Measurement-Theoretic Foundations of Preference Aggregation Logic for Weighted Utilitarianism

Satoru Suzuki

Faculty of Arts and Sciences, Komazawa University,
1-23-1, Komazawa, Setagaya-ku, Tokyo 154-8525, Japan
bxs05253@nifty.com

Abstract. Harsanyi [4, 5] develops expected utility theory of von Neumann and Morgenstern [21] to provide two formalizations of utilitarianism. Weymark [22, 23] refers to these results as Harsanyi's Aggregation and Impartial Observer Theorems. In this paper, we are concerned only with Aggregation Theorem. Sen [6] argues that von Neumann-Morgenstern expected utility theory is an ordinal theory and, therefore, any increasing transform of an expected utility function is a satisfactory representation of an individual's preference relation. However, utilitarianism requires a cardinal theory of utility and so Harsanyi is not justified in giving his theorems utilitarian interpretations. Sen's informal discussion of these issues is formalized by Weymark [22]. Broome [2] calls this argument the "standard objection" to Harsanyi's theorems. The aims of this paper are as follows: (1) As our response to the standard objection, we show in terms of measurement theory that Harsanyi's Aggregation Theorem plus Bernoulli Hypothesis and Probability Agreement Hypothesis can imply weighted utilitarianism. (2) We propose a new version of complete logic for preference aggregation represented by a weighted utilitarian rule—Preference Aggregation Logic for Weighted Utilitarianism (PALU) by means of measurement theory.

1 Motivation

Harsanyi [4, 5] develops expected utility theory of von Neumann and Morgenstern [21] to provide *two formalizations* of *utilitarianism*. Weymark [22, 23] refers to these results as Harsanyi's *Aggregation* and *Impartial Observer Theorems*.[1] Weymark [23] states Harsanyi's Aggregation Theorem in the following informal way.

> *Individual* and *social preferences* on the set of alternatives are assumed to satisfy the *axioms of expected utility theory*. Furthermore, two alternatives are *socially indifferent* if every individual is *indifferent* between them (*Pareto Indifference*). With these assumptions, if the preferences are represented by *expected utility functions*, then the social utility function is a *linear function* of the individual utility functions. Hence, alternatives are *socially* ranked according to a *weighted utilitarian* rule.

[1] In this paper, we are not concerned with Impartial Observer Theorem because the model of Preference Aggregation Logic for Weighted Utilitarianism (PALU) is based only on Aggregation Theorem.

Sen [6] argues that von Neumann-Morgenstern expected utility theory is an *ordinal* theory and, therefore, *any increasing* transform of an expected utility function is a satisfactory representation of an individual's preference relation. However, utilitarianism requires a *cardinal* theory of utility and so Harsanyi is not justified in giving his theorems utilitarian interpretations. Sen's informal discussion of these issues is formalized by Weymark [22]. Broome [2] calls this argument the "*standard objection*" to Harsanyi's theorems. The aims of this paper are as follows:

1. As our response to the standard objection, we show in terms of *measurement theory* that Harsanyi's Aggregation Theorem plus *Bernoulli Hypothesis* and *Probability Agreement Hypothesis* can imply *weighted utilitarianism*.

2. We propose a new version of complete logic for preference aggregation represented by a weighted utilitarian rule—Preference Aggregation Logic for Weighted Utilitarianism (PALU) by means of *measurement theory*.

The structure of this paper is as follows. In Section 2, we argue on the standard objection to Harsanyi's theorems and offer a response in terms of measurement theory. In Section 3, we define the language $\mathscr{L}_{\mathsf{PALU}}$ of PALU, define a model $\mathfrak{M}$ of $\mathscr{L}_{\mathsf{PALU}}$, provide PALU with satisfaction, truth and validity definitions, provide PALU with its proof system, give some examples of the theorems of PALU that are characteristic of preference aggregation represented by weighted utilitarian rule, and touch upon the soundness and completeness theorems of PALU. In Section 4, we finish with brief concluding remarks.

