Strategic Reasoning in Extensive Games with Short Sight

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Strategic Reasoning in Extensive Games with Short Sight

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A calculation

In a game like Chess, game tree’s size grows exponentially with both its depth and its branching factor.
Time complexity: \( O(b^d) \) (\( b \) for branching factor, \( d \) for depth)

\[
\begin{align*}
  b &= 3 \\
  d &= 4 
\end{align*}
\]

Branching factor: \( b \approx 35 \), depth: \( d \approx 100 \).
Number of paths in the game tree: \( 35^{100} \approx 10^{135} \). Much too big for a normal game tree search.

Comparison: Number of particles in the universe \( \approx 10^{87} \)

Strong assumption:
Entire structure of a game is common knowledge to all players.

Solution:
Grossi and Turrini proposed the concept of games with short sight (Grossi and Turrini, 2012), in which players can only see part of the game tree.

Figure: Short-sighted people

Contribution:
A modal Logic system for reasoning about games with short sight.
**Extensive game: an example**

**Figure:** Tic-Tac-Toe game

Rule: Two players take turns to mark the spaces in a 3 × 3 grid. The player who succeeds in placing three respective marks in a horizontal, vertical, or diagonal row wins the game.

- $\Sigma_i$ is a non-empty set of strategies. $\Sigma_i = \{\sigma_i\}$.
- $\sigma_i$ is a strategy of player $i$, which is a function $\{v \in V \setminus Z | t(v) = i\} \rightarrow V|v$, assigning a successor $v'$ of $v$ to each non-terminal node $v$ when it is $i$'s turn to move. (where $V|v$ is the set of nodes extending $v$.)
- $\sigma = (\sigma_i)_{i \in N}$ represents a strategy profile which is a combination of strategies from all players and $\Sigma$ represents the set of all strategy profiles.
- $\sigma_{-i}$ denotes the collection of strategies in $\sigma$ excluding those for player $i$.
- $O(\sigma)$ is the outcome if the strategy profile $\sigma$ is followed by all players.
- $O(\sigma', \sigma_{-i})$ is the outcome if player $i$ use strategy $\sigma'$ while all other players employ $\sigma$.
- $\succeq_i$ is a preference relation over $V^2$ for each player $i$.

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**Extensive game (Con’d)**

**Definition**

An extensive game is a tuple $G=(N, V, A, t, \Sigma_i, \succeq_i)$, where
- $N$ is a non-empty set of the players,
- $V$ a set of nodes or vertices including a root $v_0$
- $A \subseteq V^2$ a set of arcs. If $(v, v') \in A$, we call $v'$ a successor of $v$.

Leaves are the nodes that have no successors, denoted by $Z$.
- $t$ is turn function assigning a member of $N$ to each non-terminal node. $t(v) = i$;

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**Example**

Two players: player 1(×) and player 2(○). Solid arrows: the moves of player 1, dotted arrows: moves of player 2. The initial state is $v_0$.

$v_1, v_2, v_3$ are all successors of $v_0$.
$v_{10}, v_{11}, v_{12}$ are the terminal nodes (leaves).
Example
Formally, $G = (N, V, A, t, \Sigma, \geq)$
- $N = \{1, 2\}$;
- $V = \{v_0, v_1, v_2, \ldots\}$;
- $(v_0, v_1), (v_0, v_2), (v_0, v_3) \in A$;
- $V_{10}, V_{11}, V_{12} \in \mathbb{Z}$;
- $t(v_0) = 1, t(v_2) = 2, j \cdots$.

Let $s_1$ such that $s_1(v_0) = v_2, s_1(v_5) = v_8, \cdots$.
Let $s_2$ such that $s_2(v_2) = v_5, \cdots$.

Thus, a strategy profile $\sigma = (s_1, s_2)$ such that $O(\sigma) = v_{11}$.

- $V_{12} \geq 1, V_{11} \geq 1, V_{10}$. (since player 1 wins the game in $v_{12}$, loses it in $v_{10}$, and gains a draw in $v_{11}$).

Sight function

Definition
(sight function). Let $G = (N, V, A, t, \Sigma, \geq)$ be an extensive game. A short sight function for $G$ is a function $s : V \setminus \mathbb{Z} \to 2^V \setminus \emptyset$, associating to each non-terminal node $v$ a finite subset of all the available nodes at $v$.

