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20th SAPPORO SYMPOSIUM ON PARTIAL DIFFERENTIAL EQUATIONS

February 13～15, 1996

Department of Mathematics
Hokkaido University
060 Sapporo, Japan

Chairman Rentaro Agemi
(Hokkaido Univ.)

Program

Feb.13 (Room 3-508)
9:30～10:30 Kiyoshi Mochizuki(Tokyo Metroplitan Univ.)
Energy decay of solutions to the wave equation with dissipation localized near infinity

11:00～12:00 Yi Zhou(Fudan Univ.)
Local existence with minimal regularity for quadratic nonlinear wave equations

1:30～2:00 (*)

2:00～3:00 Kunihiko Kajitani(Tsukuba Univ.)
Global solutions of the Cauchy problem for Kirchhoff equations

3:30～4:30 Akitaka Matsumura(Osaka Univ.)
A bifurcation phenomenon for the periodic solutions of the Duffin equation

Feb. 14 (Room 4-508)
9:30～10:30 Tatsuo Nishitani(Osaka Univ.)
Regurarity of solutions to characteristic boundary value problem for systems

11:00～12:00 J. Vaillant(Paris VI)
Hyerbolic systems and Gevrey class

1:30～2:00 (*)

2:00～2:50 Jian Zhai(Hokkaido Univ.)
The existence of non-trivial solutions to Ginzburg-Landau type equations

3:00～3:50　Kunio Hidano(Waseda Univ.)
On the existence of wave operators and asymptotic completeness for nonlinear wave equations in R^{4+1} with small data

4:00～4:50　Kazuyoshi Yokoyama(Hokkaido Univ.)
Global existence for systems of wave equations with different speeds

5:00～5:30　(*)

6:00～8:00　Banquet (懇親会)
Faculty House, Trillium

Feb.15　(Room 4-508)
9:30～10:30　Xu-Yan Chen(Hiroshima Univ. and Georgia Inst. of Tech.)
Local Asymptotic forms of solutions to elliptic and parabolic boundary value problems

11:00～12:00　Toshitaka Nagai(Kyushu Inst. of Tech.)
Blow-up of solutions to a system of partial differential equations modelling chemotaxis

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1.(*) indicates the discussion time with speakers at Room 4-507

Energy decay of solutions to the wave equation with dissipation localized near infinity

Kiyoshi Mochizuki (Tokyo Metropolitan University)

Let Ω be an exterior domain in $\mathbf{R}^N$ with smooth boundary. We consider the initial boundary value problem

$$w_{tt} - \Delta w + b(x,t)|w_t|^{p-1}w_t = 0 \quad in\ \Omega \times (0,\infty),$$

$$w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) \quad in\ \Omega,$$

$$w(x,t) = 0 \quad on\ \partial\Omega \times (0,\infty),$$

where $p \geq 1$ and $b(x,t) \geq 0$. We define the energy of solutions at time t by

$$\|w(t)\|_E^2 = \frac{1}{2}\int_\Omega \{w_t(x,t)^2 + |\nabla w(x,t)|^2\}dx.$$

Then the following energy equation holds for any t.

$$\|w(t)\|_E^2 + \int_0^t \int_\Omega b(x,\tau)w_t(x,\tau)^p dx d\tau = \|w(0)\|_E^2.$$

Since $b(x,t) \geq 0$, the energy $\|w(t)\|_E^2$ is decreasing in $t > 0$. Thus, a question naturally rises whether it decays or not as t goes to infinity.

The first results is obtained by Mizohata-Mochizuki [1] in 1966, where is considered the case $\Omega = \mathbf{R}^3$, $b = b(x) = O(|x|^{-3-\delta})$ $(\delta > 0)$ and $p = 1$. The principle of limiting amplitude is proved, and so, the nondecay of energy is suggested there.

Nondecay results of energy are obtained by Mochizuki [4] in 1976 (see also [5]) in case $p = 1$ and $b(x,t) \leq c_1(1+|x|)^{-1-\delta}$. Moreover, if $N \geq 3$ and Ω is exterior of starshaped domain, then every solution is proved to be asymptotically free. Note that the energy of solution decays as $t \to \infty$ provided $b(x,t) \geq c_2(1+t+|x|)^{-1}$ (see Matsumura [2] in 1977). These show that if $b(x,t) = O(|x|^{-\gamma})$, then $\gamma = 1$ is the critical exponent of energy decay. In the recent work Mochizuki-Nakazawa [7] we considered the case $b(x,t) = o(|x|^{-1})$, and obtained the critical exponent of logrithmic order.

Similar results are obtained for the nonlinear case $p > 1$ if $\Omega = \mathbf{R}^N$. It is proved in Mochizuki-Motai [6] that the energy decay occurs provided $0 \leq \gamma \leq 1 - N(p-1)/2$, and energy does not decay for small solutions provided $\gamma > 1 - (N-1)(p-1)/2$ (the nondecay result is anounced in the earlier work Mochizuki [3]). The problem is unsolved so far in the interval $1 - N(p-1)/2 < \gamma \leq 1 - (N-1)(p-1)/2$.

In this talk we shall report an energy decay result for localized dissipation near infinity. We consider the linear problem with $p = 1$. Assume

$$N \geq 3 \text{ and } \mathbf{R}^N \backslash \Omega \text{ is starshaped w.r.t. the origin;}$$

$$b(x,t) \text{ is bounded, nonnegative, and } b_t(x,t) \leq 0;$$

there exists an $R > 0$ such that

$$b_0(1 + t + |x|)^{-1} \leq b(x,t) \leq b_1 \text{ in } \{|x| > R\} \times (0, \infty)$$

for some $b_0, b_1 > 2$. Then the energy of solutions decays like

$$\|w(t)\|_E^2 \leq K(1+t)^{-1}.$$

The proof will be based on weighted energy inequalities (Mochizuki [8]). Similar problem is already considered by Zuazua [9] for the Klein-Gordon equation. However, his method is not applicable to our classical wave equation.

Note that our proof can be applied to the quasilinear wave equation

$$w_{tt} - \{\alpha + \beta \|w(t)\|^2\} \Delta w + b(x,t) w_t = 0 \text{ in } \Omega \times (0, \infty),$$

without any essential modification.

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LOCAL EXISTENCE WITH MINIMAL REGULARITY FOR QUADRATIC NONLINEAR WAVE EQUATIONS

YI ZHOU

Dedicated to the 60-th anniversary of Professor Koji Kubota

0. INTRODUCTION

Consider nonlinear wave equations of the type

$$\Box\phi^I = F^I(\phi, \partial\phi) \tag{0.1}$$

where $\Box$ denotes the standard D'Alembertian in R^{n+1} and the nonlinear terms F are quadratic in $\partial\phi$, i.e:

$$F^I(\phi, \partial\phi) = \sum_{J,K,l,m} \Gamma^I_{J,K,l,m}(\phi)\partial_l\phi^J\partial_m\phi^K \tag{0.2}$$

We shall study the problem of minimal regularity of initial conditions for which the corresponding initial value problem

$$\varphi(0,x) = f_0(x), \quad \partial_t\phi(0,x) = f_1(x) \tag{0.3}$$

is well posed.

We shall distinguish between the following cases:

1) The general case.

2) The case when the equations satisfy the null condition, that is

$$F^I(\phi, \partial\phi) = \sum_{J,K} \Gamma^I_{J,K}(\phi)B^I_{J,K}(\partial\phi^J, \partial\phi^K) \tag{0.4}$$

with $B^I_{J,K}$ any of the null forms.

$$Q_0(\phi, \varphi) = -\partial_t\phi\partial_t\varphi + \sum_{i=1}^{n} \partial_i\phi\partial_i\varphi \tag{0.5}$$

$$Q_{\alpha\beta}(\phi, \varphi) = \partial_\alpha\phi\partial_\beta\varphi - \partial_\beta\phi\partial_\alpha\varphi \quad 0 \le \alpha, \beta \le n \tag{0.6}$$

proposed running head: Local regularity of wave equation .

3) the case when the equation is of wave maps type, that is, only the null form Q_0 is present in case 2.

4) the case when the equation is a real wave map, that is, $\Gamma^I_{J,K}(\phi)$ is further the Christoffell symbol of a Riemannian manifold.

The classical local existence theorem rely only on the energy estimates and Sobolev inequality. Thus, it requires

$$f_0 \in H^{s+1}, \quad f_1 \in H^s \tag{0.7}$$

for $s > \frac{n}{2}$. However, Klainerman-Machedon [2] proved that the problem is locally well-posed in H^{s+1} for $s \geq \frac{n-1}{2}$ provided that the nonlinear terms satisfy the "null condition". Starting from this work, there are many other works following this. I shall summarize the recent progress in the following table.

H^{s+1}	general case	null condition	wave maps type	"real" wave maps
$n \geq 4$	$s \geq \frac{n-1}{2}$, sharp	$s \geq \frac{n-1}{2}$, sharp	$s \geq \frac{n-1}{2}$, sharp	$s \geq \frac{n-1}{2}$, sharp
$n = 3$	$s > 1$, sharp	$s > \frac{3}{4}$, ?	$s > \frac{1}{2}$, sharp	? ?
$n = 2$	$s > \frac{3}{4}$, sharp	$s > \frac{1}{4}$, sharp	$s \geq \frac{1}{5}$, ?	? ?

In this table, the local well-posedness result in $n \geq 4$ is due to Beals&Bezard [1], the sharpness is due to J.Shatah [6].

The local well-posedness for $n = 3$ in the general case is due to Ponce&Sideris [5] the sharpness is due to H.Lindblad [4]; the local well-posedness result when the equation satisfy the "null condition" is due to the author [10]; the local well-posedness result for wave maps type is due to Klainerman& Machedon [3], the sharpness is also due to them.

The local well-posedness for $n = 2$ in the general case follows from the same argument as in Ponce-Sideris [5], the sharpness is due to H.Lindblad [4]. the local well-posedness as well as sharpness result when the equation satisfy the "null condition" is due to the author [10]; The local well-posedness result for wave maps type is also due to the author [9].

For the local well-posedness of wave maps in the covariant case, see Shatah & Tahvildar-Zadeh [7].

In this talk, I shall focus on the case n=2,3 and the equation satisfies the null condition.

1. Regularity properties of the first iterate

We consider

$$\Box\Psi = Q(\phi, \varphi) \tag{1.1}$$

$$\Psi(0,x) = 0, \Psi_t(0,x) = 0 \tag{1.2}$$

where Q is any of the null forms (0.5)(0.6) and ϕ, φ are solutions to homogeneous wave equations:

$$\Box\phi = 0 \tag{1.3}$$

$$\phi(0,x) = f(x), \phi_t(0,x) = 0 \tag{1.4}$$

$$\Box\varphi = 0 \tag{1.5}$$

$$\varphi(0,x) = g(x), \varphi_t(0,x) = 0 \tag{1.6}$$

A. $n = 3$, $Q = Q_0$.

It follows from the identity

$$2Q_0(\phi,\varphi) = \Box(\phi\varphi) - \phi\Box\varphi - \varphi\Box\phi \tag{1.7}$$

that $\Psi = \phi\varphi$ up to a solution of the free wave equation, so it follows trivially that $\Psi(t,\cdot) \in H^{s+1}$ for $s > \frac{1}{2}$ provided

$$f, g \in H^{s+1} \tag{1.8}$$

for $s > \frac{1}{2}$.

B. $n = 3$, $Q = Q_{\alpha\beta}$

We have

Proposition 1.1. *Consider in R^{3+1} the Cauchy problem (1.1)(1.2) with ϕ, φ satisfying respectively (1.3)(1.4) and (1.5)(1.6), If (1.8) is satisfied for $\frac{1}{2} < s < 1$. then the first iterate Ψ belongs to H^{s+1} and moreover, there hold*

$$\|\Psi_t(t,\cdot)\|_{H^s} + \|\Psi_x(t,\cdot)\|_{H^s} \leq C_\gamma t^\gamma \|f\|_{H^{s+1}} \|g\|_{H^{s+1}} \tag{1.9}$$

for any γ with $0 < \gamma < s - \frac{1}{2}$.

C.$n = 2$, $Q = Q_0$

Similar to the case $n = 3$. $\Psi = \phi\varphi$ up to a solution of the free wave equation, so it follows trivially that $\Psi(t,\cdot) \in H^{s+1}$ for any $s > 0$ if (1.8) is satisfied for $s > 0$.