2 Standard Objection and Our Response

We define a prospect and an ordered mixture space as follows:

Definition 1 (Prospect and Ordered Mixture Space). *Let*

- $\mathscr{A}$ *be a nonempty set of alternatives,*
- $\mathscr{J}$ *a real unit interval* $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$,
- $[\ \ ] : \mathscr{A} \times \mathscr{J} \times \mathscr{A} \to \mathscr{A}$ *a mixture operation such that* $[\mathbf{a}, 1, \mathbf{b}] = \mathbf{a}, [\mathbf{a}, \alpha, \mathbf{b}] = [\mathbf{b}, 1-\alpha, \mathbf{a}]$ *and* $[[\mathbf{a}, \beta, \mathbf{b}], \alpha, \mathbf{b}] = [\mathbf{a}, \alpha\beta, \mathbf{b}]$,
- $\mathscr{S} := \{1, \ldots, n\}$ *a* ***society set,***
- $\precsim_i$ *a* $[\ \ ]$*-monotonic ordering of an agent* $i \in \mathscr{S}$ *on* $\mathscr{A}$*,and*
- $\precsim$ *a* $[\ \ ]$*-monotonic ordering of* $\mathscr{S}$ *on* $\mathscr{A}$*, where* $\mathbf{a}, \mathbf{b} \in \mathscr{A}$ *and* $\alpha, \beta \in \mathscr{J}$.

We call

- $[\mathbf{a}, \alpha, \mathbf{b}]$ *a* ***prospect,***

- $(\mathscr{A}, [\quad], \precsim_i)$ *an **ordered mixture space of** i, and*

- $(\mathscr{A}, [\quad], \precsim)$ *an **ordered mixture space of** $\mathscr{S}$.*

We define a weak order, etc. as follows:

Definition 2 (Weak Order, etc.).

1. Weak Order
$\precsim$ is a weak order (connected and transitive) on $\mathscr{A}$. (Similarly for $\precsim_i$.)

2. Continuity
For any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathscr{A}$ for which $\mathbf{a} \prec \mathbf{b} \prec \mathbf{c}$, there exists $\alpha \in \mathscr{J}$ such that $[\mathbf{a}, \alpha, \mathbf{c}] \sim \mathbf{b}$, where $\mathbf{a} \prec \mathbf{b} := \mathbf{b} \not\precsim \mathbf{a}$ and $\mathbf{a} \sim \mathbf{b} := \mathbf{a} \precsim \mathbf{b}$ and $\mathbf{b} \precsim \mathbf{a}$. (Similarly for $\prec_i$ and $\sim_i$.)

3. Independence
For any $\mathbf{a}, \mathbf{b} \in \mathscr{A}$ and any $\alpha \in \mathscr{J}$, if $\mathbf{a} \sim \mathbf{b}$, then $\mathbf{a} \sim [\mathbf{a}, \alpha, \mathbf{b}]$.
For any distinct $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathscr{A}$ and any $\alpha \in \mathscr{J}$, if $\mathbf{a} \sim \mathbf{b}$, then $[\mathbf{a}, \alpha, \mathbf{c}] \sim [\mathbf{b}, \alpha, \mathbf{c}]$. (Similarly for $\sim_i$.)

4. Pareto Indifference
For any $a, b \in \mathscr{A}$, if, for any $i \in \mathscr{S}$, ($\mathbf{a} \sim_i \mathbf{b}$), then $\mathbf{a} \sim \mathbf{b}$.

