Toward sheaf semantics for a multi-agent substructural modal logic

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1. Introduction

Kitamura, Nakatogawa and Fukayama (2007) chose a certain substructural logic, as an initial and basic tool to analyze the two wise girls puzzle discussed in Yasugi and Oda (2002)[4]. This substructural logic is named as $\mathbf{CFL_eKD4^2}$ by Fukayama, and is based on the two systems, $\mathbf{CFL_eKD}$ and $\mathbf{CFL_eKT4}$, which are introduced in Watari, Nakatogawa and Ueno (1999). $\mathbf{CFL_eKD4^2}$ is a Classical Full Lambek with exchange rules and the axioms K, D and 4 about two modal operators. The 2007 paper contains a detailed logical analysis, due to the effort of Fukayama, of the possibility for a solution of that puzzle, by replacing the connectives in the ordinary sentential logic with the ones in a substructural logic. In this study, we will offer an overview of several semantics relevant to that study, before we spell out possible world semantics to the system in question. In particular, we will focus on a development of the semantics specified on the basis of the notion of a presheaf.

2. Previous studies

By the term "substructural logics" we mean logics which are conscious of the

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[4] Kaneko and Nagashima (1997) proved the cut elimination theorem and its lemmas for their epistemic logics formulated as sequent calculi. Their systems contain infinitary formulae and deal with the notion of common knowledge. Yasugi and Oda (1999) used the results of Kaneko and Nagashima omitting those features, and proved the cut elimination theorem and its lemmas for the system of their own. Yasugi and Oda (2002) used the theorem and the lemmas to analyze the two wise girls puzzle.

number and/or the order of resources used in the inference. As the corresponding formal system, we investigate an extension of the sequent calculus **CFL.** This system contains a set of inference rules for resource-conscious connectives, and the so-called structural inference rules, in addition to the inference rules for the familiar logical connectives.

We shall note some of the inference rules of $\mathbf{CFL_eKD4^2}$ which are characteristic of the system and illustrate resource-consciousness. (For the full set of the axioms and the inference rules of $\mathbf{CFL_eKD4^2}$, see Kitamura et al. (2007).) Hereafter, Γ's and Δ's denote (possibly empty) lists of sentences, and σ and τ denote sentences. First, the inference rules for the *multiplicative conjunction* * and the *multiplicative disjunction* + are as follows:

$$\frac{\Gamma, \sigma, \tau, \Gamma' \longrightarrow \Delta}{\Gamma, \sigma * \tau, \Gamma' \longrightarrow \Delta}\ (*\,\text{left}) \qquad \frac{\Gamma \longrightarrow \sigma, \Delta \quad \Gamma' \longrightarrow \tau, \Delta'}{\Gamma, \Gamma' \longrightarrow \sigma * \tau, \Delta, \Delta'}\ (*\,\text{right})$$

$$\frac{\Gamma, \sigma \longrightarrow \Delta \quad \Gamma', \tau \longrightarrow \Delta'}{\Gamma, \Gamma', \sigma + \tau \longrightarrow \Delta, \Delta'}\ (+\,\text{left}) \qquad \frac{\Gamma \longrightarrow \Delta, \sigma, \tau, \Delta'}{\Gamma \longrightarrow \Delta, \sigma + \tau, \Delta'}\ (+\,\text{right})$$

Below are the inference rules for the familiar, *additive*, conjunction ∧ and disjunction ∨:

$$\frac{\Gamma, \sigma, \Gamma' \longrightarrow \Delta}{\Gamma, \sigma \wedge \tau, \Gamma' \longrightarrow \Delta}\ (\wedge\,\text{left}) \qquad \frac{\Gamma \longrightarrow \Delta, \sigma, \Delta' \quad \Gamma \longrightarrow \Delta, \tau, \Delta'}{\Gamma \longrightarrow \Delta, \sigma \wedge \tau, \Delta'}\ (\wedge\,\text{right})$$

$$\frac{\Gamma, \tau, \Gamma' \longrightarrow \Delta}{\Gamma, \sigma \wedge \tau, \Gamma' \longrightarrow \Delta}\ (\wedge\,\text{left})$$

$$\frac{\Gamma, \sigma, \Gamma' \longrightarrow \Delta \quad \Gamma, \tau, \Gamma' \longrightarrow \Delta'}{\Gamma, \sigma \vee \tau, \Gamma' \longrightarrow \Delta, \Delta'}\ (\vee\,\text{left}) \qquad \frac{\Gamma \longrightarrow \Delta, \sigma, \Delta'}{\Gamma \longrightarrow \Delta, \sigma \vee \tau, \Delta'}\ (\vee\,\text{right})$$

$$\frac{\Gamma \longrightarrow \Delta, \tau, \Delta'}{\Gamma \longrightarrow \Delta, \sigma \vee \tau, \Delta'}\ (\vee\,\text{right})$$