D. $n = 2$, $Q = Q_{\alpha\beta}$

Proposition 1.2. *Consider in R^{2+1} the Cauchy problem (1.1)(1.2) with ϕ, φ satisfying respectively (1.3)(1.4) and (1.5)(1.6), and (1.8) is satisfied .*

i)If $\frac{1}{4} < s < \frac{1}{2}$, then the first iterate Ψ belongs to H^{s+1} and moreover, there hold

$$\|\Psi_t(t,\cdot)\|_{H^s} + \|\Psi_x(t,\cdot)\|_{H^s} \leq C_\gamma t^\gamma \|f\|_{H^{s+1}} \|g\|_{H^{s+1}} \tag{1.10}$$

for any γ with $\frac{1}{4} < \gamma < s$.

ii)If $0 < s \leq \frac{1}{4}$, then the first iterate Ψ fails to be in H^{s+1} and moreprecisely the following estimates fail

$$\|\Psi_t(t,\cdot)\|_{H^s} + \|\Psi_x(t,\cdot)\|_{H^s} \leq C(t) \|f\|_{H^{s+1}} \|g\|_{H^{s+1}} \tag{1.11}$$

It turns out that only the knowledge of local regularity of the first iterate is not enough for our purpose, we need to know the microlocal regularity of the first iterate as well

A. $n = 3$, $Q = Q_0$

proposition 2.1. *Consider $\Psi = \phi\varphi$ where ϕ, φ satisfy respectively (1.3)(1.4) and (1.5)(1.6) with (1.8) being satisfied for $\frac{1}{2} < s < 1$. Let $\tilde{\Psi}(\tau, \xi)$ be the space-time Fourier transform of Ψ, then microlocally at noncharacteristic point*

$$\tau^2 - \xi^2 \neq 0$$

Ψ belongs to H^{2s+1}. Moreprecisely, the following estimate holds

$$\iint (|\tau| + |\xi|)^{2s+2} ||\tau| - |\xi||^{2s} \tilde{\Psi}(\tau, \xi) d\tau d\xi \leq C \|f\|^2_{H^{s+1}} \|g\|^2_{H^{s+1}} \tag{2.1}$$

B. $n = 3$, $Q = Q_{\alpha\beta}$

proposition 2.2. *Under the assumptions of proposition 1.1. Let $\tilde{\Psi}(\tau, \xi)$ be the space-time Fourier transform of Ψ, then microlocally at noncharacteristic point*

$$\tau^2 - \xi^2 \neq 0$$

Ψ belongs to H^{2s+1}. Moreprecisely, the following estimate holds

$$\iint (|\tau| + |\xi|)^{2s} ||\tau| - |\xi||^{-2(1-s)} \tilde{Q_{12}}(\tau, \xi) d\tau d\xi \leq C \|f\|^2_{H^{s+1}} \|g\|^2_{H^{s+1}} \tag{2.2}$$

C. $n = 2$, $Q = Q_0$

proposition 2.3. *Consider $\Psi = \phi\varphi$ where ϕ, φ satisfy respectively (1.3)(1.4) and (1.5)(1.6) with (1.8) being satisfied for $0 < s < \frac{1}{2}$. Let $\tilde{\Psi}(\tau, \xi)$ be the space-time Fourier transform of Ψ, then microlocally at noncharacteristic point*

$$\tau^2 - \xi^2 \neq 0$$

Ψ belongs to $H^{2s+\frac{3}{2}}$. Moreprecisely, the following estimate holds

$$\iint (|\tau| + |\xi|)^{2s+2} ||\tau| - |\xi||^{2s+1} \tilde{\Psi}(\tau, \xi) d\tau d\xi \leq C \|f\|^2_{H^{s+1}} \|g\|^2_{H^{s+1}} \tag{2.3}$$

D. $n = 2$, $Q = Q_{\alpha\beta}$

proposition 2.4. *Under the assumptions of proposition 1.2. Let $\tilde{\Psi}(\tau, \xi)$ be the space-time Fourier transform of Ψ, then microlocally at noncharacteristic point*

$$\tau^2 - \xi^2 \neq 0$$

Ψ belongs to $H^{2s+\frac{3}{2}}$ for $\frac{1}{4} < s < \frac{1}{2}$. Moreprecisely, the following estimate holds

$$\iint (|\tau| + |\xi|)^{2s} ||\tau| - |\xi||^{-(1-2s)} \tilde{Q_{12}}(\tau, \xi) d\tau d\xi \leq C \|f\|^2_{H^{s+1}} \|g\|^2_{H^{s+1}} \tag{2.4}$$

3. Smoothing estimates

proposition 3.1. *(Strichartz estimate) Let $u(t,x)$ be a function defined on R^{3+1} satisfying the homogeneous wave equation*

$$\Box u(t,x) = 0 \tag{3.1}$$

$$u(0,x) = f_0, u_t(0,x) = f_1 \tag{3.2}$$

Then

$$\|u\|_{L^4(R^4)} \leq C(\|f_0\|_{H^{\frac{1}{2}}(R^3)} + \|f_1\|_{H^{-\frac{1}{2}}(R^3)}) \tag{3.3}$$

proposition 3.2. *Let $u(t,x)$ be a function defined on R^{2+1}, then there hold*

$$\|u\|^2_{L^4(R^3)} \leq C \iint |\xi|^{\frac{1}{4}} |\tau + |\xi||^{\frac{1}{4}} \tilde{u}^2(\tau,\xi) d\tau d\xi \tag{3.4}$$

and

$$\|u\|^2_{L^4(R^3)} \leq C \iint |\xi|^{\frac{1}{4}} |\tau - |\xi||^{\frac{1}{4}} \tilde{u}^2(\tau,\xi) d\tau d\xi \tag{3.5}$$

where $\tilde{u}$ is the space-time Fourier transform of u.

Remark. *By Sobolev inequality, it can only be proved that*

$$\|u\|^2_{L^4(R^3)} \leq C \iint |\xi||\tau + |\xi||^{\frac{1}{2}} \tilde{u}^2(\tau,\xi) d\tau d\xi \tag{3.6}$$

Smoothing estimates for the null forms

Introduce the space-time norms

$$N_{a,b}(\phi) = \left(\iint w_+^{2a}(\tau,\xi) w_-^{2b}(\tau,\xi) |\tilde{\phi}(\tau,\xi)|^2 d\tau d\xi \right)^{\frac{1}{2}} \tag{4.1}$$

where $\tilde{\phi}$ denotes the space-time Fourier transform of ϕ and

$$w_\pm(\tau,\xi) = 1 + ||\tau| \pm |\xi|| \tag{4.2}$$

A. $n = 3$, $Q = Q_0$

Theorem 4.1. *Consider the space-time norms (4.1) and functions ϕ, φ defined in R^{3+1}. The estimates*

$$N_{s,s-1}(Q_0(\phi,\varphi)) \leq C N_{s+1,s}(\phi) N_{s+1,s}(\varphi) \tag{4.3}$$

hold true for any $\frac{1}{2} < s < 1$

B. $n = 3$, $Q = Q_{\alpha\beta}$

Theorem 4.2. *Consider the space-time norms (4.1) and functions ϕ, φ defined in R^{3+1}. The estimates*

$$N_{s,s-1}(Q_{12}(\phi,\varphi)) \leq CN_{s+1,s}(\phi)N_{s+1,s}(\varphi) \tag{4.3}$$

hold true for any $\frac{3}{4} < s < 1$

C. $n = 2$, $Q = Q_0$

Theorem 4.3. *Consider the space-time norms (4.1) and functions ϕ, φ defined in R^{2+1}. The estimates*

$$N_{s,s-\frac{1}{2}}(Q_0(\phi,\varphi)) \leq CN_{s+1,s+\frac{1}{2}}(\phi)N_{s+1,s+\frac{1}{2}}(\varphi) \tag{4.3}$$

hold true for any $\frac{1}{8} \leq s < \frac{1}{2}$

D. $n = 2$, $Q = Q_{\alpha\beta}$

Theorem 4.4. *Consider the space-time norms (4.1) and functions ϕ, φ defined in R^{2+1}. The estimates*

$$N_{s,s-\frac{1}{2}}(Q_{12}(\phi,\varphi)) \leq CN_{s+1,s+\frac{1}{2}}(\phi)N_{s+1,s+\frac{1}{2}}(\varphi) \tag{4.3}$$

hold true for any $\frac{1}{4} < s < \frac{1}{2}$

5. Concuding Remarks

By the smoothing estimates proved in section 4, we can easily prove the local well-posedness result stated in the table. The open problem is that if we can prove global existence under the conditions that the weighted H^{s+1} norm of f_0 as well as the weighted H^s norm of f_1 are small, with s restricted to the values in the table.

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KIRCHHOFF 方程式の大域解について

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1. Introduction

Kirchhoff 方程式は弾性体の振動を記述する方程式として１８８３年に Kirchhoff によって導入された。その後１９４０年に Bernstein がこの方程式に対する初期値問題を数学的にとりあつかった。彼は 初期値として周期関数を与えて、それらが C^∞ ならば局所解、又それらが実解析的ならば時間大域解が存在することを証明した。以後多数の研究者がこの問題の周辺で研究しているが未だに完全な解決をみていないようである。

Kirchhoff 方程式は次の形であたえられる。

$$\frac{\partial^2 u}{\partial t^2} - (a^2 + \int_{R^n} |\nabla u(t,x)|^2)dx)\Delta u(t,x) = 0, t > 0, x \in R^n \tag{1}$$

$$u(0,x) = u_0(x), u_t(o,x) = u_1(x), x \in R^n. \tag{2}$$

この初期値問題が、初期値が実解析な時、時間大域解もつことを証明してみよう。筆者が興味を持ったのは、この問題は完全に線形問題として取り扱うことが出来るということにあります。以下このことを説明したい。

初期値 u_0, u_1 が Sobolev 空間に属しているとき（１）、（２）が時間局所解を持つことは、例えば逐次近似法で簡単に証明できる。　（１）、（２）の強解 $u(t,x) \in \cap_{j=0}^{2} C^{2-j}((0,T_0) : H^j)$（以下この空間を X_{T_0} とかく）が存在ような T_0 で最大なものを T とする。$T = \infty$ を背理法によってしめす。T が有限であると仮定する。

補題 1。$u(t,x) \in X_T$ とする。エネルギーを次の式で与える。

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + a(t)\|\nabla u(t)\|^2, \qquad a(t) = a^2\|\nabla u(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^4。$$

このとき $E(t) = E(0), 0 \le t < T$ が成り立つ。

この補題より $a(t) = a^2 + \|\nabla u(t)\|^2 \in L^1(0,T)$ がわかる。そこで係数 $a(t)$ をもつつぎの波動方程式を考える。

$$\frac{\partial^2}{\partial t^2} w(t,x) - a(t)\Delta w(t,x) = 0, t \in (0,T), x \in R^n, \tag{3}$$

$$w(0,x) = u_0(x), w_t(0,x) = u_1(x), x \in R^n. \tag{4}$$

初期値 $u_i (i = 0,1)$ はある正数 ρ_0 に対して

$$e^{\rho_0 <\xi>} \widehat{u_i}(\xi) \in L^2(R^n), i = 0,1 \tag{5}$$

を満たしているとする。$\hat{u}$ は u の Fourier 変換を表す。このとき（3）、（4）次の2つの命題が成り立つ。

命題1。初期値問題（3）、（4）の X_T における解は一意的である。

命題2。初期値 $u_i(i=0,1)$ が（5）を満たしているとき、（3）、（4）は解

$$w(t,x) \in X_T \cap_{j=0}^{1} C^j([0,T];H^{1-j})$$

をもつ。

命題1、2より $u(t,x)=w(t,x), t\in(0,T)$ を得る。従って

$$\lim_{t\to T-0} u(t,x)=w(T,x), \lim_{t\to T-0} u_t(t,x)=w_t(T,x)$$

を初期値、t＝Tを初期面とする（1）、（2）は局所解を持つことになり、u(t,x) は t＝Tを越えて延長可能となりTの定義に反する。

2. 命題1、2の証明

$\rho\in R^1$ にたいして作用素 $e^{\rho<D>}$ を

$$e^{\rho<D>}u(x)=\int_{R^n} e^{ix\xi+\rho<\xi>}\hat{u}(\xi)d\xi, \tag{7}$$

によって定義する。（3）をみたす $w\in X_T$ にたいして $v(t,x)=e^{-\rho<D>}w(t,x)$ とおいて、$(\partial_t+\rho'(t)<D>)v=e^{-\rho<D>}\partial_t w$ に注意すると v は

$$(\partial_t+\rho'(t)<D>)^2 v(t,x)-a(t)\Delta v(t,x)=0, t\in(0,T), x\in R^n, \tag{8}$$

をみたすことがわかる。（8）にたいしてエネルギーを

$$e(t)^2=\frac{1}{2}\|(\partial_t+\rho'(t)<D>)v(t)\|_{L^2}^2+b(t)\|\nabla v(t)\|^2, \tag{9}$$

を導入する。ここで$\rho(t), b(t)$ はつぎのようにとる。

$$\rho(t)=-\rho_1+\int_0^t \frac{|a(s)-b(s)|}{b(s)^{\frac{1}{2}}}ds, \tag{10}$$

ρ_1は定数、b(t) は正の関数である。

方程式（8）は$\rho'(t)>0$ ならば parabolic type になることに注意すると次の補題を得る。

補題2。もし$\rho'(t)>0$ であり,b(t) が $C^2([0,T))$ でかつ $b'(t)b(t)^{-1}\in L^1(0,T)$ ならばある正数 C があって $e(t)\le Ce(0), t\in(0,T)$ がなりたつ。

証明。

$$2e(t)e'(t)=(b(t)-a(t))\Re((\partial_t+\rho'(t)<D>)\nabla v,\nabla v)_{L^2}$$

$$+b'(t)\|\nabla v\|_{L^2}^2+\rho'(t)\|<D>^{\frac{1}{2}}(\partial_t+\rho'(t)<D>)v\|_{L^2}^2$$

$$+a(t)\rho'(t)\| < D >^{\frac{1}{2}} \nabla v\|_{L^2}^2$$

$$\leq \frac{|b'(t)|}{b(t)}e(t)^2 + (\frac{|b(t)-a(t)|^2}{|\rho'(t)|} + b(t)\rho'(t))\| < D >^{\frac{1}{2}} \nabla v\|^2$$

$$\leq \frac{|b'(t)|}{b(t)}e(t)^2, t \in (0,T).$$

命題１の証明；（１０）において$\rho_1 = 0, b(t) = 1$ととると、$\rho(t) > 0$だから$w \in X_T$より$v \in X_T$となり補題２より初期値u_0, u_1が零ならばe(t)=0となり一意性がしめせた。

