Linearly compact modules and cogenerators

Dedicated to Professor Yoshie Katsurada on her 60th birthday

By Takeshi ONODERA

§ 1. Introduction.

Let R be a ring and ${}_RM$ a left R-module. ${}_RM$ is called *linearly compact* if any finitely solvable system of congruences $x \equiv x_\alpha$ (mod M_α) where the M_α are submodules of ${}_RM$, is solvable. In his paper [6] B. Müller has shown that this class of modules plays an essential role in Morita duality without chain conditions.

The purpose of this paper is to investigate linearly compact modules in detail from the viewpoint of their relationship with cogenerators.

Let ${}_RQ$ be a cogenerator (in the category of left R-modules) and $S = End({}_RQ)$ the endomorphism ring of ${}_RQ$. For a left R-module ${}_RM$ we denote the right S-module $\mathrm{Hom}_R(M, Q)_S$, the Q-dual of ${}_RM$, by M_S^*. Then, in § 3, we have the following

THEOREM (THEOREM 2). *For a left R-module ${}_RM$ the following conditions are equivalent*:

(1) *${}_RM$ is linearly compact.*

(2) *For each submodule $\mathfrak{u}$ of M_S^* and S-homomorphism g of $\mathfrak{u}$ into Q_S, there exists an element a of ${}_RM$ such that $g(f) = f(a)$ for all $f \in \mathfrak{u}$.*

(3) *${}_RM$ is Q-reflexive and Q_S is M_S^*-injective.*

With the aid of this characterizations, we can show that a linearly compact module is necessarily complemented (Theorem 5), and, as a special case of this, that a linearly compact ring is semiperfect. Also, as a corollary of our theorem, we see that a ring R is a two-sided cogenerator (= both ${}_RR$ and R_R are cogenerators) if and only if R is a one-sided linearly compact cogenerator.

In § 2, as a preliminary, we show that linearly compact modules are finite dimensional. In § 4 we consider the relations between linearly compact modules and injective cogenerators with essential socles and obtain more detailed results (Theorem 6, Theorem 7). In § 5 we shall state some applications of our considerations.

Throughout this paper we assume that all rings have an identity element and all modules are unital.

§ 2. Preliminaries.

We start with the following

PROPOSITION 1. *A submodule of a linearly compact modules is linearly compact.*

PROOF. *This is clear from the definition of a linearly compact module.*

PROPOSITION 2[1]. *A linearly compact module can not be an infinite direct sum of nonzero submodules.*

PROOF. Let ${}_RM$ be a linearly compact left R-module and suppose that ${}_RM=\bigoplus_{\alpha\in\Lambda} N_\alpha$, an infinite direct sum of nonzero submodulse N_α's. For each $\alpha\in\Lambda$, let $N^\alpha=\bigoplus_{\beta\neq\alpha} N_\beta$ and x_α be a nonzero element of N_α. Now consider the following system of congruences $x\equiv x_\alpha(\text{mod } N^\alpha)$. This is finitely solvable, because for any finite number of $\alpha_1, \alpha_2, \cdots, \alpha_n$ in Λ $x=\sum_{i=1}^{n} x_{\alpha_i}$ is a solution of $x\equiv x_{\alpha_i}(\text{mod } N^{\alpha_i})$ $i=1, 2, \cdots, n$. Let a be a solution of our system of congruences. Then it follows that, for each $\alpha\in\Lambda$, the α-component of a is $x_\alpha(\neq 0)$. But this is a contradiction.

PROPOSITION 3. *A linearly compact module is finite dimensional, that is, it contains no infinite direct sum of nonzero submodules.*

PROOF. This is a direct consequence of Propositions 1 and 2.

§ 3. Linearly compact modules and cogenerators.

Let ${}_RM$ and ${}_RN$ be two left R-modules and $S=\text{End}({}_RN)$ the endomorphism ring of ${}_RN$. ${}_RN_S$ is then naturally a two-sided R-S-module. We shall denote the right S-module $\text{Hom}_R(M, N)_S$ by M_S^* and the left R-module ${}_R\text{Hom}_S(M^*, N)$ by ${}_RM^{**}$ and call them, respectively, the N-dual and the N-bidual of ${}_RM$. There is a natural R-homomorphism φ of ${}_RM$ into ${}_RM^{**}$ which is defined by

$$\varphi(a)(f)=f(a), \quad a\in M, \quad f\in M^{**}.$$

when φ is a monomorphism, we say that ${}_RM$ is N-torsionless, and, when φ is an isomorphism, we say that ${}_RM$ is N-reflexive. It is easy to see that ${}_RM$ is N-torsionless if and only if, for each nonzero element a of ${}_RM$, there exists $f\in M^*$ such that $f(a)\neq 0$.