To observe difference between the rules for multiplicative connectives and additive connectives, let us take the rules (* left) and (∧ left), for example. Each of the rules yields the conjunctive sentence of σ and τ on the left side of the lower sequent. They differ in that (* left) requires both σ and τ on the left side of the upper sequent and that ($\wedge$ left) requires σ or τ (and not both). So according to the first (second) rule of ($\wedge$ left), one can infer the lower sequent containing τ (σ) although the upper sequent does not contain it. If we regard the sentences appearing in the inference as the resources produced or consumed in it, we can say that ($\wedge$ left) produces a part of a conjunctive sentence without any relevant resources. In contrast, (* left) can be said to produce a conjunctive sentence by use of resources fully relevant to it. This is a typical example of what we mean by "resource-consciousness" of a connective. The connective + is also resource-conscious because, according to the rules (+ right), one can infer a sequent containing the disjunctive sentence $\sigma + \tau$ only if the upper sequent contains both resources σ and τ of the sentence.

$\mathbf{CFL_eKD4^2}$ has three structural inference rules. One is the cut rule that is also seen in other sequent calculi. The others are the following *exchange* rules. (The suffix "$\mathbf{_e}$" comes from this.)

$$\frac{\Gamma, \sigma, \tau, \Gamma' \longrightarrow \Delta}{\Gamma, \tau, \sigma, \Gamma' \longrightarrow \Delta} \ (\text{e left}) \qquad\qquad \frac{\Gamma \longrightarrow \Delta, \sigma, \tau, \Delta'}{\Gamma \longrightarrow \Delta, \tau, \sigma, \Delta'} \ (\text{e right})$$

This rule allows us to exchange the order of adjacent two sentences.

$\mathbf{CFL_eKD4^2}$ has three inference rules for the two modal operators B_i (i = 1, 2). Let Γ be a list σ_1, σ_2, ..., σ_n of sentences. Then we abbreviate the list of modal sentences $B_i\sigma_1$, $B_i\sigma_2$, ..., $B_i\sigma_n$ as $B_i\Gamma$. Using that notation, we can describe the three rules as follows:

$$\frac{\Gamma \longrightarrow \sigma}{B_i\Gamma \longrightarrow B_i\sigma} \ (B_i\text{-}\mathbf{K}) \qquad \frac{\Gamma \longrightarrow}{B_i\Gamma \longrightarrow} \ (B_i\text{-}\mathbf{D}) \qquad \frac{B_i\Gamma \longrightarrow \sigma}{B_i\Gamma \longrightarrow B_i\sigma} \ (B_i\text{-}\mathbf{4})$$

Let us focus on the interpretations of a modal substructural logic by some algebraic structures. Watari, Ueno and Nakatogawa (1999) supply some algebraic semantics to various substructural modal logics, and they contain the semantics to the sequent calculi $\mathbf{CFL_eKD}$ and $\mathbf{CFL_eKT4}$ close to our system[5]. The algebraic structure they use for such an interpretation is a lattice with the multiplication of monoid, and two functions are defined on the lattice. The additive logical connectives are interpreted as the lattice operations. The multiplicative logical connectives are interpreted by use of the multiplication of monoid. In addition, the necessity operator and the possibility operator are interpreted as the two functions mentioned above.

As an attempt to develop some possible world semantics for the sequent calculus in substructural logics, Ono and Komori (1985) define the so-called Kripke model via some algebraic semantics. The basic device is a monoid. In particular, their apparatus is a monoid with a partial order compatible with its multiplication, that is, a partially ordered monoid, abbreviated as PO-monoid. This PO-monoid is called "SO-monoid" if it is a meet-semilattice in the sense that the operation of meet compatible with the multiplication of monoid is defined on it. A frame is defined to be a pair of SO-monoid and its subset satisfying several properties.