命題２の証明；方程式（８）をxにかんしてフーリエ変換するとtに関する線形常微分方程式となり解の存在は明らかである。その解$w(t,\xi)$がξの関数としてL^2に属することは補題２の証明と同様に出来る。実際、$w(t,\xi)$エネルギーを

$$e(t,\xi)^2 = \frac{1}{2}|(\partial_t - \rho'(t) < \xi >)v(t,\xi)|^2 + b(t)|\nabla v(t,\xi)|$$

とおくと、$e(t,\xi) \leq Ce(0,\xi), t \in (0,T), \xi \in R^n$　をえる。初期値u_0, u_1が（５）を満たしているので、(10)において$\rho_1 = \rho_0$ととり、b(t)を$\rho(t) < 0, t \in [0,T]$となるようにとれる。このとき$e(0,\xi) \in L^2(R^n_\xi)$となり、従って$v(t,\xi)$もそうである。w(t,x)を$e^{\rho(t)<\xi>}v(t,\xi)$のフーリエ逆変換として定義すると閉区間「０、Ｔ］で$\rho(t) < 0$だから$t \in [0,T]$に対してw(t,x)はxに関して実解析的となり命題２が示せた。

注意：上に述べた線形問題の議論はMizohata[22],Kajitani[21],Kinoshita[23]等で採用されたものである。

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文献についての解説

Kirchhoff方程式が初めて導入されたのは１８８２年で、Kirchhoffの本「８」に見ることができる。この本の中で方程式の導き方が説明されているが、ドイツ語のわからない人にはSpagnolo[15]が便利である。その後、１９４０年になって

Bernstein[2]がこの方程式を初めて数学的に取り扱った。彼は初期値として周期関数を与えてそれらがSobolev空間に属するときは時間局所解、またそれらが実解析的ならば時間大域解が存在することを証明した。さらにこの方向で方程式を一般化した論文としてはPohozaev[13]がある。また係数が退化した場合をとりあつかっている論文としてはNishida[10], D'Ancona&Spagnolo[5], さらにLaplacianを退化楕円型作用素に置き換えた方程式をあつかっている論文としてはKajitani&Yamaguti[7], Yamaguti[18]がある。係数の滑らかさが連続性しかない場合はArosio&Spagnolo[1], Hirosawa[6]を参照。初期値をGevrey classまたはquasianalytic class に与えて大域解を求めた論文としてはNishihara[12], Yamaguti[18]がある。

小さな初期値を与えてSobolev spaceで 大域解を求めたのはGreenberg&Hu[29]が最初であろう。その後D'Ancona&Spagnolo[25, 27]が続き、さらにYamazaki[35]はScatteringを試みている。

Kirchhoff方程式にdessipative term やnonlinear lower order term をつけ加えて小さな初期値を与えてSobolev 空間で大域解を求めたり、あるいは解の爆発を証明している論文は多数あり筆者の力不足もあり個々に解説出来ないので文献リストを参照してください。

局所解の存在定理に関した論文も数多くあり文献リストを参照してください。

ここにリストアップ文献は筆者の手元にあるものだけなので、文献漏れがあると思います。お気付き方はお知らせください。

A bifurcation phenomenon for the periodic solutions of the Duffing equations

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ABSTRACT

This talk is a brief survey of our recent work [2] on a bifurcation phenomenon for the periodic solutions of the Duffing Equation

$$u''(t) + \mu u'(t) + u^3(t) = f_\lambda(t), \quad t \in R \tag{1}$$

where μ is a positive constant, and f_λ is a given family of T-periodic external forces parametrized by λ which somehow represents the magunitude of f_λ (e.g., $f_\lambda = \lambda sin(t)$). It is well-known that for any λ there exists at least one T-periodic solution of (1), and furthermore if the magunitude λ is suitably small, then the periodic solution is unique and asymptotically stable. As λ increases, we can observe by numerical computations that the solution loses its stability and various bifurcation phenomena take place. In particular, the period-doubling bifurcations are observed as very important phenomena along the route toward a so called "Chaos". However, it is surprising that there have been no rigorous proofs of these bifurcation phenomena. Recently, Komatsu-Kano-Matsumura [1] tried to detect a bifurcation phenomenon around a "linear probe" $\{(\lambda, u_\lambda)\}_{\lambda>0}$ inserted into the product space (λ, u), which is defined by

$$u_\lambda(t) := \lambda U(t), \quad U(t) : \text{given T-periodic function}$$
$$f_\lambda(t) := u''_\lambda(t) + \mu u'_\lambda(t) + u^3_\lambda(t).$$

Here we should note that $u = u_\lambda$ is a trivial solution of (1) corresponding to $f_\lambda(t)$ for any $\lambda > 0$. Then, in the particular case $U(t) = sin(2\pi t)$ (T = 1), studying the linearized equation of (1) at $u = u_\lambda$

$$v''(t) + \mu v'(t) + 3\lambda^2 U^2(t) v(t) = 0 \tag{2}$$

by the arguments of continued fractions, they showed that if $\mu \leq 4\pi/5$, then the T-periodic solution bifurcates at least at three points from the probe $\{u_\lambda\}_{\lambda>0}$. They also noticed by numerical computations that there might be infinely many bifurcation points of T-periodic solution. However, they could not detect a period-doubling bifurcation. In the coming paper [2], treating more general T-periodic functions $U(t)$, we show that only T-periodic or 2T-periodic solution can bifurcate from $\{u_\lambda\}_{\lambda>0}$, and under some condition on μ there do exist infinitely many bifurcation points of T-periodic solution, and also generically exist infinitely many bifurcation points of 2T-periodic solution (period-doubling bifurcations) except some paticular cases of $U(t)$. Furthermore, we show the asymptotic stability and unstability of the trivial solution u_λ alternates at each bifurcation point. We also show that the case $U(t) = sin(2\pi t)$ is really a particular one where only T-periodic solution can bifurcate from $\{u_\lambda\}_{\lambda>0}$. The proofs are given by investigating the eigen-values of the Poincare map of the linearized equation (2) by means of the expansion theory by generalized eigen-functions established by Titchmarsh-Kodaira, and asymptotic analysis with respect to λ.

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REGULARITY OF SOLUTIONS TO CHARACTERISTIC BOUNDARY VALUE PROBLEM FOR SYSTEMS

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1. INTRODUCTION

Let Ω be a bounded open subset of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. In Ω, we consider a first order symmetric system

$$Lu = \sum_{j=1}^{n} A_j(x)\partial_j u + B(x)u, \quad A_j(x), B(x) \in C^\infty(\bar{\Omega}), \quad A_j^*(x) = A_j(x)$$

with $u = (u_1, ..., u_N)$ where $\partial_j = \partial/\partial x_j$. For $x \in \partial\Omega$ we denote by

$$A_b(x) = \sum_{j=1}^{n} \nu_j A_j(x)$$

the boundary matrix where $\nu = (\nu_1, ..., \nu_n)$ is the unit outward normal to Ω.

Let $\mathcal{O}^\pm$ be disjoint two open subsets of $\partial\Omega$ with smooth boundary $\partial\mathcal{O}^\pm$. Let us set $\mathcal{O} = \mathcal{O}^+ \cup \mathcal{O}^-$ and assume that the boundary matrix $A_b(x)$ is non-singular in $\partial\Omega \setminus \partial\mathcal{O}$ and is negative definite in $\mathcal{O}^-$ and positive definite in $\mathcal{O}^+$. We study the following boundary value problem

$$\text{(BVP)} \qquad \begin{cases} (L+\lambda)u = f & \text{in} \quad \Omega \\ u \in M & \text{at} \quad \partial\Omega \end{cases}$$

where λ is a large positive parameter and for each $x \in \partial\Omega$, $M(x)$ is a linear subspace of $\mathbf{C}^N$. We assume that $M(x)$ is smooth in $\partial\Omega \setminus \partial\mathcal{O}$ up to the boundary and for each $\bar{x} \in \partial\mathcal{O}$ there is a relatively open subset $\bar{x} \in U \subset \partial\Omega$ and a vector subbundle $N_{\text{reg}}(x)$ of $U \times \mathbf{C}^N$ such that $\text{Ker}A_b(x) = N_{\text{reg}}(x)$ in $U \cap \partial\mathcal{O}$. We assume that the boundary condition is maximal positive in the sense that

$$\begin{cases} \langle A_b(x)v, v\rangle \geq 0, \quad \forall v \in M(x), \quad \forall x \in \partial\Omega, \\ \dim M(x) = \text{the number of nonnegative eigenvalues of } A_b(x) \end{cases}$$

and hence in particular

$$\begin{cases} M(x) = \mathbf{C}^N, & x \in \mathcal{O}^+ \\ M(x) = \{0\}, & x \in \mathcal{O}^-. \end{cases}$$

Let $\bar{x} \in \partial\mathcal{O}$ and we work in a neighborhood $\bar{x} \in U \subset \partial\Omega$. Let us take $h(x) \in C^\infty(U)$, a defining function of $\partial\mathcal{O} \cap U$, so that A_b is definite on $\{x \in U | h(x) > 0\}$. Let $v_1(x), ..., v_\nu(x)$ be a smooth basis for the bundle $N_{\text{reg}}(x)$ defined in U. Let us set

$$A_{b\partial\mathcal{O}}(x) = h(x)^{-1}(A_b(x)v_i(x), v_j(x)),\ A_{\partial\mathcal{O}}(x) = -(A_{\tilde{h}}(x)v_i(x), v_j(x))$$

which are $\nu \times \nu$ matrices and

$$A_{\tilde{h}}(x) = \sum_{j=1}^{n} \frac{\partial \tilde{h}}{\partial x_j} A_j(x)$$

where $\tilde{h}(x)$ is an extension of $h(x)$ to a neighborhood of $\partial\Omega \cap U$ in $\mathbf{R}^n$. Note that both $A_{b\partial\mathcal{O}}(x)$ and $A_{\partial\mathcal{O}}(x)$ are well defined on $\partial\mathcal{O}$ up to positive multiple scalar factor. We assume that

(H) $\quad A_{b\partial\mathcal{O}}(x)$ and $A_{\partial\mathcal{O}}(x)$ are definite with the opposite definiteness on $\partial\mathcal{O}$

Take $r(x) \in C^\infty(\bar{\Omega})$ such that $\Omega = \{x | r(x) > 0\}$ and $dr(x) \neq 0$ on $\partial\Omega$ and let $h_\pm(x) \in C^\infty(\bar{\Omega})$ be such that $\mathcal{O}^\pm = \{h_\pm(x) > 0\}$ and $dh_\pm(x) \neq 0$ on $\partial\mathcal{O}^\pm$. Let us set $\tilde{h}_\pm(x) = h_\pm(x) - \kappa r(x)$ and

$$m_\pm(x) = \sqrt{\tilde{h}_\pm(x)^2 + \mu r(x)^2}, \quad \phi_\pm(x) = \sqrt{\tilde{h}_\pm(x)^2 + \mu r(x)^2} - \tilde{h}_\pm(x)$$

where $\kappa > 0$, $\mu > 0$ are positive constants which will be determined later. By $D_0^\infty(\Omega)$ we denote the set of all $u \in C^\infty(\bar{\Omega})$ with $\text{supp} u \cap \bar{\mathcal{O}}^- = \emptyset$. The space $D_0(\Omega)$ consists of $u \in L^2(\Omega)$ verifying the same support condition. The space $H_1(\Omega; \partial\Omega)$ is the set of $u \in L^2(\Omega)$ such that $Vu \in L^2(\Omega)$ for all $C^1(\bar{\Omega})$ vector fields V which are tangent to $\partial\Omega$. For each $s \in \mathbf{Z}_+$ the space $H_s(\Omega; \partial\Omega)$ can be defined similarly. We denote by $X_{p,q}(\Omega)$ the completion of $D_0^\infty(\Omega)$ under the norm

$$||u||^2_{X_{p,q}(\Omega)} = \sum_{j=1}^{p} ||\phi_+^{2j+2q}\phi_-^{-2p-2q+2j}u||^2_{H_j(\Omega;\partial\Omega)} + ||m_-^{2q-5}\phi_-^{-2p-4q}u||^2_{L^2(\Omega)}.$$

Theorem 1.1. *There are $q_0 > 0$ and $\lambda_0(p,q) > 0$ such that if p, $q \in \mathbf{Z}_+$, $q \geq q_0$ and $\text{Re}\lambda \geq \lambda_0(p,q)$ then for every $f \in X_{p,q}(\Omega)$ there is a solution $u \in X_{p,q}(\Omega)$ to* (BVP) *verifying*

$$||u||_{X_{p,q}(\Omega)} \leq C||f||_{X_{p,q}(\Omega)}$$

2. A PRIORI ESTIMATES

Lemma 2.1. *Suppose that $\phi_\pm^{\pm s}u \in L^2(\Omega)$, $m_\pm\phi_\pm^{\pm s}(L+\lambda)u \in L^2(\Omega)$ and $\phi_\pm^{\pm s}u$ verifies the boundary condition. Then we have*

$$(\text{Re}\lambda - \lambda_0)||\sqrt{m_\pm}\phi_\pm^{\pm s}u||^2 + s(\delta - s^{-1}c_0)||\phi_\pm^{\pm s}u||^2 \leq Cs^{-1}||m_\pm\phi_\pm^{\pm s}(L+\lambda)u||^2$$

with some $\delta > 0$ where $||u|| = ||u||_{L^2(\Omega)}$.