Weymark [23] formulates Harsanyi's Aggregation Theorem as follows:

Theorem 1 (Harsanyi's Aggregation Theorem). *Suppose that $\precsim_i$ $(i \in \mathscr{S})$ and $\precsim$ are binary relation on $\mathscr{A}$ that satisfies **Weak Order**, **Continuity** and **Independence** and also suppose that **Pareto Indifference** is satisfied by these relations. Let U_i be an expected utility representation of $\precsim_i$, and U be an expected utility representation of $\precsim$. Then, there exist $\alpha_i (i \in \mathscr{S}), \beta \in \mathbb{R}$ such that, for any $\mathbf{a} \in \mathscr{A}$,*

$$(1)\ U(\mathbf{a}) = \sum_{i=1}^{n} \alpha_i U_i(\mathbf{a}) + \beta.$$

An implication of (1) is that, for any $\mathbf{a}, \mathbf{b} \in \mathbf{A}$,

$$(2)\ U(\mathbf{a}) \leq U(\mathbf{b}) \quad \textit{iff} \quad \sum_{i=1}^{n} \alpha_i U_i(\mathbf{a}) \leq \sum_{i=1}^{n} \alpha_i U_i(\mathbf{b}).$$

Remark 1. The conclusion Harsanyi intends to draw is that alternatives are socially ranked using an *weighted utilitarian* rule.

We define an increasing transform, etc. as follows:

Definition 3 (Increasing Transform, etc.).

- *$f : \mathbb{R} \to \mathbb{R}$ is an **increasing transform** if for any $x, y \in \mathbb{R}$, $f(x) \leq f(y)$ iff $x \leq y$.*

– *A utility function that is unique up to an increasing transform is said to be **ordinal**.*

– $f : \mathbb{R} \to \mathbb{R}$ *is a **positive linear transform** if there exist* $\alpha \in \mathbb{R}^+$ *and* $\beta \in \mathbb{R}$ *such that* $f(x) = \alpha x + \beta$ *for any* $x \in \mathbb{R}$.

– *A utility function that is unique up to a positive linear transform is said to be **cardinal**.*

– *The n-tuple of positive linear transforms* $F = (f_1, \ldots, f_n)$*, where* $f_i(x) = \alpha x + \beta_i$ *for any* $x \in \mathbb{R}$ *for some* $\alpha \in \mathbb{R}^+$ *and* $\beta_1, \ldots, \beta_n \in \mathbb{R}$*, is called **co-cardinal**.*

According to Weymark [23], the underlying reason for the problems identified by Sen is that in order for utilitarianism to be meaningful, it must be possible to compare *utility differences* (gains and losses) both intrapersonally and interpersonally. The need of difference comparability can be seen most clearly rewriting (2) as follows:

$$(3)\ U(\mathbf{a}) \leq U(\mathbf{b}) \quad \text{iff} \quad \sum_{i=1}^{n} \alpha_i (U_i(\mathbf{a}) - U_i(\mathbf{b})) \leq 0.$$

The utility difference sum in (3) does not change if the utility profile $\mathbf{U}$ is replaced by the profile $\mathbf{V} = F \circ \mathbf{U} := (f_1 \circ U_1, \ldots, f_n \circ U_n)$ for some *co-cardinal* n-tuple of transform F. Let $\mathscr{U}^C$ denote the set of such profile of utility functions. The profiles in $\mathscr{U}^C$ are the *only* profiles that preserve the utility difference sum in (3). However, nothing in the version of expected utility theory that Harsanyi employed in his theorems rules out the use of *non-linear* increasing transform of U_i. So, because the set of admissible profiles is not always a subset of $\mathscr{U}^C$, $\precsim$ is *not always weighted utilitarian*. As our response to the standard objection, we would like to show in terms of *measurement theory* that Harsanyi's Aggregation Theorem plus *Bernoulli Hypothesis* and *Probability Agreement Hypothesis* can imply *weighted utilitarianism*. There are two main problems in *measurement theory*:

1. the representation problem: justifying the assignment of numbers to objects,
2. the uniqueness problem: specifying the transformation up to which this assignment is unique.

