A note on Sazonov's theorem

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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§ 1. Introduction

Sazonov has shown that a cylinder set measure μ on the Hilbert space H is countably additive iff μ is continuous relative to the nuclear topology. In this note, we shall show that this fact is true in countable Hilbert spaces. For this purpose, we shall define the nuclear topology in countable Hilbert spaces.

Throughout this note, we shall suppose that Hilbert spaces and countable Hilbert spaces are separable with real coefficients.

Let Φ be a countable Hilbert space and $(\varphi, \psi)_n$ $(n=1, 2, \cdots)$ be its scalar products, Φ_n be the completion of Φ with respect to $(\varphi, \psi)_n$.

Let $\mathfrak{S}_n$ denote the family of all positive definite nuclear operators in Φ_n. The class of sets $\{[\varphi\in\Phi\,;(T\varphi, \varphi)_n<1]\ T\in\mathfrak{S}_n\}$ $(n=1, 2, \cdots)$ defines a system of neighborhoods at the origin for a certain topology. We shall call this topology the nuclear topology.

Now we introduce a condition for the countable additivity of measures on the cylinder sets in adjoint spaces of countable Hilbert spaces.

THEOREM 1. (c.f. [1], [2].)

If μ is a countably additive cylinder set measure on the adjoint space Φ^* of a countable Hilbert space Φ, then for any $\varepsilon>0$ there is a ball $S_n(R)$ $=\{\|F\|_{-n}\leqq R\}$ such that the μ-measure of its complement is less than ε.

THEOREM 1′. (c.f. [1], [2].)

Suppose that μ is a cylinder set measure on the adjoint space Φ^* of a countable Hilbert space Φ. If for any $\varepsilon>0$ there is a ball $S_n(R)$ in Φ^* such that the measure of any cylinder set lying outside $S_n(R)$ is less than ε, then μ is countably additive.

Next we introduce the most important lemma for the proof of our main theorem.

LEMMA (Minlos). (c.f. [1], [2].)

Let μ be a cylinder set measure on the adjoint space $\Phi^*=\bigcup_{n=1}^{\infty}\Phi_n^*$ of a countable Hilbert space $\Phi=\bigcap_{n=1}^{\infty}\Phi_n$. Let Q be an ellipsoid in the Hilbert *space Φ_n^* such that the sum of the squares of its principal semiaxes is equal to H^2, and the measure of any half space in Φ^*, not containing Q, is less than ε. If $S_n(R)=\{\|F\|_{-n}\leqq R\}$ is any ball in Φ_n^* containing Q, then the measure of any cylinder set Z, lying outside $S_n(R)$, is less than $C\left(\varepsilon+\frac{H^2}{R^2}\right)$, where C is the absolute constant.*

§ 2. Main theorem

In this section, we shall prove the following theorem.

THEOREM A. *In order that a cylinder set measure μ on the adjoint space $\Phi^*=\bigcup_{n=1}^{\infty}\Phi_n^*$ of a countable Hilbert space $\Phi=\bigcap_{n=1}^{\infty}\Phi_n$ is countably additive, it is necessary and sufficient that μ is continuous relative to the nuclear topology.*

The continuity of μ means the following: For any $\varepsilon>0$ there exist $\delta>0$, n and positive definite nuclear operator T in Φ_n such that the inequality $(T\varphi, \varphi)_n\leqq\delta$ implies that *$\mu(\Gamma_\varphi)\leqq\varepsilon$, where Γ_φ denotes the strip defined by $|F(\varphi)|\geqq 1$.*

PROOF. First we prove the necessity of the condition. Suppose that μ is countably additive. By Theorem 1, for any $\varepsilon>0$ there is a ball $S_n(R)=\{\|F\|_{-n}\leqq R\}$ such that the measure of its complement is less than $\frac{\varepsilon}{2}$. We define T by setting

$$(T\varphi, \varphi)_n=\int_{S_n(R)}|F(\varphi)|^2\,d\mu(F).$$

Obviously T is a positive definite operator in Φ_n. To show that it is nuclear, we note that for any orthonormal basis $\{\varphi_k\}$ in Φ_n one has

$$\sum_{k=1}^{\infty}(T\varphi_k, \varphi_k)_n=\int_{S_n(R)}\sum_{k=1}^{\infty}|F(\varphi_k)|^2\,d\mu(F)$$
$$=\int_{S_n(R)}\|F\|_{-n}^2\,d\mu(F)\leqq R^2.$$

In other words, the series $\sum_{k=1}^{\infty}(T\varphi_k, \varphi_k)_n$ converges for any orthonormal basis $\{\varphi_k\}$ in Φ_n. It follows that T is a nuclear operator in Φ_n.