Set $\phi = \phi_+\phi_-$ and $m = m_+m_-$ and define $H(s,t)$ by

$$(L + H(s,t))\sqrt{m}\phi_+^s\phi_-^{-t} = \sqrt{m}\phi_+^s\phi_-^{-t}L.$$

Then we get

Lemma 2.2. *Suppose that* $\sqrt{m}\phi^s u \in L^2(\Omega)$, $\sqrt{m}\phi^s(L+\lambda)u \in L^2(\Omega)$ *and* $\sqrt{m}\phi^s u \in M$ *at* $\partial\Omega$. *Then we have*

$$\begin{aligned}&\mathrm{Re}((L+H(t_1,t_2)+\lambda)\sqrt{m}\phi^s u, \sqrt{m}\phi^s u)\\ &\quad\geq (\mathrm{Re}\lambda-\lambda_0)||\sqrt{m}\phi^s u||^2+\delta||\phi^s u||^2\end{aligned}$$

with some $\delta>0$ *where* $\min(t_1,t_2)\geq q_0$ *and* $s\geq 1$.

3. Tangential regularity

For $u\in L^2(\mathbf{R}^n_+)$ we set

$$u^\sharp(x)=u(e^{x_1},x')e^{x_1/2},\quad u^\natural(x)=u(e^{x_1},x')$$

where $x=(x_1,x')=(x_1,x_2,...,x_n)$. It is clear that for $u\in L^2(\mathbf{R}^n_+)$ we have $u^\sharp\in L^2(\mathbf{R}^n)$ and $||u^\sharp||_{L^2(\mathbf{R}^n)}=||u||_{L^2(\mathbf{R}^n_+)}$. Let us write $\Omega=\mathbf{R}^n_+$ and $Z_1=x_1D_1$, $Z_j=D_j$, $2\leq j\leq n$. Note that

$$D_1u^\sharp(x)=(x_1D_1u)^\sharp(x)-2^{-1}iu^\sharp(x)=\{(x_1D_1-\frac{i}{2})u\}^\sharp(x).$$

Lemma 3.1. *Let* $u\in L^2(\mathbf{R}^n_+)$. *Then*

$$u^\sharp\in H_s(\mathbf{R}^n)\Longleftrightarrow Z_{i_1}\cdots Z_{i_p}u\in L^2(\mathbf{R}^n_+),\quad p\leq s\Longleftrightarrow u\in H_s(\Omega;\partial\Omega).$$

Moreover we have $||u^\sharp||_{H_s(\mathbf{R}^n)}\sim||u||_{H_s(\Omega;\partial\Omega)}$.

For $u\in C_0^\infty(\bar{\mathbf{R}}^n_+)$ we set

$$||u||^2_{s-1,\delta(t)}=\int|\hat{u}^\sharp(\xi)|^2\langle\xi\rangle^{2s}(1+|\delta\xi|^2)^{-1}d\xi=||u^\sharp||^2_{s-1,\delta},\ ||u||^2_{s(t)}=||u^\sharp||^2_s$$

and

$$J_\varepsilon u=\int u(x_1e^{\varepsilon y_1},x'+\varepsilon y')e^{\varepsilon y_1/2}\rho(y)dy$$

where $\hat{\rho}(\xi)=O(|\xi|^k)$ as $\xi\to 0$ and $\hat{\rho}(t\xi)=0$ for all real t implies $\xi=0$ if $\xi\in\mathbf{R}^n$. Then for $u\in L^2(\mathbf{R}^n)$ we have $(J_\varepsilon u)^\sharp=\rho_\varepsilon * u^\sharp$ where $\rho_\varepsilon(y)=\varepsilon^{-n}\rho(y/\varepsilon)$. From [3] it follows that

$$C_1||u||^2_{s-1,\delta(t)}\leq\int_0^1||J_\varepsilon u||^2_{L^2(\mathbf{R}^n_+)}\varepsilon^{-2s}(1+\frac{\delta^2}{\varepsilon^2})^{-1}\frac{d\varepsilon}{\varepsilon}+||u||^2_{s-1(t)}\leq C_2||u||^2_{s-1,\delta(t)}$$

for $\forall u\in H_{s-1}(\Omega;\partial\Omega)$.

Lemma 3.2. *Let* $s\geq 1$ *and* $a\in C_0^\infty(\bar{\mathbf{R}}^n_+)$. *Then we have*

$$\int_0^1||[aZ,J_\varepsilon]u||^2_{L^2(\mathbf{R}^n_+)}\varepsilon^{-2s}(1+\frac{\delta^2}{\varepsilon^2})^{-1}\frac{d\varepsilon}{\varepsilon}\leq C_s||u||^2_{s-1,\delta(t)}$$

for $u\in C_0^\infty(\bar{\mathbf{R}}^n_+)$ *where* $Z=Z_j$.

4. Weak and strong solutions

Lemma 4.1. *Assume that $u \in D_0(\Omega)$ is a weak solution to* (BVP). *Let $w = \phi_-^{-t-s}\phi_+ u$. Then there is a sequence $w_n \in C^\infty(\bar{\Omega})$ such that*

$$L^2\text{–}\lim \sqrt{m}\phi^s J_\varepsilon w_n = \sqrt{m}\phi^s J_\varepsilon w,$$
$$L^2\text{–}\lim (L+\lambda)\sqrt{m}\phi^s J_\varepsilon w_n = (L+\lambda)\sqrt{m}\phi^s J_\varepsilon w.$$

Lemma 4.2. *Suppose that $u \in D_0(\Omega)$ is a weak solution to* (BVP). *Then we have*

$$\begin{aligned}&\mathrm{Re}((L+H(s+1,t)+\lambda)\sqrt{m}\phi^s J_\varepsilon w, \sqrt{m}\phi^s J_\varepsilon w)\\ &\geq (\mathrm{Re}\lambda - \lambda_0)||\sqrt{m}\phi^s J_\varepsilon w||^2 + \delta||\phi^s J_\varepsilon w||^2\end{aligned}$$

where $w = \phi_-^{-s-t}\phi_+ u$ and $s,\ t \geq q_0$.

5. Commutator estimates

We note that $x_2\partial_1 u = A(x)Lu$. Then it follows that

Lemma 5.1. *We have*

$$\begin{aligned}(\partial_1 u)^\sharp(x) &= (x_1 m^{-2})^\natural(x)(\partial_1 - 2^{-1})u^\sharp(x)\\ &+(x_2 m^{-2})^\natural(x)\left((ALu)^\sharp(x) - \sum_{j=2} A_j^\natural(x)\partial_j u^\sharp(x)\right)\end{aligned}$$

Proposition 5.2. *We have*

$$\begin{aligned}&\int_0^1 ||m(x)\phi^s[L, J_\varepsilon]u||^2_{L^2(\Omega)}\varepsilon^{-2l}(1+\frac{\delta^2}{\varepsilon^2})^{-1}\frac{d\varepsilon}{\varepsilon}\\ &\leq \int_0^1 ||\phi^s J_\varepsilon u||^2_{L^2(\Omega)}\varepsilon^{-2l}(1+\frac{\delta^2}{\varepsilon^2})^{-1}\frac{d\varepsilon}{\varepsilon} + C\sum_{j=0}^{l-1}(||\phi^{s-l+j}u||^2_{j(t)} + ||\phi^{s-l+1+j}Lu||^2_{j(t)})\\ &+C(||m^{s-l-4}u||^2_{L^2(\Omega)} + ||m^{s-l-3}Lu||^2_{L^2(\Omega)})\end{aligned}$$

6. Proof of Theorem

We first remark that

$$(L+H(s+1,t)+\lambda)\sqrt{m}\phi^s J_\varepsilon w = \sqrt{m}\phi_+^{s+1}\phi_-^{-t}(L+\lambda)\phi_-^{s+t}\phi_+^{-1}J_\varepsilon w.$$

Let $u \in D_0(\Omega)$ be a weak solution to (BVP). Thus it follows from Lemma 4.2 that

$$C||m\phi_+^{s+1}\phi_-^{-t}(L+\lambda)\phi_-^{s+t}\phi_+^{-1}J_\varepsilon w||^2 \geq (\mathrm{Re}\lambda - \lambda_0)||\sqrt{m}\phi^s J_\varepsilon w||^2 + \delta||\phi^s J_\varepsilon w||^2$$

if $s,\ t \geq q_0$ where $w = \phi_-^{-s-t}\phi_+ u$.

Lemma 6.1. *Let $u \in D_0(\Omega)$ be a weak solution to* (BVP). *Then with $w = \phi_-^{-s-t}\phi_+ u$ we have*

$$\int_0^1 ||m\phi_+^{s+1}\phi_-^{-t}(L+\lambda)\phi_+^{-1}\phi_-^{s+t}J_\varepsilon w||^2\varepsilon^{-2l}(1+\frac{\delta^2}{\varepsilon^2})^{-1}\frac{d\varepsilon}{\varepsilon}$$

$$\leq C(s,t)\sum_{j=0}^{l-1}||\phi_+^{s-l+j}\phi_-^{-t-l-1+j}u||_{j(t)}^2 + C\sum_{j=0}^{l}||m\phi_+^{s-l+1+j}\phi_-^{-t-l+j}(L+\lambda)u||_{j(t)}^2$$

$$+C(||m^{s-l-4}\phi_-^{-s-t-1}u||^2 + ||m^{s-l-3}\phi_+\phi_-^{-s-t}(L+\lambda)u||^2)$$

On the other hand

Lemma 6.2. *Let $u \in D_0(\Omega)$ and $w = \phi_-^{-s-t}\phi_+ u$ then*

$$||\phi_+^{s+1}\phi_-^{-t}u||_{l-1,\delta(t)}^2 \leq \int_0^1 ||\phi^s J_\varepsilon w||^2\varepsilon^{-2l}(1+\frac{\delta^2}{\varepsilon^2})^{-1}\frac{d\varepsilon}{\varepsilon}$$

$$+C\sum_{j=0}^{l-1}||\phi_+^{s-l+1+j}\phi_-^{-t-l+j}u||_{j(t)}^2 + C||m^{s-l-1}\phi_+\phi_-^{-s-t}u||^2$$

Proposition 6.3. *Assume that $u \in D_0(\Omega)$ is a weak solution to* (BVP). *Then*

$$\delta||\phi_+^{s+1}\phi_-^{-t}u||_{l(t)}^2 \leq C(s,t)\sum_{j=0}^{l-1}||\phi_+^{s-l+j}\phi_-^{-t-l-1+j}u||_{j(t)}^2$$

$$+C\sum_{j=0}^{l}||m\phi_+^{s-l+1+j}\phi_-^{-t-l+j}(L+\lambda)u||_{j(t)}^2$$

$$+C(||m^{s-l-4}\phi_-^{-s-t-1}u||^2 + ||m^{s-l-3}\phi_+\phi_-^{-s-t}(L+\lambda)u||^2)$$

Take $s = 2l+2q-1$ and $t = 2(p+q)-2l$, $2q \geq q_0+1$ in Proposition 6.3 to get

$$\delta||\phi_+^{2l+2q}\phi_-^{-2p-2q+2l}u||_{l(t)}^2 \leq C(l)\sum_{j=0}^{l-1}||\phi_+^{l+2q-1+j}\phi_-^{-2p-2q-1+l+j}u||_{j(t)}^2$$

$$+C\sum_{j=0}^{l}||m\phi_+^{l+2q+j}\phi_-^{-2p-2q+l+j}(L+\lambda)u||_{j(t)}^2$$

$$+C(||m^{l+2q-5}\phi_-^{-2p-4q}u||^2 + ||m^{l+2q-4}\phi_+\phi_-^{-2p-4q+1}(L+\lambda)u||^2)$$

because $s,\ t \geq q_0$ for $l = 0,...,p$. We multiply a_l to the inequality and sum up over $l = 0,...,p$. Choosing a_j, $j = 0,...,p$ so that $a_j - \sum_{l=j+1}^p a_l C(l) \geq c > 0$ the inequality results

$$\sum_{j=0}^{p}||\phi_+^{2j+2q}\phi_-^{-2p-2q+2j}u||_{j(t)}^2 \leq C\sum_{j=0}^{p}||m\phi_+^{2j+2q}\phi_-^{-2p-2q+2j}(L+\lambda)u||_{j(t)}^2$$

$$+C(||m^{2q-5}\phi_-^{-2p-4q}u||^2 + ||m^{2q-4}\phi_-^{-2p-4q+1}(L+\lambda)u||^2).$$

On the other hand if $u \in D_0(\Omega)$ it is clear that

$$m_-^{2q-5}\phi_-^{-2p-4q}u \in D_0(\Omega), \quad m_-^{2q-4}\phi_-^{-2p-4q}(L+\lambda)u \in L^2(\Omega)$$

and $m_-^{2q-5}\phi_-^{-2p-4q}u \in M$ at $\partial\Omega$. Then by Lemma 2.1 it follows that

$$c(p)||m_-^{2q-5}\phi_-^{-2p-4q}u||^2 \leq C||m_-^{2q-4}\phi_-^{-2p-4q}(L+\lambda)u||^2.$$

Using this inequality we get

Proposition 6.4. *Assume that $u \in D_0(\Omega)$ is a weak solution to* (BVP). *Suppose that*

$$m_-^{2q-4}\phi_-^{-2p-4q}f \in L^2(\Omega), \quad m\phi_+^{2j+2q}\phi_-^{-2p-2q+2j}f \in H_j(\Omega;\partial\Omega),\ 1 \leq j \leq p.$$

Then it follows that

$$m_-^{2q-5}\phi_-^{-2p-4q}u \in L^2(\Omega), \quad \phi_+^{2j+2q}\phi_-^{-2p-2q+2j}u \in H_j(\Omega;\partial\Omega),\ 1 \leq j \leq p$$

and moreover

$$\sum_{j=0}^{p}||\phi_+^{2j+2q}\phi_-^{-2p-2q+2j}u||_{j(t)}^2 + ||m_-^{2q-5}\phi_-^{-2p-4q}u||^2$$
$$\leq C\sum_{j=0}^{p}||m\phi_+^{2j+2q}\phi_-^{-2p-2q+2j}f||_{j(t)}^2 + C||m_-^{2q-4}\phi_-^{-2p-4q}f||^2.$$

Let $f \in X_{p,q}(\Omega)$. Take $f_n \in D_0^\infty(\Omega)$ so that $f_n \to f$ in $X_{p,q}(\Omega)$. Then there exists a weak solution $u_n \in D_0(\Omega)$ to the boundary value problem $(L+\lambda)u_n = f_n$ with $u_n \in M$ at $\partial\Omega$. From Proposition 6.4 it follows that $u_n \in X_{p,q}(\Omega)$ and

$$||u_n||_{X_{p,q}(\Omega)} \leq C||f_n||_{X_{p,q}(\Omega)}$$

Thus letting $n \to \infty$ we get Theorem 1.1. □

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SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS AND CLASSES OF GEVREY

JEAN VAILLANT

I. Definitions. $x = (x_0, x') = (x_0, x_1, \dots, x_n) \in \Omega$; Ω is an open neighborhood of 0 in $\mathbb{R}^n$. We consider a matrix $m \times m$ differential operator h of order 1, with analytic coefficients in Ω. We denote a its principal part and b its part of order 0, such that: $h = a + b$.