A solution to the former can be furnished by a *representation theorem*, which establishes that the specified conditions on a qualitative relational system are (necessary and) sufficient for the assignment of numbers to objects that represents (or preserves) all the relations in the system. A solution to the latter can be furnished by a *uniqueness theorem*, which specifies the transformation up to which this assignment is unique. We define a strictly positive social utility structure as follows:

Definition 4 (Strictly Positive Social Utility Structure). *A finite collection of relational structures* $\mathbb{M}(\mathscr{S})$ *is called **strictly positive social utility structure** of the society* $\mathscr{S}$ *if the following conditions are met:*

1. Weak Order, Continuity, Independence, and Pareto Indifference
$\precsim_i$ *and* $\precsim$ *satisfy Weak Order, Continuity, Independence, and Pareto Indifference.*

2. Strong Pareto

For any $\mathbf{a},\mathbf{b} \in \mathscr{A}$*, if, for some* $j \in \mathscr{S}$ *for any* $i \neq j \in \mathscr{S}$*, (*$\mathbf{a} \precsim_i \mathbf{b}$ *and* $\mathbf{a} \prec_j \mathbf{b}$*), then* $\mathbf{a} \prec \mathbf{b}$.

Domotor [3] proves Harsanyi's Aggregation Theorem as the following representation and uniqueness theorems in a precise way:

Theorem 2 (Representation). *$\mathbb{M}(\mathscr{S})$ is a strictly positive social utility structure iff there exist utility functions $U_i, U : \mathscr{A} \to \mathbb{R}$ and $\gamma \in \mathbb{R}^+$ such that, for any $\mathbf{a},\mathbf{b} \in \mathscr{A}$, $\alpha \in \mathscr{J}$ and $i \in \mathscr{S}$,*

1. $\mathbf{a} \precsim_i \mathbf{b}$ *iff* $U_i(\mathbf{a}) \leq U_i(\mathbf{b})$,
2. $U_i([\mathbf{a},\alpha,\mathbf{b}]) = \alpha U_i(\mathbf{a}) + (1-\alpha)U_i(\mathbf{b})$,
3. $U'(\mathbf{a}) = \sum_{i\in\mathscr{S}} \gamma_i U_i(\mathbf{a})$*, where* $U' = g \circ U$ *for some* $g \in \mathscr{G}$ *(*$\mathscr{G}$ *:= the group of* ***positive linear transforms****).*

Theorem 3 (Uniqueness). *The constants γ_i are given* ***uniquely*** *by the choice of U_i in a fixed scale $g_i \in \mathscr{G}(i \in \mathscr{S})$, where g_i is a* ***transform*** *of U_i.*

According to Broome [1], we define Bernoulli and Probability Agreement Hypotheses as follows:

Definition 5 (Bernoulli Hypothesis). *$P : \mathscr{A} \to \mathscr{J}$ and $V : \mathscr{A} \to \mathbb{R}$ satisfy* ***Bernoulli Hypothesis*** *iff, for any $\mathbf{a} \in \mathscr{A}$ and $\alpha \in \mathscr{J}$,*

$$\begin{cases} V([\mathbf{a},\alpha,\mathbf{b}]) = P(\mathbf{a})V(\mathbf{a}) + P(\mathbf{b})V(\mathbf{b}), \\ P(\mathbf{a}) + P(\mathbf{b}) = 1, \\ \alpha = P(\mathbf{a}) \end{cases}$$

hold.

Definition 6 (Probability Agreement Hypothesis). *A probability function relative to $i \in \mathscr{S}$ $P_i : \mathscr{A} \to \mathscr{J}$ satisfies* ***Probability Agreement Hypothesis*** *iff, for any $\mathbf{a} \in \mathscr{A}$,*

$$P_1(\mathbf{a}) = P_2(\mathbf{a}) = \cdots = P_n(\mathbf{a})$$

holds.

From Theorems 2 and 3 and Definitions 5 and 6, the next essential corollaries follow:

Corollary 1 (Representation). *Suppose that $P_i : \mathscr{A} \to \mathscr{J}$ is* ***given*** *for any $i \in \mathscr{S}$ and that $U_i, U : \mathscr{A} \to \mathbb{R}$ and $P_i : \mathscr{A} \to \mathscr{J}$ satisfy* ***Bernoulli*** *and* ***Probability Agreement Hypotheses****. Then $\mathbb{M}(\mathscr{S})$ is a strictly positive social utility structure iff there exist utility functions $U_i, U : \mathscr{A} \to \mathbb{R}$ and $\gamma_i \in \mathbb{R}^+$ such that, for any $\mathbf{a},\mathbf{b} \in \mathscr{A}$, $\alpha \in \mathscr{J}$ and $i \in \mathscr{S}$,* *1.* $\mathbf{a} \precsim_i \mathbf{b}$ *iff* $U_i(\mathbf{a}) \leq U_i(\mathbf{b})$,