Now consider any element φ such that $(T\varphi, \varphi)_n\leqq\frac{\varepsilon}{2}$, and let us estimate the measure of the strip Γ_φ defined by $|F(\varphi)|\geqq 1$. Obviously

$$\mu(\Gamma_\varphi)=\mu(\Gamma_\varphi')+\mu(\Gamma_\varphi'')$$

where Γ_φ' is that part of Γ_φ contained in the ball $S_n(R)$, and Γ_φ'' is that part lying outside $S_n(R)$. In view of the choice of $S_n(R)$ we have $\mu(\Gamma''_\varphi)\leqq\frac{\varepsilon}{2}$. On the other hand, from the inequality $|F(\varphi)|\geqq 1$, which holds for all $F\in\Gamma_\varphi$ and therefore for all $F\in\Gamma'_\varphi$, it follows that

$$\mu(\Gamma'_\varphi)=\int_{\Gamma'_\varphi}d\mu(F)\leqq\int_{\Gamma'_\varphi}|F(\varphi)|^2\,d\mu(F)$$

$$\leqq\int_{S_n(R)}|F(\varphi)|^2\,d\mu(F)=(T\varphi,\varphi)_n\leqq\frac{\varepsilon}{2}\,.$$

Hence $\mu(\Gamma_\varphi)\leqq\varepsilon$.

Thus we have the assertion.

Next we prove the sufficiency of the condition. Suppose that μ is continuous relative to the nuclear topology. By Theorem 1′, to prove the countable additivity of μ it suffices to show that for any $\varepsilon>0$ one can find n and R such that the measure of any cylinder set lying outside the ball $S_n(R)=\{\|F\|_{-n}\leqq R\}$ is less than ε.

Since μ is continuous relative to the nuclear topology, for any $\varepsilon>0$ there exist n, $a>0$ and positive definite nuclear operator T in Φ_n such that

$$\mu\left\{|F(\varphi)|\geqq 1\right\}<\frac{\varepsilon}{2C}\quad\text{for}\quad\varphi\in U=\left\{(T\varphi,\varphi)_n^{\frac{1}{2}}<a\right\}$$

where C is the same constant in Lemma (Minlos).

Case 1. Let T be the strictly positive definite nuclear operator. In this case, $\|\varphi\|_n^T=(T\varphi,\varphi)_n^{\frac{1}{2}}$ is a Hilbertian norm. Putting $\rho=\frac{1}{a}$, there exists a ball $S_n^T(\rho)=\{\|F\|_{-n}^T\leqq\rho\}$ such that the measure of any half space in Φ^* which does not intersect $S_n^T(\rho)$ has measure less than $\frac{\varepsilon}{2C}$. Let Φ_n^T be the completion of Φ with respect to $\|\varphi\|_n^T$. Let j be a canonical mapping of Φ_n into Φ_n^T. Since T is a nuclear operator, j is a Hilbert-Schmidt operator of Φ_n into Φ_n^T. Therefore its adjoint j^* is a Hilbert-Schmidt operator of $(\Phi_n^T)^*$ into Φ_n^*. Thus $j^*S_n^T(\rho)$ is an ellipsoid, and the sum of the squares of its principal semiaxes is finite.

Let H^2 be denote the sum of the squares of the principal semiaxes of the ellipsoid $j^*S_n^T(\rho)$ in Φ_n^*, and choose R so large that the ball $S_n(R)$ in Φ_n^* contains the ellipsoid $j^*S_n^T(\rho)$, and also $\frac{H^2}{R^2}\leqq\frac{\varepsilon}{2C}$. By Lemma (Minlos), for any cylinder set Z in Φ^* lying outside $S_n(R)$ one has the estimate

$$\mu(Z)\leqq C\left(\frac{\varepsilon}{2C}+\frac{H^2}{R^2}\right)\leqq\varepsilon\,.$$

Thus we have found a ball $S_n(R)$ such that the measure of any cylinder set Z which lies outside $S_n(R)$ has μ-measure not exceeding the given value $\varepsilon>0$. Hence Theorem 1′ implies that the measure μ is countably additive.

Case 2. Let T be not necessarily strictly positive definite. In this case, by considering its associated Hilbert space instead of Φ_n^T, we can prove as case 1. Q.E.D.

Let Φ be a countable Hilbert space and μ be a cylinder set measure on the adjoint space Φ^*. We define the Fourier transform of μ as the functional $\hat{\mu}(\varphi)$ defined on Φ by

$$\hat{\mu}(\varphi)=\int e^{iF(\varphi)}\,d\mu(F).$$

REMARK. In the above theorem, we can suppose that the nuclear topology is metrizable. In this case, the following holds: μ is countably additive iff $\hat{\mu}(\varphi)$ is continuous relative to the nuclear topology.

In general case, in order that μ is countably additive, it is necessary that $\hat{\mu}(\varphi)$ is continuous relative to the nuclear topology.