We denote $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \dots, \xi_n) \in$ the dual variable of x and we consider the characteristic determinant $\det a(x, \xi)$; we assume that it is hyperbolic with constant multiplicity with respect to the direction $(1, 0)$: the m characteristics roots in ξ_0 are real, with constant multiplicity for $\xi' \neq 0$. We can decompose $\det a(x, \xi)$ in irreducible polynomials in $\mathcal{O}[\xi]$, the ring of polynomials in ξ with coefficients the analytic germs at 0; to semplify the notations, we assume there is only one multiple factor H; we denote m_1 its multiplicity so that $\det a(x, \xi) = H^{m_1} K$. We denote A the cofactor matrix such that: $aA = Aa = \det aI$.

We consider the localized ring of $\mathcal{O}[\xi]$ with respect to the prime ideal defined by H; it is a principal ring and in this ring, a is equivalent to the diagonal matrix [5]

$$\operatorname{diag}[H^p, H^{q_1}, \dots, H^{q_l}, 1, \dots, 1],$$

where the integers $p = q_0$, q_1, $\dots$, q_l are such that

$$p \geq q_1 \geq \dots \geq q_l > 0; \quad p + q = m_1 \quad q = q_1 + \dots + q_l.$$

A is divisible by H^q; we denote $\mathcal{A} = A/H^q$; so that: $a\mathcal{A} = \mathcal{A}a = H^p K I$.

For every matricial operator $\Lambda'(x, D)$, of order $\leq \mu$, we denote $\Lambda(x, \xi) = \sigma_\mu(\Lambda')$ the homogeneous symbol of order μ of Λ'. Invertly, if $\Lambda(x, \xi)$ is a matrix of symbols of order μ, we denote $\Lambda'(x, D)$ any operator such that $\sigma_\mu(\Lambda') = \Lambda$.

We consider the Cauchy problem

$$\begin{cases} h(x, D)u = f(x) \\ u|_{x_0 = t} = g(x') \end{cases}.$$

f is given, g_t is the Cauchy data on $x_0 = t$ and u is unknown; the hyperplans $x_0 = t$ are not characteristics at each point.

II. Conditions LG. We want to define conditions on the operator h, for $m_1 = 5$, and to give the corresponding Gevrey classes γ^d, for which the Cauchy problem is well posed in γ^d. We recall the conditions L ([4][6]), such that the Cauchy problem is locally well posed in C^∞; we introduce supplementary conditions L^+ which define sub-conditions in L. We give the results in the more interesting and difficult cases, for $m_1 = 5$.

Type (3,2). That is the case where a is equivalent to $\operatorname{diag}[H^3, H^2, 1, \dots, 1]$: $m_1 = 5$, $p = 3$, $q_1 = 2$, $l = 1$.

L_1. There exist $\mathcal{A}'$, H', K' et Λ_1 such that

$$S_0 \equiv \mathcal{A}\sigma_{\mu_0}(h\mathcal{A}' - H'^3 K') = H\Lambda_1;$$

we have

$$a\Lambda_1 = H^2 K \sigma_{\mu_0}(h\mathcal{A}' - H'^3 K') = H\Lambda_1.$$

L_2. There exist Λ_1' and Λ_2 such that

$$S_1 \equiv \mathcal{A}\sigma_{\mu_1}(h\Lambda_1' - h\mathcal{A}' H'^2 K' + H'^3 K' H'^2 K') = H^2 \Lambda_2.$$

L_3. There exist Λ_2' and Λ_3 such that

$$S_2 \equiv \mathcal{A}\sigma_{\mu_2}(h\Lambda_2' - h\Lambda_1' H'K' + h\mathcal{A}' H'^2 K' H' K' - H'^3 K' H'^2 K' H' K') = H^2\Lambda_3.$$

L_4. There exist Λ_3' and Λ_4 such that,

$$S_3 \equiv \mathcal{A}\sigma_{\mu_3}(h\Lambda_3' - h\Lambda_2' H'K' + h\Lambda_1'(H'K')^2$$
$$- h\mathcal{A}' H'^2 K'(H'K')^2 + (H'^2K')^2(H'K')^2) = H^2\Lambda_4,$$

L_5. There exist Λ_4' and Λ_5 such that, with analogues definitions, $S_4 = H^2\Lambda_5$.
L_6. There exist Λ_5' and Λ_6 such that, $S_5 = H^2\Lambda_6$.
L_7. There exist Λ_6' and Λ_7 such that, $S_6 = H^2\Lambda_7$.
L_8. There exist Λ_7' and Λ_8 such that, $S_7 = H^2\Lambda_8$.

We introduce supplementary conditions
(i) L_1^+. There exist $\mathcal{A}'$, H', K' and Λ_1^+ such that $S_0 = H^2\Lambda_1$.
We remark that L_1^+ implyes L_1 and L_2; L_1^+, L_3 and L_4 implyes L_5; L_1^+, L_3, L_4 and L_6 implyes L_7; if L_1^+ is not satisfied L_1 to L_5 implyes L_6 to L_8.
(ii) L_1 is satisfied. We define
L_2^+. There exist Λ_1' and Λ_2^+ such that $S_1 = H\Lambda_2^+$. We remark that L_2 implyes L_2^+.
(iii) We remark that L_1, L_2 implyes
L_3^+. $S_2 = H\Lambda_3^+$ which definies Λ_3^+.
We define
L_4^+. There exist $\Lambda_3^{+\prime}$ and Λ_4^+ such that

$$S_3^+ \equiv \mathcal{A}\sigma_{\mu_3^+}(h\Lambda_3' - h\Lambda_2' H'^2 K' + h\Lambda_1' H'K'H'^2K'$$
$$- h\mathcal{A}' H'^2 K' H' K' H'^2 K' + H'^3 K' H'^2 K' H' K' H'^2 K') = H^2\Lambda_4^+,$$

where $\mu_3^+ = \mu_3 + s$. We remark that L_3 implyes L_4^+.

The conditions $(LG)_d$ are the followings

$(LG)_{\frac{3}{2}=\frac{p}{p-1}}$. L_1 is not verified.
- $(LG)_{\frac{5}{3}}$ L_1 is verified and L_2^+ is not verified.
- $(LG)_2$ (L_1 and L_2^+ are verified and L_2 is not verified) or (L_1^+ is verified and L_3 is not verified).
- $(LG)_{\frac{5}{2}}$ L_1^+ is not verified, L_1 and L_2 are verified, L_4^+ is not verified.
- $(LG)_3$ (L_1^+ is not verified, L_1, L_2 and L_4^+ are verified, L_3 is not verified) or (L_1^+ and L_3 are verified, L_4 is not verified).
- $(LG)_4$ (L_1^+ is not verified, L_1, L_2 and L_3 are verified, L_4 is not verified) or (L_1^+ L_3 and L_4 are verified, L_6 is not verified).
- $(LG)_5$ (L_1^+ is not verified, L_1, L_2, L_3 and L_4 are verified, L_5 is not verified) or (L_1^+ L_3, L_4 and L_6 are verified, L_8 is not verified).

Remark. The maximum value of $1/d$ is $2/3$; we have considered all d corresponding to $1/d$ smaller than $2/3$: 1/5,2/5,3/5, 1/4, 2/4, 1/3, 1/2, with denominator $\leq m_1 = 5$.

Proposition. *The conditions* $(LG)_d$ *are independents from the operators* H', K', $\mathcal{A}'$, Λ_1, ...

Theorem. *The conditions* $L \equiv (L_1, \dots L_8)$ *are necessary and sufficient in order the Cauchy problem is well posed in* C^∞.

We give explicit formulas for conditions $(LG)_d$ *and we prove that conditions* $(LG)_d$ *are sufficient in order the Cauchy problem is well posed in* $\gamma^{d'}$, $d' \leq d$.

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THE EXISTENCE OF NON-TRIVIAL SOLUTIONS TO GINZBURG-LANDAU TYPE EQUATIONS

JIAN ZHAI

Dedicated to Professor Kôji Kubota on the occasion of his sixtieth birthday

ABSTRACT. We prove the existence of the solutions which converge in C^0 to a harmonic map for an elliptic system depending on a large parameter.

1. Introduction

We study the existence of the solutions of the Ginzburg-Landau type elliptic equations and the relations between the solutions and harmonic maps.

Let $\Omega \subset \mathbb{R}^n (n \geqq 2)$ be a bounded domain with $C^{2+\alpha}$ $(0 < \alpha < 1)$ boundary. We consider the following Ginzburg-Landau type elliptic system:

$$\Delta u - \lambda W_u(u) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

with the first boundary condition where

$$u = (u_1, \dots, u_m) : \Omega \to \mathbb{R}^m (m \geqq 2);$$
$$W(u) = \frac{1}{4}(a(u) - 1)^2, \quad W_u(u) = \left(\frac{\partial W(u)}{\partial u_1}, \dots, \frac{\partial W(u)}{\partial u_m} \right),$$

$a(u) \in C^\infty(\mathbb{R}^m, \mathbb{R})$ satisfies some growth condition (cf. Assumption 1) and $N := \{u \in \mathbb{R}^m | a(u) - 1 = 0\}$ is a $(m-1)$-dimensional orientable compact connected Riemannian manifold without boundary and $\lambda > 0$ is a large parameter.

Our purpose is to study the existence of the solutions u_λ of (1.1) for large λ and the relation between u_λ and a harmonic map $\bar{u} : \Omega \to N$.

When $W(u) = \frac{1}{4}(|u|^2 - 1)^2$, $m = 2$, (1.1) is the well-known Ginzburg-Landau equation which comes from phase transition problems occurring in superconductivity and superfluidity.

In the case of that $n = 2$ and Ω is a smooth bounded simply connected domain, Bethuel, Brezis and Hélein in [BBH93] proved that the minimizer of the Ginzburg-Landau functional with the first boundary condition $u|_{\partial\Omega} = g$ converges in $C^{1+\alpha}(\overline{\Omega})$

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to a harmonic map $\bar{u} : \Omega \to S^1$ provided that the boundary value g satisfies $\deg(g, \partial\Omega) = 0$.

In the case of Ginzburg-Landau equation with the Neumann boundary condition, it is known that if there exists a continuous map $\theta_0 : \overline{\Omega} \to S^1$ which is not homotopy equivalent to a constant value map, then there exists stable non-constant steady state solution u_λ provided that λ is large. Moreover, $\frac{u_\lambda}{|u_\lambda|}$ is homotopic to θ_0 and u_λ converges in $C^{1+\alpha}(\overline{\Omega})$ to a harmonic map (cf. [JMZ94]).

In this paper, we want to consider more general potential W which includes the case of Ginzburg-Landau equation. Precisely, we shall prove the existence of solutions u_λ of (1.1) in Hölder space $C^\alpha(\overline{\Omega})$ which converge in $C^0(\overline{\Omega})$ to a harmonic map $\bar{u} : \Omega \to N$ as $\lambda \to \infty$ provided that $\bar{u}$ satisfies a certain assumption (cf. section 2, Assumption 2-4).

Methods developed in [BBH93] and [JMZ94] seems not to be applicable to general case. In this paper, we first solve a variational inequality with an obstacle. Second, we use the maximum principle to prove that the solutions of the variational inequality with obstacle are, in fact, the solutions of the Euler-Lagrange equation of the original variational problem.

The difficulty is how to remove the obstacle. The key step is that we calculate W carefully and find that the solutions of the variational inequality with obstacle satisfy a scale elliptic differential inequality provided that λ is large enough. Using a maximum principle which is due to Stampacchia, we obtain the estimate for the bounds of the solutions under an assumption for $\bar{u}$ and Ω. Furthermore, we find that the coincidence set of the solutions of the variational inequality with obstacle is empty for large λ. It is to say that they are solutions of the original variational problem without obstacle.