In contrast to this, Restall (2000, pp.239-248) employs a ternary relation as an accessibility relation in addition to binary ones, and gives a kind of possible world semantics. He distinguishes between the accessibility relations when the logical connectives vary. He interprets the modal operators $\Box$ and $\Diamond$ of the alethic modality using the accessibility relation S. In addition, he interprets two kinds of negation using the accessibility relation C. Moreover, he interprets multiplicative connectives using the ternary relation R. Here each of the relations S, C and R is dominated by mutually different semantic principles. Although his semantics is more or less complicated, it is possible to apply it to various modal substructural logics. We would rather like to seek a more natural extension of the semantic notions specified on the basis of the Kripke structure.

Let us turn to modal logics. Shehtman and Skvortsov (1990) and Awodey and

[5] The modal logic **KT4** accords with the one **S4**.

Kishida (2008) describe a Kripke structure using the notion of a sheaf over a topological Boolean algebra. The topological Boolean algebra has the function I satisfying several properties on the underlying set. If I is an operation on the power set $P(X)$ of a set X, I typically assigns to each subset of X its interior. In addition, using the operation I, we can define a topology on X. In this sense, we can say that the Boolean algebra with the operation I has a topological structure. If an element a of the Boolean algebra is an interpretation of the sentence σ, then $I(a)$ is an interpretation of the modal sentence $\Box\sigma$. Thus one can use a topological Boolean algebra for the interpretation of modal sentential logic.

A notion of a sheaf over a topological space X is defined in terms of the notion of presheaf. Consider an open set U of X. A *presheaf* F over X assigns to an open set U the set $F(U)$. Consider a pair of open sets under the inclusion relation $V \subseteq U$. Then F assigns to the pair a function from $F(U)$ to $F(V)$. It is required that $F(U, U)$ be the identity function on $F(U)$, and when $W \subseteq V \subseteq U$, the composite function $F(W, V) \circ F(V, U)$ must be equal to $F(W, U)$. For example, let $F(U)$ be the set of continuous functions from U to the set R of real numbers. Then $F(V, U) : F(U) \to F(V)$ assign to an element f its restriction $f|_V$ to V. Thus a presheaf on a space X gives local information on open subsets of X. A sheaf is a presheaf which enables us to obtain the total information by gluing local information. Let F be a presheaf on X. F is called a *sheaf* if the following condition holds. Let U be an open subset of X, $\{U_i\}_i$ the open covering of U, and $\{f_i\}_i$ the family of their elements. Suppose that an element f_i of $F(U_i)$ and an element f_j of $F(U_j)$ are equal within the intersection of U_i and U_j, that is, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Then there uniquely exists an element f of $F(U)$ such that its restriction to U_i is f_i. That is the condition for sheaf. To understand it more clearly, imagine a graph paper whose X-axis is the space X and whose Y-axis is the set R of real numbers. U is an open subset of X, which is covered by open sets U_i's. Then the conditions states that if continuous functions f_i's defined on any two of U_i's have the same value on their intersection, there is exactly one function defined on U and its restriction to U_i is f_i. In a word, a total function on U is uniquely obtainable by gluing local functions defined on its open covers U_i's. A sheaf F is essentially a choice of open subsets of X which makes the above situation true.

There is an alternative definition of sheaf over X. Awodey and Kishida (2008) adopt this way. Let F be a topological space and π be a function from F to X. Then (F, π) is called a sheaf over X if the following conditions hold: for each element a of F, there is a neighborhood U such that $\pi(U)$ is an open subset of X and the restriction $\pi|_U$ is a homeomorphism. Here π is called a *projection*. Consider an element p of X. The inverse image of $\{p\}$ under π is called a *fiber* of F over p. In constructing denotational semantics of modal predicate logics, one can take the domain of interpretation as F. If we interpret n-place predicates in sheaf semantics, it becomes necessary to use the projection π^n from D^n to X. It involves the notion of the product of sheaves. Each sentence has the sum of fiberwise extensions as its extension. For a sentence σ, the extension of $\Box\sigma$ can be provided as the interior of the extension of σ.

