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Basis construction for the Shi and Catalan arrangements

(Shi 配置と Catalan 配置における基底の構成)

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Preface

A hyperplane arrangement is a finite set of hyperplanes in a finite dimensional vector space. For a Weyl group W, The Weyl arrangement is the set of all reflecting hyperplanes of reflections in W. In particular, the Weyl arrangement with respect to the Weyl group of the type A_ℓ is called the braid arrangement. The Shi arrangement is originally defined as an affine arrangement of hyperplanes consisting of the hyperplanes of the braid arrangement and their parallel translations. The Shi arrangement was introduced by J. Y. Shi in [11] in the study of the Kazhdan-Lusztig representation theory of the affine Weyl groups. One of the remarkable properties of the Shi arrangement is the fact that its number of chambers is equal to $(\ell+2)^\ell$. A good number of articles, including [5, 6, 8, 13, 21], study this intriguing property. Because of Zaslavsky's chamber counting formula [22], the property follows from the formula

$$\pi(\mathcal{S}(A_\ell), t) = (1+t)(1+(\ell+1)t)^\ell$$

for the Poincaré polynomial [9] of the cone over the Shi arrangement $\mathcal{S}(A_\ell)$. Ch. Athanasiadis proved that $D(\mathcal{S}(A_\ell))$ is a free S_z-module with exponents $(0, 1, \ell{+}1, \ldots, \ell{+}1)$ in [5]. He consequently proved the formula above thanks to the factorization theorem in [18] which asserts that if the logarithmic derivation module $D(\mathcal{A})$ is a free S-module with a basis $\theta_1, \ldots, \theta_\ell$ then the Poincaré polynomial of $\mathcal{A}$ is equal to $\prod_{i=1}^{\ell}(1 + (\deg \theta_i)t)$. His proof of the freeness in [5] uses the addition-deletion theorem [16, 17]. Later M. Yoshinaga extended this result in [21] to the extended Shi and Catalan arrangements and affirmatively settled the Edelman-Reiner conjecture [6] by using algebro-geometric method. However, even in the case of Shi arrangements, no basis was constructed until [14].

This doctoral thesis is based on [1, 14, 15]. In this thesis we construct bases for the logarithmic derivation modules of the cones over the Shi arrangements of the types A_ℓ, B_ℓ, C_ℓ, and the extended Shi and Catalan arrangements of the type A_2. For the type D_ℓ, an explicit basis formula for the Shi arrangement was constructed by R. Gao, D. Pei and H. Terao in [7]. In the construction for the Shi arrangements of the types A_ℓ, B_ℓ, C_ℓ, the most important ingredients of our recipe are the Bernoulli polynomial $B_k(x)$ and their relatives $B_{p,q}(x)$. In the construction for the Shi and Catalan arrangements of the type A_2, the simple-root basis [3] which is a special basis of the extended Shi arrangement and the multiarrangement theory play the important role. In particular, as for the multiarrangement theory, explicit bases for the restriction of the Shi and Catalan arrangements onto the infinite hyperplane are constructed by T. Abe, L. Solomon, H. Terao, and M. Yoshinaga [2, 12, 20].

The organization of this thesis is as follows: In chapter 1, we recall definitions of arrangement theory and define the extended Shi and Catalan arrangement. In chapter 2, we give an explicit construction of bases for the Shi arrangement of the type A_ℓ. In chapter 3, we give an explicit construction of bases for the Shi arrangement of the type B_ℓ. In chapter 4, we give an explicit construction of bases for the Shi arrangement of the type C_ℓ. In chapter 5, we give an explicit construction of bases for the extended Shi and Catalan arrangement of the type A_2.

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Contents

Chapter 1

Preliminaries

1.1 Arrangements

In this section we give some basic definitions of the theory of hyperplane arrangements.

Let $\mathbb{K}$ be a field and V an ℓ-dimensional vector space over $\mathbb{K}$.

Definition 1.1.1. *A **hyperplane** H in V is an $(\ell-1)$-dimensional affine subspace of V. A **hyperplane arrangement** $\mathcal{A}$ is a finite set of hyperplanes in V. We call $\mathcal{A}$ an ℓ-**arrangement** when we would like to emphasize the dimension of V. If each hyperplane H in $\mathcal{A}$ passes through the origin O_V, that is $O_V \in \cap_{H\in\mathcal{A}} H$, we call $\mathcal{A}$ **central**.*

Let $S = S(V^*)$ be the symmetric algebra of the dual space V^* and $\{x_1, \ldots, x_\ell\} \subset V^*$ a basis for V^*. S can be identified with a polynomial ring $\mathbb{K}[x_1, \ldots, x_\ell]$. Each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial α_H of degree 1 defined up to constant multiple.

Definition 1.1.2. *For a hyperplane $\mathcal{A}$, we define the **defining polynomial** $Q(\mathcal{A})$ of $\mathcal{A}$ by*

$$Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} \alpha_H.$$

We agree that if $\mathcal{A}$ is the empty arrangement, then the defining polynomial is $Q(\mathcal{A}) = 1$.

Definition 1.1.3. *Let U be an $(\ell+1)$-dimensional vector space containing V as an affine subspace $\{z = 1\}$ of U, where z is an element of the dual space U^*. Then we may regard $U^* = V^* \oplus \langle z\rangle = \langle x_1, \ldots, x_\ell, z\rangle$, and let S_z denote the symmetric algebra $S(U^*) = \mathbb{K}[x_1, \ldots, x_\ell, z]$ of the dual space U^*. Let H be a hyperplane in V. The **cone** $\mathbf{c}H$ over H is the hyperplane in U which* *passes through the origin O_U of U and H. Let $\mathcal{A}$ be an affine arrangement in V. Then the* ***cone*** $\mathbf{c}\mathcal{A}$ *over $\mathcal{A}$ is defined by*

$$\mathbf{c}\mathcal{A} = \{\{z = 0\}\} \cup \{\mathbf{c}H \mid H \in \mathcal{A}\}.$$

Since the cone $\mathbf{c}H$ is the kernel of homogenization $z\alpha_H(x_1/z, \ldots, x_\ell/z)$ of $\alpha_H(x_1, \ldots, x_\ell)$, the defining polynomial of the cone $\mathbf{c}\mathcal{A}$ is

$$Q(\mathbf{c}\mathcal{A}) = z \cdot z^{\deg Q(\mathcal{A})} Q(\mathcal{A})(\frac{x_1}{z}, \ldots, \frac{x_\ell}{z}).$$

Note that the cone $\mathbf{c}\mathcal{A}$ is a central arrangement for any affine arrangement $\mathcal{A}$.

Definition 1.1.4. *The S-module*

$$\begin{aligned}\mathrm{Der}(S) = \{\theta : S \to S \mid \theta \textit{ is } \mathbb{K}\textit{-linear}, \\ \theta(fg) = \theta(f)g + f\theta(g) \textit{ for any } f, g \in S\}\end{aligned}$$

is called the ***derivation module*** *of S over $\mathbb{K}$. We call an element of* $\mathrm{Der}(S)$ *a* ***derivation****. It is well known that*

$$\mathrm{Der}(S) = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_\ell} \right\rangle_S.$$

Definition 1.1.5. *Let $\mathcal{A}$ be a central arrangement in V. Then the* ***logarithmic derivation module*** *$D(A)$ of $\mathcal{A}$ is defined by*

$$\begin{aligned} D(\mathcal{A}) &= \{\theta \in \mathrm{Der}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\} \\ &= \{\theta \in \mathrm{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S \textit{ for any } H \in \mathcal{A}\}.\end{aligned}$$

We call an element of $D(\mathcal{A})$ a ***logarithmic derivation****.*

Definition 1.1.6. *The* ***Euler derivation*** *$\theta_E \in \mathrm{Der}(S)$ is defined by*

$$\theta_E = \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i}.$$

It is easy to see that $\theta_E \in D(\mathcal{A})$ for any arrangement $\mathcal{A}$.