2. $U_i([\mathbf{a}, \alpha, \mathbf{b}]) = P_i(\mathbf{a})U_i(\mathbf{a}) + P_i(\mathbf{b})U_i(\mathbf{b})$,

3. $P_i(\mathbf{a}) + P_i(\mathbf{b}) = 1$,

4. $\alpha = P_1(\mathbf{a}) = P_2(\mathbf{a}) = \cdots = P_n(\mathbf{a})$,

5. $U'(\mathbf{a}) = \sum_{i \in \mathscr{S}} \delta_i U_i(\mathbf{a})$*, where* $U' = h \circ U$ *for some* $h \in \mathscr{G}$ *(*$\mathscr{G}$ *:= the group of positive linear transforms).*

Remark 2. The following is an implication of Corollary 1. Suppose that $P_i : \mathscr{A} \to \mathscr{J}$ is given for any $i \in \mathscr{S}$ and that $U_i, U : \mathscr{A} \to \mathbb{R}$ and $P_i : \mathscr{A} \to \mathscr{J}$ satisfy Bernoulli and Probability Agreement Hypotheses. Then, for any $\mathbf{a} \in \mathscr{A}$ and $i \in \mathscr{S}$ and for some $\alpha \in \mathscr{J}$,

$$\begin{cases} f_i(U_i(\mathbf{a})) = \alpha U_i(\mathbf{a}) + \beta_i, \\ \alpha = P_i(\mathbf{a}) \end{cases}$$

hold. Then the permissible transform f_i of U_i is only a *positive linear* one, that is, the n-tuple of transforms $(f_1, \ldots, f_n)$ is *co-cardinal*. So Harsanyi's Aggregation Theorem plus *Bernoulli Hypothesis* and *Probability Agreement Hypothesis* can imply *weighted utilitarianism*.

Corollary 2 (Uniqueness). *The constants* δ_i *are given uniquely by the choice of* U_i *in a fixed scale* $h_i \in \mathscr{G} (i \in \mathscr{S})$*, where* h_i *is a transform of* U_i*.*

3 Preference Aggregation Logic for Weighted Utilitarianism PALU

Next we will construct a preference aggregation logic for weighted utilitarianism PALU on the basis of Corollaries 1 and 2. So in PALU, both any agent $i \in \mathscr{S}$ and the society are supposed to behave as expected utility maximizers because of *Bernoulli Hypothesis* and agree on the probability assignment for any alternative because of *Probability Agreement Hypothesis*.

3.1 Language of PALU

We define the language $\mathscr{L}_{\mathsf{PALU}}$ of PALU as follows:

Definition 7 (Language of PALU).

– *Let* $\mathscr{S}$ *denote a nonempty* ***society*** *set of* ***agents****,* $\mathscr{V}$ *a set of individual variables,* $\mathscr{C}$ *a set of individual constants,* $\leqslant_i$ *a* ***weak preference relation symbol*** *of* i*,* $\leqslant$ *a* ***social weak preference relation symbol****.*

– *The language* $\mathscr{L}_{\mathsf{PALU}}$ *of* PALU *is given by the following BNF grammar:*

$$\varphi ::= t = t \mid t \leqslant_i t \mid t \leqslant t \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x \varphi$$
$$t ::= x \mid a,$$

where $x \in \mathscr{V}$ *and* $a \in \mathscr{C}$.

– $\bot, \vee, \rightarrow, \leftrightarrow$ *and* $\exists$ *are introduced by the standard definitions.*

– $t_1 \leqslant_i t_2$ *means that **an agent** i **does not prefer*** t_1 ***to*** t_2.