3. Category theory

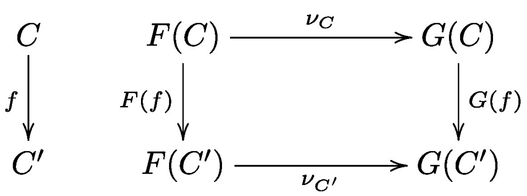
One can give a more general consideration to this concept, by restating the notion of a presheaf in terms of category theory. We will briefly introduce several concepts of category theory and its related devices. A *category* consists of objects and arrows among them. The central notion is not an object but an arrow. An arrow has objects as its source and its target. The former is called the domain of the arrow and the latter the codomain. If an arrow f has the domain A and the codomain B, we write $f: A\to B$. If $f: A\to B$ and $g: B\to C$, their *composition* $g\circ f: A\to C$ is defined. The axioms of category theory state the following two conditions on composition: first, the composition is associative, that is, $h\circ(g\circ f)=(h\circ g)\circ f$; second, a unit of composition exists for each object A. It is called the *identity arrow* on A, written as id_A. So the unit laws require that $\mathrm{id}_B \circ f = f$ and $g \circ \mathrm{id}_B = g$ for f: $A \to B$ and g: $B \to C$. A category has various examples due to its abstractness. For example, all sets and the functions between them constitute **Set**, the category of sets. All topological spaces and the continuous functions between them constitute **Top**, the category of topological spaces. All groups and the homomorphisms between them constitute **Grp**, the category of groups. They are categories consisting of some kind of mathematical objects and the structure-preserving functions between them.

In addition to this, a single mathematical object is often regarded as a category. Let W be a set. A binary relation on W is called a *preorder* if it is reflexive and transitive. We can regard W with a preorder R as a category. The objects of the category are the elements of W, and the arrows are the pairs $\langle a, b \rangle$ of elements of W such that aRb holds. The composition of arrows $\langle a, b \rangle$ and $\langle b, c \rangle$ is defined as $\langle a, c \rangle$. This definition is supported by the transitivity of R. The existence of identity arrows is shown by the reflexivity of R.

In the context of possible world semantics for modal logic, a preorder structure $\langle W, R \rangle$ is a **KT4**-frame in the sense that $\langle W, R \rangle$ satisfies the axioms of the system **KT4** for modal logic. For a system **S** which is stronger than **KT4**, **S**-frame is a category. On the contrary, frames for systems weaker than **KT4** are not always a category. The system **KD4** of our interest is weaker than **KT4** because the axiom D $\Box\sigma \to \Diamond\sigma$ is provable in **KT4** and one can easily construct a **KT4**-frame which is not a **KD4**-frame (Note that R in a **KD4**-frame is reflexive and *serial,* in the sense that for every w in W, there is an element v in W such that wRv (Garson, 2006, p.96).) Consequently a **KD4**-frame is not always a category. It means that category theory is applicable to possible world semantics for epistemic logic as long as its frame is transitive.

As there are structure-preserving mappings between groups, there are assignments between categories, which are called functors. A *functor* assigns an arrow in a category to an arrow in another category. A functor must preserve composition and the identity arrows. The notion of presheaf is generalized by the notion of functor. Let **C** be a category. A *presheaf* *F* on **C** is a functor from **C** to **Set**, which assigns to a **C**-arrow *f*: *A*→*B* a function *F*(*f*): *F*(*B*)→*F*(*A*). The conditions to be obeyed by a presheaf are equivalent to the ones to be obeyed by a functor. If **C** is the set *O*(*X*) of open subsets of a topological space *X*, the definition stated here is equivalent to the original one. We take for **C** a preorder structure <*W*, *R*> consisting of possible worlds and the accessibility relation between them. We have the following two reasons why *R* is a preorder: first, we can regard <*W*, *R*> as a category immediately; second, <*W*, *R*> constitutes a frame for the familiar modal logic **KT4**. In what follows, we often abbreviate <*W*, *R*> as *W*.

The presheaves on a category themselves constitute a category by defining the appropriate concept of arrows between them. It is the concept of a natural transformation. Let **C** and **D** be a category. Let *F* and *G* be a functor from **C** to **D**. A *natural transformation* ν from *F* to *G* is a family of arrows $\nu_C : F(C) \to G(C)$ for each object *C* in the category **C**. ν_C's must satisfy the following condition: for each arrow *f* from *C* to *C′* in the category **C**, the composed arrows $G(f) \circ \nu_C$ and $\nu_{C'} \circ F(f)$ coincide.



4. Topos-theoretic approach

Here we will observe how the category of presheaves over *W* gives semantics for a sentential modal logic. We write *W*^ to denote the category. Those who construct the semantics mentioned above intend to give semantics for a *predicate* modal logic using their approach. One can obtain semantics of a sentential modal logic by partially simplifying their semantics. Moerdijk and van Oosten (2007, p.15) describes this way.

