A calculation

In a game like Chess, game tree’s size grows exponentially with both its depth and its branching factor.

Time complexity: $O(b^d)$ (b for branching factor, d for depth)

Branching factor: $b \approx 35$, depth: $d \approx 100$.
Number of paths in the game tree: $35^{100} \approx 10^{135}$. Much too big for a normal game tree search.

Comparison: Number of particles in the universe $\approx 10^{87}$
Extensive game: an example

Figure: Tic-Tac-Toe game

Rule: Two players take turns to mark the spaces in a 3 × 3 grid. The player who succeeds in placing three respective marks in a horizontal, vertical, or diagonal row wins the game.

- $\Sigma_i$ is a non-empty set of strategies. $\Sigma_i = \{\sigma_i\}$.
- $\sigma_i$ is a strategy of player $i$, which is a function $\{v \in V \setminus Z \mid t(v) = i\} \rightarrow V|v$, assigning a successor $v'$ of $v$ to each non-terminal node $v$ when it is $i$’s turn to move. (where $V|v$ is the set of nodes extending $v$.)
- $\sigma = (\sigma_i)_{i \in N}$ represents a strategy profile which is a combination of strategies from all players and $\Sigma$ represents the set of all strategy profiles.
- $\sigma_{-i}$ denotes the collection of strategies in $\sigma$ excluding those for player $i$.
- $O(\sigma)$ is the outcome if the strategy profile $\sigma$ is followed by all players.
- $O(\sigma_i', \sigma_{-i})$ is the outcome if player $i$ use strategy $\sigma'$ while all other players employ $\sigma$.
- $\succeq_i$ is a preference relation over $V^2$ for each player $i$. 

Definition

An extensive game is a tuple $G = (N, V, A, t, \Sigma, \succeq_i)$, where
- $N$ is a non-empty set of the players,
- $V$ a set of nodes or vertices including a root $v_0$,
- $A \subseteq V^2$ a set of arcs. If $(v, v') \in A$, we call $v'$ a successor of $v$.
- Leaves are the nodes that have no successors, denoted by $Z$.
- $t$ is turn function assigning a member of $N$ to each non-terminal node. $t(v) = i$;

Example

Two players: player 1(×) and player 2(○). Solid arrows: the moves of player 1, dotted arrows: moves of player 2. The initial state is $v_0$. $v_1, v_2, v_3$ are all successors of $v_0$. $v_{10}, v_{11}, v_{12}$ are the terminal nodes (leaves).
**Example**

Formally, $G = (N, V, A, t, \Sigma, \succeq)$:

- $N = \{1, 2\}$;
- $V = \{v_0, v_1, v_2, \ldots\}$;
- $(v_0, v_1), (v_0, v_2), (v_0, v_3) \in A$;
- $v_{10}, v_{11}, v_{12} \in Z$;
- $t(v_0) = 1, t(v_2) = 2, j \cdots$.

- A $\sigma_1$ such that $\sigma_1(v_0) = v_2, \sigma_1(v_5) = v_8, \cdots$
- A $\sigma_2$ such that $\sigma_2(v_2) = v_5, \cdots$

Thus, a strategy profile $\sigma = (\sigma_1, \sigma_2)$ such that $O(\sigma) = v_{11}$.

- $v_{12} \succeq v_{11} \succeq v_{10}$, (since player 1 wins the game in $v_{12}$, loses it in $v_{10}$, and gains a draw in $v_{11}$).

**Sight function**

**Definition**

(sight function). Let $G = (N, V, A, t, \Sigma, \succeq)$ be an extensive game. A short sight function for $G$ is a function $s: V \setminus Z \to 2^{V \setminus \emptyset}$, associating to each non-terminal node $v$ a finite subset of all the available nodes at $v$.

**Definition**

(Extensive game with short sight). An extensive game with short sight (Egss) is a tuple $S = (G, s)$ where $G$ is a finite extensive game and $s$ a sight function.

- Endowing an extensive game with a sight function.
Sight-filtrated extensive game

At each node $v$, players can see a subgame $S[v]$ of the whole game. This subgame is determined by their sight:

$$S[v] = (N[v], V[v], A[v], t[v], \Sigma[v], \geq[v])$$

- $N[v] = N$;
- $V[v] = s(v)$, which is the set of nodes within the sight of $s(v)$.
- $A[v] = A \cap (V[v])^2$;
- $t[v] = V[v] \setminus Z[v] \rightarrow N$ so that $t[v](v') = t(v')$;
- $\Sigma[v]$ is the set of strategies for each player available at $v$ and restricted to $s(v)$. It consists of elements $\sigma[v]$ such that $\sigma[v](v') = \sigma[v']$ for each $v' \in V[v]$ with $t[v](v') = i$;
- $\geq[v] = \geq \cap (V[v])^2$.

Syntax and Semantics

$$\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi' \mid (\leq)_i \varphi \mid (\hat{\sigma}) \varphi \mid (\hat{\sigma}_{-i}) \varphi \mid (\hat{\sigma} \; \hat{\sigma}) \varphi \mid (\hat{\sigma} \; \hat{\sigma}_{-i}) \varphi$$

- The label $\leq_i$ denotes player $i$'s preference relation.
- The label $\hat{\sigma}$ stands for the outcomes of strategy profiles. $(v, v') \in R_{\hat{\sigma}}$ if $v'$ is the terminal node reached from $v$ by following $\sigma$.
- $(v_0, v_1) \in R_{\hat{\sigma}}$
Syntax and Semantics

\( \varphi ::= p | \neg \varphi | \varphi_0 \land \varphi_1 | (\leq)\varphi | (\sigma)\varphi | (\delta\sigma)\varphi | (\delta^2)\varphi | (\delta^3)\varphi \)

- The label \( \leq \) is sight function for the current node, and
  
- \((v, v') \in R_v \) means “node \( v' \) is within the sight at the present node \( v \).”

