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# Vitali's theorem in vector lattices

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Shozo KOSHI

## § 1. Introduction

Let  $\Omega$  be a measure space with finite measure  $\mu$ . Then, Vitali's theorem announces: *let  $f_n (n=1, 2, \dots)$  be a sequence of summable functions on  $\Omega$  with equi-absolutely continuous integrals and  $f_n$  converges to  $f$  in measure. Then,*

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu.$$

In this note, we shall generalize the Vitali's theorem to vector lattices.

## § 2. Convergences in vector lattices.

Let  $R$  be a  $\sigma$ -complete vector lattice i. e.  $\bigcap_{n=1}^{\infty} a_n$  exists for every positive elements  $0 \leq a_n \in R (n=1, 2, \dots)$ . In the sequel, we assume that  $R$  is  $\sigma$ -complete. For  $a_n \in R (n=1, 2, \dots)$ , if  $\bigcap_{m=1}^{\infty} (\bigcup_{n \geq m} a_n)$  and  $\bigcup_{m=1}^{\infty} (\bigcap_{n \geq m} a_n)$  exist and equal to  $a$ , then we denote that

$$o\text{-}\lim_{n \rightarrow \infty} a_n = a.$$

In this case, we say that the sequence  $\{a_n\}$  is *order-convergent* to  $a$ . It is easy to see that  $o\text{-}\lim a_n = a$  iff there exist  $b_n \downarrow 0$  (i. e.  $b_1 \geq b_2 \geq \dots$  with  $\bigcap_{n=1}^{\infty} b_n = 0$ ) such that

$$|a_n - a| \leq b_n$$

where  $|x| = x \cup (-x) = x^+ + x^-$  for  $x \in R$ .

We shall define star-order-convergence as follows: a sequence  $a_n (n=1, 2, \dots)$  is said to be *star-order-convergent* to  $a$  if for every subsequence of  $\{a_n\}$ , there exists its subsequence which is order-convergent to  $a$ . We shall denote

$$s\text{-}o\text{-}\lim a_n = a$$

if  $a_n (n=1, 2, \dots)$  is star-order-convergent to  $a$ .

For a subset  $M$  of  $R$ , we denote  $M^\perp = \{a : |a| \cap |b| = 0 \text{ for all } b \in M\}$ .

If we can decompose  $a \in R$  into as follows :

$$(A) \quad a = a_1 + a_2$$

with  $a_1 \in M$  and  $a_2 \in M^\perp$ , then  $M$  is called normal.

If  $M$  is normal, then  $M = M^{\perp\perp}$  and the decomposition (A) of  $a$  is uniquely determined. Namely,

$$a = a_1 + a_2, \quad a = a'_1 + a'_2, \quad a_1, a'_1 \in M, \quad a_2, a'_2 \in M^\perp$$

imply  $a_1 = a'_1$  and  $a_2 = a'_2$ .

We see that the operator  $[M]a = a_1$  is linear and lattice-homomorphic.  $[M]$  is called a projection operator.

The normal subsets of  $R$  (or equivalently projection operators) constitutes a Boolean lattice by the usual order.

Let  $R$  be  $\sigma$ -complete. For  $0 \leq p \in R$ , the subset  $\{p\}^{\perp\perp}$  is normal and

$$[\{p\}^{\perp\perp}]a = (\text{denoted by } [p]a) = \bigcup_{n=1}^{\infty} (np \cap a) \quad \text{for } a \geq 0.$$

In general,  $[p]a = [p]a^+ - [p]a^-$ .

$\{[N]; [p] \geq [N]\}$  is  $\sigma$ -complete as a Boolean lattice and for every  $[N]$  with  $[p] \geq [N]$  there exists  $q \in R$  with  $[N] = [q]$ .

For arbitrary  $p \in R$ ,  $[p] = [|p|]$ .

Let  $[p_n]$  be a sequence of projection operators.  $[P_n]$  is order-convergent to 0 if  $\bigcap_{m=1}^{\infty} (\bigcup_{n \geq m} P_n) = 0$ , i.e. there exists  $[Q_n] \geq [P_n]$  with  $[Q_1] \geq [Q_2] \geq \dots$  and  $\bigcap_{n=1}^{\infty} [Q_n] = 0$ .

We shall write  $[P_n] \downarrow 0$  if  $[P_1] \geq [P_2] \geq \dots$  and  $\bigcap_{n=1}^{\infty} [P_n] = 0$ .