Definition 1.1.7. *A central arrangement $\mathcal{A}$ is called a* ***free arrangement*** *if the logarithmic derivation module $D(\mathcal{A})$ is a free S-module.*

If $\mathcal{A}$ is a free arrangement, then there exists a homogeneous basis $\{\theta_1, \ldots, \theta_\ell\}$ of $D(\mathcal{A})$, and the multiset of degrees of $\{\theta_1, \ldots, \theta_\ell\}$ is uniquely determined independent of the choice of a homogeneous basis. We call the multiset the **exponents** of $\mathcal{A}$ and write $\exp \mathcal{A} = (\deg \theta_1, \ldots, \deg \theta_\ell)$.

Let $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$. There is a very useful criterion for checking whether $\theta_1, \ldots, \theta_\ell$ form a basis for $D(\mathcal{A})$.

Theorem 1.1.8. *(Saito's criterion [10]) Let $\theta_1, \ldots, \theta_\ell$ be homogeneous logarithmic derivations of $\mathcal{A}$. Then the following two conditions are equivalent:*

(1) $\det \mathrm{M}(\theta_1, \ldots, \theta_\ell) \doteq Q(\mathcal{A})$,

(2) $\theta_1, \ldots, \theta_\ell$ form a basis for $D(\mathcal{A})$,

(3) $\theta_1, \ldots, \theta_\ell$ are linearly independent over S and $\sum_{i=1}^{\ell} \deg \theta_i = |\mathcal{A}|$,

where $\mathrm{M}(\theta_1, \ldots, \theta_\ell)$ is the coefficient matrix

$$\mathrm{M}(\theta_1, \ldots, \theta_\ell) = \begin{bmatrix} \theta_1(x_1) & \cdots & \theta_\ell(x_1) \\ \vdots & \ddots & \vdots \\ \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell) \end{bmatrix},$$

and the notation $f \doteq g$ $(f, g \in S)$ expresses that $f = cg$ for some $c \in \mathbb{K}^*$.

Let $\mathcal{A}$ be an affine arrangement. For the cone $\mathbf{c}\mathcal{A}$ over $\mathcal{A}$, we define the S_z-module $D_0(\mathbf{c}\mathcal{A})$ by

$$D_0(\mathbf{c}\mathcal{A}) = \{\theta \in D(\mathbf{c}\mathcal{A}) \mid \theta(z) = 0\}.$$

Proposition 1.1.9. *The logarithmic derivation module $D(\mathbf{c}\mathcal{A})$ can be decomposed as a direct sum of S_z-modules as follows:*

$$D(\mathbf{c}\mathcal{A}) = S_z\theta_E \oplus D_0(\mathbf{c}\mathcal{A}),$$

where

$$\theta_E = z\frac{\partial}{\partial z} + \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i}$$

is the Euler derivation.

Proof. Let $\theta \in D(\mathbf{c}\mathcal{A})$. By definition of the cone, we can write $\theta(z) = fz$ for some $f \in S_z$. Here we express $\theta = f\theta_E + (\theta - f\theta_E)$, then $\theta - f\theta_E \in D_0(\mathbf{c}\mathcal{A})$. Hence $D(\mathbf{c}\mathcal{A}) = S_z\theta_E + D_0(\mathbf{c}\mathcal{A})$. Let $\theta \in S_z\theta_E \cap D_0(\mathbf{c}\mathcal{A})$. If $\theta = g\theta_E$ for some $g \in S_z$, then $0 = \theta(z) = gz$, hence $g = 0$. Therefore $D(\mathbf{c}\mathcal{A}) = S_z\theta_E \oplus D_0(\mathbf{c}\mathcal{A})$. $\square$

Hence $\mathbf{c}\mathcal{A}$ is free if and only if $D_0(\mathbf{c}\mathcal{A})$ is a free S_z-module, and $\theta_1, \ldots, \theta_\ell$ form a basis for $D_0(\mathbf{c}\mathcal{A})$ if and only if $\theta_E, \theta_1, \ldots, \theta_\ell$ form a basis for $D(\mathbf{c}\mathcal{A})$. Thus in order to construct a basis for $D(\mathbf{c}\mathcal{A})$, it is sufficient to construct a basis for $D_0(\mathbf{c}\mathcal{A})$.

1.2 The extended Shi and Catalan arrangements

In this section we introducce the extended Shi and Catalan arrangements. Then we recall a result of a freeness for the cones over the extended Shi and Catalan arrangements obtained by Yoshinaga.

Let E be an ℓ-dimensional Euclidean space over $\mathbb{R}$. Let Φ be a crystallographic irreducible root system in the dual space E^* and Φ^+ a positive system of Φ. For $\alpha \in \Phi^+$ and $i \in \mathbb{Z}$, define the affine hyperplane $H_{\alpha,i}$ by

$$H_{\alpha,i} = \{v \in V \mid \alpha(v) = i\}.$$

Definition 1.2.1. *The arrangement $\mathcal{A}(\Phi) = \{H_{\alpha,0} \mid \alpha \in \Phi^+\}$ is called the* ***Weyl arrangement*** *of the type Φ.*

Definition 1.2.2. *Let $k \in \mathbb{Z}_{\geq 0}$. Then the* ***extended Shi arrangement*** Shi^k *of the type Φ and the* ***extended Catalan arrangment*** Cat^k *of the type Φ are affine arrangements defined by*

$$\mathrm{Shi}^k = \{H_{\alpha,i} \mid \alpha \in \Phi^+, -k+1 \leq i \leq k\},$$
$$\mathrm{Cat}^k = \{H_{\alpha,i} \mid \alpha \in \Phi^+, -k \leq i \leq k\}.$$

In particular, the arrangement Shi^1 is called **Shi arrangement** which was introduced by J. Y. Shi in [11] in the study of the Kazhdan-Lusztig representation theory of the affine Weyl groups. Yoshinaga [21] proved the freeness of the cones over the extended Shi and Catalan arrangements and affirmatively settled the Edelman-Reiner conjecture [6].

Theorem 1.2.3. *(M. Yoshinaga [21]) Let $k \in Z_{\geq 0}$. Then*

(1) the cone over the extended Shi arrangement $\mathbf{c}\mathrm{Shi}^k$ is free with

$$\exp(\mathbf{c}\mathrm{Shi}^k) = (1, kh, kh, \ldots, kh),$$

(2) the cone over the extended Catalan arrangement $\mathbf{c}\mathrm{Cat}^k$ is free with

$$\exp(\mathbf{c}\mathrm{Cat}^k) = (1, e_1 + kh, e_2 + kh, \ldots, e_\ell + kh),$$

where h is the Coxeter number of Φ and $e_1, \ldots, e_\ell$ are the exponents of Φ.

Chapter 2

The Shi arrangements of the type A_ℓ

In this chapter, we construct a basis for the logarithmic derivation module of the cone over the Shi arrangement of the type A_ℓ. This chapter is based on [14].