As usally, for a subset $X(Y)$ of ${}_RM(M_S^*)$, we denote the submodule $\{f\in M^*|f(a)=0 \text{ for all } a\in X\}$ $\{(a\in M|f(a)=0 \text{ for all } f\in Y)\}$ of $M_S^*({}_RM)$ by

---

1) See, also, [3], p. 111.

$\mathrm{Ann}_{M^*}(X)$ $(\mathrm{Ann}_M(Y))$. By definition it holds $\mathrm{Ann}_M(\mathrm{Ann}_{M^*}(X)) \supseteq X$ $(\mathrm{Ann}_{M^*}(\mathrm{Ann}_M(Y)) \supseteq Y)$.

PROPOSITION 4. *Let ${}_RM$ and ${}_RN$ be two left R-modules and M_S^* the N-dual of ${}_RM$, where $S=End({}_RN)$. Then, for a submodule $\mathfrak{u}$ of ${}_RM$, it holds $Ann_M(Ann_{M^*}(\mathfrak{u}))=\mathfrak{u}$ if and only if $M/\mathfrak{u}$ is N-torsionless.*

PROOF. This is more or less known. But, for completeness, we shall give here a proof for it. Let $M/\mathfrak{u}$ be N-torsionless. Suppose $\mathrm{Ann}_M(\mathrm{Ann}_{M^*}(\mathfrak{u})) \supsetneqq \mathfrak{u}$ and let a be an element of $\mathrm{Ann}_M(\mathrm{Ann}_{M^*}(\mathfrak{u}))$ which does not belong to $\mathfrak{u}$. Then, by assumption, there exists R-homomorphism f of $M/\mathfrak{u}$ into ${}_RN$ such that $f(\bar{a})=0$, where $\bar{a}$ denotes the coset of a in $M/\mathfrak{u}$. Let ν be the natural homomorphism of ${}_RM$ onto $M/\mathfrak{u}$. Then $f\nu \in \mathrm{Ann}_{M^*}(\mathfrak{u})$ and $f\nu(a) \neq 0$. But this is a contradiction.

Conversely, let $\mathrm{Ann}_M(\mathrm{Ann}_{M^*}(\mathfrak{u}))=\mathfrak{u}$ and a be an element of ${}_RM$ which does not belong to $\mathfrak{u}$. Then there exists $f \in \mathrm{Ann}_{M^*}(\mathfrak{u})$ such that $f(a) \neq 0$. Let $\bar{f}$ be the induced homomorphism of $M/\mathfrak{u}$ into ${}_RN$. Then we have $\bar{f}(\bar{a}) = f(a) \neq 0$. Thus $M/\mathfrak{u}$ is N-torsionless.

DEFINITION 1 ([1], [13]). *Let ${}_RM$ and ${}_RN$ be two left R-modules. ${}_RN$ is called M-injective if every R-homomorphism of any submodule of ${}_RM$ into ${}_RN$ is extended to that of ${}_RM$ into ${}_RN$.*

DEFINITION 2. *A left R-module ${}_RQ$ is called a cogenerator* (*in the category of left R-modules*) *if every left R-module is Q-torsionless.*

It is known that a left R-module ${}_RQ$ is a cogenerator if and only if, for each minimal left R-module ${}_R\mathfrak{m}$, ${}_RQ$ contains a submodule which is isomorphic to the injective envelope of ${}_R\mathfrak{m}$([9], Lemma 1).