We shall denote  $[P_n] \downarrow\downarrow 0$  if for every subsequence of  $[P_n]$  there exists its subsequence order-convergent to 0.

Now, we shall consider a special convergence in a  $\sigma$ -complete vector lattice.

We shall denote

$$\circledast - \lim_{n \rightarrow \infty} a_n = a$$

if there exists  $[P_m] \downarrow\downarrow 0$  such that  $(I - [P_m])a_n$  is star-order-convergent to  $(I - [P_m])a$  ( $m = 1, 2, \dots$ ). In the case of  $L_1$ -space (the totality of summable functions)  $f_n \rightarrow f$  (in measure) implies  $f_n \rightarrow f$  (in above sense).

It is easy to see that if  $\circledast - \lim_{n \rightarrow \infty} a_n = a$ , then there exists  $[p_m] \downarrow\downarrow 0$  such that  $p_m \in R$  ( $m = 1, 2, \dots$ ) and  $(I - [p_m])a_n$  is star-order-convergent to  $(I - [p_m])a$  for all  $m = 1, 2, \dots$ .

We see easily  $[p_n] \downarrow 0$  and  $[q_n] \downarrow 0$  imply  $[p_n] \cup [q_n] \downarrow 0$ . Hence, we see

that if  $\bigotimes\text{-}\lim_{n \rightarrow \infty} a_n = a$  and  $\bigotimes\text{-}\lim_{n \rightarrow \infty} b_n = b$ , then

$$\begin{aligned}\bigotimes\text{-}\lim_{n \rightarrow \infty} (a_n + b_n) &= \bigotimes\text{-}\lim_{n \rightarrow \infty} a_n + \bigotimes\text{-}\lim_{n \rightarrow \infty} b_n, \\ \bigotimes\text{-}\lim_{n \rightarrow \infty} (a_n \cup b) &= a \cup b, \quad \bigotimes\text{-}\lim_{n \rightarrow \infty} (a_n \cap b) = a \cap b, \\ \bigotimes\text{-}\lim_{n \rightarrow \infty} |a_n - a| &= 0.\end{aligned}$$

### § 3. Equi-continuous subsets

Let  $\bar{a}$  be a linear functional on  $R$ .  $\bar{a}$  is said to be *o-continuous linear functional* if  $o\text{-}\lim a_n = a$  implies  $\bar{a}(a_n) \rightarrow \bar{a}(a)$ .

It is easy to see that if  $\bar{a}$  is an *o-continuous linear functional* on  $R$ , then

$$\sup_{0 \leq b \leq |a|} |\bar{a}(b)| < +\infty \quad \text{for all } a \in R.$$

The totality of *o-continuous linear functionals* is denoted by  $\bar{R}$  and is called an *order-conjugate space* of  $R$ . Since  $R$  is reduced to 0 in some occasion, we assume that  $R$  is not trivial in the sense that for every  $a \neq 0$ , there exists  $\bar{a} \in \bar{R}$  with  $\bar{a}(a) \neq 0$ .

$\bar{a} \geq \bar{b}$  means that  $\bar{a}(a) \geq \bar{b}(a)$  for all  $a \geq 0$ .

By this order,  $\bar{R}$  is a complete vector lattice. In the sequel, we assume that  $R$  is not trivial. A subset  $\Gamma$  of  $R$  is *equi-continuous* if for  $\bar{a}_n \downarrow 0$ ,  $\bar{a}_n \in \bar{R}$  and  $\varepsilon > 0$ , there exists a natural number  $n_0$  with

$$\bar{a}_{n_0}(|a|) \leq \varepsilon \quad \text{for all } a \in \Gamma.$$

By definition, if  $\Gamma$  is an equi-continuous subset of  $R$ , then  $N[\Gamma] = \{b; |b| \leq |a| \text{ for some } a \in \Gamma\}$  is also an equi-continuous subset. It is known that  $\Gamma$  is equi-continuous iff  $\Gamma$  is relative compact by the weak topology induced by  $\bar{R}$ .

For  $[P_n] \downarrow 0$  and  $\bar{a} \in R$ ,  $\bar{a}[P_n](a) = \bar{a}([P_n]a)$  is also *o-continuous linear functional* for all  $n$  and  $\bar{a}[P_n] \downarrow 0$ .

### § 4. Vitali's theorem

Vitali's theorem for summable functions can be formulated as follows in the case of vector lattices.