2.1 Notations

Let E be an ℓ-dimensional Euclidean space and Φ_A be the root system of the type A_ℓ. Let Φ_A^+ denote the set of positive roots. In this chapter we explicitly choose E and Φ_A as follows: let $V = \mathbb{R}^{\ell+1}$ and $x_1, \dots, x_{\ell+1}$ be an orthonormal basis for the dual space V^*. Define

$$
\begin{aligned}
E &:= \left\{ \sum_{i=1}^{\ell+1} c_i x_i \in V^* \;\middle|\; \sum_{i=1}^{\ell+1} c_i = 0 \right\}, \\
\Phi_A &:= \{x_i - x_j \in E \mid 1 \le i \le \ell+1, 1 \le j \le \ell+1, i \ne j\}, \\
\Phi_A^+ &:= \{x_i - x_j \in \Phi \mid i < j\}.
\end{aligned}
$$

Then $\mathcal{A}(\Phi_A)$ is called a **braid arrangement**, which is undoubtedly the most-studied arrangement of hyperplanes in various contexts. The Shi arrangement of the type A_ℓ is given by

$$\mathcal{A}(\Phi_A) \cup \{H_{\alpha,1} \mid \alpha \in \Phi_+\} = \bigcup_{\substack{1 \le i \le \ell+1 \\ 1 \le j \le \ell+1}} \{\{x_i - x_j = 0\}, \{x_i - x_j = 1\}\}.$$

Let $\mathcal{S}(A_\ell)$ denote the cone over the Shi arrangement $\mathbf{c}\mathrm{Shi}^1$ of the type A_ℓ. It is a central arrangement defined by

$$Q(\mathcal{S}(A_\ell)) = z \prod_{1\le p<q\le \ell+1} (x_p - x_q) \prod_{1\le p<q\le \ell+1} (x_p - x_q - z) = 0.$$

It follows from Theorem 1.2.3 that $\mathcal{S}(A_\ell)$ is free with

$$\exp(\mathcal{S}(A_\ell)) = (0, 1, \ell+1, \dots, \ell+1).$$

Here there appears 0 in $\exp(\mathcal{S}(A_\ell))$ because the Weyl group W of the type A_ℓ is not essential for $V = \mathbb{R}^{\ell+1}$.

The organization of this chapter is as follows: In Section 2.2, we will define the polynomials $B_{p,q}(x)$ which includes the Bernoulli polynomials. In Section 2.3, Theorem 2.3.5 proves that the derivations constructed in Definition 2.3.1 form a basis for the derivation module $D(\mathcal{S}(A_\ell))$.

2.2 The Bernoulli polynomials and $B^A_{p,q}(x)$

Let $B^A_k(x)$ denote the k-th **Bernoulli polynomial**. Let $B^A_k(0) = B_k$ denote the k-th **Bernoulli number**. The most important property of the Bernoulli polynomial in this paper is the following elementary formula (e.g., [4]):

Theorem 2.2.1.

$$B^A_k(x+1) - B^A_k(x) = kx^{k-1}.$$

Definition 2.2.2. *For $(p,q) \in (\mathbb{Z}_{\ge 0})^2$, consider a polynomial $B^A_{p,q}(x)$ in x satisfying the following two conditions:*

(1) $B^A_{p,q}(x+1) - B^A_{p,q}(x) = (x+1)^p x^q$,

(2) $B^A_{p,q}(0) = 0$.

It is easy to see that $B^A_{p,q}(x)$ is uniquely determined by these two conditions.

Example 2.2.3. *(1) When $(p,q) = (0,q)$, we have*

$$B^A_{0,q}(x) = \frac{1}{q+1}\left\{B^A_{q+1}(x) - B^A_{q+1}\right\}$$

because of Theorem 2.2.1.

(2) When $(p,q) = (p,0)$, we obtain

$$B^A_{p,0}(x) = \frac{(-1)^{p+1}}{p+1}\left\{B^A_{p+1}(-x) - B^A_{p+1}\right\} = (-1)^{p+1} B^A_{0,p}(-x)$$

because

$$\begin{aligned}&(-1)^{p+1}B^A_{0,p}(-x-1)-(-1)^{p+1}B^A_{0,p}(-x)\\ =&(-1)^p\left\{B^A_{0,p}(-x)-B^A_{0,p}(-x-1)\right\}=(-1)^p(-x-1)^p=(x+1)^p.\end{aligned}$$

(3) For a general $(p,q)\in(\mathbb{Z}_{\geq 0})^2$, it easily follows from Theorem 2.2.1 that the polynomial has an expression in terms of the Bernoulli polynomials as

$$B^A_{p,q}(x)=\sum_{i=0}^{p}\frac{1}{q+i+1}\binom{p}{i}\left\{B^A_{q+i+1}(x)-B^A_{q+i+1}\right\}=\sum_{i=0}^{p}\binom{p}{i}B^A_{0,q+i}(x).$$

For example, $B^A_{1,1}(x)=B^A_{0,1}(x)+B^A_{0,2}(x)=\frac{1}{3}(x^3-x)$.

Note that the polynomial $B^A_{p,q}(x)$ is a polynomial of degree $p+q+1$. The homogenization $\overline{B}^A_{p,q}(x,z)$ of $B^A_{p,q}(x)$ is defined by

$$\overline{B}^A_{p,q}(x,z):=z^{p+q+1}B^A_{p,q}\left(\frac{x}{z}\right).$$

2.3 A basis construction

Let $1\leq j\leq \ell$. Define

$$I_1=\{x_1,x_2,\ldots,x_{j-1}\},\ \ I_2=\{x_{j+2},x_{j+3},\ldots,x_{\ell+1}\}.$$

Let $\sigma^{(s)}_k$ denote the elementary symmetric function in the variables in I_s of degree k ($s=1,2$, $k\in\mathbb{Z}_{\geq 0}$). Recall the homogeneous polynomials $\overline{B}^A_{p,q}(x,z)$ of degree $p+q+1$ defined at the end of the previous section.

Definition 2.3.1. *Let ∂_i $(1\leq i\leq \ell+1)$ denote $\partial/\partial x_i$. Define homogeneous derivations*

$$\eta:=\sum_{i=1}^{\ell+1}\partial_i\in D_0(\mathcal{S}(A_\ell)),$$

and

$$\varphi^A_j:=(x_j-x_{j+1}-z)\sum_{i=1}^{\ell+1}\sum_{\substack{0\leq k_1\leq j-1\\ 0\leq k_2\leq \ell-j}}(-1)^{k_1+k_2}\sigma^{(1)}_{j-1-k_1}\sigma^{(2)}_{\ell-j-k_2}\overline{B}^A_{k_1,k_2}(x_i,z)\partial_i$$

for $1\leq j\leq \ell$.

We will prove that the derivations $\eta, \varphi_1^A, \ldots, \varphi_\ell^A$ form a basis for $D_0(\mathcal{S}(A_\ell))$. First we will verify the following Proposition:

Proposition 2.3.2. *The derivations φ_j^A $(1 \leq j \leq \ell)$ belong to the module $D_0(\mathcal{S}(A_\ell))$.*

Proof. We first have

$$\varphi_j^A(x_p - x_q) = (x_j - x_{j+1} - z) \sum_{\substack{0 \leq k_1 \leq j-1 \\ 0 \leq k_2 \leq \ell - j}} (-1)^{k_1+k_2} \sigma^{(1)}_{j-1-k_1} \sigma^{(2)}_{\ell-j-k_2} \left\{ \overline{B}^A_{k_1,k_2}(x_p, z) - \overline{B}^A_{k_1,k_2}(x_q, z) \right\}.$$

Since the right hand side equals zero if we set $x_p = x_q$, we may conclude that $\varphi_j^A(x_p - x_q)$ is divisible by $x_p - x_q$ for all pairs (p, q) with $1 \leq p < q \leq \ell + 1$.