Definition
(Extensive game with short sight). An extensive game with short sight ($E_{\text{gss}}$) is a tuple $S = (G, s)$ where $G$ is a finite extensive game and $s$ a sight function.

- Endowing an extensive game with a sight function.
Sight-filtrated extensive game

At each node $v$, players can see a subgame $S^i_v$ of the whole game. This subgame is determined by their sight:

$$S^i_v = (N^i_v, V^i_v, A^i_v, t^i_v, \Sigma^i_v, \geq^i_v)$$

- $N^i_v = N$;
- $V^i_v = s(v)$, which is the set of nodes within the sight of $s(v)$.
- $A^i_v = A \cap (V^i_v)^2$;
- $t^i_v = V^i_v \setminus Z^i_v \rightarrow N$ so that $t^i_v(v') = t(v')$;
- $\Sigma^i_v$ is the set of strategies for each player available at $v$ and restricted to $s(v)$. It consists of elements $\sigma^i_v$ such that $\sigma^i_v(v') = \sigma(v')$ for each $v' \in V^i_v$ with $t^i_v(v') = i$;
- $\geq^i_v = \geq \cap (V^i_v)^2$.

### Example

- $N^i_v = N$;
- $V^i_v = \{v_0, v_1, v_2, v_3, v_5\}$;
- $Z^i_v = \{v_1, v_3, v_4, v_5\}$;
- $A^i_v = \{(v_0, v_1), (v_2, v_4), (v_2, v_4), \cdots\}$;
- $\sigma^i_v = (\sigma^i v_0, \sigma^i v_0)$ such that $O^i v_0 (\sigma^i v_0) = v_5$, with $\sigma^i v_0 (v_0) = v_2$ and $\sigma^i v_2 (v_2) = v_5$

### Syntax and Semantics

$\varphi ::= p \mid \neg \varphi \mid \varphi_0 \land \varphi_1 \mid (\leq)_i \varphi \mid (\sigma)\varphi \mid (\sigma_i)\varphi \mid (\sigma_i^\leq)\varphi \mid (\sigma_i^\geq)\varphi$

- The label $\leq_i$ denotes player $i$’s preference relation.
- The label $\sigma_i$ stands for the outcomes of strategy profiles. $(v, v') \in R^i_{\sigma_i}$ if $v'$ is the terminal node reached from $v$ by following $\sigma$.
- $(v_0, v_1) \in R^i_{\sigma}$

### Syntax and Semantics

$\varphi ::= p \mid \neg \varphi \mid \varphi_0 \land \varphi_1 \mid (\leq)_i \varphi \mid (\sigma)\varphi \mid (\sigma_i)\varphi \mid (\sigma_i^\leq)\varphi \mid (\sigma_i^\geq)\varphi$

- $(v, v') \in R^i_{\sigma_i}$ if $v'$ is one of the leaf nodes extending $v$ that player $i$ can enforce provided that the other players strictly follow their strategies in $\sigma$.
- $O(\sigma_{-1}, \sigma'_1) = v_12$,
- $(v_0, v_12) \in R^i_{\sigma_i}$
Syntax and Semantics

\[ \varphi ::= p \mid \neg \varphi \land \varphi_1 \mid \langle \leq \rangle \varphi \mid \langle \delta \rangle \varphi \mid \langle \delta_{\downarrow} \rangle \varphi \mid \langle \varepsilon \rangle \varphi \mid \langle \delta^2 \rangle \varphi \mid \langle \delta_{\uparrow} \rangle \varphi \]

- The label \( \langle \leq \rangle \) is the sight function for the current node, and 
  \((v, v') \in R_v\) means "node \( v' \) is within the sight at the present node \( v \)."