– $t_1 \leqslant t_2$ *means that **the society does not prefer*** t_1 ***to*** t_2.

– *We define a **strict preference relation symbol*** $\lessdot_i$ *and an **indifference relation symbol*** $\approx_i$ *as follows (Similarly with* $\lessdot$ *and* $\approx$*):*

$$t_1 \lessdot_i t_2 := \neg(t_2 \leqslant_i t_1),$$
$$t_1 \approx_i t_2 := t_1 \leqslant_i t_2 \text{ and } t_2 \leqslant_i t_1.$$

– *The set of all well-formed formulae of* $\mathscr{L}_{\mathsf{PALU}}$ *is denoted by* $\Phi_{\mathscr{L}_{\mathsf{PALU}}}$.

3.2 Semantics of PALU

3.2.1 Model of $\mathscr{L}_{\mathsf{PALU}}$ We define a model $\mathfrak{M}$ of $\mathscr{L}_{\mathsf{PALU}}$ as follows:

Definition 8 (Model $\mathfrak{M}$ of $\mathscr{L}_{\mathsf{PALU}}$). $\mathfrak{M}$ *is a tuple* $(\mathscr{S}, \mathscr{A}, a^{\mathfrak{M}}, b^{\mathfrak{M}}, \ldots, [\quad], \precsim_i, \precsim)$*, where:*

– $\mathscr{S} := \{1, \ldots, n\}$ *is a society set,*

– $\mathscr{A}$ *is a nonempty set of alternatives,*

– $a^{\mathfrak{M}}, b^{\mathfrak{M}}, \ldots \in \mathscr{A}$,

– $\{(\mathscr{A}, [\quad], \precsim_i, \precsim) : i \in \mathscr{S}\}$ *is a **strictly positive social utility structure** of* $\mathscr{S}$ *of Definition 4.*

3.2.2 Truth in PALU We define an (extended) assignment function as follows:

Definition 9 ((Extended) Assignment Function). *Let* $\mathscr{V}$ *denote a set of individual variables,* $\mathscr{C}$ *a set of individual constants and* $\mathscr{A}$ *a set of alternatives.*

– *We call* $s : \mathscr{V} \rightarrow \mathscr{A}$ *an assignment function.*

– $\tilde{s} : \mathscr{V} \cup \mathscr{C} \rightarrow \mathscr{A}$ *is defined by recursion as follows:*
  1. *For any* $x \in \mathscr{V}$, $\tilde{s}(x) = s(x)$.
  2. *For any* $a \in \mathscr{C}$, $\tilde{s}(a) = a^{\mathfrak{M}}$.

We call $\tilde{s}$ an extended assignment function.

We provide $\mathscr{L}_{\mathsf{PALU}}$ with the following satisfaction definition relative to $\mathfrak{M}$, define the truth in $\mathfrak{M}$ by means of satisfaction, and then define validity:

Definition 10 (Satisfaction, Truth and Validity). *What it means for $\mathfrak{M}$ to satisfy $\varphi \in \Phi_{\mathscr{L}_{\mathsf{PALU}}}$ with s, in symbols $\mathfrak{M} \models_{\mathsf{PALU}} \varphi[s]$, is inductively defined as follows:*