The congruent notation $\equiv$ in the following calculation is modulo the ideal $(x_p - x_q - z)$:

$$\begin{aligned}
&\varphi_j^A(x_p - x_q - z) \\
&\equiv (x_j - x_{j+1} - z) \\
&\sum_{\substack{0 \leq k_1 \leq j-1 \\ 0 \leq k_2 \leq \ell - j}} (-1)^{k_1+k_2} \sigma^{(1)}_{j-1-k_1} \sigma^{(2)}_{\ell-j-k_2} \left\{ \overline{B}^A_{k_1,k_2}(x_p, x_p - x_q) - \overline{B}^A_{k_1,k_2}(x_q, x_p - x_q) \right\} \\
&= (x_j - x_{j+1} - z) \sum_{\substack{0 \leq k_1 \leq j-1 \\ 0 \leq k_2 \leq \ell - j}} (-1)^{k_1+k_2} \sigma^{(1)}_{j-1-k_1} \sigma^{(2)}_{\ell-j-k_2} \\
&\qquad (x_p - x_q)^{k_1+k_2+1} \{ B^A_{k_1,k_2}(\frac{x_p}{x_p - x_q}) - B^A_{k_1,k_2}(\frac{x_q}{x_p - x_q}) \} \\
&= (x_j - x_{j+1} - z) \\
&\quad \sum_{\substack{0 \leq k_1 \leq j-1 \\ 0 \leq k_2 \leq \ell - j}} (-1)^{k_1+k_2} \sigma^{(1)}_{j-1-k_1} \sigma^{(2)}_{\ell-j-k_2} (x_p - x_q)^{k_1+k_2+1} (\frac{x_p}{x_p - x_q})^{k_1} (\frac{x_q}{x_p - x_q})^{k_2} \\
&= (x_j - x_{j+1} - z)(x_p - x_q) \sum_{\substack{0 \leq k_1 \leq j-1 \\ 0 \leq k_2 \leq \ell - j}} (-1)^{k_1+k_2} \sigma^{(1)}_{j-1-k_1} \sigma^{(2)}_{\ell-j-k_2} x_p^{k_1} x_q^{k_2} \\
&= (x_j - x_{j+1} - z)(x_p - x_q) \sum_{k_1=0}^{j-1} \sigma^{(1)}_{j-1-k_1} (-x_p)^{k_1} \sum_{k_2=0}^{\ell-j} \sigma^{(2)}_{\ell-j-k_2} (-x_q)^{k_2} \\
&= (x_j - x_{j+1} - z)(x_p - x_q) \prod_{s=1}^{j-1} (x_s - x_p) \prod_{s=j+2}^{\ell+1} (x_s - x_q) \equiv 0
\end{aligned}$$

for all pairs (p, q) with $1 \leq p < q \leq \ell + 1$. $\square$

Lemma 2.3.3. *Suppose $\ell \geq 1$. Let N be the $\ell \times \ell$-matrix whose (i,j)-entry is equal to the elementary symmetric function of degree $\ell - i$ in the variables $x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{\ell+1}$. Then*

$$\det N = (-1)^{\ell(\ell-1)/2} \prod_{\substack{1 \leq p < q \leq \ell \\ q-p>1}} (x_p - x_q).$$

Proof. Note that we have the equality

$$\begin{aligned} &\begin{bmatrix} 1 & -x_p & (-x_p)^2 \dots (-x_p)^{\ell-2} & (-x_p)^{\ell-1} \end{bmatrix} N \\ = &\begin{bmatrix} \prod_{\substack{1 \leq s \leq \ell+1 \\ s \notin \{1,2\}}} (x_s - x_p) & \prod_{\substack{1 \leq s \leq \ell+1 \\ s \notin \{2,3\}}} (x_s - x_p) & \dots & \prod_{\substack{1 \leq s \leq \ell+1 \\ s \notin \{\ell,\ell+1\}}} (x_s - x_p) \end{bmatrix} \end{aligned}$$

for any $1 \leq p \leq \ell$. Suppose that

$$1 \leq p < q \leq \ell + 1, \ \ q - p > 1.$$

Set $x_p = x_q$ in N, and we get N_{pq}. Then we may conclude that

$$\begin{bmatrix} 1 & -x_p & (-x_p)^2 \dots (-x_p)^{\ell-2} & (-x_p)^{\ell-1} \end{bmatrix} N_{pq} = \mathbf{0}.$$

This implies that $\det N_{pq} = 0$ and that $\det N$ is divisible by $x_p - x_q$. Since

$$\deg(\det N) = \ell(\ell-1)/2 = \deg \prod_{\substack{1 \leq p < q \leq \ell+1 \\ q-p>1}} (x_p - x_q),$$

there exists a constant C such that

$$\det N = C\,(-1)^{\ell(\ell-1)/2} \prod_{\substack{1 \leq p < q \leq \ell+1 \\ q-p>1}} (x_p - x_q) = C \prod_{\substack{1 \leq p < q \leq \ell+1 \\ q-p>1}} (x_q - x_p).$$

By comparing the coefficients of $x_3 x_4^2 \dots x_\ell^{\ell-2} x_{\ell+1}^{\ell-1}$ on both sides, we obtain $C = 1$. $\square$

Proposition 2.3.4. *The derivations $\eta, \varphi_1^A, \dots, \varphi_\ell^A$ are linearly independent over S_z.*

Proof. Set $z = 0$ in φ_j^A and we get ϕ_j as follows:

$$\phi_j := \varphi_j^A|_{z=0} = (x_j - x_{j+1}) \sum_{i=1}^{\ell+1} \sum_{\substack{0 \le k_1 \le j-1 \\ 0 \le k_2 \le \ell-j}} \frac{(-1)^{k_1+k_2}}{k_1+k_2+1} \sigma^{(1)}_{j-1-k_1} \sigma^{(2)}_{\ell-j-k_2}\, x_i^{k_1+k_2+1} \partial_i$$

$$= (x_j - x_{j+1}) \sum_{k=1}^{\ell} \frac{(-1)^{k-1}}{k} \left(\sum_{\substack{k_1+k_2+1=k \\ 0 \le k_1 \le j-1 \\ 0 \le k_2 \le \ell-j}} \sigma^{(1)}_{j-1-k_1} \sigma^{(2)}_{\ell-j-k_2} \right) \sum_{i=1}^{\ell+1} x_i^k \partial_i$$

$$= (x_j - x_{j+1}) \sum_{k=1}^{\ell} \frac{(-1)^{k-1}}{k}\, \sigma_{\ell-k}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{\ell+1}) \sum_{i=1}^{\ell+1} x_i^k \partial_i.$$

Here $\sigma_{\ell-i}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{\ell+1})$ stands for the elementary symmetric function of degree $\ell - i$ in the variables $x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{\ell+1}$. This is equal to the (i, j)-entry N_{ij} of the matrix N in Lemma 2.3.3. Thus we have

$$\phi_j(x_i) = (x_j - x_{j+1}) \sum_{k=1}^{\ell} \frac{(-1)^{k-1}}{k}\, x_i^k N_{kj}. \tag{2.1}$$

Define two $(\ell+1) \times (\ell+1)$-diagonal matrices D_1 and D_2 by

$$D_1 := [1] \oplus [1] \oplus \left[(-1)^1/2\right] \oplus \left[(-1)^2/3\right] \oplus \cdots \oplus [(-1)^{\ell-1}/\ell],$$
$$D_2 := [1] \oplus [x_1 - x_2] \oplus [x_2 - x_3] \oplus \cdots \oplus [x_\ell - x_{\ell+1}],$$

where $\oplus$ stands for the direct sum of matrices. Also define two $(\ell+1)\times(\ell+1)$-matrices $\tilde{N}$ and M by

$$\tilde{N} := [1] \oplus N, \quad M := \left[x_i^{j-1}\right]_{1 \le i \le \ell+1, 1 \le j \le \ell+1}.$$