Let M, N be compact Riemannian manifolds with metrics g, h respectively. In local coordinates $x = (x_1, \ldots, x_n)$ and $u = (u_1, \ldots, u_m)$ we denote

$$g = (g_{\alpha\beta})_{1 \leqq \alpha, \beta \leqq n}, \quad h = (h_{ij})_{1 \leqq i, j \leqq m}, \quad \text{and} \quad (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}.$$

A smooth map $u : M \to N$ is harmonic iff u satisfies

$$-\Delta_M u + \Gamma_N(u)(\nabla u, \nabla u) = 0 \tag{1.2}$$

where

$$\Delta_M = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x_\beta} \right)$$

is the Laplace-Beltrami operator on M and

$$(\Gamma_N(\nabla u, \nabla u))^k = g^{\alpha\beta} \Gamma^k_{ij}(u) \frac{\partial u_i}{\partial x_\alpha} \frac{\partial u_j}{\partial x_\beta}, \quad 1 \leqq k \leqq m.$$

Eells and Sampson [ES64] proved that if the Riemannian curvature of N is non-positive then there exists a harmonic map in every homotopy class. R.Hamilton [H75] extended the above result to compact manifolds M and N with boundary. Recently, a significant progress has been made without the assumption on the Riemannian curvature of N by Struwe (cf. [S90] and references given in there).

This paper consists of five sections. In section 2 we state assumptions and main results which are proved in section 4 and section 5. Section 3 is devoted to some preliminaries which are used in section 4 and section 5.

Notation: Let $u(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $n \geqq 2$, $m \geqq 2$. Let B_1 denote the unit ball in $\mathbb{R}^m$. The notation

$$u(\Omega) \subset \overline{B}_1$$

means that $u(x) \in \overline{B}_1$ for a.e.$x \in \Omega$.

Let Δ denote the Laplace operator on $\mathbb{R}^n$, i.e.

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial x_i}.$$

If W satisfies Assumption 1 (cf. section 2), N is an orientable $m-1$ dimensional compact connected Riemannian manifold without boundary. Let $\gamma(u)$ denote the unit outer normal vector of N at u.

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On the existence of wave operators and asymptotic completeness for nonlinear wave equations in $\mathbf{R}^{4+1}$ with small data

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ABSTRACT

This talk is concerned with the scattering problem for the following nonlinear wave equations in four space dimensions

$$(NLW) \qquad \Box u = F(u), \qquad t \in \mathbf{R},\ x \in \mathbf{R}^4.$$

Here $\Box = \partial_t^2 - \Delta = \partial_t^2 - \sum_{j=1}^4 \partial_j^2$, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ and $F(u) = \lambda|u|^{\rho-1}u$ or $\lambda|u|^\rho$ with some $\lambda \in \mathbf{R} \setminus \{0\}$, $\rho > 1$.

First of all we introduce some spaces of functions and explain several notations. We define

$$\begin{aligned} E &:= \{\ (f_1, f_2) \mid f_1 \in L^4(\mathbf{R}^4),\ \nabla f_1,\ f_2 \in L^2(\mathbf{R}^4)\ \}, \\ \Sigma &:= \{\ (f_1, f_2) \in L^2(\mathbf{R}^4) \times L^2(\mathbf{R}^4) \mid \\ &\qquad ||(f_1, f_2)||_\Sigma \equiv \sum_{|\alpha|\le 2} ||<\cdot>^{|\alpha|} \partial_x^\alpha f_1||_{L^2} + \sum_{|\alpha|=3} ||<\cdot>^2 \partial_x^\alpha f_1||_{L^2} \\ &\qquad\qquad + \sum_{|\alpha|\le 1} ||<\cdot>^{|\alpha|+1} \partial_x^\alpha f_2||_{L^2} + \sum_{|\alpha|=2} ||<\cdot>^2 \partial_x^\alpha f_2||_{L^2} < \infty\ \}. \end{aligned}$$

Here $<x>= \sqrt{1+|x|^2}$, $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_4^{\alpha_4}$ for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_4)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_4$. We put $\Sigma_\delta = \{\ (f_1, f_2) \in \Sigma \mid ||(f_1, f_2)||_\Sigma < \delta\ \}$ for $\delta > 0$. For any (not necessarily bounded) interval I and any Banach space X $BC(I; X)$ means the set of all bounded and continuous functions on I with values in X.

Following Klainerman, we introduce the set of partial differential operators

$$\Gamma = \{ \Gamma_j \mid j = 0, \cdots, 15 \} = \{ \partial_0, \cdots, \partial_4, L_1, \cdots, L_4, \Omega_{12}, \cdots, \Omega_{34}, L_0 \}.$$

Here $\partial_0 = \partial_t$, $L_j = x_j\partial_t + t\partial_j$, $\Omega_{k\ell} = x_k\partial_\ell - x_\ell\partial_k$ $(1 \le k < \ell \le 4)$ and $L_0 = t\partial_t + x_1\partial_1 + \cdots + x_4\partial_4$. For a multi-index $\alpha = (\alpha_0, \cdots, \alpha_{15})$ we set $\Gamma^\alpha := \Gamma_0^{\alpha_0} \cdots \Gamma_{15}^{\alpha_{15}}$. For a non-negative integer N and p with $1 \le p < \infty$ we introduce the norm

$$||u(t,\cdot)||_{\Gamma,N,p} := \sum_{|\alpha| \le N} \left(\int_{\mathbf{R}^4} |\Gamma^\alpha u(t,x)|^p dx \right)^{1/p}.$$

We also define the norm

$$||Du(t,\cdot)||_{\Gamma,N,p} := \sum_{k=0}^{4} ||\partial_k u(t,\cdot)||_{\Gamma,N,p}.$$

for a vector $Du = (\partial_t u,\ \partial_1 u, \cdots,\ \partial_4 u)$. Let $||u(t,\cdot)||_e$ mean the energy norm. Namely,

$$||u(t,\cdot)||_e := \frac{1}{\sqrt{2}} \sqrt{||\partial_t u(t,\cdot)||_{L^2}^2 + ||\nabla u(t,\cdot)||_{L^2}^2}.$$

We set $\omega := \sqrt{-\Delta}$. Now we are ready to state our results. The first result could be termed "Cauchy problem at $\pm\infty$". This is an improvement of the previous works due to W.A.Strauss, K.Mochizuki & T.Motai.

Theorem 1 Let $\rho > 2$. There exists a $\delta > 0$ depending on λ, ρ with the following properties $(I)_\pm$, $(II)_\pm$:

$(I)_-$ For any $(f_-, g_-) \in \Sigma_\delta$ the equation (NLW) has a unique solution $u = u(t,x)$ satisfying

$$\Gamma^\alpha u \in BC((-\infty, 0]\,; L^2(\mathbf{R}^4)) \ \text{ for any } \alpha \text{ with } |\alpha| \le 2,$$
$$\partial_k \Gamma^\alpha u \in L^\infty((-\infty, 0); L^2(\mathbf{R}^4)) \ \text{ for any } \alpha \text{ with } |\alpha| = 2 \text{ and } k = 0, \cdots, 4,$$
$$||u(t,\cdot) - u_-(t,\cdot)||_e \to 0 \ \text{ as } t \to -\infty,$$

where $u_-(t) = (\cos\omega t) f_- + (\omega^{-1} \sin\omega t) g_-$. Moreover, this solution u satisfies

$$\partial_k \Gamma^\alpha u \in BC((-\infty, 0]\,; L^2(\mathbf{R}^4)) \ \text{ for any } \alpha \text{ with } |\alpha| = 2 \text{ and } k = 0, \cdots, 4,$$

$$\sup_{t<0} ||u(t,\cdot)||_{\Gamma,2,2} + \sum_{|\alpha|=2} \sup_{t<0} ||D\Gamma^\alpha u(t,\cdot)||_{L^2} \le C_1 ||(f_-, g_-)||_\Sigma \text{ for some constant } C_1 > 0,$$
$$||(u(0), \partial_t u(0))||_\Sigma \le C_2 ||(f_-, g_-)||_\Sigma \text{ for some constant } C_2 > 0,$$
$$||u(t,\cdot) - u_-(t,\cdot)||_{\Gamma,2,2} = O(|t|^{-3(\rho-2)/2}),$$
$$||D\{u(t,\cdot) - u_-(t,\cdot)\}||_{\Gamma,2,2} = O(|t|^{-3(\rho-1)/2+1}) \text{ as } t \to -\infty.$$

$(II)_-$(*Continuous dependence*) Let $(f_-^{(j)}, g_-^{(j)}) \in \Sigma_\delta$ $(j = 1, 2)$. Let $u^{(j)} = u^{(j)}(t,x)$ be the two corresponding solutions to (NLW) in $(I)_-$. If $||(f_-^{(1)} - f_-^{(2)}, g_-^{(1)} - g_-^{(2)})||_\Sigma \to 0$, then it holds that

$$\sup_{t<0} ||u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)||_{\Gamma,2,2} + \sum_{|\alpha|=2} \sup_{t<0} ||D\Gamma^\alpha \{u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\}||_{L^2} \to 0,$$
$$||(u^{(1)}(0,\cdot) - u^{(2)}(0,\cdot), \partial_t u^{(1)}(0,\cdot) - \partial_t u^{(2)}(0,\cdot))||_\Sigma \to 0.$$

$(I)_+$For any $(f_+, g_+) \in \Sigma_\delta$ the equation (NLW) has a unique solution $u = u(t,x)$ satisfying

$$\Gamma^\alpha u \in BC([0,\infty); L^2(\mathbf{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| \le 2,$$
$$\partial_k \Gamma^\alpha u \in L^\infty((0,\infty); L^2(\mathbf{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| = 2 \text{ and } k = 0, \cdots, 4,$$
$$||u(t,\cdot) - u_+(t,\cdot)||_e \to 0 \text{ as } t \to +\infty,$$

where $u_+(t) = (\cos \omega t) f_+ + (\omega^{-1} \sin \omega t) g_+$. Moreover, this solution u satisfies

$$\partial_k \Gamma^\alpha u \in BC([0,\infty); L^2(\mathbf{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| = 2 \text{ and } k = 0, \cdots, 4,$$
$$\sup_{t>0} ||u(t,\cdot)||_{\Gamma,2,2} + \sum_{|\alpha|=2} \sup_{t>0} ||D\Gamma^\alpha u(t,\cdot)||_{L^2} \le C_1 ||(f_+, g_+)||_\Sigma,$$
$$||(u(0), \partial_t u(0))||_\Sigma \le C_2 ||(f_+, g_+)||_\Sigma,$$
$$||u(t,\cdot) - u_+(t,\cdot)||_{\Gamma,2,2} = O(t^{-3(\rho-2)/2}),$$
$$||D\{u(t,\cdot) - u_+(t,\cdot)\}||_{\Gamma,2,2} = O(t^{-3(\rho-1)/2+1}) \text{ as } t \to +\infty.$$

$(II)_+$(*Continuous dependence*) Let $(f_+^{(j)}, g_+^{(j)}) \in \Sigma_\delta$ $(j = 1, 2)$. Let $u^{(j)} = u^{(j)}(t,x)$ be the two corresponding solutions to (NLW) in $(I)_+$. If $||(f_+^{(1)} - f_+^{(2)}, g_+^{(1)} -$

$g_+^{(2)})||_\Sigma \to 0$, then it holds that

$$\sup_{t>0} ||u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)||_{\Gamma,2,2} + \sum_{|\alpha|=2} \sup_{t>0} ||D\Gamma^\alpha\{u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\}||_{L^2} \to 0,$$

$$||(u^{(1)}(0,\cdot) - u^{(2)}(0,\cdot), \partial_t u^{(1)}(0,\cdot) - \partial_t u^{(2)}(0,\cdot))||_\Sigma \to 0.$$

The next result is concerned with the ordinary Cauchy problem. This is an improvement of recent result of Y.Zhou on a class of Cauchy data. Our lower bound of ρ is optimal in view of the blow up theorem due to T.C.Sideris.

Theorem 2 Let $\rho > 2$. Then there exists an $\varepsilon > 0$ depending on λ, ρ with the following properties:

(I) (*Existence and Uniqueness*) For any $(f,g) \in \Sigma_\varepsilon$ the equation (NLW) has a unique solution $u = u(t,x)$ satisfying

$$(u(0), \partial_t u(0)) = (f,g),$$

$$\Gamma^\alpha u \in BC(\mathbf{R}; L^2(\mathbf{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| \le 2,$$

$$\partial_k \Gamma^\alpha u \in L^\infty(\mathbf{R}; L^2(\mathbf{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| = 2 \text{ and } k = 0, \cdots, 4.$$

Moreover, this solution u satisfies

$$\partial_k \Gamma^\alpha u \in BC(\mathbf{R}; L^2(\mathbf{R}^4)) \text{ for any } \alpha \text{ with } |\alpha| = 2 \text{ and } k = 0, \cdots, 4,$$

$$\sup_{t\in\mathbf{R}} ||u(t,\cdot)||_{\Gamma,2,2} + \sum_{|\alpha|=2} \sup_{t\in\mathbf{R}} ||D\Gamma^\alpha u(t,\cdot)||_{L^2} \le C_3 ||(f,g)||_\Sigma \text{ for some constant } C_3 > 0.$$

(II) (*Asymptotic behavior*) There exists a unique pair of functions (f^+, g^+), (f^-, g^-) satisfying

$$(f^\pm, g^\pm) \in E,$$

$$||u(t,\cdot) - u^\pm(t,\cdot)||_e \to 0 \text{ as } t \to \pm\infty \text{ (the double sign in the same order).}$$

Here $u^\pm(t) = (\cos\omega t) f^\pm + (\omega^{-1}\sin\omega t) g^\pm$. These $u^\pm$ satisfy

$$(u^\pm(0), \partial_t u^\pm(0)) \in \Sigma,$$

$$||(u^\pm(0), \partial_t u^\pm(0))||_\Sigma \le C_4 ||(f,g)||_\Sigma \text{ for some constant } C_4 > 0,$$

$$||u(t,\cdot) - u^\pm(t,\cdot)||_{\Gamma,2,2} = O(|t|^{-3(\rho-2)/2}),$$

$$||D\{u(t,\cdot) - u^\pm(t,\cdot)\}||_{\Gamma,2,2} = O(|t|^{-3(\rho-1)/2+1}) \text{ as } t \to \pm\infty.$$

(III) (*Continuous dependence*) Let $u^{(j)} = u^{(j)}(t,x)$ $(j = 1,2)$ be the two solutions to (NLW) with $(u^{(j)}(0), \partial_t u^{(j)}(0)) = (f^{(j)}, g^{(j)}) \in \Sigma_\varepsilon$. Let $(f^{(j)+}, g^{(j)+})$, $(f^{(j)-}, g^{(j)-})$ be the corresponding pairs of functions in (II). If $||(f^{(1)} - f^{(2)}, g^{(1)} - g^{(2)})||_\Sigma \to 0$, then it holds that

$$\sup_{t\in\mathbf{R}} ||u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)||_{\Gamma,2,2} + \sum_{|\alpha|=2} \sup_{t\in\mathbf{R}} ||D\Gamma^\alpha\{u^{(1)}(t,\cdot) - u^{(2)}(t,\cdot)\}||_{L^2} \to 0,$$
$$||(f^{(1)+} - f^{(2)+}, g^{(1)+} - g^{(2)+})||_\Sigma, \quad ||(f^{(1)-} - f^{(2)-}, g^{(1)-} - g^{(2)-})||_\Sigma \to 0.$$

The next theorem follows immediately from Theorem 1.