- $\mathfrak{M} \models_{\mathsf{PALU}} t_1 = t_2[s]$ *iff* $\tilde{s}(t_1) = \tilde{s}(t_2)$,
- $\mathfrak{M} \models_{\mathsf{PALU}} \top$,
- $\mathfrak{M} \models_{\mathsf{PALU}} \neg\varphi[s]$ *iff* $\mathfrak{M} \not\models_{\mathsf{PALU}} \varphi[s]$,
- $\mathfrak{M} \models_{\mathsf{PALU}} \varphi \wedge \psi[s]$ *iff* $\mathfrak{M} \models_{\mathsf{PALU}} \varphi[s]$ *and* $\mathfrak{M} \models_{\mathsf{PALU}} \psi[s]$,
- $\mathfrak{M} \models_{\mathsf{PALU}} t_1 \leqslant_i t_2[s]$ *iff* $\tilde{s}(t_1) \precsim_i \tilde{s}(t_2)$,
- $\mathfrak{M} \models_{\mathsf{PALU}} t_1 \leqslant t_2[s]$ *iff* $\tilde{s}(t_1) \precsim \tilde{s}(t_2)$,
- $\mathfrak{M} \models_{\mathsf{PALU}} \forall x\varphi[s]$ *iff for any* $\mathfrak{d} \in \mathscr{A}$, $\mathfrak{M} \models_{\mathsf{PALU}} \varphi[s(x|\mathfrak{d})]$,
  where $s(x|\mathfrak{d})$ is the function that is exactly like s except for one thing: for the individual variable x, it assigns the individual $\mathfrak{d}$. This can be expressed as follows:

$$s(x|\mathfrak{d})(y) := \begin{cases} s(y) & \text{if} \quad y \neq x \\ \mathfrak{d} & \text{if} \quad y = x. \end{cases}$$

If $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{PALU}}} \varphi[s]$ for all s, we write $\mathfrak{M} \models_{\mathscr{L}_{\mathsf{PALU}}} \varphi$ and say that φ is true in $\mathfrak{M}$. If φ is true in all models of $\mathscr{L}_{\mathsf{PALU}}$, we write $\models_{\mathscr{L}_{\mathsf{PALU}}} \varphi$ and say that φ is valid.

The next essential corollary follows from Corollary 1 and Definition 10.

Corollary 3 (Weighted Utilitarian Rule). *In $\mathfrak{M}$ of $\mathscr{L}_{\mathsf{PALU}}$, there exist utility functions $U_i : \mathscr{A} \to \mathbb{R}$ such that for any $\tilde{s}(t_1), \tilde{s}(t_2) \in \mathscr{A}$,*
$\mathfrak{M} \models_{\mathscr{L}_{\mathsf{PALU}}} t_1 \leqslant t_2[s]$ *iff* $\sum_{i=1}^{n} \delta_i U_i(\tilde{s}(t_1)) \leq \sum_{i=1}^{n} \delta_i U_i(\tilde{s}(t_2))$.

> *Remark 3.* This corollary indicates that we can reason about preference aggregation represented by an *weighted utilitarian rule* in terms of PALU.

3.3 Syntax of PALU

3.3.1 Proof System of PALU We extend a proof system of *first-order logic with an equality symbol* in such a way as to add the syntactic counterparts of the *Connectedness* of $\precsim_i$ and $\precsim$, the *Transitivity* of $\precsim_i$ and $\precsim$, *Pareto Indifference*, and *Strong Pareto*:

Definition 11 (Proof System of PALU).

– *all valid formulae of first-order logic with an equality symbol,*

– $\forall x \forall y((x \leqslant_i y) \vee (y \leqslant_i x))$
(Syntactic Counterpart of ***Connectedness of*** $\precsim_i$*),*

– $\forall x \forall y((x \leqslant y) \vee (y \leqslant x))$
(Syntactic Counterpart of ***Connectedness of*** $\precsim$*),*

– $\forall x \forall y \forall z((x \leqslant_i y) \wedge (y \leqslant_i z)) \to (x \leqslant_i z))$
(Syntactic Counterpart of ***Transitivity of*** $\precsim_i$*),*

– $\forall x \forall y \forall z((x \leqslant y) \wedge (y \leqslant z)) \to (x \leqslant z))$
(Syntactic Counterpart of ***Transitivity of*** $\precsim$*),*

– $\forall x \forall y \left(\bigwedge_{1 \leq i \leq n} (x \approx_i y) \to (x \approx y) \right)$
(Syntactic Counterpart of ***Pareto Indifference****),*

– $\forall x \forall y \left(\bigvee_{1 \leq j \leq n} \bigwedge_{1 \leq i \leq n, i \neq j} ((x \leqslant_i y) \wedge (x \lessdot_j y)) \to (x \lessdot y) \right),$
(Syntactic Counterpart of ***Strong Pareto****),*

