Strategic Reasoning in Extensive Games with Short Sight

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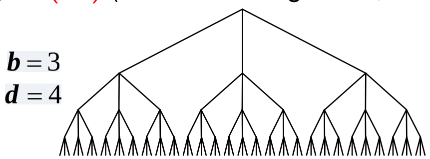
SOCREAL 2013
Hokkaido University, Sapporo, Japan
October 27th, 2013

Outline

A calculation

In a game like Chess, game tree's size grows exponentially with both its depth and its branching factor.
Time complexity: $O(b^d)$ (b for branching factor, d for depth)



Branching factor: $b \approx 35$, depth: $d \approx 100$.
Number of paths in the game tree: $35^{100} \approx 10^{135}$- Much too big for a normal game tree search.
Comparison: Number of particles in the universe $\approx 10^{87}$

• Strong assumption:
Entire structure of a game is common knowledge to all players.

• Solution:
Grossi and Turrini proposed the concept of *games with short sight* (Grossi and Turrini, 2012), in which players can only see part of the game tree.

Figure: Short-sighted people

• Contribution:
A modal Logic system for reasoning about games with short sight.

Extensive game: an example

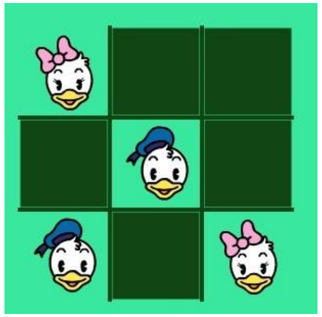


Figure: Tic-Tac-Toe game

Rule: Two players take turns to mark the spaces in a 3×3 grid. The player who succeeds in placing three respective marks in a horizontal, vertical, or diagonal row wins the game.

Extensive game(Con'd)

Definition

An extensive game is a tuple G=($N, V, A, t, \Sigma_i, \succeq_i$), where
- N is a non-empty set of the players,
- V a set of nodes or vertices including a root v_0
- $A \subseteq V^2$ a set of arcs. If $(v, v') \in A$, we call v' a *successor* of v. Leaves are the nodes that have no successors, denoted by Z.
- t is turn function assigning a member of N to each non-terminal node. $t(v) = i$;

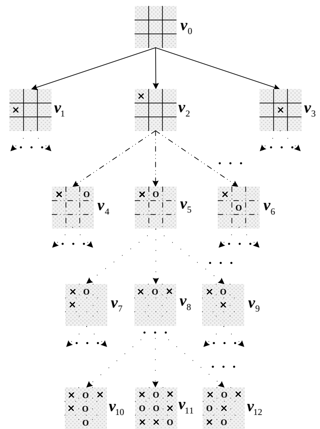
- Σ_i is a non-empty set of strategies. $\Sigma_{\mathrm{i}} = \{\sigma_i\}$.

σ_i is a strategy of player i, which is a function : $\{v \in V \backslash Z | \ t(v) = i\} \rightarrow V|v$, assigning a successor v' of v to each non-terminal node v when it is i's turn to move. (where $V|v$ is the set of nodes extending v.)

$\sigma = (\sigma_i)_{i \in N}$ represents a strategy profile which is a combination of strategies from all players and Σ represents the set of all strategy profiles.
σ_{-i} denotes the collection of strategies in σ excluding those for player i.
$O(\sigma)$ is the outcome if the strategy profile σ is followed by all players.
$O(\sigma'_i, \sigma_{-i})$ is the outcome if player i use strategy σ' while all other players employ σ .
- $\succeq_i$ is a preference relation over V^2 for each player i.