**THEOREM 1.** *Let  $a_n \in R$  ( $n=1, 2, \dots$ ) be an equi-continuous sequences and  $\bigotimes\text{-}\lim_{n \rightarrow \infty} a_n = a$ . Then,  $a_n$  is weakly convergent to  $a$  (i.e.  $\bar{a}(a_n) \rightarrow \bar{a}(a)$  for all  $\bar{a} \in \bar{R}$ ).*

**PROOF.** Let  $[p_n] \downarrow \downarrow 0$  and  $0 \leq \bar{a} \in \bar{R}$ . We shall prove that we can find

a natural number  $n_0$  such that

$$\bar{a}([p_n]|a_m|) \leq \varepsilon \quad \text{for } n \geq n_0 \quad \text{and } m=1, 2, \dots.$$

If not, we find  $\varepsilon > 0$  and  $m_\nu \leq m_{\nu+1} \leq \dots$  and  $n_\nu \leq n_{\nu+1} \leq \dots$  such that

$$\bar{a}([p_{n_\nu}]|a_{m_\nu}|) \geq \varepsilon \quad \nu = 1, 2, \dots.$$

By assumption, there exists a subsequence  $[q_\nu]$  of  $[p_{n_\nu}]$  ( $\nu=1, 2, \dots$ ) order-convergent to 0 such that

$$\bar{a}([q_\nu]|a_{m_\nu}|) \geq \varepsilon.$$

This contradicts to the equi-continuity of  $\Gamma = \{a_n\}$ .

We shall prove that  $s-0-\lim a_n = a$  implies  $\bar{a}(a_n) \rightarrow \bar{a}(a)$ .

If not, there exists  $n_\nu$  ( $\nu=1, 2, \dots$ ) such that

$$|\bar{a}(a_{n_\nu} - a)| \geq \varepsilon \quad \text{for some } \varepsilon > 0.$$

By assumption, we find a subsequence of  $\{a_{n_\nu}\}$  order-convergent to  $a$ . This is a contradiction, since  $\bar{a}$  is continuous by order-convergence.

Let  $\otimes\text{-}\lim_{n \rightarrow \infty} a_n = a$ . There exists  $[a] \geq [p_m] \downarrow \downarrow 0$  such that

$$s-o-\lim_{n \rightarrow \infty} (I - [p_m]) a_n = (I - [p_m]) a.$$

Hence, choosing  $m$  such that  $|\bar{a}([p_m](a_n - a))| < \varepsilon$ , we have

$$|\bar{a}(a_n) - \bar{a}(a)| \leq |\bar{a}([p_m](a_n - a))| + |\bar{a}((I - [p_m])(a_n - a))| \leq 2\varepsilon$$

for sufficient large  $n$ .

This proves Theorem 1.

COROLLARY. Let  $\otimes\text{-}\lim_{n \rightarrow \infty} a_n = a$ .  $\{a_n\}$  is weakly convergent to  $a$  if and only if  $\{a_n\}$  is equi-continuous.

### § 5. $|w|$ -convergence.

By definition,

$|w|-\lim_{n \rightarrow \infty} a_n = a$  iff  $\lim_{n \rightarrow \infty} \bar{a}(|a_n - a|) = 0$  for all  $a \in \bar{R}$ .  $|w|-\lim_{n \rightarrow \infty} a_n = a$  implies that  $a_n$  is weakly convergent to  $a$ . But, in general the converse is not true.

THEOREM 2. Under the assumption of Theorem 1, we have

$$|w|-\lim_{n \rightarrow \infty} a_n = a.$$

PROOF. If  $\{a_n\}$  is equi-continuous, then  $\{x_n\}$  is equi-continuous where  $x_n = |a_n - a|$ . We see easily

$$\otimes\text{-}\lim_{n \rightarrow \infty} a_n = a \quad \text{iff} \quad \otimes\text{-}\lim_{n \rightarrow \infty} x_n = \otimes\text{-}\lim_{n \rightarrow \infty} |a_n - a| = 0.$$

By Theorem 1,  $\{x_n\}$  is weakly convergent to 0. But this means that

$$|\omega| - \lim_{n \rightarrow \infty} a_n = a.$$

REMARK. If  $R$  is a space of summable functions defined on a finite measure space, under the assumption of Theorem 1, we have  $\lim_{n \rightarrow \infty} \|a_n - a\| = 0$  by Theorem 2.

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