THEOREM 1. *Let ${}_RM$ be a left R-module, ${}_RQ$ a cogenerator and M_S^* the Q-dual of ${}_RM$, where $S=End({}_RQ)$. Then, for each finitely generated submodule $\mathfrak{u}$ of M_S^* and S-homomorphismg of $\mathfrak{u}$ into Q_S, there exists an element a of ${}_RM$ such that $g(f)=f(a)$ for all $f \in \mathfrak{u}$.*

PROOF. Let $\mathfrak{u}=\sum_{i=1}^n f_iS$. It suffices to prove that there exists an element a of ${}_RM$ such that $g(f_i)=f_i(a)$ $i=1, 2, \cdots, n$. Let $\mathfrak{n}=\{(f_1(m), f_2(m), \cdots, f_n(m)) \mid m \in M\}$ $(\subseteq {}_RQ^{(n)}$, the direct sum of n-copies of ${}_RQ)$ and suppose that $z:=(g(f_1), g(f_2), \cdots, g(f_n)) \notin \mathfrak{n}$. Since ${}_RQ$ is a cogenerator, there exists an R-homomorphism α of ${}_RQ^{(n)}$ into ${}_RQ$ such that $\alpha(z) \neq 0$ and $\alpha(n)=0$ for all $n \in \mathfrak{n}$. This implies that there exists $(s_1, s_2, \cdots, s_n) \in S^{(n)}$ such that $\sum_{i=1}^n g(f_i)s_i \neq 0$ and $\sum_{i=1}^n f_i(m)s_i=0$ for all $m \in M$. Thus we have $g(\sum_{i=1}^n f_is_i) \neq 0$ and $\sum_{i=1}^n f_is_i=0$. But this is a contradiction.

THEOREM 2. *Let $_RM$, $_RQ$ and M_S^* be as in Theorem 1. Then the following conditions are equivalent*:

(1) *$_RM$ is linearly compact.*

(2) *For each submodule $\mathfrak{u}$ of M_S^* and S-homomorphism g of $\mathfrak{u}$ into Q_S, there exists an element a of $_RM$ such that $g(f)=f(a)$ for all $f\in\mathfrak{u}$.*

(3) *$_RM$ is Q-reflexive and Q_S is M_S^*-injective.*

PROOF. Since the equivalence (2)$\Longleftrightarrow$(3) is almost clear from the definitions, we prove here only the equivalenbe (1)$\Longleftrightarrow$(2). (1)$\Longrightarrow$(2). Let $\mathfrak{u}$ be a submodule of M_S^* and g a S-homomorphism of $\mathfrak{u}$ into Q_S. For each $f\in\mathfrak{u}$ there exists, by Theorem 1, an element $x_f\in M$ such that $g(f)=f(x_f)$. Then, again by Theorem 1, we see that the system of congruences $x\equiv x_f$ (mod ker (f)) is finitely solvable. Let a be a solution for it. Then we have $a\equiv x_f$ (mod ker (f)), that is, $f(a)=f(x_f)=g(f)$ for all $f\in\mathfrak{u}$. This completes the proof of our assertion. (2)$\Longrightarrow$(1). Let $x\equiv x_\alpha$ (mod M_α), where the M_α are submodules of $_RM$, be a finitely solvable system. Then, as is easily verified, the mapping

$$g:\sum \mathrm{Ann}_{M^*}(M_\alpha)\ni\sum_{\text{finite}} f_{\alpha_i}\longrightarrow\sum f_{\alpha_i}(x_{\alpha_i})\in Q$$

where $f_{\alpha_i}\in\mathrm{Ann}_{M^*}(M_{\alpha_i})$, is an well defined S-homomorphism. By assumption, there exists an element $a\in M$ such that $g(\sum f_{\alpha_i})=\sum f_{\alpha_i}(x_{\alpha_i})=(\sum f_{\alpha_i})(a)$. It follows that, for each α, $f_\alpha(x_\alpha)=f_\alpha(a)$ for all $f_\alpha\in\mathrm{Ann}_{M^*}(M_\alpha)$, and, by Proposition 4, this means that $a\equiv x_\alpha$(mod M_α). Thus $_RM$ is linearly compact.

COROLLARY 1. *Let $_RQ$ be a cogenerator and $S=End(_RQ)$. Then $_RQ$ is linearly compact if and only if the right S-module Q_S is injective.*

PROOF. Since $_RQ$ is Q-reflexive and $\mathrm{Hom}_R(Q,Q)=S$, we see, by the equivalence (1)$\Longleftrightarrow$(3) in Theorem 2, that $_RQ$ is linearly compact if and only if Q_S is (S-injective, whence) injective.