Frame \( F \) (\( (V, R_{\leq}, R_{\delta}, R_{\delta_{\downarrow}}, R_{\varepsilon}, R_{\delta^2}, R_{\delta_{\uparrow}}) \)), where

- \( vR_{\leq} v' \) iff \( v' \geq v \)
- \( vR_{\delta} v' \) iff \( v' = O_v(\sigma|_{v}) \)
- \( vR_{\delta_{\downarrow}} v' \) iff \( v' \in O_v(\sigma_{\downarrow}|_{v}) \)
- \( vR_{\varepsilon} v' \) iff \( v' \in s(v) \)
- \( vR_{\delta^2} v' \) iff \( v' = O_v(\sigma|_{v}) \)
- \( vR_{\delta_{\uparrow}} v' \) iff \( v' \in O_v(\sigma_{\uparrow}|_{v}) \)

Syntax and Semantics

Model: \( M = (V, R, I) \)

- \( M, v \models \langle \leq \rangle \varphi \) iff \( M, u \models \varphi \) for some \( u \in V \) with \( vR_{\leq} u \).
- \( M, v \models \langle \delta \rangle \varphi \) iff \( M, u \models \varphi \) for some \( u \in V \) with \( vR_{\delta} u \).
- \( M, v \models \langle \delta_{\downarrow} \rangle \varphi \) iff \( M, u \models \varphi \) for some \( u \in V \) with \( vR_{\delta_{\downarrow}} u \).
- \( M, v \models \langle \varepsilon \rangle \varphi \) iff \( M, u \models \varphi \) for some \( u \in V \) with \( vR_{\varepsilon} u \).
- \( M, v \models \langle \delta^2 \rangle \varphi \) iff \( M, u \models \varphi \) for some \( u \in V \) with \( vR_{\delta^2} u \).
- \( M, v \models \langle \delta_{\uparrow} \rangle \varphi \) iff \( M, u \models \varphi \) for some \( u \in V \) with \( vR_{\delta_{\uparrow}} u \).
Games with Short Sight

Logic

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References

Syntax and Semantics

Axiomatization

Validities

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Solution concepts for traditional extensive games

Definition

(Best response and Nash equilibrium)
A best response for player $i$ of an extensive game is a strategy profile $σ^*$ such that $O(σ^*_i, σ^*_{-i}) ≥_i O(σ_i, σ^*_{-i})$ for every strategy $σ_i$ of player $i$. A strategy profile $σ^*$ is a Nash equilibrium of an extensive game if it is a best response for every player $i$.

Soundness and Completeness

LEGs is sound and complete w.r.t. the class of all games with short sight.

Solution concepts for extensive games with short sight

Definition

(Sight-compatible best response and Nash equilibrium).
Let $S = (G, s)$ be an Egss and $S|_v$ be the sight-filtrated extensive game at $v$. A strategy profile $σ^*$ is a sight-compatible best response for $i$ if for every nonterminal node $v$, it holds that $O|_v(σ^*_i|_v, σ^*_{-i}|_v) ≥_v O|_v(σ_i|_v, σ^*_{-i}|_v)$ for any strategy $σ_i|_v$ available to $i$. $σ^*$ is a sight-compatible Nash equilibrium (SCNE) of $S$ if it is a sight-compatible best response for every player $i ∈ N$. 
Theorem

Let $S$ be an Egss given by $(N, V, A, t, \Sigma_i, \succeq_i, s)$. Then for any player $i$, any strategy profiles $\sigma$ in $S$ and any formulas $\varphi$ of $\mathcal{LS}$:

(a) $\sigma$ is a sight-compatible best response (SCBR) of $S$ for $i$ iff $F_S \models [\hat{\sigma}^s] \varphi \rightarrow [\hat{\sigma}^s_i] (\leq_i) \varphi$.

(b) $\sigma$ is a sight-compatible Nash equilibrium (SCNE) of $S$ iff $F_S \models \bigwedge_{i \in N} ([\hat{\sigma}^s] \varphi \rightarrow [\hat{\sigma}^s_i] (\leq_i) \varphi)$.

(c) $\sigma$ is a subgame perfect equilibrium (SPE) of $S$ iff for any $u \in V \setminus Z$, $F_{S|u} \models \bigwedge_{i \in N} ([\hat{\sigma}] \varphi \rightarrow [\hat{\sigma}^s_i] (\leq_i) \varphi)$.

(d) A strategy profile $\sigma$ is a sight-compatible SPE of $S$ iff for all $v \in V \setminus Z$, $F_{S|v} \models \bigwedge_{i \in N} ([\hat{\sigma}] \varphi \rightarrow [\hat{\sigma}^s_i] (\leq_i) \varphi))$. 

Thank you!