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A calculation

In a game like Chess, game tree’s size grows exponentially with both its depth and its branching factor.
Time complexity: $O(b^d)$ (b for branching factor, d for depth)

```
b = 3
\text{Branching factor: } b \approx 35, \text{ depth: } d \approx 100.
```

Number of paths in the game tree: $35^{100} \approx 10^{35}$.

Much too big for a normal game tree search.

Comparison: Number of particles in the universe $\approx 10^{87}$
Extensive game: an example

Rule: Two players take turns to mark the spaces in a $3 \times 3$ grid. The player who succeeds in placing three respective marks in a horizontal, vertical, or diagonal row wins the game.

- $\Sigma_i$ is a non-empty set of strategies. $\Sigma_i = \{\sigma_i\}$.
- $\sigma_i$ is a strategy of player $i$, which is a function $\{v \in V \setminus Z | t(v) = i\} \rightarrow V|v$, assigning a successor $v'$ of $v$ to each non-terminal node $v$ when it is $i$'s turn to move. (where $V|v$ is the set of nodes extending $v$.)
- $\sigma = (\sigma_i)_{i \in N}$ represents a strategy profile which is a combination of strategies from all players and $\Sigma$ represents the set of all strategy profiles.
- $\sigma_{-i}$ denotes the collection of strategies in $\sigma$ excluding those for player $i$.
- $O(\sigma)$ is the outcome if the strategy profile $\sigma$ is followed by all players.
- $O(\sigma', \sigma_{-i})$ is the outcome if player $i$ use strategy $\sigma'$ while all other players employ $\sigma$.
- $\succeq_i$ is a preference relation over $V^2$ for each player $i$.

Example

Two players: player 1 (x) and player 2 (o). Solid arrows: the moves of player 1, dotted arrows: moves of player 2. The initial state is $v_0$.
$V_1, V_2, V_3$ are all successors of $v_0$.
$V_{10}, V_{11}, V_{12}$ are the terminal nodes (leaves).
Example

Formally, $G = (N, V, A, t, \Sigma, \succeq)$
- $N = \{1, 2\}$
- $V = \{v_0, v_1, v_2, \ldots\}$
- $(v_0, v_1), (v_0, v_2), (v_0, v_3) \in A$
- $v_{10}, v_{11}, v_{12} \in \mathbb{Z}$
- $t(v_0) = 1, t(v_2) = 2, j \ldots$
- $a \sigma_1$ such that $\sigma_1(v_0) = v_2, \sigma_1(v_5) = v_8, \ldots$
- $a \sigma_2$ such that $\sigma_2(v_2) = v_5, \ldots$

Thus, a strategy profile $\sigma = (\sigma_1, \sigma_2)$ such that $O(\sigma) = v_{11}$.
- $v_{12} \succeq v_{11} \succeq v_{10}$. (since player 1 wins the game in $v_{12}$, loses it in $v_{10}$, and gains a draw in $v_{11}$).

Definition

(sight function). Let $G = (N, V, A, t, \Sigma, \succeq)$ be an extensive game. A short sight function for $G$ is a function $s : V \setminus Z \rightarrow 2^V \setminus \emptyset$, associating to each non-terminal node $v$ a finite subset of all the available nodes at $v$.

Definition

(Extensive game with short sight). An extensive game with short sight (Egss) is a tuple $S = (G, s)$ where $G$ is a finite extensive game and $s$ a sight function.

- Endowing an extensive game with a sight function.
Sight-filtrated extensive game

At each node \( v \), players can see a subgame \( S[v] \) of the whole game. This subgame is determined by their sight:

\[
S[v] = (N[v], V[v], A[v], t[v], \Sigma[v], \geq[v])
\]

- \( N[v] = N \);
- \( V[v] = s(v) \), which is the set of nodes within the sight of \( s(v) \);
- \( A[v] = A \cap (V[v])^2 \);
- \( t[v] = V[v] \setminus Z[v] \rightarrow N \) so that \( t[v](v') = t(v') \);
- \( \Sigma[v] \) is the set of strategies for each player available at \( v \) and restricted to \( s(v) \). It consists of elements \( \sigma[v] \) such that \( \sigma[v](v') = \sigma[v'] \) for each \( v' \in V[v] \) with \( t[v](v') = i \);
- \( \geq[v] = \geq \cap (V[v])^2 \).

Example

- \( N[v] = N \);
- \( V[v] = \{v_0, v_1, v_2, v_3, v_5\} \);
- \( Z[v] = \{v_1, v_3, v_4, v_5\} \);
- \( A[v] = \{(v_0, v_1), (v_2, v_3), (v_2, v_4), \cdots\} \);
- \( \sigma[v] = (\sigma_1[v], \sigma_2[v]) \) such that \( O[v_0] = v_5 \), with \( \sigma_1[v](v_0) = v_2 \) and \( \sigma_2[v](v_2) = v_5 \)

Syntax and Semantics

\[
\varphi ::= p | \neg \varphi | \phi_0 \land \phi_1 | (\leq)_i \varphi | (\approx)_i \varphi | (\prec)_i \varphi | (\preceq)_i \varphi | (\approx)_i \varphi | (\preceq)_i \varphi
\]

- The label \( \leq_i \) denotes player \( i \)'s preference relation.
- The label \( \approx_i \) stands for the outcomes of strategy profiles. \( (v, v') \in R_i \) if \( v' \) is the terminal node reached from \( v \) by following \( \sigma \).
- \( (v_0, v_1) \in R_i \)
Syntax and Semantics

\[ \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi_1 \mid (\leq)\varphi \mid (\delta)\varphi \mid (\delta_{-j})\varphi \mid (\leq_s)\varphi \mid (\delta_s)\varphi \]

- The label \( \leq \) is sight function for the current node, and \((v, v') \in R_s\) means ‘node \( v' \) is within the sight at the present node \( v \).”