COROLLARY 2. *Let $_RM$ be a left R-module and $_RQ$ a linearly compact cogenerator. Then $_RM$ is linearly compact if and only if $_RM$ is Q-reflexive.*

PROOF. Since then Q_S, where $S=End(_RQ)$ is injective, our assertion is also a direct consequence of the equivalence (1)$\Longleftrightarrow$(3) in Theorem 2.

COROLLARY 3. *For a ring R the following conditions are equivalent*:

(1) *R is a two-sided* (*injective*) *cogenerator* (*=both $_RR$ and R_R are* (*injective*) *cogenerators*).

(2) *$_RR$ is a cogenerator and R_R is injective.*

(3) *$_RR$ is a linearly compact cogenerator.*

PROOF. The equivalence (1)$\Longleftrightarrow$(2) follows from [[5], Theorem 1 and Theorem 2], while the equivalence (2)$\Longleftrightarrow$(3) is a special case of our Corollary 1.

PROPOSITION 5. *An epimorphic image of a linearly compact module is linearly compact.*

PROOF. This follows direct from definition. But we give here an another proof for it with the use of Theorem 2. Let ${}_RM$ be a linearly compact left R-module and N a submodule of ${}_RM$. Let, further, ${}_RQ$, S and M_S^* be as in Theorem 1. $\mathrm{Hom}_R(M/N, Q)_S$ is naturally isomorphic to $\mathrm{Ann}_{M^*}(N)$. Since ${}_RM$ is linearly compact, for each submodule $\mathfrak{u}$ of $\mathrm{Ann}_{M^*}(N)$ and S-homomorphism g of $\mathfrak{u}$ into Q_S, there exists an element a of ${}_RM$ such that $g(f)=f(a)$ for all $f\in\mathfrak{u}$. But this means, by isomorphism, that for each submodule $\bar{\mathfrak{u}}$ of $\mathrm{Hom}_R(M/N, Q)_S$ and S-homomorphism $\bar{g}$ of $\bar{\mathfrak{u}}$ into Q_S there exists an element $\bar{a}$ of M/N such that $\bar{g}(\bar{f})=\bar{f}(\bar{a})$ for all $\bar{f}\in\bar{\mathfrak{u}}$. It follows, by Theorem 2, that M/N is linearly compact.

DEFINITION 3. *A left R-module ${}_RM$ is called cofinitely generated if for each family of submodules $\{M_\alpha\}_{\alpha\in A}$ of ${}_RM$ such that $\bigcap_{\alpha\in A} M_\alpha=0$, there exists a finite number of $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $\bigcap_{i=1}^{n} M_{\alpha_i}=0$.*

We quote here two theorems about cofinitely generated modules without proof from [7] ([7], Satz 1 and Satz 2).

THEOREM 3. *For a left R-module ${}_RM$ the following conditions are equivalent:*

(1) *${}_RM$ is cofinitely generated.*

(2) *The socle of ${}_RM$ is finitely generated and an essential submodule of ${}_RM$.*

THEOREM 4. *A left R-module ${}_RM$ is artinian if and only if every epimorphic image of ${}_RM$ is cofinitely generated.*

PROPOSITION 6. *Let R be a ring such that every nonzero left R-module has a nonzero socle. Then every linearly compact left R-module is artinian.*

PROOF. Let ${}_RM$ be a linearly compact module and ${}_R\bar{M}$ be an epimorphic image of ${}_RM$. By Proposition 5, ${}_R\bar{M}$ is linearly compact. It follows, by Proposition 3 and the assumption on R, that the socle of ${}_R\bar{M}$ is finitely generated and an essential submodule of ${}_R\bar{M}$, that is, ${}_R\bar{M}$ is cofinitely generated. Since this is true for every epimorphic image of ${}_RM$, we see, by Theorem 4, that ${}_RM$ is artinian.

