Minlos' lemma and Umemura's lemma

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Yasuji TAKAHASHI

§ 1. Introduction

Minlos has shown that all cylinder set measures in the adjoint spaces of nuclear countable Hilbert spaces, satisfying the continuity condition, is countably additive. (c.f. [1], [2].)
The central point in the proof of this theorem is Minlos' Lemma.

But the proof of this Lemma is not so easy.

On the other hand, Umemura has shown that all cylinder set measures in the adjoint spaces of nuclear spaces, satisfying the continuity condition, is countably additive. In this case, nuclear spaces are not necessarily metrizable. (c.f. [3].)
The central point in the proof of this theorem is Umemura's Lemma.

In this note, using Umemura's Lemma, which can be proved easily, we shall prove Minlos' Lemma.

First we shall introduce these two lemmas.

Minlos' LEMMA. (c.f. [1], [2].)

Let μ be a probability Borel measure in N-dimensional vector space R^N and let Q be an ellipsoid such that the measure of any half space not containing the ellipsoid is less than ε. Then the measure of the regin outside any given sphere of radius R and center at origin does not exceed $C\left(\varepsilon+\frac{H^2}{R^2}\right)$, where H^2 is the sum of the squares of the semiaxes of Q, and C is a constant which depend neither upon N nor upon the choice of sphere or ellipsoid Q.

Umemura's LEMMA. (c.f. [3].)

Let R^N be an N-dimensional vector space and R'^N be its adjoint space. Namely,

$$R^N=\left\{x=(x_1,\cdots,x_N)\,\middle|\,x_i\,;\,real\right\},$$

$$R'^N=\left\{\xi=(\xi_1,\cdots,\xi_N)\,\middle|\,\xi_i\,;\,real\right\}\quad and\quad \xi(x)=\sum_{i=1}^{N}\xi_i x_i\,.$$

Let S be the unit ball in R^N, and F be an ellipsoid in R'^N.

$$S=\left\{x\in R^N \,\middle|\, \sum_{k=1}^{N} x_k^2 \leqq 1\right\}, \qquad F=\left\{\xi\in R'^N \,\middle|\, \sum_{k=1}^{N} b_k^2\xi_k^2 \leqq 1\right\}.$$

Then, for any probability Borel measure μ on R^N, the following condition (a) implies the condition (b).

(*a*) $\xi\in F \Longrightarrow \mu\left(\left\{x \,\middle|\, \xi(x)\geqq 1\right\}\right)<\varepsilon$.

(*b*) $\mu(S)\geqq 1-\gamma_0(\varepsilon+D)$ *where* $D=\sum_{k=1}^{N} b_k^2$.

Here γ_0 is an absolute constant which does not depend on N.

REMARK. In Umemura's Lemma, we can take the following condition (a′) instead of the condition (a).

(a′) $\xi\in F \Longrightarrow \mu\left(\left\{x \,\middle|\, \xi(x)>1\right\}\right)<\varepsilon$. (c.f. [3].)

§ 2. Proof of Minlos' Lemma.

In this section, using Umemura's Lemma, we shall prove Minlos' Lemma.

Proof of Minlos' Lemma. First we shall define the polar set $\mathring{F}$ of the ellipsoid F by the relation;

$$\mathring{F}=\left\{x \,\middle|\, |\xi(x)|\leqq 1 \ \text{ for } \ ^{\forall}\xi\in F\right\}.$$

By easy calculations, we have

$$\mathring{F}=\left\{x \,\middle|\, \sum_{k=1}^{N} \frac{x_k^2}{b_k^2} \leqq 1\right\}.$$

From Umemura's Lemma and its Remark, we obtain that the following condition (c) implies the condition (d).

(∗) (c) The μ-measure of any half space in R^N which does not intersect the ellipsoid $\mathring{F}$ is less than ε.

(d) $\mu(S)\geqq 1-\gamma_0(\varepsilon+D)$

Next we define T by setting

$$T(x)=\frac{x}{R} \quad \text{for} \quad ^{\forall}x\in R^N.$$

Obviously T is a one-to-one onto homeomorphic transformation, so we shall define another probability Borel measure μ^* by setting

$$\mu^*(A)=\mu(T^{-1}A) \ \text{ for any Borel set } \ A\subset R^N.$$

By easy calculations, we have

$$T\mathring{F}=\left\{x \,\middle|\, \sum_{k=1}^{N} \frac{x_k^2}{b_k^2/R^2} \leqq 1\right\}.$$

Now suppose that the μ-measure of any half space in R^N which does not intersect ellipsoid $\mathring{F}$ is less than ε. Then the μ^*-measure of any half space in R^N which does not intersect ellipsoid $T\mathring{F}$ is less than ε.

In $(*)$, using μ^* and $T\mathring{F}$ instead of μ and $\mathring{F}$ respectively, we have

$$\mu^*(S) \geqq 1-\gamma_0\left(\varepsilon+\frac{D}{R^2}\right).$$

Since

$$\mu^*(S)=\mu(T^{-1}S)=\mu\big(S(R)\big),$$

we have

$$\mu\big(S(R)\big) \geqq 1-\gamma_0\left(\varepsilon+\frac{D}{R^2}\right).$$

Thus we have the assertion. Q.E.D.

§ 3. Proof of Umemura's Lemma.

In this section, using Minlos' Lemma, we shall prove Umemura's Lemma.

Proof of Umemura's Lemma. From Minlos' Lemma, we have $(*)$ in the previous section 2.

Now suppose that the condition (a) is fulfilled. Then, in order to prove Umemura's Lemma it suffices to show that the condition (c) is fulfilled.

Let Z be any half space which does not intersect $\mathring{F}$. Since half space Z does not contain origin, we have

$$Z=\big\{x\,;\,\xi(x)\geqq 1\big\} \quad \text{for some} \quad \xi\in R'^N.$$

Then it suffices to show that $\xi\in F$.

If not so, and therefore $\sum_{k=1}^{N} b_k^2\xi_k^2>1$.

Putting

$$x=\left(\frac{b_i^2\xi_i}{\rho}\right)_{i=1,\cdots,N} \quad \text{where} \quad \rho=\sqrt{\sum_{k=1}^{N} b_k^2\xi_k^2}\,,$$

we have

$$\xi(x)=\sqrt{\sum_{k=1}^{N} b_k^2\xi_k^2}>1, \qquad x\in\mathring{F}.$$

Which is a contradiction.

Thus we have the assertion. Q.E.D.

Department of Mathematics
Hokkaido University

References

[1] I. M. GELFAND and N. J. VILENKIN: Generalized Functions, Vol. 4 (1961).

[2] R. A. MINLOS: Generalized random processes and their extension to measures. (in Russian) Trudy Moskov. Mat. Obsc. 8 (1959) pp. 497–518.

[3] Y. UMEMURA: Measures on infinite dimensional vector spaces. Publ. Res. Inst. Mat. Sci. 1 (1965) 1–47.

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