Frame \( F \): \( (V, R_{\leq}, R_{\sigma}, R_{\sigma\sigma}, R_v, R_{\delta\sigma}, R_{\delta^2\sigma}) \), where

- \( vR_{\leq} v' \) if \( v' \geq_i v \)
- \( vR_{\sigma} v' \) if \( v' = O_{\sigma}(v) \)
- \( vR_{\sigma\sigma} v' \) if \( v' \in O_{\sigma}(v) \)
- \( vR_{\delta\sigma} v' \) if \( v' \in S(v) \)
- \( vR_{\delta^2\sigma} v' \) if \( v' = O_{\delta}(v) \)
- \( vR_{\delta^3\sigma} v' \) if \( v' \in O_{\delta}(v) \)

Syntax and Semantics

- \( \varphi ::= p | \neg \varphi | \varphi_0 \land \varphi_1 | (\leq)\varphi | (\sigma)\varphi | (\delta\sigma)\varphi | (\delta^2)\varphi | (\delta^3)\varphi \)

- \((v, v') \in R_v \) if “state \( v' \) is the outcome of \( \sigma_{i=1} \) in \( S_{i=1} \) that is reachable from the starting point \( v \), i.e., \( v' = O_{\sigma}(v) \).

- The interpretation for \( R_{\delta^2} \) is similar.

Model: \( M = (V, R, I) \)

- \( M, v \models (\leq)\varphi \) if \( v \models \varphi \) for some \( u \in V \) with \( vR_{\leq} u \).
- \( M, v \models (\sigma)\varphi \) if \( v \models \varphi \) for some \( u \in V \) with \( vR_{\sigma} u \).
- \( M, v \models (\delta\sigma)\varphi \) if \( v \models \varphi \) for some \( u \in V \) with \( vR_{\delta\sigma} u \).
- \( M, v \models (\delta^2)\varphi \) if \( v \models \varphi \) for some \( u \in V \) with \( vR_{\delta^2} u \).
- \( M, v \models (\delta^3)\varphi \) if \( v \models \varphi \) for some \( u \in V \) with \( vR_{\delta^3} u \).
Validities

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Solution concepts for traditional extensive games

**Definition**

(Best response and Nash equilibrium)

A best response for player \( i \) of an extensive game is a strategy profile \( \sigma^* \) such that \( O(\sigma^*_i, \sigma^*_{-i}) \succeq_i O(\sigma_i, \sigma^*_{-i}) \) for every strategy \( \sigma_i \) of player \( i \). A strategy profile \( \sigma^* \) is a Nash equilibrium of an extensive game if it is a best response for every player \( i \).

Soundness and Completeness

**Definition**

(Sight-compatible best response and Nash equilibrium).

Let \( S = (G, s) \) be an Egss and \( S^*_v \) be the sight-filtrated extensive game at \( v \). A strategy profile \( \sigma^* \) is a sight-compatible best response for \( i \) if for every nonterminal node \( v \), it holds that \( O^*_v(\sigma^*_i|v, \sigma^*_{-i}|v) \succeq_i O^*_v(\sigma_i|v, \sigma^*_{-i}|v) \) for any strategy \( \sigma_i|v \) available to \( i \).

\( \sigma^* \) is a sight-compatible Nash equilibrium (SCNE) of \( S \) if it is a sight-compatible best response for every player \( i \in N \).
Characterize solution concepts

**Theorem**

Let $S$ be an $Egss$ given by $(N, V, A, t, \Sigma, \geq, s)$. Then for any player $i$, any strategy profiles $\sigma$ in $S$ and any formulas $\varphi$ of $LS$:

(a) $\sigma$ is a sight-compatible best response (SCBR) of $S$ for $i$ iff $F_S \models [\hat{\sigma}_i^s] \varphi \rightarrow [\hat{\tilde{\sigma}}_i^s] (\leq_i) \varphi$.

(b) $\sigma$ is a sight-compatible Nash equilibrium (SCNE) of $S$ iff $F_S \models \bigwedge_{i \in N} ([\hat{\sigma}] \varphi \rightarrow [\hat{\tilde{\sigma}}_i^s] (\leq_i) \varphi)$.

(c) $\sigma$ is a subgame perfect equilibrium (SPE) of $S$ iff for any $u \in V \setminus Z$, $F_{S|u} \models \bigwedge_{i \in N} ([\hat{\sigma}] \varphi \rightarrow [\hat{\tilde{\sigma}}_i^s] (\leq_i) \varphi)$.

(d) A strategy profile $\sigma$ is a sight-compatible SPE of $S$ iff for all $v \in V \setminus Z$, $F_{S|v} \models [\hat{\sigma}] (\bigwedge_{i \in N} ([\hat{\sigma}] \varphi \rightarrow [\hat{\tilde{\sigma}}_i^s] (\leq_i) \varphi))$.

Thank you!