The converse to Proposition 6 holds without any assumption on R, that is, PROPOSITION 7[2]. *Every artinian module over a ring R is linearly compact.*

PROOF. Let ${}_RQ$ be an injective cogenerator, $S=\mathrm{End}({}_RQ)$ and M_S^* the Q-dual of an artinian left R-module ${}_RM$. At first, we want to show that M_S^* is noetherian. Let $\mathfrak{u}$ be a submodule of M_S^* and $N=\mathrm{Ann}_M(\mathfrak{u})=\bigcap_{u\in\mathfrak{u}}\ker(u)$. Since M/N is cofinitely generated, there exists a finite number of elements $u_1, u_2, \cdots, u_n$ of $\mathfrak{u}$ such that $N=\bigcap_{i=1}^{n} \ker\ (u_i)=\mathrm{Ann}_M(\sum_{i=1}^{n} u_iS)$. Since ${}_RQ$ is injective[3], it follows that $\mathfrak{u}\subseteqq \mathrm{Ann}_{M^*}(\mathrm{Ann}_M(\mathfrak{u}))=\mathrm{Ann}_{M^*}(N)=\sum_{i=1}^{n} u_iS\subseteq\mathfrak{u}$, whence $\mathfrak{u}=\sum_{i=1}^{n} u_iS$. Thus $\mathfrak{u}$ is finitely generated. Since this is true for every submodule $\mathfrak{u}$ of M_S^*, M_S^* is noetherian as asserted. By Theorem 1, for each (finitely generated) submodule $\mathfrak{u}$ of M_S^* and S-homomorphis g of $\mathfrak{u}$ into Q_S, there exists an element a of ${}_RM$ such that $g(f)=f(a)$ for all $f\in\mathfrak{u}$. This implies, by Theorem 2, that ${}_RM$ is linearly compact. Since there exists always an injective cogenerator ${}_RQ$, our proof is thus complete.

PROPOSITION 8[4]. *Let ${}_RM$ be a left R-module and N a submodule of ${}_RM$. If both N and M/N are linearly compact, then ${}_RM$ is linearly compact.*

PROOF. Let ${}_RQ$ be an injective cogenerator, $S=\mathrm{End}({}_RQ)$ and M_S^* the Q-dual of ${}_RM$. $\mathrm{Hom}_R(M/N, Q)_S$ and $\mathrm{Hom}_R(N, Q)_S$ are then naturally isomorphic to $\mathrm{Ann}_{M^*}(N)$ and $M^*/\mathrm{Ann}_{M^*}(N)$, respectively. For a given submodule $\mathfrak{u}$ of M_S^* and S-homomorphism g of $\mathfrak{u}$ into Q_S, let g be the restriction of g onto $\mathfrak{u}\cap\mathrm{Ann}_{M^*}(N)$. Since M/N is linearly compact, there exists an element a of ${}_RM$ such that $g(u)=u(a)$ for all $u\in\mathfrak{u}\cap\mathrm{Ann}_{M^*}(N)$. Further, since N is linearly compact, for the well defined S-homomorphism.

$$\alpha : \mathfrak{u}+\mathrm{Ann}_{M^*}(N)/\mathrm{Ann}_{M^*}(N) \ni \bar{u} \longmapsto g(u)-u(a)\in Q$$

where $u\in\mathfrak{u}$, there exists an element n_0 of N such that $\alpha(\bar{u})=g(u)-u(a)=\bar{u}(n_0)=u(n_0)$ for all $u\in\mathfrak{u}$. It follows that $g(u)=u(a+n_0)$ for all $u\in\mathfrak{u}$. This means, by theorem 2, that ${}_RM$ is linearly compact.

LEMMA 1. *Let ${}_RM$ be a linearly compact left R-module, ${}_RQ$ a cogenerator, $S=End({}_RQ)$ and M_S^* the Q-dual of ${}_RM$. Then for any two submodules $\mathfrak{u}$ and $\mathfrak{v}$ of M_S^* we have $Ann_M(\mathfrak{u}\cap\mathfrak{v})=Ann_M(\mathfrak{u})+Ann_M(\mathfrak{v})$.*

PROOF. Clearly $\mathrm{Ann}_M(\mathfrak{u}\cap\mathfrak{v})\supseteq\mathrm{Ann}_M(\mathfrak{u})+\mathrm{Ann}_M(\mathfrak{v})$. Let $x\in\mathrm{Ann}_M(\mathfrak{u}\cap\mathfrak{v})$. Then, as is easily seen, the mapping

2) See [12], Proposition 5.

3) See [10], Theorem 1.1.