COROLLARY. *In order that the Gaussian measure μ, defined in the adjoint space Φ^* of a countable Hilbert space Φ by a continuous scalar product (φ, ψ), is countably additive, it is necessary and sufficient that there exist n and positive definite nuclear operator T in Φ_n such that*

$$(\varphi, \psi)=(T\varphi, \psi)_n \quad \textit{for all} \quad \varphi, \psi\in\Phi.$$

PROOF. First we prove the necessity of the condition. Suppose that μ is countably additive. By the above Remark, $\hat{\mu}(\varphi)=\exp\left[-\frac{\|\varphi\|^2}{2}\right]$ is continuous relative to the nuclear topology, and therefore $\|\varphi\|=(\varphi, \varphi)^{\frac{1}{2}}$ is continuous relative to the nuclear topology. From this, there exist n, $C>0$ and positive definite nuclear operator S in Φ_n such that

$$(\varphi, \varphi)\leqq C(S\varphi, \varphi)_n \quad \text{for all} \quad \varphi\in\Phi.$$

From this, we can easily show that there exists a positive definite nuclear operator T in Φ_n such that

$$(\varphi, \psi)=(T\varphi, \psi)_n \quad \text{for all} \quad \varphi, \psi\in\Phi.$$

Sufficiency is obvious. Q.E.D.

§3. Other theorem

The following theorem is well known.

THEOREM 2. *Let H_1 and H_2 be Hilbert spaces, and let T be a Hilbert-*

Schmidt operator of H_1 *into* H_2. *Suppose that* μ *is a cylinder set measure, satisfying the continuity condition in* H_1. *Then the measure in* H_2 *induced by* T *and* μ *is countably additive.*

In countable Hilbert spaces, the above theorem is true in the following sense. First we shall define Hilbert-Schmidt operators in countable Hilbert spaces.

Let Φ and Ψ be countable Hilbert spaces, and let T be a continuous linear operator of Φ into Ψ. If for any m there exists n such that operator T of Φ_n into Ψ_m is a Hilbert-Schmidt operator in Hilbert spaces, we call such operator T the Hilbert-Schmidt operator in countable Hilbert spaces.

Then, we can prove the following theorem.

THEOREM B. *Let* Φ *and* Ψ *be countable Hilbert spaces, and let* T *be a continuous linear operator of* Ψ *into* Φ. *Then the following three conditions are equivalent.*

(1) *T is a Hilbert-Schmidt operator of Ψ into Φ.*

(2) *For any continuous cylinder set measure μ in Φ^*, the measure $T^*\mu$ in Ψ^* induced by T and μ is countably additive.*

(3) *Let μ_n be the Gaussian measure, defined in Φ^* by $(\varphi, \psi)_n^{\Phi}$, then for any n, the measure $T^*\mu_n$ in Ψ^* induced by T and μ_n is countably additive.*

PROOF. (1)$\Rightarrow$(2)

As in the proof of Theorem A, from the continuity condition of μ, for any $\varepsilon>0$ there exists a ball $S_m(\rho)=\{\|F\|_{-m}\leqq\rho\}$ such that the measure of any half space in Φ^* which does not intersect $S_m(\rho)$ is less than $\frac{\varepsilon}{2C}$. Since T is a Hilbert-Schmidt operator of Ψ into Φ, there exists n such that T is a Hilbert-Schmidt operator of Ψ_n into Φ_m, and therefore its adjoint T^* is a Hilbert-Schmidt operator of Φ_m^* into Ψ_n^*. Then, $T^*S_m(\rho)$ is an ellipsoid in Ψ_n^*, and the sum of the squares of its principal semiaxes is finite. From this, using Lemma (Minlos), we can easily show that $T^*\mu$ is countably additive.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(1)

Suppose that $T^*\mu_m$ is countably additive for any m. Then, by the Remark of Theorem A, its Fourier transform $\widehat{T^*\mu_m}(\varphi)$ is continuous relative to the nuclear topology. By easy caluculations, we have

$$\widehat{T^*\mu_m}(\varphi)=\exp\left[-\frac{(T\varphi, T\varphi)_m^{\Phi}}{2}\right]$$

where $(\varphi, \psi)_m^{\Phi}$ is a scalar product in Φ.

Therefore there exist n, $C>0$ and positive definite nuclear operator S in Ψ_n such that

$$(T\varphi, T\varphi)_m^{\Phi} \leqq C(S\varphi, \varphi)_n^{\Psi} \quad \text{for all} \quad \varphi \in \Psi .$$

To show that T is a Hilbert-Schmidt operator of Ψ_n into Φ_m, we note that for any orthonormal basis $\{\varphi_k\}$ in Ψ_n one has

$$\sum_{k=1}^{\infty}(T\varphi_k, T\varphi_k)_m^{\Phi} \leqq C\sum_{k=1}^{\infty}(S\varphi_k, \varphi_k)_n^{\Psi} < \infty .$$

In other words, the series $\sum_{k=1}^{\infty}(T\varphi_k, T\varphi_k)_m^{\Phi}$ converges for any orthonormal basis $\{\varphi_k\}$. It follows that T is a Hilbert-Schmidt operator of Ψ_n into Φ_m. By definition, T is a Hilbert-Schmidt operator of Ψ into Φ. Q.E.D.

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