Vitali's theorem in vector lattices

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction

Let \( \Omega \) be a measure space with finite measure \( \mu \). Then, Vitali's theorem announces: let \( f_n(n=1, 2, \cdots) \) be a sequence of summable functions on \( \Omega \) with equi-absolutely continuous integrals and \( f_n \) converges to \( f \) in measure. Then,

\[
\lim_{n \to \infty} \int_{\rho} f_n \, d\mu = \int_{\rho} f \, d\mu.
\]

In this note, we shall generalize the Vitali's theorem to vector lattices.

§ 2. Convergences in vector lattices.

Let \( R \) be a \( \sigma \)-complete vector lattice i.e. \( \bigcap_{n=1}^{\infty} a_n \in R \) (\( n=1, 2, \cdots \)). In the sequel, we assume that \( R \) is \( \sigma \)-complete. For \( a_n \in R \) (\( n=1, 2, \cdots \)), if \( \bigcap_{m=1}^{\infty} (\bigcup_{n \geq m} a_n) \) and \( \bigcup_{m=1}^{\infty} (\bigcap_{n \geq m} a_n) \) exist and equal to \( a \), then we denote that

\[
o-lim_{n \to \infty} a_n = a.
\]

In this case, we say that the sequence \( \{a_n\} \) is order-convergent to \( a \). It is easy to see that \( o-lim a_n = a \) iff there exist \( b_n \downarrow 0 \) (i.e. \( b_1 \geq b_2 \geq \cdots \) with \( \bigcap_{n=1}^{\infty} b_n = 0 \)) such that

\[
|a_n - a| \leq b_n
\]

where \( |x| = x \cup (-x) = x^+ + x^- \) for \( x \in R \).

We shall define star-order-convergence as follows: a sequence \( a_n(n=1, 2, \cdots) \) is said to be star-order-convergent to \( a \) if for every subsequence of \( \{a_n\} \), there exists its subsequence which is order-convergent to \( a \). We shall denote

\[
s-o-lim a_n = a
\]

if \( a_n(n=1, 2, \cdots) \) is star-order-convergent to \( a \).

For a subset \( M \) of \( R \), we denote \( M^\perp = \{a : |a| \cap |b| = 0 \text{ for all } b \in M\} \).
If we can decompose $a \in R$ into as follows:

$$(A) \quad a = a_1 + a_2$$

with $a_1 \in M$ and $a_2 \in M^\perp$, then $M$ is called normal.

If $M$ is normal, then $M = M^\perp\perp$ and the decomposition (A) of $a$ is uniquely determined. Namely,

$$a = a_1 + a_2, \quad a = a'_1 + a'_2, \quad a_1, a'_1 \in M, \quad a_2, a'_2 \in M^\perp$$

imply $a_1 = a'_1$ and $a_2 = a'_2$.

We see that the operator $[M]a = a_1$ is linear and lattice-homomorphic. $[M]$ is called a projection operator.

The normal subsets of $R$ (or equivalently projection operators) constitutes a Boolean lattice by the usual order.

Let $R$ be $\sigma$-complete. For $0 \leq p \in R$, the subset $(p)^{\perp\perp}$ is normal and

$$\left[(p)^{\perp\perp}\right]a = \left(\text{denoted by } [p]a\right) = \bigcup_{n=1}^{\infty}(np \cap a) \quad \text{for } a \geq 0.$$


Let $[p] \leq [N]$; $[p] \geq [N]$ is $\sigma$-complete as a Boolean lattice and for every $[N]$ with $[p] \geq [N]$ there exists $q \in R$ with $[N] = [q]$.

For arbitrary $p \in R$, $[p] = [\{p\}]$.

Let $[p_n]$ be a sequence of projection operators. $[p_n]$ is order-convergent to 0 if $\bigcap_{m=1}^{\infty}(\bigcup_{n \geq m}P_n) = 0$, i.e. there exists $[Q_n] \geq [P_n]$ with $[Q_1] \geq [Q_2] \geq \cdots$ and $\bigcap_{n=1}^{\infty}[Q_n] = 0$.