4) See [12], Proposition 9.

$$g : \mathfrak{u}+\mathfrak{v} \ni u+v \longrightarrow u(x) \in Q$$

where $u \in \mathfrak{u}$ and $v \in \mathfrak{v}$, is an well defined S-homomorphism. Since ${}_RM$ is linearly compact, there exists an element m_0 of ${}_RM$ such that $g(u+v)=u(x)=(u+v)(m_0)$ for all $u+v \in \mathfrak{u}+\mathfrak{v}$. Then, since $0=g(v)=v(m_0)$ for all $v \in \mathfrak{v}$, $m_0 \in \mathrm{Ann}_M(\mathfrak{v})$. On the other hand, since $u(x)=g(u)=u(m_0)$ for all $u \in \mathfrak{u}$, $x-m_0$ $\mathrm{Ann}_M(\mathfrak{u})$. It follows that $x \in \mathrm{Ann}_M(\mathfrak{u})+\mathrm{Ann}_M(\mathfrak{v})$. Thus our proof is complete.

DEFINITION 4 (see [4]). *A left R-module ${}_RM$ is called complemented if for each submodule N of ${}_RM$ there exists a submodule N' of ${}_RM$ such that $M=N+N'$ and N' is minimal with respect to this property.*

THEOREM 5. *A linearly compact module is complemented.*

PROOF. Let ${}_RM$ be a linearly compact left R-module, ${}_RQ$ a cogenerator, $S=\mathrm{End}({}_RQ)$ and M_S^* the Q-dual of ${}_RM$. For a submodule N of ${}_RM$ let $\mathfrak{u}$ be a submodule of M_S^* such that $\mathrm{Ann}_{M^*}(N) \cap \mathfrak{u}=0$ and $\mathfrak{u}$ is maximal with respect to this property. By Zorn's lemma there exists such a submodule $\mathfrak{u}$. By Proposition 4 and Lemma 1, it holds then $M=N+\mathrm{Ann}_M(\mathfrak{u})$. We want to show that $\mathrm{Ann}_M(\mathfrak{u})$ is minimal with respect to this property. Let N' be a submodule of ${}_RM$ such that $N' \subseteq \mathrm{Ann}_M(\mathfrak{u})$ and $M=N+N'$. Then we have $0=\mathrm{Ann}_{M^*}(N) \cap \mathrm{Ann}_{M^*}(N')$. Since $\mathfrak{u} \subseteq \mathrm{Ann}_{M^*}(\mathrm{Ann}_M(\mathfrak{u})) \subseteq \mathrm{Ann}_{M^*}(N')$, the maximality of $\mathfrak{u}$ implies that $\mathfrak{u}=\mathrm{Ann}_{M^*}(N')$. Hence $N'=\mathrm{Ann}_M(\mathfrak{u})$ and $\mathrm{Ann}_M(\mathfrak{u})$ is minimal with respect to $M=N+\mathrm{Ann}_M(\mathfrak{u})$ as asserted.

COROLLARY. *Let R be a ring such that ${}_RR$ is linearly compact. Then R is a semiperfect ring.*

PROOF. This follows from Theorem 5 and [[4], Satz].

§ 4. Linearly compact modules and injective cogenerators with essential socles.

In this section we consider the relations between linearly compact modules and injective cogenerators with essential socles.

LEMMA 2. *Let ${}_RM$ be a linearly compact left R-module and ${}_RQ$ an injective cogenerator with an essential socle, that is, an injective cogenerator whose socle is an essential submodule of it, $S=End({}_RQ)$ and M_S^* the Q-dual of ${}_RM$. Then for each nonzero element f of M_S^* and proper submodule $\mathfrak{u}$ of fS, there exists a nonzero S-homomorphism g of fS into Q_S such that $g(u)=0$ for all $u \in \mathfrak{u}$.*

PROOF. Since $M/\ker(f) \cong f(M) \subseteq Q$, $M/\ker(f)$ is, by our assumptions, cofinitely generated. Suppose $\mathrm{Ann}_M(\mathfrak{u})=\ker(f)\,(=\mathrm{Ann}_M(fS))$. Then there exists a finite number of elements $u_1, u_2, \cdots, u_n$ of $\mathfrak{u}$ such that $\mathrm{Ann}_M(\mathfrak{u})$

$=\mathrm{Ann}_M(fS)=\mathrm{Ann}_M(\sum_{i=1}^{n} u_i S)$. It follows then, since ${}_RQ$ is injective, that $\sum_{i=1}^{n} u_i S = fS$. But this is a contradiction. Let a be an element of $\mathrm{Ann}_M(\mathfrak{u})$ which does not belong to $\ker(f)$. Then the mapping $g: fS \ni fs \longrightarrow (fs)(a) \in Q$ is a desired homomorphism.