From (2.1) we obtain

$$P := \begin{bmatrix} 1 & \phi_1(x_1) & \ldots & \phi_\ell(x_1) \\ 1 & \phi_1(x_2) & \ldots & \phi_\ell(x_2) \\ 1 & \phi_1(x_3) & \ldots & \phi_\ell(x_3) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 1 & \phi_1(x_{\ell+1}) & \ldots & \phi_\ell(x_{\ell+1}) \end{bmatrix} = M D_1 \tilde{N} D_2.$$

Thus, by applying the Vandermonde determinant formula and Lemma 2.3.3, we deduce

$$\begin{aligned}\det P &= (\det M)(\det D_1)(\det \tilde{N})(\det D_2)\\ &= \left(\prod_{1\le p<q\le \ell+1}(x_q-x_p)\right)\left(\frac{(-1)^{\ell(\ell-1)/2}}{\ell!}\right)(\det N)\prod_{1\le p\le \ell}(x_p-x_{p+1})\\ &= \left(\frac{(-1)^{\ell(\ell+1)/2}}{\ell!}\right)\prod_{1\le p<q\le \ell+1}(x_p-x_q)^2\neq 0.\end{aligned}$$

Thus $\eta, \phi_1, \dots, \phi_\ell$ are linearly independent. This implies that $\eta, \varphi_1^A, \dots, \varphi_\ell^A$ are linearly independent. □

Remark. The derivations $\phi_1, \dots, \phi_\ell$ are a basis for the derivation module of the double Coxeter arrangement of the type A_ℓ studied in [12] (cf. [19]).

Theorem 2.3.5. *The derivations $\eta, \varphi_1^A, \dots, \varphi_\ell^A$ form a basis for $D_0(\mathcal{S}(A_\ell))$.*

Proof. We may apply Theorem 1.1.8 (Saito's criterion) thanks to Propositions 2.3.2 and 2.3.4 because

$$\deg \eta + \sum_{j=1}^{\ell} \deg \varphi_j^A = \ell(\ell+1) = |\mathcal{S}(A_\ell)| - 1.$$

□

Remark. The Bernoulli polynomials explicitly appear in the first derivation φ_1^A and the last one φ_ℓ^A because of Example 2.2.3 (1) and (2):

$$\begin{aligned}\varphi_1^A &= (x_1-x_2-z)\sum_{i=1}^{\ell+1}\sum_{k_2=0}^{\ell-1}(-1)^{k_2}\sigma^{(2)}_{\ell-1-k_2}\,\overline{B}^A_{0,k_2}(x_i,z)\partial_i\\ &= (x_1-x_2-z)\sum_{i=1}^{\ell+1}\sum_{k=1}^{\ell}\frac{(-1)^{k-1}}{k}\sigma^{(2)}_{\ell-k}z^k\left(B_k^A(x_i/z)-B_k\right)\partial_i,\end{aligned}$$

and

$$\begin{aligned}\varphi_\ell^A &= (x_\ell-x_{\ell+1}-z)\sum_{i=1}^{\ell+1}\sum_{k_1=0}^{\ell-1}(-1)^{k_1}\sigma^{(1)}_{\ell-1-k_1}\,\overline{B}^A_{k_1,0}(x_i,z)\partial_i\\ &= (x_\ell-x_{\ell+1}-z)\sum_{i=1}^{\ell+1}\sum_{k=1}^{\ell}\frac{(-1)^{k-1}}{k}\sigma^{(1)}_{\ell-k}(-z)^k\left(B_k^A(-x_i/z)-B_k\right)\partial_i.\end{aligned}$$

Here $\sigma_d^{(1)}$ and $\sigma_d^{(2)}$ are the elementary symmetric functions of degree d in the variables $x_1, \dots, x_{\ell-1}$ and $x_3, \dots, x_{\ell+1}$ respectively.

Example 2.3.6. *For A_3, we have*

$$
\begin{aligned}
\eta &= \partial_1 + \partial_2 + \partial_3 + \partial_4,\\
\varphi_1^A &= x_1(x_1 - x_2 - z)\left\{x_3x_4 - \frac{1}{2}(x_3 + x_4)(x_1 - z) + \frac{1}{3}\left(x_1^2 - \frac{3}{2}x_1z + \frac{1}{2}z^2\right)\right\}\partial_1\\
&\quad + x_2(x_1 - x_2 - z)\left\{x_3x_4 - \frac{1}{2}(x_3 + x_4)(x_2 - z) + \frac{1}{3}\left(x_2^2 - \frac{3}{2}x_2z + \frac{1}{2}z^2\right)\right\}\partial_2\\
&\quad - \frac{1}{6}x_3(x_1 - x_2 - z)(x_3 + z)(x_3 - 3x_4 - z)\partial_3\\
&\quad - \frac{1}{6}x_4(x_1 - x_2 - z)(x_4 + z)(x_4 - 3x_3 - z)\partial_4,\\
\varphi_2^A &= -\frac{1}{6}x_1(x_2 - x_3 - z)(x_1 - z)(x_1 - 3x_4 - 2z)\partial_1\\
&\quad + x_2(x_2 - x_3 - z)\left\{x_1x_4 - \frac{1}{2}x_1(x_2 - z) - \frac{1}{2}x_4(x_2 + z) + \frac{1}{3}(x_2^2 - z^2)\right\}\partial_2\\
&\quad + x_3(x_2 - x_3 - z)\left\{x_1x_4 - \frac{1}{2}x_1(x_3 - z) - \frac{1}{2}x_4(x_3 + z) + \frac{1}{3}(x_3^2 - z^2)\right\}\partial_3\\
&\quad + \frac{1}{6}x_4(x_2 - x_3 - z)(x_4 + z)(3x_1 - x_4 - 2z)\partial_4,\\
\varphi_3^A &= -\frac{1}{6}x_1(x_3 - x_4 - z)(x_1 - z)(x_1 - 3x_2 + z)\partial_1\\
&\quad - \frac{1}{6}x_2(x_3 - x_4 - z)(x_2 - z)(x_2 - 3x_1 + z)\partial_2\\
&\quad + x_3(x_3 - x_4 - z)\left\{x_1x_2 - \frac{1}{2}(x_1 + x_2)(x_3 + z) + \frac{1}{3}\left(x_3^2 + \frac{3}{2}x_3z + \frac{1}{2}z^2\right)\right\}\partial_3\\
&\quad + x_4(x_3 - x_4 - z)\left\{x_1x_2 - \frac{1}{2}(x_1 + x_2)(x_4 + z) + \frac{1}{3}\left(x_4^2 + \frac{3}{2}x_4z + \frac{1}{2}z^2\right)\right\}\partial_4.
\end{aligned}
$$

Chapter 3

The Shi arrangements of the type B_ℓ

In this chapter, we construct a basis for the logarithmic derivation module of the cone over the Shi arrangement of the type B_ℓ. This chapter is based on [15].