Example

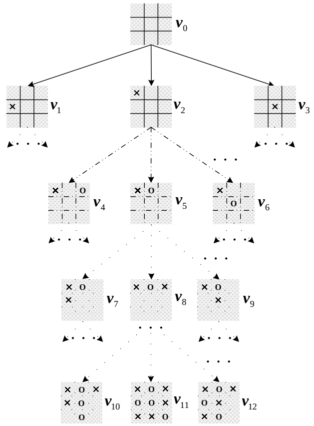
Two players: player 1($\times$) and player 2($\circ$).
Solid arrows: the moves of player 1,
dotted arrows: moves of player 2.
The initial state is v_0.
v_1,v_2,v_3 are all successors of v_0.
v_{10},v_{11},v_{12} are the terminal nodes (leaves).



Example

Formally, G=$(N, V, A, t, \Sigma_i, \succeq_i)$

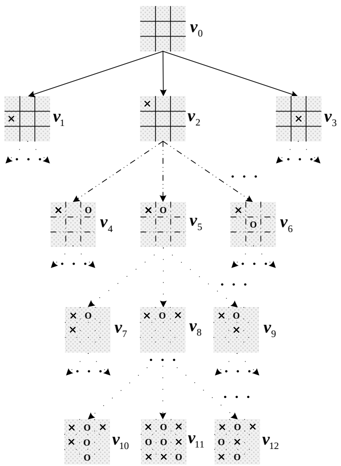
- $N = \{1, 2\}$;
- $V = \{v_0, v_1, v_2 \cdots\}$;
- (v_0, v_1), (v_0, v_2),$(v_0, v_3) \in A$;
- $v_{10}, v_{11}, v_{12} \in Z$;
- $t(v_0) = 1, t(v_2) = 2, j \cdots$.
- a σ_1 such that $\sigma_1(v_0) = v_2$, $\sigma_1(v_5) = v_8, \cdots$
  a σ_2 such that $\sigma_2(v_2) = v_5, \cdots$
  Thus, a strategy profile $\sigma = (\sigma_1, \sigma_2)$ such that $O(\sigma) = v_{11}$.
- $v_{12} \succeq_1 v_{11} \succeq_1 v_{10}$. (since player 1 *wins* the game in v_{12}, *loses* it in v_{10}, and gains a *draw* in v_{11}).



Sight function

Definition

(sight function). Let $G = (N, V, A, t, \Sigma_i, \succeq_i)$ be an *extensive game*. A short sight function for G is a function $s : V \backslash Z \to 2^{V|_v} \backslash \emptyset$, associating to each non-terminal node v a finite subset of all the available nodes at v.



- Two steps: $s(v_0) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$

Definition

(Extensive game with short sight). An extensive game with short sight (Egss) is a tuple $S = (G, s)$ where G is a finite extensive game and s a sight function.

- Endowing an extensive game with a sight function.

Sight-filtrated extensive game

At each node v, players can see a subgame $S\lceil_v$ of the whole game. This subgame is determined by their sight:
$S\lceil_v = (N\lceil_v, V\lceil_v, A\lceil_v, t\lceil_v, \Sigma_i\lceil_v, \succeq_i\lceil_v)$

- $N\lceil_v = N$;
- $V\lceil_v = s(v)$, which is the set of nodes within the sight of $s(v)$.
- $A\lceil_v = A \cap (V\lceil_v)^2$;
- $t\lceil_v = V\lceil_v \backslash Z\lceil_v \to N$ so that $t\lceil_v(v') = t(v')$;
- $\Sigma_i\lceil_v$ is the set of strategies for each player available at v and restricted to $s(v)$. It consists of elements $\sigma_i\lceil_v$ such that $\sigma_i\lceil_v(v') = \sigma_i(v')$ for each $v' \in V\lceil_v$ with $t\lceil_v(v') = i$;
- $\succeq_i\lceil_v = \succeq_i \cap (V\lceil_v)^2$.