THEOREM 6. *Under the assumptions in Lemma 2, we have, for any submodule* $\mathfrak{u}$ *of* M_S^*, $Ann_{M^*}(Ann_M(\mathfrak{u}))=\mathfrak{u}$.

PROOF. Suppose $\mathrm{Ann}_{M^*}(\mathrm{Ann}_M(\mathfrak{u})) \supsetneqq \mathfrak{u}$ and let f be an element of $\mathrm{Ann}_{M^*}(\mathrm{Ann}_M(\mathfrak{u}))$ which does not belong to $\mathfrak{u}$. Since $\mathfrak{u}+fS/\mathfrak{u} \cong fS/\mathfrak{u} \cap fS$, there exists, by Lemma 2, a nonzero S-homomorphism of $\mathfrak{u}+fS$ into Q_S which annihilates $\mathfrak{u}$. Then, since ${}_RM$ is linearly compact, there exists an element a of ${}_RM$ such that $f(a) \neq 0$ and $u(a)=0$ for all $u \in \mathfrak{u}$. But this is a contradiction, because we have then $a \in \mathrm{Ann}_M(\mathfrak{u})$ and $f \in \mathrm{Ann}_{M^*}(\mathrm{Ann}_M(\mathfrak{u}))$.

COROLLARY 1. *Under the assumptions in Lemma 2, there exists a lattice anti-isomorphism between the lattice of submodules of* ${}_RM$ *and that of submodules of* M_S^* *under the annihilator relations.*

PROOF. This follows from Proposition 4 and Theorem 6.

REMARK. Since for any ring R there exists always an injective cogenerator ${}_RQ$ with an essential socle, one can deduce Theorem 5 direct from the above corollary.

COROLLARY 2. *Under the assumptions in Lemma 2, the right S-module* M_S^* *is linearly compact.*

PROOF. With the aid of Theorem 6, one can prove this corollary just as in the proof for the implication $(2) \Longrightarrow (1)$ Theorem 2.

COROLLARY 3. *Let* ${}_RQ$ *be a linearly compact injective cogenerator with an essential socle and* $S=End({}_RQ)$. *Then the right S-module* Q_S *is an injective cogenerator* (*in the category of right S-modules*).

PROOF. By Corollary 1 to Theorem 2, Q_S is injective. Moreover, by Theorem 6, we see that, for each maximal right ideal $\mathfrak{r}$ of S, $\mathrm{Ann}_Q(\mathfrak{r}) \neq 0$. This means that, for each minimal right S-module $\mathfrak{n}_S$, Q_S contains an isomorphic image of $\mathfrak{n}_S$. Thus Q_S is an injective cogenerator.

From our observations on linearly compact modules and cogenerators we have the following

THEOREM 7. *Let* ${}_RQ$ *be a balanced*$(={}_RR$ *is Q-reflexive*) *linearly compact injective cogenrator with an essential socle and* $S=End({}_RQ)$. *Then so is the right S-module* Q_S, *and, the contravariant functors* $Hom_R(\ ,Q)$ *and* $Hom_S(\ ,Q)$ *are mutually inverse category-isomorphism between the category* *of all linearly compact left R-modules and that of all lnearly compact right S-modules. Moreover, a left R-module (a right S-module) is Q-reflexive if and only if it is linearly compact.*

§ 5. Applications.

PROPOSITION 9. *Let ${}_RM$ and ${}_RN$ be two R-modules, $S=End({}_RN)$* and *M_S^* be the N-dual of ${}_RM$. Suppose that, for each submodule $\mathfrak{u}$ and S-homomorphism g of $\mathfrak{u}$ into N_S, there exists an element a of ${}_RM$ such that $g(f)=f(a)$ for all $f\in\mathfrak{u}$, or, in other words, ${}_RM$ is N-refiexive and N_S is M_S^*-injective. Then ${}_RM$ can not be an infinite direct sum of nonzero submodules M_α's such that M_α's are N-torsionless.*