Theorem 3 The wave operators W_+, W_-

$$W_\pm : (u_\pm(0), \partial_t u_\pm(0)) \mapsto (u(0), \partial_t u(0))$$

can be defined as one–one and continuous mappings from Σ_δ into $\Sigma_{C_2\delta}$.

We take δ small so that $C_2\delta < \varepsilon$ may hold. Combining Theorem 2 with Theorem 3, we conclude

Theorem 4 The scattering operator S

$$S : (u_-(0), \partial_t u_-(0)) \mapsto (u^+(0), \partial_t u^+(0))$$

can be defined as a one–one and continuous mapping from Σ_δ into $\Sigma_{C_2C_4\delta}$.

Global Existence for Systems of Wave Equations with Different Speeds

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次の連立波動方程式の初期値問題を考える。

$$(CP)\quad \begin{cases} \partial_t^2 u^i - c_i^2 \Delta u^i = F_i(u') \quad \text{in} \quad (0,\infty)\times \boldsymbol{R}^2 \quad (i=1,2,\cdots,m) \\ u(0,\cdot)=\epsilon f,\ \partial_t u(0,\cdot)=\epsilon g \quad \text{in} \quad \boldsymbol{R}^2 \end{cases}$$

ただし

$$\begin{aligned} &u=(u^1,u^2,\cdots,u^m) \\ &u^i:[0,\infty)\times \boldsymbol{R}^2 \to \boldsymbol{R} \quad (i=1,2,\cdots,m) \\ &u'=(\partial_t u,\partial_1 u,\partial_2 u)\ ,\ \partial_t u=\partial_0 u \\ &c_i>0 \quad (i=1,2,\cdots,m)\ ,\ \epsilon>0 \\ &f,g\in C_0^\infty(\boldsymbol{R}^2;\boldsymbol{R}^m) \end{aligned}$$

また

$$\begin{aligned} &F_i\in C^\infty(\boldsymbol{R}^{3m}) \\ &|F_i(u')|\le M|u'|^p \quad (i=1,2,\cdots,m\,;\ p\text{ は 3 以上の整数}) \end{aligned}$$

とする。

初期値問題 (CP) について、大域解の存在定理を得ることが目標である。一般に初期値問題 (CP) に関して、次のことが知られている。

- $p\ge 4$ ならば、(CP) は small data に対して大域解を持つ
- $p=3$ ならば、(CP) には small data であっても大域解が存在するとは限らない。Life-span T_ϵ については、$T_\epsilon \ge A\exp\frac{B}{\epsilon^2}$ という下からの評価がある

しかし、M. Kovalyov は F_i の形に条件を課することにより、$p=3$ についても次の大域解の存在定理を得た：

c_i $(i=1,2,\cdots,m)$ がすべて相異なり

$$\left.\frac{\partial^3 F_i}{\partial(\partial_a u^j)\partial(\partial_b u^j)\partial(\partial_c u^j)}\right|_{u'=0}=0 \quad (i,j=1,2,\cdots,m\ ;\ a,b,c=0,1,2)$$

ならば、 small data に対して初期値問題 (CP) は滑らかな大域解を持つ

以上の M. Kovalyov の結果を拡張して、次の定理が得られた。

定理

c_i $(i=1,2,\cdots,m)$ はすべて相異なり

$$\left.\frac{\partial^3 F_i}{\partial(\partial_a u^i)\partial(\partial_b u^i)\partial(\partial_c u^i)}\right|_{u'=0} = 0 \quad (i=1,2,\cdots,m\ ;\ a,b,c=0,1,2)$$

が成り立つとする。このとき、f,g および F_i, c_i $(i=1,2,\cdots,m)$ のみに依存して適当な正の数 ϵ_0 をとれば、$0<\epsilon\leq\epsilon_0$ なるすべての ϵ に対して (CP) は $[0,\infty)\times \boldsymbol{R}^2$ において C^∞ 級の解 u をもつ。

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Local Asymptotic Forms of Solutions to Elliptic and Parabolic Boundary Value Problems

by

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The analytic functions satisfies, among other things, two distinguished properties: (*Unique Continuation*) an analytic function having a zero of infinite order vanishes identically; and (*Polynomial Local Asymptotics*) the asymptotics of a nonzero analytic function near a zero point is given by a nonzero homogeneous polynomial.

The aim of this talk is to discuss the above properties for solutions of elliptic and parabolic equations.

The unique continuation is best understood for second order elliptic operators. In his seminal paper of 1939, Torsten Carleman proved a strong unique continuation theorem for second order elliptic operators with continuously differentiable coefficients. This result indicates that the behavior of a solution of an elliptic equation near a zero point resembles that of analytic functions, even in the case where the solution may not be smooth. The technical influence of Carleman's paper has also been tremendous. Over the years, various generalized unique continuation theorems have been shown by different versions of the so-called "Carleman inequalities". The understanding has been increased significantly in recent years from the efforts of proving

the unique continuation theorem for elliptic operators with unbounded coefficients.

The polynomial local asymptotic property has been studied for solutions of elliptic equations since 1950s. Lipman Bers has shown that near a zero point of finite order, a solution of an elliptic equation with Hölder continuous coefficients is asymptotic to a homogeneous polynomial, which satisfies the elliptic equation with leading coefficients evaluated at the zero point and with lower order terms dropped. .

It is well known that a rich body of knowledge for elliptic equations is very helpful in the analysis of parabolic equations as well, in the sense that it at least raises natural conjectures and can suggest useful technical tools. Actually the basic existence and regularity theories for elliptic and parabolic equations are almost parallel to each other. However, some careful considerations are needed for finding the parabolic analogue of unique continuation. Liouville constructed a nontrivial classical solution of $u_t = \Delta u$ on $\mathbf{R}^N \times \mathbf{R}$, which vanishes for all $t \leq 0$. So, even the classical heat equation does not obey purely local versions of the unique continuation theorem.

One possibility of avoiding the above pathology is to limit ourselves to solutions of parabolic boundary value problems. To be more precise, consider

$$u_t = \Delta u + \sum b_j(x,t)\partial_j u + c(x,t)u \qquad x \in \Omega, T_1 < t < T_2,$$

with the Dirichlet boundary condition

$$u(x,t) = 0 \qquad\qquad x \in \partial\Omega, T_1 < t < T_2,$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary and the coefficients $b_j(x,t)$ and $c(x,t)$ are bounded measurable functions. We conjecture that the strong unique continuation theorem holds for solutions of this boundary value problem. Unfortunately, the difficulty remains formidable. The previous example of Liouville's indicates that some naive guesses of

parabolic analogues of the Carleman inequalities do not hold. At the present time, it is unclear what should be the correct parabolic version of Carleman estimate good enough for the parabolic unique continuation theory.

In the particular case where the coefficients are time independent, the strong unique continuation property is (essentially) established by Fang-Hua Lin (weaker results were also obtained by Hidehiko Yamabe and Seizo Ito decades ago). The proof relies heavily on the observation that when coefficients are time-independent, any solution of the parabolic boundary value problem can be expressed explicitly in terms of eigenfunctions and eigenvalues of the elliptic operator $\Delta + \sum b_j(x)\partial_j + c(x)$. Using such formula of Fourier series expansion, one can deduce the parabolic unique continuation theorem from the elliptic counterpart. This reduction technique fails to work for equations with time-dependent coefficients.

Here, I suggest a new approach to general equations with time-dependent coefficients, without any use of the Carleman inequality. The method will be mainly based on recasting the equation in terms of parabolic self similar variables and on deriving some appropriate energy estimates.

For any $x_0 \in \overline{\Omega}$ and $t_0 \in (T_1, T_2)$, let

$$y = (x - x_0)/\sqrt{t_0 - t}, \qquad t_0 - t = e^{-s}.$$

The domain Ω is transformed into $\Omega(s) := e^{s/2}(\Omega - x_0)$. (Notice that as $s \to \infty$, $\Omega(s) \to \mathbf{R}^N$ in the case $x_0 \in \Omega$ while it approaches a half space if $x_0 \in \partial\Omega$.) Then, the function $v(y, s) := u(x, t)$ satisfies

$$v_s - \Delta v + \frac{1}{2} y \cdot \nabla v = e^{-s/2} \sum b_j \partial_j v + e^{-s} c v,$$

for $y \in \Omega(s)$ and $s > -\log(t_0 - T_1)$, with the boundary condition $v = 0$ on $\partial\Omega(s)$. The local asymptotics of $u(x, t)$ as $t \uparrow t_0$ can be determined by studying the long-term behavior of $v(\cdot, s)$ as $s \to \infty$. Since the right hand side — the perturbation term — decays exponentially as $s \to \infty$, one expects

that solution v should asymptotically behave very much like a solution to the unperturbed equation as $s \to \infty$. In terms of the original variables (x, t), this makes a natural (and correct) link between the local asymptotics of $u(x, t)$ and the classical heat equation.

The above idea has recently been successfully applied to the Cauchy problem. The analysis of our boundary value problem can be carried out similarly, although the boundary terms cause a little bit complication. A technical key step is to get a fine estimate of the following quotient:

$$Q(s) := \frac{\int_{\Omega(s)} |\nabla v(y,s)|^2 \exp(-|y|^2/4)dy}{\int_{\Omega(s)} v(y,s)^2 \exp(-|y|^2/4)dy}$$

as $s \to \infty$. As a matter of fact, we prove that $Q(s)/Q(s_0)$ is bounded by a constant depending only on the coefficients of the equation. It is done by differentiating $Q(s)$ in s, integrating by parts and then applying the Cauchy-Schwarz inequality and the Gronwall argument. Boundary integral terms arising in the process can be dealt with by using the star-shapedness or the convexity of the domain. (It would be interesting to know whether such geometric conditions are essential or merely technical for the unique continuation.) The boundedness of $Q(s)$ implies the relative compactness of the normalized solution $v(\cdot, s)/\|v(\cdot, s\|$ in the topolgy of $L^2(\mathbf{R}^N, \exp(-|y|^2/4)dy)$ and thus allows us to take various nontrivial scaling limits.

This leads to the following result:

THEOREM 1 *Let $u(x,t)$ be a solution of the above boundary value problem and let $(x_0, t_0) \in \overline{\Omega} \times (T_1, T_2)$. Assume that Ω is star-shaped with respect to x_0.*

(i) Strong Unique Continuation: *If $u(x, t_0) = O(|x - x_0|^k)$ as $x \in \Omega$ approaches x_0 for all $k > 0$, then $u(x,t) \equiv 0$ on $\Omega \times (T_1, T_2)$;*

(ii) Polynomial Local Asymptotics: *If $u \not\equiv 0$ on $\Omega \times (T_1, T_2)$ and $u(x_0, t_0) = 0$, then there is a positive integer k such that*

$$u(x,t) = W(x - x_0, t - t_0) + o\left((|x - x_0|^k + |t - t_0|^{k/2}\right),$$

as $(x,t) \to (x_0,t_0)$ *in* $\overline{\Omega} \times (T_1,T_2)$. *Here* $W(z,\tau) \not\equiv 0$ *is a polynomial satisfying the classical heat equation* $W_\tau = \Delta W$ *and is homogeneous with respect to the parabolic scaling:* $W(\lambda z, \lambda^2 \tau) = \lambda^k W(z,\tau)$.

Combining the above local asymptotics result with some geometric measure theoretic arguments, one can also get a bound on Hausdorff dimensions of zero sets of solutions.

THEOREM 2 *Assume that* Ω *is convex. If a solution* $u(x,t)$ *is not identically zero on* $\Omega \times (T_1,T_2)$, *then the nodal set* $Z(t) = \{x \in \Omega | u(x,t) = 0\}$ *has Hausdorff dimension not higher than* $\dim \Omega - 1$ *for every* t.

As mentioned above, a standard technique in the unique continuation theory is the Carleman inequality, which usually requires difficult analysis. In marked contrast, our approach is very simple and totally elementary. It is remarkable that we can even get quantitative estimates of the frequency quotient $Q(s)$ without using any heavy technologies. This turns out to be useful in the analysis of other problems including some nonlinear parabolic equations.