Example

- $N\lceil_{v_0} = N$;
- $V\lceil_{v_0} = \{v_0, v_1, v_2, v_3, v_4, v_5\}$,
- $Z\lceil_{v_0} = \{v_1, v_3, v_4, v_5\}$;
- $A\lceil_{v_0} = \{(v_0, v_1), (v_2, v_4), (v_2, v_4), \cdots\}$;
- $\sigma\lceil_{v_0} = (\sigma_1\lceil_{v_0}, \sigma_2\lceil_{v_0})$ such that $O\lceil_{v_0}(\sigma\lceil_{v_0}) = v_5$, with $\sigma_1\lceil_{v_0}(v_0) = v_2$ and $\sigma_2\lceil_{v_0}(v_2) = v_5$

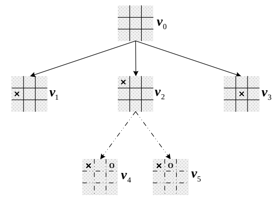
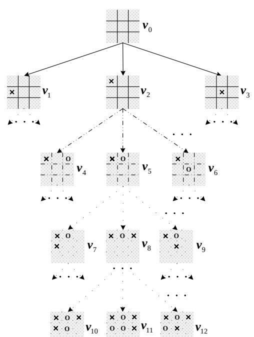


Figure: Sight-filtrated extensive game $S\lceil_{v_0}$

Syntax and Semantics

$\varphi ::= p | \neg\varphi | \varphi_0 \wedge \varphi_1 | \langle \leq_i \rangle\varphi | \langle \mathring{\sigma} \rangle\varphi | \langle \mathring{\sigma}_{-i} \rangle\varphi | \langle \prec \rangle\varphi | \langle \mathring{\sigma}^S \rangle\varphi | \langle \mathring{\sigma}^S_{-i} \rangle\varphi$

- The label $\leq_i$ denotes player i's preference relation.
- The label $\mathring{\sigma}$ stands for the outcomes of strategy profiles. $(v, v') \in R_{\mathring{\sigma}}$ iff v' is the terminal node reached from v by following σ.
  $(v_0, v_{11}) \in R_{\mathring{\sigma}}$



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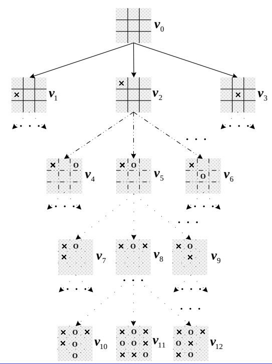
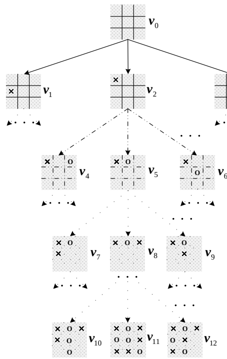
- $(v, v') \in R_{\mathring{\sigma}_{-i}}$ iff v' is one of the leaf nodes extending v that player i can enforce provided that the other players strictly follow their strategies in σ.
  $O(\sigma_{-1}, \sigma'_1) = v_{12}$,
  $(v_0, v_{12}) \in R_{\mathring{\sigma}_{-1}}$



Syntax and Semantics

$\varphi ::= p |\ \neg\varphi |\ \varphi_0 \wedge \varphi_1 |\ \langle \leq_i \rangle \varphi |\ \langle \mathring{\sigma} \rangle \varphi |\ \langle \mathring{\sigma}_{-i} \rangle \varphi |\ \langle \lessdot \rangle \varphi |\ \langle \mathring{\sigma}^S \rangle \varphi |\ \langle \mathring{\sigma}^S_{-i} \rangle \varphi$

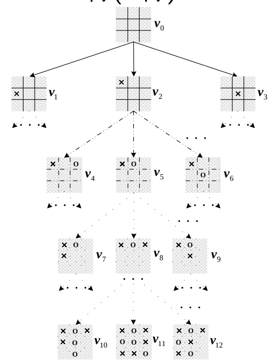
- The label $\lessdot$ is sight function for the current node, and $(v, v') \in R_\lessdot$ means "node v' is within the sight at the present node v."