3.1 Notations

Let E be an ℓ-dimensional Euclidean space. Let $x_1, \dots, x_\ell$ be an orthonormal basis for the dual space E^*. In this chapter we explicitly choose root systems Φ_B and positive root system Φ_B^+ of the type B_ℓ as follows:

$$\Phi_B := \{\pm x_i, \pm x_p \pm x_q \in E^* \mid 1 \le i \le \ell, 1 \le p < q \le \ell\},$$
$$\Phi_B^+ := \{x_i, x_p \pm x_q \in \Phi_B \mid 1 \le i \le \ell, 1 \le p < q \le \ell\}.$$

We express the cones over the Shi arrangements Shi[1] of the type B_ℓ by $\mathcal{S}(B_\ell)$. Then the defining polynomial of $\mathcal{S}(B_\ell)$ is

$$Q(\mathcal{S}(B_\ell)) = z \prod_{i=1}^{\ell} x_i(x_i - z) \prod_{1 \le p < q \le \ell} \{(x_p + x_q)(x_p - x_q)$$
$$(x_p + x_q - z)(x_p - x_q - z)\}$$

It follows from Yoshinaga's Theorem 1.2.3 that $\mathcal{S}(B_\ell)$ is free with

$$\exp(\mathcal{S}(B_\ell)) = (1, 2\ell, 2\ell, \dots, 2\ell).$$

The organization of this chapter is as follows: In Section 3.2, we will construct ℓ derivations $\varphi_1^B, \dots, \varphi_\ell^B$ belonging to $D_0(\mathcal{S}(B_\ell))$. In Section 3.3, we will prove that they form a basis of $D_0(\mathcal{S}(B_\ell))$.

3.2 A basis construction for the type B_ℓ

Definition 3.2.1. *For $(r,s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$, define a polynomial $B^B_{r,s}(x)$ in x satisfying the following two conditions:*

(i) $B^B_{r,s}(x+1) - B^B_{r,s}(x) = \dfrac{(x+1)^r - (-x)^r}{(x+1)-(-x)}(x+1)^s(-x)^s,$

(ii) $B^B_{r,s}(0) = 0.$

Note that $\frac{(x+1)^r-(-x)^r}{(x+1)-(-x)}$ is a polynomial either of degree $r-1$ (when r is odd) or of degree $r-2$ (when r is even). It is thus easy to see that $B^B_{r,s}(x)$ uniquely exists and

$$\deg B^B_{r,s}(x) = \begin{cases} r+2s & \text{if } r \text{ is odd,} \\ r+2s-1 & \text{if } r \text{ is even.} \end{cases}$$

Lemma 3.2.2. *$B^B_{r,s}(x)$ is an odd function.*

Proof. Replacing x with $-x-1$ in 3.2.1 (i), we have

$$\begin{aligned} B^B_{r,s}(-x) - B^B_{r,s}(-x-1) &= \frac{(-x)^r - (x+1)^r}{(-x)-(x+1)}(-x)^s(x+1)^s \\ &= B^B_{r,s}(x+1) - B^B_{r,s}(x). \end{aligned}$$

Then we get $F(x) = F(x+1)$ where $F(x) := B^B_{r,s}(x) + B^B_{r,s}(-x)$. Thus we obtain

$$F(n) = F(n-1) = \cdots = F(0) = 0 \ \ (n \in \mathbb{Z}_{\geq 0})$$

and

$$B^B_{r,s}(x) + B^B_{r,s}(-x) = F(x) = 0.$$

□

Definition 3.2.3. *The homogenization $\overline{B}^B_{r,s}(x,z)$ of $B^B_{r,s}(x)$ is defined by*

$$\overline{B}^B_{r,s}(x,z) := z^{r+2s} B^B_{r,s}(x/z).$$

Let $1 \leq j \leq \ell$. Define

$$I^{(j)}_1 = \{x_1, \ldots, x_{j-1}\}, \ \ I^{(j)}_2 = \{x_j\}, \ \ I^{(j)}_3 = \{x_{j+1}, \ldots, x_\ell\}$$

Let $\sigma_k(y_1, y_2, \ldots)$ $(k \in \mathbb{Z}_{\geq 0})$ denote the elementary symmetric polynomials in $y_1, y_2, \ldots$ of degree k. Then define

$$\sigma^{(2,j)}_k := \sigma_k(x_j), \ \ \tau^{(3,j)}_k := \sigma_k(x^2_{j+1}, \ldots, x^2_\ell).$$

The following construction of φ^B_j is inspired by the basis of the type A_ℓ in Chapter 2. The definition of φ^B_j is a suitable variation of φ^A_j which is defined in Chapter 2 for the type A_ℓ.

Definition 3.2.4. *Let ∂_i $(1 \le i \le \ell)$ and ∂_z denote $\partial/\partial x_i$ and $\partial/\partial z$ respectively. Define the following homogeneous derivations*

$$\varphi_j^B := (-1)^j \sum_{i=1}^{\ell} \Bigg\{ \sum_{\substack{N_1, N_2 \subset I_1^{(j)} \\ N_1 \cap N_2 = \emptyset}} \Big(\prod_{x_t \in N_1} x_t^2 \Big) \Big(\prod_{x_t \in N_2} (-x_t z) \Big)$$

$$\sum_{\substack{0 \le k_2 \le 1 \\ 0 \le k_3 \le \ell - j}} (-1)^{k_2+k_3} \sigma_{k_2}^{(2,j)} \tau_{k_3}^{(3,j)} \, \overline{B}_{r,s}^B(x_i, z) \Bigg\} \partial_i,$$

where

$$r := 2\ell - 2j - k_2 - 2k_3 + 2 \ge 1, \quad s := |I_1^{(j)} \setminus (N_1 \cup N_2)| = (j-1) - |N_1| - |N_2| \ge 0$$

for $1 \le j \le \ell$.

It is easy to see that each φ_j^B is a homogeneous derivation of degree 2ℓ which is equal to the Coxeter number for B_ℓ. We will prove that the derivations θ_E and $\varphi_1^B, \ldots, \varphi_\ell^B$ form a basis for $D(\mathcal{S}(B_\ell))$. First we will verify the following

Proposition 3.2.5. *Let $\varepsilon \in \{-1, 0, 1\}$. Then we have the following congruence relations:*

$$\overline{B}_{r,s}^B(x_p, z) + \varepsilon \overline{B}_{r,s}^B(x_q, z) \equiv 0 \mod (x_p + \varepsilon x_q),$$

$$\overline{B}_{r,s}^B(x_p, z) + \varepsilon \overline{B}_{r,s}^B(x_q, z) \equiv (x_p + \varepsilon x_q) \frac{x_p^r - (\varepsilon x_q)^r}{x_p - \varepsilon x_q} (x_p \cdot \varepsilon x_q)^s \mod (x_p + \varepsilon x_q - z).$$

Proof. The first congruence follows from Definition 3.2.1 (ii) and Lemma 3.2.2. Let the congruent notation $\equiv$ in the following calculation be modulo the ideal $(x_p + \varepsilon x_q - z)$. By Definition 3.2.1 and Lemma 3.2.2, we have

$$\begin{aligned}
&\overline{B}_{r,s}^B(x_p, z) + \varepsilon \overline{B}_{r,s}^B(x_q, z) = \overline{B}_{r,s}^B(x_p, z) + \overline{B}_{r,s}^B(\varepsilon x_q, z) \\
&= z^{r+2s} \{ B_{r,s}^B \left(\frac{x_p}{z} \right) + B_{r,s}^B \left(\frac{\varepsilon x_q}{z} \right) \} \\
&\equiv (x_p + \varepsilon x_q)^{r+2s} \left\{ B_{r,s}^B \left(\frac{x_p}{x_p + \varepsilon x_q} \right) + B_{r,s}^B \left(\frac{\varepsilon x_q}{x_p + \varepsilon x_q} \right) \right\} \\
&= (x_p + \varepsilon x_q)^{r+2s} \left\{ B_{r,s}^B \left(\frac{x_p}{x_p + \varepsilon x_q} \right) - B_{r,s}^B \left(-\frac{\varepsilon x_q}{x_p + \varepsilon x_q} \right) \right\}
\end{aligned}$$

$$= (x_p+\varepsilon x_q)^{r+2s}\frac{\left(\frac{x_p}{x_p+\varepsilon x_q}\right)^r-\left(\frac{\varepsilon x_q}{x_p+\varepsilon x_q}\right)^r}{\frac{x_p}{x_p+\varepsilon x_q}-\frac{\varepsilon x_q}{x_p+\varepsilon x_q}}\left(\frac{x_p}{x_p+\varepsilon x_q}\right)^s\left(\frac{\varepsilon x_q}{x_p+\varepsilon x_q}\right)^s$$
$$= (x_p+\varepsilon x_q)\frac{x_p^r-(\varepsilon x_q)^r}{x_p-\varepsilon x_q}(x_p\cdot\varepsilon x_q)^s.$$