We shall write $[P_n] \downarrow 0$ if $[P_1] \geq [P_2] \geq \cdots$ and $\bigcap_{n=1}^{\infty}[P_n] = 0$.

We shall denote $[P_n] \downarrow\downarrow 0$ if for every subsequence of $[P_n]$ there exists its subsequence order-convergent to 0.

Now, we shall consider a special convergence in a $\sigma$-complete vector lattice.

We shall denote

$$\text{lim}_{n \to \infty} a_n = a$$

if there exists $[p_n] \downarrow\downarrow 0$ such that $(I - [P_n])a_n$ is star-order-convergent to $(I - [P_n])a$ ($m = 1, 2, \cdots$). In the case of $L_1$-space (the totality of summable functions) $f_n \to f$ (in measure) implies $f_n \to f$ (in above sense).

It is easy to see that if $\text{lim}_{n \to \infty} a_n = a$, then there exists $[p_n] \downarrow\downarrow 0$ such that $p_n \in R$ ($m = 1, 2, \cdots$) and $(I - [P_n])a_n$ is star-order-convergent to $(I - [P_n])a$ for all $m = 1, 2, \cdots$.

We see easily $[p_n] \downarrow 0$ and $[q_n] \downarrow 0$ imply $[p_n] \cup [q_n] \downarrow 0$. Hence, we see
that if $\bigotimes_{n=1}^{\infty}a_{n}=a$ and $\bigotimes_{n=1}^{\infty}b_{n}=b$, then
\[
\bigotimes_{n=1}^{\infty}(a_{n}+b_{n}) = \bigotimes_{n=1}^{\infty}a_{n} + \bigotimes_{n=1}^{\infty}b_{n},
\]
\[
\bigotimes_{n=1}^{\infty}(a_{n} \cup b) = a \cup b,
\]
\[
\bigotimes_{n=1}^{\infty}(a_{n} \cap b) = a \cap b,
\]
\[
\bigotimes_{n=1}^{\infty}|a_{n}-a| = 0.
\]

\section{Equi-continuous subsets}

Let $a$ be a linear functional on $R$. $a$ is said to be \emph{o-continuous linear functional} if $o\lim a_{n}=a$ implies $a(a_{n}) \to a(a)$.

It is easy to see that if $a$ is an o-continuous linear functional on $R$, then
\[
\sup_{0 \leq b \leq |a|} |a(b)| < +\infty \quad \text{for all } a \in \overline{R}.
\]
The totality of o-continuous linear functionals is denoted by $\overline{R}$ and is called an order-conjugate space of $R$. Since $R$ is reduced to 0 in some occasion, we assume that $R$ is not trivial in the sense that for every $a \neq 0$, there exists $a \in \overline{R}$ with $a(a) \neq 0$.

$a \geq b$ means that $a(a) \geq b(a)$ for all $a \geq 0$.

By this order, $\overline{R}$ is a complete vector lattice. In the sequel, we assume that $R$ is not trivial. A subset $\Gamma$ of $R$ is \emph{equi-continuous} if for $a_{n} \downarrow 0$, $a_{n} \in \overline{R}$ and $\varepsilon > 0$, there exists a natural number $n_{0}$ with
\[
a_{n_{0}}(|a|) \leq \varepsilon \quad \text{for all } a \in \Gamma.
\]
By definition, if $\Gamma$ is an equi-continuous subset of $R$, then $N[\Gamma] = \{b; |b| \leq |a| \text{ for some } a \in \Gamma\}$ is also an equi-continuous subset. It is known that $\Gamma$ is equi-continuous iff $\Gamma$ is relative compact by the weak topology induced by $\overline{R}$.

For $[P_{n}] \downarrow 0$ and $a \in R$, $a[P_{n}](a) = a([P_{n}]a)$ is also o-continuous linear functional for all $n$ and $a[P_{n}] \downarrow 0$.

\section{Vitali's theorem}

Vitali's theorem for summable functions can be formulated as follows in the case of vector lattices.