PROOF. Suppose that $M=\bigoplus_{\alpha\in\Lambda}M_\alpha$, an infinite direct sum of nonzero submodules M_α's such that M_α's are N-torsionless. For each $\alpha\in\Lambda$, let $M^\alpha=\bigoplus_{\beta\neq\alpha}M_\beta$ and x_α be a nonzero element of M_α. Further, for the (well difined) S-homomorphism

$$g:\sum \mathrm{Ann}_{M^*}(M^\alpha)\ni\sum_{\text{finite}}f_{\alpha_i}\longrightarrow\sum f_{\alpha_i}(x_{\alpha_i})\in N$$

where $f_{\alpha_i}\in\mathrm{Ann}_{M^*}(M^{\alpha_i})$, let a be an element of ${}_RM$ such that $g(\sum f_{\alpha_i})=\sum f_{\alpha_i}(x_{\alpha_i})=(\sum f_{\alpha_i})(a)$ for all $\sum f_{\alpha_i}\in\sum\mathrm{Ann}_{M^*}(M^\alpha)$. Then, for each $\alpha\in\Lambda$, we have $f_\alpha(x_\alpha)=f_\alpha(a)$, that is $a-x_\alpha\in\mathrm{Ann}_M(\mathrm{Ann}_{M^*}(M^\alpha))=M^\alpha$. It follows that, for each $\alpha\in\Lambda$, the α-component of a is $x_\alpha(\neq 0)$. But this is a contradiction.

COROLLARY (Sandomierski). *Let ${}_RM$ be a left R-module and $S=End({}_RM)$. If the right S-module M_S is injective, then ${}_RM$ can not be an infinite direct sum of nonzero submodules.*

PROOF. Since ${}_RM$ is M-reflexive, this is a direct consequence of Proposition 2.

A ring R is called a left V-ring if every minimal left R-module is injective. It is known that a ring R is a left V-ring if and only if every left R-module has a zero radical. (See, for example, [8] Theorem 35).

PROPOSITION 10. *Let R be a left V-ring and $Q=\bigoplus\mathfrak{m}_\alpha$, where $\mathfrak{m}_\alpha$ ranges over all minimal left R-modules. If ${}_RQ$ is balanced, then R is semisimple artinian.*

PROOF. Let S be the endomorphism ring of ${}_RQ$. Then, since ${}_RQ$ is a cogenerator and Q_S is a completely reducible right S-module, there exists a lattice anti-isomorphism between the lattice of left ideals of R and that of submodules of Q_S under the annihilator relation. Moreover, since Q_S is completely reducible, one can see that every left ideal of R is a direct summand of ${}_R R$. It follows that R is semisimple artinian.

Department of Mathematics
Hokkaido University

References

[1] G. AZUMAYA: Completely faithful modules and self-injective rings, Nagoya Math. J. 27 (1966), 697–708.

[2] G. AZUMAYA: M-projective and M-injective modules (to appear).

[3] N. BOURBAKI: Eléments de Mathématique, Algébre Commutative, Chapitre 3, Paris, 1967.

[4] F. KASCH and E. A. MARES: Eine Kenzeichung semi-perfekter Moduln, Nagoya Math. J. 27 (1966), 525–529.

[5] F. KASCH, H.-J. SCHNEIDER and H. J. STOLBERG: On injective modules and cogenerators, Report 69–23, Dept. Math., Carnegie Institute of Technology, Carnegie Mellon Univ., 1969.

[6] B. MÜLLER: Linear compactness and Morita duality, J. Algebra 16 (1970), 60–66.

[7] T. ONODERA: Koendlich erzeugten Moduln und Kogeneratoren (to appear).

[8] T. ORNSTEIN: Rings with chain conditions, Ph. D. Thesis, Rutgers, 1966.

[9] B. OSOFSKY: A generalizalization of quasi-Frobenius rings, J. Algebra 4 (1966), 373–387.

[10] A. ROSENBERG and D. ZELINSKY: Annihilators, Portugalle Math. 20 (1961), 53–65.

[11] F. L. SANDOMIERSKI: Some examples of right self-injective rings which are not left self-injective, Proc. Amer. Math. Soc. 26 (1970), 244–245.

[12] D. ZELINSKY: Linearly compact modules and rings, Amer. J. Math. 75 (1953), 79–90.

[13] W. ZIMMERMANN: Injektive Strukturen und M-injektive, Objekte, Dissertation, Univ. München, 1969.

(Received August 25, 1971)