W^ satisfies the axioms of elementary topos. An *elementary topos* is a category which can be seen as a generalization of the category of sets. A topos has special objects such as the terminal object, the pullback of two arrows, the exponential B^A and the subobject classifier. In the category of sets, they are a singleton, the fibered product, the set of functions from *A* to *B*, and the set of truth values, respectively. The axioms of an elementary topos are stated by using the first-order language, that is, the elementary language on arrows in the category.

Above all, the existence of the subobject classifier is characteristic of a topos. The *subobject classifier* consists of the *truth-value object* Ω and the *truth arrow t* from the terminal object 1 to Ω.

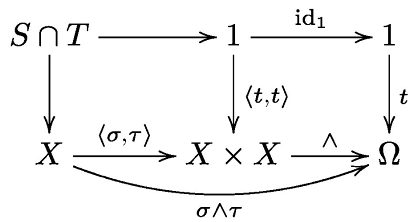
$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[\chi]{} & \Omega \end{array}$$

This is a diagram which explains what the subject classifier is. The axiom for the subobject classifier states that for any *X*, *A* and *i*, there is an unique arrow $\chi : X \to \Omega$ such that this diagram is a pullback diagram. It implies that the left side arrow *i*, which is in a sense "part of *X*," corresponds to the downside arrow χ in a unique way. Note that the codomain of the downside arrow is the truth-value object Ω, so we can see that in the category of sets it is a characteristic function for some set, actually, *A* in this diagram. In this way, χ classifies *A* in *X*. So χ is called *the classifying arrow* for *A*. Conversely, we can see that in some cases *A* is the set of the elements *x* of *X* such that χ is true of *x*. In this sense, *A* works as the extension of χ.

One can see that a logical connective is a classifying arrow. For example, a conjunction takes two truth values and gives one back. So a conjunction is a function from $\Omega \times \Omega$ to Ω. It is a classifying arrow which classifies the part $<t, t>$ of $\Omega \times \Omega$. (The product of objects exists because it is a special case of pullback.) The other connectives also follow the pattern of the relevant classifying arrow.

If there are arrows σ and τ both from an object *X* to Ω, each of them has its extension *S* and *T*, respectively.

By composing the arrow pair $<\sigma, \tau>$ and the conjunction arrow $\wedge$, we have the arrow $\sigma \wedge \tau$ from *X* to Ω.



This is again a classifying arrow. We can show that it classifies the intersection of *S* and *T*. Under the abbreviation $\sigma \circ i$ as σ^* and $t\circ!$ as "true", it follows that $(\sigma \wedge \tau)^*$ = true if and only if σ^* = true and τ^* = true. This is the familiar truth condition of a conjunctive sentence. Thus the concept of a topos is closely related to logic. Many studies, including Mac Lane and Moerdijk (1992), Goldblatt (1984/2006) and Bell (1988/2008), deal with the relation between topos and logic.

For any category **C**, including our *W*, there is a common way to construct the subobject classifier in **C**^. Let *C* be an object in **C**. A *cosieve* on *C* is a set of arrows whose codomain is *C* with the property that if *f* belongs to the set and *g* is an arrow composable with *f*, *f*∘*g* also belongs to the same set. In a word, a cosieve is closed under composition from the right. The set of *all* arrows whose codomain is *C* is called the *maximal cosieve* on *C* and written as max(*C*). The truth-value object Ω is defined as a presheaf on **C** which assigns to each object *C* in **C** the set of all cosieves on *C*. Furthermore, the truth arrow *t* from 1 to Ω is defined as a family $\{t_C\}_{C \in \mathbf{C}}$ of functions t_C from 1(*C*) to $\Omega(C)$. The terminal object 1 in **C**^ constantly assigns the singleton {*} to each object *C*. t_C assigns to * the maximal cosieve on *C*. We do not go into the proof that Ω and t thus defined constitute the subobject classifier in **C**^. Rather, we should work out the interpretation of sentences.

An atomic sentence is interpreted as a subobject of 1 in **C**^. If you want to interpret some atomic sentence as true, its interpretation is the identity arrow on 1. It is a special subobject of 1 that is classified by the truth arrow. In fact, a subobject of 1 in *W*^ corresponds to a downward-closed subset of *W*. For an atomic sentence *p*, we write [*p*] to denote a downward-closed subset of *W*. Then for a world *w* in *W*, that *p* is true in *w*, in symbols *w* |= *p*, is naturally defined as $w \in [p]$. The conditions for compound sentences, such as w |= $\sigma \wedge \tau$ if and only if *w* |= σ and *w* |= τ, follow from the general framework of topos.