\textbf{Theorem 1.} Let $a_{n} \in R$ ($n=1, 2, \cdots$) be an equi-continuous sequences and $\bigotimes_{n=1}^{\infty}a_{n}=a$. Then, $a_{n}$ is weakly convergent to $a$ (i.e. $a(a_{n}) \to a(a)$ for all $a \in \overline{R}$).

\textbf{Proof.} Let $[p_{n}] \downarrow 0$ and $0 \leq a \in \overline{R}$. We shall prove that we can find
a natural number \(n_0\) such that
\[
a([p_n]|a_m)| \leq \varepsilon \quad \text{for} \quad n \geq n_0 \quad \text{and} \quad m = 1, 2, \ldots.
\]
If not, we find \(\varepsilon > 0\) and \(m_1 \leq m_2 \leq \cdots \) and \(n_1 \leq n_2 \leq \cdots\) such that
\[
a([p_{n_\nu}]|a_{m_\nu}) \geq \varepsilon \quad \nu = 1, 2, \ldots.
\]
By assumption, there exists a subsequence \([q_{\nu}]\) of \([p_{n_\nu}]\) \((\nu = 1, 2, \ldots)\) order-convergent to 0 such that
\[
a([q_{\nu}]|a_{m_\nu}) \geq \varepsilon.
\]
This contradicts the equi-continuity of \(\Gamma = \{a_n\}\).

We shall prove that \(s-0-\lim a_n = a\) implies \(a(a_n) \to a(a)\).
If not, there exists \(n_\nu(\nu = 1, 2, \ldots)\) such that
\[
|a(a_{n_\nu} - a)| \geq \varepsilon \quad \text{for some} \quad \varepsilon > 0.
\]
By assumption, we find a subsequence of \(\{a_{n_\nu}\}\) order-convergent to 0. This is a contradiction, since \(a\) is continuous by order-convergence.

Let \(\otimes-\lim_{n \to \infty} a_n = a\). There exists \([a] \geq [p_m] \downarrow 0\) such that
\[
s-o-\lim_{n \to \infty} (I - [p_m]) a_n = (I - [p_m]) a.
\]
Hence, choosing \(m\) such that \(|a([p_m](a_n - a)| < \varepsilon\), we have
\[
|a(a_n) - a(a)| \leq |a([p_m](a_n - a)| + |a((I - [p_m])(a_n - a)| \leq 2\varepsilon
\]
for sufficient large \(n\).
This proves Theorem 1.

**Corollary.** Let \(\otimes-\lim_{n \to \infty} a_n = a\). \(\{a_n\}\) is weakly convergent to \(a\) if and only if \(\{a_n\}\) is equi-continuous.

**§ 5.** \(|w|\)-convergence.

By definition,
\[
|w| - \lim_{n \to \infty} a_n = a \iff \lim_{n \to \infty} a(|a_n - a|) = 0 \quad \text{for all} \quad a \in \overline{R}.
\]
\(|w| - \lim_{n \to \infty} a_n = a\) implies that \(a_n\) is weakly convergent to \(a\). But, in general the converse is not true.

**Theorem 2.** Under the assumption of Theorem 1, we have
\[
|w| - \lim_{n \to \infty} a_n = a.
\]

**Proof.** If \(\{a_n\}\) is equi-continuous, then \(\{x_n\}\) is equi-continuous where \(x_n = |a_n - a|\). We see easily
\[
\otimes - \lim_{n \to \infty} a_n = a \iff \otimes - \lim_{n \to \infty} x_n = \otimes - \lim_{n \to \infty} |a_n - a| = 0.
\]
By Theorem 1, \( \{x_n\} \) is weakly convergent to 0. But this means that

\[
|w| - \lim_{n \to \infty} a_n = a.
\]

**Remark.** If \( R \) is a space of summable functions defined on a finite measure space, under the assumption of Theorem 1, we have \( \lim_{n \to \infty} \|a_n - a\| = 0 \) by Theorem 2.

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**References**


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