Syntax and Semantics

$\varphi ::= p |\ \neg\varphi |\ \varphi_0 \wedge \varphi_1 |\ \langle \leq_i \rangle \varphi |\ \langle \mathring{\sigma} \rangle \varphi |\ \langle \mathring{\sigma}_{-i} \rangle \varphi |\ \langle \lessdot \rangle \varphi |\ \langle \mathring{\sigma}^S \rangle \varphi |\ \langle \mathring{\sigma}^S_{-i} \rangle \varphi$

- $(v, v') \in R_{\mathring{\sigma}^S}$ if "state v' is the outcome of $\sigma\lceil_v$ in $S\lceil_v$ that is reachable from the starting point v, i.e., $v' = O\lceil_v(\sigma\lceil_v)$."
  $(v_0, v_5) \in R_{\mathring{\sigma}^S}$
- The interpretation for $R_{\mathring{\sigma}^S_{-i}}$ is similar.



Syntax and Semantics

Frame F: $(V, R_{\leq_i}, R_{\mathring{\sigma}}, R_{\mathring{\sigma}_{-i}}, R_\lessdot, R_{\mathring{\sigma}^S}, R_{\mathring{\sigma}^S_{-i}})$, where

$vR_{\leq_i}v'$	iff	$v' \geq_i v$
$vR_{\mathring{\sigma}}v'$	iff	$v' = O\vert_v(\sigma\vert_v)$
$vR_{\mathring{\sigma}_{-i}}v'$	iff	$v' \in O\vert_v(\sigma_{-i}\vert_v)$
$vR_\lessdot v'$	iff	$v' \in s(v)$
$vR_{\mathring{\sigma}^S}v'$	iff	$v' = O\lceil_v(\sigma\lceil_v)$
$vR_{\mathring{\sigma}^S_{-i}}v'$	iff	$v' \in O\lceil_v(\sigma_{-i}\lceil_v)$

Syntax and Semantics

Model: $M = (V, R, I)$

$M, v \models \langle \leq_i \rangle \varphi$	iff	$M, u \models \varphi$ for some $u \in V$ with $vR_{\leq_i}u$.
$M, v \models \langle \mathring{\sigma} \rangle \varphi$	iff	$M, u \models \varphi$ for some $u \in V$ with $vR_{\mathring{\sigma}}u$.
$M, v \models \langle \mathring{\sigma}_{-i} \rangle \varphi$	iff	$M, u \models \varphi$ for some $u \in V$ with $vR_{\mathring{\sigma}_{-i}}u$.
$M, v \models \langle \lessdot \rangle \varphi$	iff	$M, u \models \varphi$ for some $u \in V$ with $vR_\lessdot u$.
$M, v \models \langle \mathring{\sigma}^S \rangle \varphi$	iff	$M, u \models \varphi$ for some $u \in V$ with $vR_{\mathring{\sigma}^S}u$.
$M, v \models \langle \mathring{\sigma}^S_{-i} \rangle \varphi$	iff	$M, u \models \varphi$ for some $u \in V$ with $vR_{\mathring{\sigma}^S_{-i}}u$.

