Therefore the problem $(P^0, B_j^0)_{(t_0, x_0')}$ with $f \in C_0^\infty((0, T) \times \boldsymbol{R}_+^n)$ and homogeneous initial-boundary conditions has a unique solution $u \in H^m((0, T) \times \boldsymbol{R}_+^n)$ satisfying (4.6). Only this fact is used in the proof of the necessity of Theorem 4.1 in [2]. Thus the proof is complete.

Finally we consider mixed problems of second order:

$$P = \partial_t^2 - 2\sum_{j=1}^n a_j(t, x)\partial_t\partial_{x_j} - \sum_{j,k=1}^n a_{jk}(t, x)\partial_{x_j}\partial_{x_k} + \text{first order term},$$

$$B = \partial_{x_n} - \sum_{j=1}^{n-1} b_j(t, x')\partial_{x_j} - c(t, x')\partial_t + h(t, x')$$

where $\sum_{j,k=1}^n a_{jk}(t, x)\xi_j\xi_k > 0$ for any non zero vector $\xi \in \boldsymbol{R}^n$ and all the coefficients are real valued.

Combining Theorem 4.1 with results in [1] we obtain

THEOREM 4.2. *Suppose that $R(t, x'; 1, 0) \neq 0$ on the boundary. Then* *a variable coefficient problem* (P, B) *is strongly L^2-well-posed if and only if each constant coefficient problem* $(P^0, B^0)_{(t,x')}$ *is L^2-well-posed (with* $\nu=0$*).*

REMARK. Let (P^0, B^0) denotes $(P^0, B^0)_{(t,x')}$ for a fixed point $(t, x', 0)$. Then the following statements (I), (II) and (III) are equivalent:

(I) (P^0, B^0) *is L^2-well-posed.*

(II) *Lopatinskii's determinant for* (P^0, B^0) *does not vanish in* $\boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$ *and* (P^0, B^0) *has no supersonic speeds.*

(III) $a_{nn}c + a_n \geq 0$ *and the quadratic form* $H(\sigma) = (a_{nn}c + a_n)^2 (a_{nn}\mathfrak{e} - \mathfrak{d}^2) - 2(a_{nn}c + a_n)(a_{nn}\mathfrak{a} - a_n\mathfrak{d})(a_{nn}\mathfrak{b} + \mathfrak{d}) - (a_{nn} + a_n^2)(a_{nn}\mathfrak{b} + \mathfrak{d})^2$ *is positive semi-definite, where*

$$\mathfrak{e} = \sum_{j,k=1}^{n-1} a_{jk}\sigma_j\sigma_k, \qquad \mathfrak{d} = \sum_{j=1}^{n-1} a_{nj}\sigma_j,$$

$$\mathfrak{a} = \sum_{j=1}^{n-1} a_j\sigma_j, \qquad \mathfrak{b} = \sum_{j=1}^{n-1} b_j\sigma_j \qquad (\sigma \in \boldsymbol{R}^{n-1}).$$

The equivalence (I) and (III) has been proved in [1] and other equivalences are proved by using results in § 2 of [1] and [6].

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