Where is modality to be located? In the previous studies mentioned, we have seen that modality in syntax is related to topology in semantics. In topos theory, topology appears as an arrow *J* from Ω to Ω satisfying the following conditions:

i) $J \circ t = t$,
ii) $J \circ J = J$
iii) $J(\_ \wedge \_) = J(\_) \wedge J(\_)$

These conditions are parallel to the axioms for modal logics and one of their consequences. i) corresponds to the axiom T $\Box\sigma \rightarrow \sigma$, or dually, $\sigma \rightarrow \Diamond\sigma$. ii) corresponds to the axiom 4 $\Box\sigma \rightarrow \Box\Box\sigma$, or dually, $\Diamond\Diamond\sigma \rightarrow \Diamond\sigma$. In the system **KT4**, both $\Box\Box\sigma \equiv \Box\sigma$ and $\Diamond\Diamond\sigma \equiv \Diamond\sigma$ are provable. iii) may correspond to $\Box(\sigma \wedge \tau) \equiv (\Box\sigma \wedge \Box\tau)$ or $\Diamond(\sigma \wedge \tau) \equiv (\Diamond\sigma \wedge \Diamond\tau)$. The former is provable in the basic modal logic **K**. The left to right direction of the latter is also provable in **K**. The opposite direction will require a strong axiom, such as $\Diamond\sigma \rightarrow \Box\sigma$, though Bell (1988/2008:163) gives this modality the name "a possibility operator." The topology we are focusing on has some equivalent notions, one of which is a *Grothendieck topology* or a *covering sieve*. Recall that a sieve on some object *C* is the set of arrows with codomain C, and it is closed under composition from the right. The set Cov(C) must satisfy the following conditions:

i) max(C)$\in$Cov(C),

ii) If $R\in$Cov(C) then for every f: $C'\rightarrow C$, the inverse image $f^*(R)\in$Cov(C'),

iii) If R is a sieve on C and S is a covering sieve on C, such that for every arrow ($\in$S) f: $C'\rightarrow C$ we have $f^*(R)\in$Cov(C'), then $R\in$Cov(C).

The way the conditions are presented varies from textbook to textbook. We follow Moerdijk and Oosten (2007, p.22). i) states that the maximal cosieve on C is an element of Cov(C). ii) states that if R is a covering sieve, then so is its inverse image. iii) states that if R is a sieve and S is a covering sieve on C, such that the inverse image of R is a covering sieve of an object below C, then R is a covering sieve of C. The point is that Cov specifies how dense an object C is "covered" with arrows. The elements different from max(C) may not exist in Cov(C). Then it is the smallest constructions of Cov(C). On the contrary, under the largest construction, Cov(C) is equal to the set $\Omega(C)$ of all cosieves on C. It is known that Cov and a topology uniquely defines each other, so the difference involved in the construction of Cov(C) is related to the logical meaning of a topology. Recall that our W is preordered. It has a topology called a tree topology (Levy, 1979, p.201). In these arguments we are still working out the understanding about the tree topology on **KT4**-frame.

We saw a topos-theoretic approach for semantics for modal sentential logic above. However, it is not clear how to construct semantics for substructural logics on the same approach. One reason for that situation is that a topos-theoretic approach strongly takes effects from the properties of the category of sets. The conjunction defined above has familiar properties like $\sigma \wedge \sigma \equiv \sigma$ and $\sigma \wedge \tau \equiv \tau \wedge \sigma$. This implies that the conjunction is not resource-conscious. The character comes from the properties of set-intersection. A formal system defined in terms of arrows in a given topos is called *topos logic* (McLarty, p. 128). So the following proposition holds:

Proposition. Topos logic contains connectives which are not resource-conscious.

In fact, when topos logic is formalized as a sequent calculus, neither the inference rule for conjunction nor the one for disjunction is resource-conscious. In short, a topos has good potential to give semantics of modal logics based on the concept of a Kripke frame, but cannot be suitable for modeling substructural logics.

To get resource-conscious connectives, we may need to use a presheaf *of some kind of algebra*, not *of sets* as we have seen in this paper. Actually the notion of (pre-)sheaf has been widely used in algebraic geometry, and a sheaf of modules or of rings rather than a sheaf of sets is employed in order to obtain from algebraic structures a topological space relevant to it. We would like to examine that line on another occasion.

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