Validities

N	Modality	Schema	Property
T	$[\leq_i]$ $[\lessdot]$	$[\leq_i]\varphi \to \varphi$ $[\lessdot]\varphi \to \varphi$	reflexivity
4	$[\leq_i]$	$[\leq_i]\varphi \to [\leq_i][\leq_i]\varphi$	transitivity
D	$[\mathring{\sigma}]$ $[\mathring{\sigma}^s]$	$[\mathring{\sigma}]\varphi \leftrightarrow \langle\mathring{\sigma}\rangle\varphi$ $[.]\varphi \leftrightarrow \langle.\rangle\varphi$	determinism
I	$([\mathring{\sigma}], [\mathring{\sigma}_{-i}])$ $([\mathring{\sigma}^s], [\mathring{\sigma}^s_{-i}])$	$[\mathring{\sigma}_{-i}]\varphi \to [\mathring{\sigma}]\varphi$ $[\mathring{\sigma}^s_{-i}]\varphi \to [\mathring{\sigma}^s]\varphi$	inclusiveness
M	$[\mathring{\sigma}]$ $[\mathring{\sigma}_{-i}]$	$[\mathring{\sigma}]([\mathring{\sigma}']\varphi \leftrightarrow \varphi)$ $[\mathring{\sigma}_{-i}]([\mathring{\sigma}'_{-i}]\varphi \leftrightarrow \varphi)$	terminating
Y	$([\lessdot], [\mathring{\sigma}^s])$ $([\lessdot], [\mathring{\sigma}^s_{-i}])$	$[\lessdot]\varphi \to [\mathring{\sigma}^s]\varphi$ $[\lessdot]\varphi \to [\mathring{\sigma}^s_{-i}]\varphi$	visibility

Soundness and Completeness

LEGS is sound and complete w.r.t. the class of all games with short sight.

Solution concepts for traditional extensive games

Definition

*(*Best response and Nash equilibrium*)*
A best response for player i of an extensive game is a strategy profile σ^* such that $O(\sigma^*_i, \sigma^*_{-i}) \succeq_i O(\sigma_i, \sigma^*_{-i})$ for every strategy σ_i of player i. A strategy profile σ^* is a Nash equilibrium of an extensive game if it is a best response for every player i.

Solution concepts for extensive games with short sight

Definition

*(*Sight-compatible best response and Nash equilibrium*)*.
Let $S = (G, s)$ be an Egss and $S\lceil_v$ be the sight-filtrated extensive game at v. A strategy profile σ^* is a sight-compatible best response for i if for every nonterminal node v, it holds that $O\lceil_v(\sigma^*_i\lceil_v, \sigma^*_{-i}\lceil_v) \succeq_i\lceil_v O\lceil_v(\sigma_i\lceil_v, \sigma^*_{-i}\lceil_v)$ for any strategy $\sigma_i\lceil_v$ available to i.
σ^* is a *sight-compatible Nash equilibrium*(SCNE) of S if it is a sight-compatible best response for every player $i \in N$.

Characterize solution concepts

Theorem

Let S be an Egss given by $(N, V, A, t, \Sigma_i, \geq_i, s)$. Then for any player i, any strategy profiles σ in S and any formulas φ of $\mathcal{LS}$:

(a) σ *is a sight-compatible best response (SCBR) of S for i iff* $\mathcal{F}_S \models [\mathring{\sigma}^s]\varphi \to [\mathring{\sigma}^s_{-i}]\langle \leq_i \rangle \varphi$.

(b) σ *is a sight-compatible Nash equilibrium (SCNE) of S iff* $F_S \models \bigwedge_{i \in N}([\mathring{\sigma}^s]\varphi \to [\mathring{\sigma}^s_{-i}]\langle \leq_i \rangle \varphi)$.

(c) σ *is a subgame perfect equilibrium (SPE) of $S\lceil v$ iff for any* $u \in V\lceil_v \backslash Z\lceil_v$, $F_{S\lceil_v}, u \models \bigwedge_{i \in N}([\mathring{\sigma}]\varphi \to [\mathring{\sigma}_{-i}]\langle \leq_i \rangle \varphi)$.

(d) *A strategy profile σ is a sight-compatible SPE of S iff for all* $v \in V \backslash Z$, $F_{S\lceil_v}, v \models [\lessdot](\bigwedge_{i \in N}([\mathring{\sigma}]\varphi \to [\mathring{\sigma}_{-i}]\langle \leq_i \rangle \varphi))$.

Davide Grossi and Paolo Turrini.
Short sight in extensive games.
In *AAMAS*, pages 805–812, 2012.

Thank you!