Blow–up of solutions to a system of partial differential equations modelling chemotaxis

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1. Introduction

In 1970 Keller and Segel [4] proposed a mathematical model describing chemotactic aggregation of cellular slime molds which move preferentially towards relatively high concentrations of a chemical secreted by the amoebae themselves. With the cell density of the cellular slime molds $b(x,t)$ and the concentration of the chemical substance $s(x,t)$ at place x and time t, one of more simplified Keller-Segel models is described as the following system :

$$
\text{(KS)} \qquad \begin{cases} \dfrac{\partial b}{\partial t} = \nabla\cdot(\nabla b - \chi b \nabla\phi(s)) & \text{in } Q_T = \Omega \times (0,\ T), \\ \varepsilon \dfrac{\partial s}{\partial t} = \Delta s - \gamma s + \alpha b & \text{in } Q_T, \\ \dfrac{\partial b}{\partial n} = \dfrac{\partial s}{\partial n} = 0 & \text{on } \Gamma_T = \partial\Omega \times (0,\ T), \\ b(\cdot,0) = b_0, \quad s(\cdot,0) = s_0 & \text{on } \Omega. \end{cases}
$$

Here, Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $\chi, \varepsilon, \gamma$ and α are positive numbers, ϕ is smooth and $\phi'(s) > 0$ on $(0,\ \infty)$, b_0 and s_0 are smooth and non-negative on Ω.

The function ϕ is called a sensitivity function, whose interesting forms are of

$$
\phi(s) = s, \quad s^p\ (p > 0), \quad \log s, \quad \frac{s}{s+k}\ (k > 0 : \text{constant}).
$$

Conjecture (Nanjundiah [8], Childress and Percus [1]). Assume $\phi(s) = s$.

(i) $N = 1$. Blow-up of solutions (b, s) to (KS) never occurs.

(ii) $N = 2$. There exists a critical number c such that if $\int_\Omega b_0(x)dx < c$ then blow-up never occurs, and if $\int_\Omega b_0(x)dx > c$ then $b(\cdot,t)$ can form a δ-function singularity in finite time. Such a blow-up phenomenon is referred to as chemotactic collapse. When Ω is a ball and (b, s) is radial in x, the critical number is given by $c = 8\pi/(\alpha\chi)$.

(iii) $N \geq 3$. $b(\cdot,t)$ can form a δ-function singularity in finite time even if $\int_\Omega b_0(x)dx$ is small.

2. Limiting system as $\varepsilon \to 0$ in the case where χ and α are of order 1 and $\gamma = O(\varepsilon)$

Let $\phi(s) = s$. Following Jäger and Luckhaus [3], by putting $\tilde{s}(x,t) = s(x,t) - \overline{s}(t)$, where $\overline{w} = \frac{1}{|\Omega|}\int_\Omega w(x)dx$, $|\Omega|$ = the volume of Ω, and letting formally $\varepsilon \to 0$, (KS) is led to the system

$$\text{(JL)} \qquad \begin{cases} \dfrac{\partial b}{\partial t} = \nabla\cdot(\nabla b - \chi b \nabla \tilde{s}) & \text{in } Q_T, \\ 0 = \Delta \tilde{s} + \alpha(b - \overline{b_0}) & \text{in } Q_T, \\ \dfrac{\partial b}{\partial n} = \dfrac{\partial \tilde{s}}{\partial n} = 0 & \text{on } \Gamma_T, \\ b(\cdot, 0) = b_0 & \text{on } \Omega. \end{cases}$$

For a given non-negative smooth function $b_0 (\not\equiv 0)$ on $\overline{\Omega}$, there exists a unique solution $(b, \tilde{s})$ to (JL) defined on a maximal interval of existence $[0, T_{max})$ such that $(b, \tilde{s})$ is smooth on $\overline{\Omega} \times [0,\ T_{\max})$ and $b(x,t) > 0$ on $\overline{\Omega} \times (0,\ T_{\max})$. If $T_{\max} < \infty$, then

$$\limsup_{t \to T_{max}} \|b(\cdot, t)\|_{L^\infty} = \infty,$$

by which we mean that the solution blows up in finite time. Under the following conditions

(A1) $\Omega = \{x \in R^N : \ |x| < L\}$,

(A2) $b_0 \not\equiv 0$ on Ω, and b_0 is radially symmetric when $N \geq 2$,

the solution $(b, \tilde{s})$ is radial in x when $N \geq 2$.

2.1. Global existence and blow-up of solutions

Theorem 1 ([3]). *Assume $N = 2$.*
(i) *There exists a number $c > 0$ such that if $\int_\Omega b_0(x)dx < c$ then $T_{max} = \infty$.*
(ii) *Let $\Omega = \{x : \ |x| < L\}$ and b_0 be radial in x. Then there exists a number $c^* > 0$ such that* (JL) *has a blow-up solution $(b, \tilde{s})$ if $\int_\Omega b_0(x)dx > c^*$.*

Define the moment of order k for $b(\cdot, t)$ by

$$M_k(t) = \int_\Omega b(x,t)|x|^k dx.$$

Theorem 2 ([5]). *Assume* (A1) *and* (A2).
(i) *Let $N = 1$. Then the solution $(b, \tilde{s})$ is* ***globally bounded****, that is, $T_{max} = \infty$ and*

$$\sup_{t \geq 0} \{ \ \|b(\cdot, t)\|_{L^\infty} + \|\tilde{s}(\cdot, t)\|_{L^\infty} \ \} < \infty.$$

(ii) *Let $N = 2$. If $\int_\Omega b_0(x)dx < 8\pi/(\alpha\chi)$, then the solution $(b, \tilde{s})$ is globally bounded. If $\int_\Omega b_0(x)dx > 8\pi/(\alpha\chi)$ and $M_2(0)$ is sufficiently small, then $(b, \tilde{s})$ blows up in finite time.*

(iii) *Let $N \geq 3$. If $M_N(0)$ is sufficiently small, then $(b, \tilde{s})$ blows up in finite time.*

2.2. Blow-up points and formation of δ-function singularity

Theorem 3 ([5]). *Let $N \geq 2$ and assume* (A1), (A2). *Then for any $L_0 \in (0, L)$ there exists a positive constant $C(L_0)$ satisfying $C(L_0) \nearrow \infty$ as $L_0 \searrow 0$ such that*

$$b(x,t) + |\tilde{s}(x,t)| \leq C(L_0) \qquad (L_0 \leq |x| \leq L, \quad 0 \leq t < T_{\max}),$$

which means that blow-up can occur only at the origin $x = 0$.

Let $T_0 \in (0, \infty)$ be such that $M_k(t) > 0$ $(0 \leq t < T_0)$ and $M_k(T_0 - 0) = 0$ for some $k > 0$. If $T_{max} = T_0$, then

$$\lim_{t \to T_{max}} b(x,t) = \int_\Omega b_0(x)dx\, \delta(x)$$

in the sense of distribution, where $\delta(x)$ is the Dirac δ–function at the origin. However it still remains an open question whether $T_{max} = T_0$.

Herrero and Velázquez [2] have shown the existence of a radial solution $(b, \tilde{s})$ of (JL) such that b develops a δ-function in finite time.

Theorem 4 ([2]). Let $N = 2$ and $\Omega = \{x : |x| < L\}$. Then, for any fixed $T > 0$ there exists a radial solution $(b, \tilde{s})$ of (JL) such that $(b, \tilde{s})$ blows up at $x = 0$, $t = T$ and

$$b(x,t) \to \frac{8\pi}{\alpha\chi}\delta(x) + f(|x|) \quad \text{as } t \to T$$

in the sense of distribution, where

$$f(r) = \frac{C}{r^2}\exp(-2\sqrt{|\log r|})(2|\log r|)^{\frac{1}{2\sqrt{2|\log r|}} - \frac{1}{2}}(1 + o(1)) \quad \text{as } r \to 0,$$

and C is a positive constant.

They have also shown in detail how chemotactic collapse develops. Let us consider the radial functions

$$u(x) = \frac{8}{\alpha\chi(1 + |x|^2)^2}, \quad v(x) = -\frac{2}{\alpha\chi}\log(1 + |x|^2) + K$$

for any constant K, which are solutions of the stationary problem

$$\nabla \cdot (\nabla u - \chi u \nabla v) = 0, \quad \Delta v + \alpha u = 0 \quad \text{in } \Omega = \{x : |x| < L\}.$$

Theorem 5 ([2]). *Let $(b, \tilde{s})$ be the solution in Theorem 4. Then*

$$b(x,t) = \left\{ \frac{1}{R(t)^2} u\left(\frac{|x|}{R(t)}\right) \right\} (1 + o(1)) + O\left(\frac{1}{|x|^2} \exp(-\sqrt{2|\log(T-t)|}) \cdot 1_{\{|x| \geq R(t)\}}(x)\right)$$

uniformly on $\{x : |x| \leq C\sqrt{T-t}\}$ as $t \uparrow T$, where $1_{\{|x| \geq R(t)\}}$ is the characteristic function on $\{x : |x| \geq R(t)\}$ and $R(t) = O(\sqrt{T-t})$ $(t \uparrow T)$.

3. Limiting system as $\varepsilon \to 0$ in the case where χ, γ and α are of order 1

Let us put $\gamma = \alpha = 1$ for simplicity and consider the following limiting system.

$$\textbf{(LS)} \qquad \begin{cases} \dfrac{\partial b}{\partial t} = \nabla \cdot (\nabla b - \chi b \nabla \phi(s)) & \text{in } Q_T, \\ 0 = \Delta s - s + b & \text{in } Q_T, \\ \dfrac{\partial b}{\partial n} = \dfrac{\partial s}{\partial n} = 0 & \text{on } \Gamma_T, \\ b(\cdot, 0) = b_0 & \text{on } \Omega. \end{cases}$$

Theorem 6 ([6]). *Assume $N = 1$ and ϕ is smooth and $\phi'(s) > 0$ on $(0, \infty)$. Then the solution (b, s) of* (LS) *is globally bounded.*

Theorem 7 ([5, 6, 9]). *Assume $\phi(s) = s^p$ $(p > 0)$ and* (A1), (A2).

(i) *Let $N = 2$. In the case of $0 < p < 1$, the solution (b, s) is globally bounded. In the case of $p = 1$, if $\int_\Omega b_0(x)dx < 8\pi/\chi$ then the solution (b, s) is globally bounded, and if $\int_\Omega b_0(x)dx > 8\pi/\chi$ and $M_2(0)$ is sufficiently small then (b, s) blows up in finite time. In the case of $p > 1$, if $M_2(0)$ is sufficiently small then (b, s) blows up in finite time.*

(ii) *Let $N \geq 3$. If $M_{(N-2)p+2}(0)$ is sufficiently small then (b, s) blows up in finite time.*

Theorem 8 ([6, 9]). *Assume $\phi(s) = \log s$ and* (A1), (A2).

(i) *Let $N = 2$. Then the solution (b, s) is globally bounded.*

(ii) *Let $N \geq 3$. In the case of $\chi < 2/(N-2)$, the solution (b, s) is globally bounded. In the case of $\chi > 2N/(N-2)$, if $M_2(0)$ is sufficiently small then (b, s) blows up in finite time.*

4. Global existence of solutions to (KS) in two space dimensions

Let $\Omega \subset R^2$ and $\phi(s) = s$.

Theorem 9 ([10]). *Assume $b_0,\ s_0 \in H^{1+\varepsilon_0}(\Omega)$ for some $0 < \varepsilon_0 \leq 1$ and $b_0 \geq 0,\ s_0 \geq 0$ on Ω.*

(i) (KS) *has a unique non-negative solution* (b, s) *satisfying*

$$b, s \in C([0,\ T_{max}) : H^{1+\varepsilon_1}(\Omega)) \cap C^1((0,\ T_{max}) : L^2(\Omega)) \cap C((0,\ T_{max}) : H^2(\Omega))$$

for any $0 < \varepsilon_1 < \min\{\varepsilon_0, 1/2\}$. *Moreover* (b, s) *has further regularity properties* :

$$b \in C^1((0,\ T_{max}) : H^1(\Omega)), \quad s \in C^{\frac{1}{4}}((0,\ T_{max}) : H^3(\Omega)) \cap C^{\frac{5}{4}}((0,\ T_{max}) : H^1(\Omega)).$$

(ii) *If* $T_{max} < \infty$, *then*

$$\lim_{t \to T_{max}} (\|b(\cdot, t)\|_{H^{1+\varepsilon_0}} + \|s(\cdot, t)\|_{H^{1+\varepsilon_0}}) = \infty,$$
$$\limsup_{t \to T_{max}} \|b(\cdot, t)\|_{L^p} = \infty \quad \textit{for any } 1 < p \leq \infty,$$
$$\limsup_{t \to T_{max}} \|s(\cdot, t)\|_{H^{1+\varepsilon}} = \infty \quad \textit{for any } 0 < \varepsilon \leq \varepsilon_0.$$

(iii) *There exists* $c > 0$ *such that if* $\int_\Omega b_0(x)dx < c$ *then the solution* (b, s) *of* (KS) *exists globally in time.*

Theorem 10 ([7]). *Assume the same conditions as in Theorem 9 on* b_0, s_0.

(i) *If* $\int_\Omega b_0(x)dx < 4\pi/(\alpha\chi)$, *then the solution* (b, s) *of* (KS) *is globally bounded.*

(ii) *Let* $\Omega = \{x : \ |x| < L\}$ *and* (b_0, s_0) *be radial in* x. *Then the same assertion as* (i) *holds under the condition* $\int_\Omega b_0(x)dx < 8\pi/(\alpha\chi)$.

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