– *Modus Ponens,*

– *Generalization.*

A proof of $\varphi \in \Phi_{\mathscr{L}_{\mathsf{PALU}}}$ *is a finite sequence of* $\mathscr{L}_{\mathsf{PALU}}$*-formulae having* φ *as the last formula such that either each formula is an instance of an axiom, or it can be obtained from formulae that appear earlier in the sequence by applying an inference rule. If there is a proof of* φ*, we write* $\vdash_{\mathsf{PALU}} \varphi$*.*

Remark 4. The proof system of PALU has neither a syntactic counterpart of **Continuity** nor that of **Independence** both of which are satisfied in $\mathfrak{M}$ of $\mathscr{L}_{\mathsf{PALU}}$ of Definition 8. For $\mathscr{L}_{\mathsf{PALU}}$ is not so *fine-grained* as to express them. However, it is not a defect of PALU. For PALU is designed to capture *behavior* about preference aggregation represented by a weighted utilitarian rule, whereas both **Continuity** and **Independence** are the mere *structural* properties required for the existence of a weighted utilitarian rule in a mixture space.

3.3.2 Characteristic Theorems We give some examples of the theorems of PALU that are *characteristic* of *preference aggregation* represented by a *weighted utilitarian rule*:

Proposition 1 (Characteristic Theorems).

– $\vdash_{\mathsf{PALU}} \forall x \forall y \left(\bigvee_{1 \leq j \leq n} \bigwedge_{1 \leq i \leq n, i \neq j} (((x \preccurlyeq_i y) \wedge (y \preccurlyeq x)) \to (y \preccurlyeq_j x)) \right)$.

– $\vdash_{\mathsf{PALU}} \forall x \forall y \left(\bigvee_{1 \leq j \leq n} \bigwedge_{1 \leq i \leq n, i \neq j} ((x \approx_i y) \to ((x \preccurlyeq_j y) \leftrightarrow (x \preccurlyeq y))) \right)$.

– $\vdash_{\mathsf{PALU}} \forall x \forall y \left(\left(\bigwedge_{1 \leq i \leq n} (x \preccurlyeq_i y) \wedge \bigvee_{1 \leq i \leq n} (x \prec_i y) \right) \to (x \prec y) \right)$.

3.4 Metalogic of PALU

We touch upon the metatheorems of PALU. It is easy to prove the soundness of PALU.

Theorem 4 (Soundness). *For any $\varphi \in \Phi_{\mathscr{L}_{\mathsf{PALU}}}$, if $\vdash_{\mathsf{PALU}} \varphi$, then $\models_{\mathsf{PALU}} \varphi$.*

We can also prove the completeness of PALU.

Theorem 5 (Completeness). *For any $\varphi \in \Phi_{\mathscr{L}_{\mathsf{PALU}}}$, if $\models_{\mathsf{PALU}} \varphi$, then $\vdash_{\mathsf{PALU}} \varphi$.*

4 Concluding Remarks

In this paper,

1. As our response to the standard objection, we have shown in terms of measurement theory that Harsanyi's Aggregation Theorem plus Bernoulli Hypothesis and Probability Agreement Hypothesis can imply weighted utilitarianism.

2. We have proposed a new version of complete logic for preference aggregation represented by a weighted utilitarian rule (PALU) by means of measurement theory.

This paper is only a part of a larger *measurement-theoretic* study. By means of measurement theory, we constructed or are trying to construct such logics as

1. (dynamic epistemic) preference logic [7, 9],
2. dyadic deontic logic [8],
3. vague predicate logic [12, 13]
4. threshold-utility-maximiser's preference logic [10, 11],
5. interadjective-comparison logic [16],
6. gradable-predicate logic [15],
7. logic for better questions and answers [14],
8. doxastic and epistemic logic [20],
9. multidimensional-predicate-comparison logic [18],
10. logic for preference aggregation represented by a Nash collective utility function [19], and
11. modal-qualitative-probability logic [17].

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