Frame \( F \): \( (V, R_{\leq}, R_{=}, R_{s-}, R_{=}, R_{\delta}, R_{\delta_{-j}}) \), where

\[ \begin{align*}
\nu R_{=} v' & \quad \text{iff} \quad v' \geq_v v \\
\nu R_{s} v' & \quad \text{iff} \quad v' = O_v(\sigma_v) \\
\nu R_{s-} v' & \quad \text{iff} \quad v' \in O_v(\sigma_{-j}v) \\
\nu R_{=} v' & \quad \text{iff} \quad v' \in s(v) \\
\nu R_{\delta} v' & \quad \text{iff} \quad v' = O_v(\sigma T_v) \\
\nu R_{\delta_{-j}} v' & \quad \text{iff} \quad v' \in O_v(\sigma_{-j} T_v)
\end{align*} \]

Syntax and Semantics

\[ \varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi_1 \mid (\leq)\varphi \mid (\delta)\varphi \mid (\delta_{-j})\varphi \mid (\leq_s)\varphi \mid (\delta_s)\varphi \]

- The interpretation for \( R_{\delta_{-j}} \) is similar.

Model: \( M = (V, R, I) \)

\[ \begin{align*}
M, v \models (\leq) \varphi & \quad \text{iff} \quad M, u \models \varphi \text{ for some } u \in V \text{ with } v R_{\leq} u. \\
M, v \models (\delta) \varphi & \quad \text{iff} \quad M, u \models \varphi \text{ for some } u \in V \text{ with } v R_{=} u. \\
M, v \models (\delta_{-j}) \varphi & \quad \text{iff} \quad M, u \models \varphi \text{ for some } u \in V \text{ with } v R_{s-} u. \\
M, v \models (\leq_s) \varphi & \quad \text{iff} \quad M, u \models \varphi \text{ for some } u \in V \text{ with } v R_{s} u. \\
M, v \models (\delta_s) \varphi & \quad \text{iff} \quad M, u \models \varphi \text{ for some } u \in V \text{ with } v R_{\delta} u. \\
M, v \models (\delta_{-j}) \varphi & \quad \text{iff} \quad M, u \models \varphi \text{ for some } u \in V \text{ with } v R_{\delta_{-j}} u.
\end{align*} \]
Validities

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<th>Schema</th>
<th>Property</th>
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<tr>
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<td>transitivity</td>
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<td>$[\varphi]\varphi \leftrightarrow (\varphi)\varphi$</td>
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<td>inclusiveness</td>
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</table>

Soundness and Completeness

LEGS is sound and complete w.r.t. the class of all games with short sight.

Solution concepts for traditional extensive games

Definition

(Best response and Nash equilibrium)
A best response for player $i$ of an extensive game is a strategy profile $\sigma^*$ such that $O((\sigma^*_i, \sigma^*_{-i}) \geq_i O(\sigma_i, \sigma^*_{-i})$ for every strategy $\sigma_i$ of player $i$. A strategy profile $\sigma^*$ is a Nash equilibrium of an extensive game if it is a best response for every player $i$.

Definition

(Sight-compatible best response and Nash equilibrium).
Let $S = (G, s)$ be an Egss and $S\mid_v$ be the sight-filtrated extensive game at $v$. A strategy profile $\sigma^*$ is a sight-compatible best response for $i$ if for every nonterminal node $v$, it holds that $O\mid_v(\sigma^*_i \mid_v, \sigma^*_{-i} \mid_v) \geq_i O\mid_v(\sigma_i \mid_v, \sigma^*_{-i} \mid_v)$ for any strategy $\sigma_i \mid_v$ available to $i$.

$\sigma^*$ is a sight-compatible Nash equilibrium (SCNE) of $S$ if it is a sight-compatible best response for every player $i \in N$. 
Characterize solution concepts

Theorem

Let $S$ be an Egss given by $(N, V, A, t, \Sigma_i, \geq, s)$. Then for any player $i$, any strategy profiles $\sigma$ in $S$ and any formulas $\phi$ of $LS$:

(a) $\sigma$ is a sight-compatible best response (SCBR) of $S$ for $i$ iff $F_S \models [\bar{\sigma}^s] \phi \rightarrow [\bar{\sigma}_i^s] (\leq) \phi$.

(b) $\sigma$ is a sight-compatible Nash equilibrium (SCNE) of $S$ iff $F_S \models \bigwedge_{i \in N} ([\bar{\sigma}^s] \phi \rightarrow [\bar{\sigma}_i^s] (\leq) \phi)$.

(c) $\sigma$ is a subgame perfect equilibrium (SPE) of $S$ iff for any $u \in V \setminus \mathcal{Z}$, $F_{S|v, u} \models \bigwedge_{i \in N} ([\bar{\sigma}] \phi \rightarrow [\bar{\sigma}_i] (\leq) \phi)$.

(d) A strategy profile $\sigma$ is a sight-compatible SPE of $S$ iff for all $v \in V \setminus \mathcal{Z}$, $F_{S|v, v} \models [\bar{\sigma}] (\bigwedge_{i \in N} ([\bar{\sigma}] \phi \rightarrow [\bar{\sigma}_i] (\leq) \phi))$.

Thank you!