□

Proposition 3.2.6. *The derivations φ_j^B $(1\le j\le \ell)$ belong to the module $D(\mathcal{S}(B_\ell))$.*

Proof. By Proposition 3.2.5, we first have

$$\begin{aligned}\varphi_j^B(x_p+\varepsilon x_q) &= (-1)^j\sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}}\left(\prod_{x_t\in N_1}x_t^2\right)\left(\prod_{x_t\in N_2}(-x_tz)\right)\\ &\qquad\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}}(-1)^{k_2+k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}(\overline{B}_{r,s}^B(x_p,z)+\varepsilon\overline{B}_{r,s}^B(x_q,z))\\ &\equiv 0 \mod (x_p+\varepsilon x_q)\end{aligned}$$

for $1\le j\le \ell$. Thus we conclude that $\varphi_j^B(x_p),\varphi_j^B(x_p\pm x_q)$ are divisible by $x_p, x_p\pm x_q$ for $1\le p\le \ell, 1\le p<q\le \ell$ respectively.

Let the congruent notation $\equiv$ in the following calculation be modulo the ideal $(x_p+\varepsilon x_q-z)$. By Proposition 3.2.5, for $1\le j\le\ell$, we also have

$$\begin{aligned}&\varphi_j^B(x_p+\varepsilon x_q-z)=\varphi_j^B(x_p+\varepsilon x_q)\\ &=(-1)^j\sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}}\left(\prod_{x_t\in N_1}x_t^2\right)\left(\prod_{x_t\in N_2}(-x_tz)\right)\\ &\quad\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}}(-1)^{k_2+k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}(\overline{B}_{r,s}^B(x_p,z)+\varepsilon\overline{B}_{r,s}^B(x_q,z))\\ &\equiv(-1)^j\sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}}\left(\prod_{x_t\in N_1}x_t^2\right)\left(\prod_{x_t\in N_2}(-x_t(x_p+\varepsilon x_q))\right)\\ &\qquad\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}}(-1)^{k_2+k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}(x_p+\varepsilon x_q)\frac{x_p^r-(\varepsilon x_q)^r}{x_p-\varepsilon x_q}(x_p\cdot\varepsilon x_q)^s\end{aligned}$$

$$= (x_p + \varepsilon x_q) \sum_{\substack{N_1, N_2 \subset I_1^{(j)} \\ N_1 \cap N_2 = \emptyset}} \left(\prod_{x_t \in N_1} x_t^2 \right) \left(\prod_{x_t \in N_2} (-x_t(x_p + \varepsilon x_q)) \right) (x_p \cdot \varepsilon x_q)^s$$

$$\frac{(-1)^{\ell+1}}{x_p - \varepsilon x_q} \Bigg\{ \sum_{\substack{0 \le k_2 \le 1 \\ 0 \le k_3 \le \ell - j}} (-1)^{\ell-j+1-k_2-k_3} \sigma_{k_2}^{(2,j)} \tau_{k_3}^{(3,j)} x_p^r$$

$$- \sum_{\substack{0 \le k_2 \le 1 \\ 0 \le k_3 \le \ell - j}} (-1)^{\ell-j+1-k_2-k_3} \sigma_{k_2}^{(2,j)} \tau_{k_3}^{(3,j)} (\varepsilon x_q)^r \Bigg\}.$$

Here,

$$\sum_{\substack{N_1, N_2 \subset I_1^{(j)} \\ N_1 \cap N_2 = \emptyset}} \left(\prod_{x_t \in N_1} x_t^2 \right) \left(\prod_{x_t \in N_2} (-x_t(x_p + \varepsilon x_q)) \right) (x_p \cdot \varepsilon x_q)^s$$

$$= \prod_{t=1}^{j-1} (x_t^2 - (x_p + \varepsilon x_q) x_t + x_p \cdot \varepsilon x_q) = \prod_{t=1}^{j-1} (x_t - x_p)(x_t - \varepsilon x_q)$$

and

$$\sum_{\substack{0 \le k_2 \le 1 \\ 0 \le k_3 \le \ell - j}} (-1)^{\ell-j+1-k_2-k_3} \sigma_{k_2}^{(2,j)} \tau_{k_3}^{(3,j)} x_p^r$$

$$= x_p \sum_{k_2=0}^{1} \sigma_{k_2}^{(2,j)} (-x_p)^{1-k_2} \sum_{k_3=0}^{\ell-j} \tau_{k_3}^{(3,j)} (-x_p^2)^{\ell-j-k_3} = x_p(x_j - x_p) \prod_{t=j+1}^{\ell} (x_t^2 - x_p^2).$$

If $1 \le p \le j-1$, then

$$\prod_{t=1}^{j-1} (x_t - x_p)(x_t - \varepsilon x_q) = 0.$$

If $j \le p < q \le \ell$, then

$$x_p(x_j - x_p) \left(\prod_{t=j+1}^{\ell} (x_t^2 - x_p^2) \right) = \varepsilon x_q (x_j - \varepsilon x_q) \left(\prod_{t=j+1}^{\ell} (x_t^2 - (\varepsilon x_q)^2) \right) = 0.$$

Therefore

$$\begin{aligned}
&\varphi_j^B(x_p + \varepsilon x_q - z)\\
&\equiv (-1)^{\ell+1}\frac{x_p+\varepsilon x_q}{x_p-\varepsilon x_q}\prod_{t=1}^{j-1}(x_t - x_p)(x_t - \varepsilon x_q)\\
&\quad\left\{x_p(x_j - x_p)\left(\prod_{t=j+1}^{\ell}(x_t^2 - x_p^2)\right) - \varepsilon x_q(x_j - \varepsilon x_q)\left(\prod_{t=j+1}^{\ell}(x_t^2 - (\varepsilon x_q)^2)\right)\right\}\\
&= 0
\end{aligned}$$

for all pairs (p, q) with $1 \leq p < q \leq \ell$ and $\varepsilon \in \{-1, 0, 1\}$. Hence $\varphi_j^B \in D(\mathcal{S}(B_\ell))$ for $1 \leq j \leq \ell$. □

3.3 The W-equivariance

Recall that $\mathcal{A}(\Phi)$ is the Weyl arrangement in E corresponding to the irreducible root system Φ. In [12] L. Solomon and H. Terao studied the S-module

$$D(\mathcal{A}(\Phi), 2) := \{\theta \in \mathrm{Der}(S) \mid \theta(\alpha_H) \in S\alpha_H^2, H \in \mathcal{A}(\Phi)\},$$

which was denoted by $E(\mathcal{A})$ in [12]. Let h be the Coxeter number for Φ. Define

$$D(\mathcal{A}(\Phi), 2)_h := \{\theta \in D(\mathcal{A}(\Phi), 2) \mid \deg\theta = h\} \cup \{0\},$$

which is a real vector space. Note that the Weyl group W corresponding to Φ naturally acts on $D(\mathcal{A}(\Phi), 2)$ and $D(\mathcal{A}(\Phi), 2)_h$. We recall the S_z-submodule

$$D_0(\mathcal{S}(\Phi)) = \{\varphi \in D(\mathcal{S}(\Phi)) \mid \varphi(z) = 0\}$$

of $D(\mathcal{S}(\Phi))$. Then by Proposition 1.1.9, $D(\mathcal{S}(\Phi))$ has a decomposition

$$D(\mathcal{S}(\Phi)) = S_z\theta_E \oplus D_0(\mathcal{S}(\Phi))$$

over S_z. Let

$$D_0(\mathcal{S}(\Phi))_h := \{\varphi \in D_0(\mathcal{S}(\Phi)) \mid \deg\varphi = h\} \cup \{0\},$$

which is a real vector space. If $\varphi \in D_0(\mathcal{S}(\Phi))$, then $\varphi(\alpha_H) \in \alpha_H(\alpha_H - z)S_z$ for any $H \in \mathcal{A}(\Phi)$. Let $\overline{\varphi} := \varphi|_{z=0}$ be the restriction of φ to $z = 0$. Then $\overline{\varphi}(\alpha_H) \in \alpha_H^2 S$ for any $H \in \mathcal{A}(\Phi)$, hence $\overline{\varphi} \in D(\mathcal{A}(\Phi), 2)$.

Theorem 3.3.1. *(1) (L. Solomon-H. Terao [12]) The S-module $D(\mathcal{A}(\Phi), 2)$ is a free module with a basis consisting of ℓ derivations homogeneous of degree h. In other words, we have an isomorphism*

$$D(\mathcal{A}(\Phi), 2) \simeq D(\mathcal{A}(\Phi), 2)_h \otimes_{\mathbb{R}} S.$$

(2) (M. Yoshinaga [21]) The S_z-module $D_0(\mathcal{S}(\Phi))$ is a free module with a basis consisting of ℓ derivations homogeneous of degree h. In other words, we have an isomorphism

$$D_0(\mathcal{S}(\Phi)) \simeq D(\mathcal{S}(\Phi))_h \otimes_{\mathbb{R}} S_z.$$

Also the restriction map

$$\rho : D_0(\mathcal{S}(\Phi))_h \longrightarrow D(\mathcal{A}(\Phi), 2)_h$$

defined by $\varphi \mapsto \overline{\varphi} = \varphi|_{z=0}$ is a linear isomorphism.

Suppose that Φ is of the type B_ℓ in the rest of this section. Then we may define an explicit $\mathbb{R}$-linear map

$$\Psi : E^* \to D_0(\mathcal{S}(B_\ell))_h$$

by

$$\Psi(x_j) = \varphi_j^B \quad (1 \le j \le \ell)$$

using the derivations $\varphi_1^B, \dots, \varphi_\ell^B$ in Definition 3.2.4.

Theorem 3.3.2. *Let Φ be a root system of the type B_ℓ.*

(1) The map

$$\Xi : E^* \to D(\mathcal{A}(B_\ell), 2)_h$$

defined by $\Xi = \rho \circ \Psi$ is a W-equivariant isomorphism.

(2) The map

$$\Psi : E^* \to D_0(\mathcal{S}(B_\ell))_h$$

is a linear isomorphism.

Proof. (1) Since

$$\overline{B}_{r,s}^B(x_i, 0) = \begin{cases} (-1)^s x_i^{r+2s}/(r+2s) & (r : \text{odd number}) \\ 0 & (r : \text{even number}) \end{cases},$$

$$\begin{aligned}\Xi(x_j)(x_i) &= (\rho\circ\Psi(x_j))(x_i) = \varphi_j^B(x_i)|_{z=0}\\ &= (-1)^j x_j \sum_{N_1\subset I_1^{(j)}}\left(\prod_{x_t\in N_1} x_t^2\right)\sum_{k_3=0}^{\ell-j}(-1)^{1+k_3}\tau_{k_3}^{(3,j)}(-1)^s\frac{x_i^{r+2s}}{r+2s}\\ &= (-1)^j x_j \sum_{m=0}^{j-1}\sum_{\substack{N_1\subset I_1^{(j)}\\ |N_1|=m}}\left(\prod_{x_t\in N_1} x_t^2\right)\sum_{k_3=0}^{\ell-j}(-1)^{1+k_3}\tau_{k_3}^{(3,j)}(-1)^s\frac{x_i^{r+2s}}{r+2s}\\ &= x_j\sum_{m=0}^{j-1}\tau_m^{(1,j)}\sum_{k_3=0}^{\ell-j}(-1)^{m+k_3}\tau_{k_3}^{(3,j)}\frac{x_i^{2\ell-2m-2k_3-1}}{2\ell-2m-2k_3-1}\\ &= x_j\sum_{k=0}^{\ell-1}(-1)^k\sigma_k(x_1^2,\dots,x_{j-1}^2,x_{j+1}^2,\dots,x_\ell^2)\frac{x_i^{2\ell-2k-1}}{2\ell-2k-1}.\end{aligned}$$

Thus we obtain

$$\Xi(x_j) = x_j\sum_{k=0}^{\ell-1}(-1)^k\sigma_k(x_1^2,\dots,x_{j-1}^2,x_{j+1}^2,\dots,x_\ell^2)\sum_{i=1}^{\ell}\left(\frac{x_i^{2\ell-2k-1}}{2\ell-2k-1}\right)\partial_i.$$

Since

$$\sum_{i=1}^{\ell}\left(\frac{x_i^{2\ell-2k-1}}{2\ell-2k-1}\right)\partial_i$$

is a W-invariant derivation and the correspondence

$$x_j\mapsto x_j\sigma_k(x_1^2,\dots,x_{j-1}^2,x_{j+1}^2,\dots,x_\ell^2)\quad (0\le k\le \ell-1)$$

is W-equivariant for every $k\in\mathbb{Z}_{\ge 0}$, we conclude that Ξ is W-equivariant. Therefore Ξ is bijective by Schur's lemma.

(2) follows from (1) because the restriction map ρ is bijective by Theorem 3.3.1 (2). □

Theorem 3.3.3. *The derivations $\varphi_1^B,\dots,\varphi_\ell^B$ form a basis for $D_0(\mathcal{S}(B_\ell))$.*

Proof. Recall that each $\Psi(x_j)=\varphi_j^B$ belongs to $D_0(\mathcal{S}(B_\ell))_h$. Theorems 3.3.1 (2) and 3.3.2 (2) complete the proof. □

Remark 3.3.4. *Since the W-equivariant isomorphism $\Xi : E^*\to D(\mathcal{A}(B_\ell),2)_h$ in Theorem 3.3.2 (1) is unique up to a nonzero constant multiple by Schur's lemma, the derivations $\varphi_1^B|_{z=0},\dots,\varphi_\ell^B|_{z=0}$ coincide with the Solomon-Terao basis in [12] up to a nonzero constant multiple. Therefore, our construction of $\varphi_1^B,\dots,\varphi_\ell^B$ can be regarded as an explicit realization of the basis existence theorem by M. Yoshinaga in [21].*

Chapter 4

The Shi arrangements of the type C_ℓ

In this chapter, we construct a basis for the logarithmic derivation module of the cone over the Shi arrangement of the type C_ℓ. This chapter is based on [15].

4.1 Notations

In this chapter we explicitly choose root systems Φ_C and positive root system Φ_C^+ of the type C_ℓ as follows:

$$\begin{aligned}
\Phi_C &:= \{\pm 2x_i, \pm x_p \pm x_q \in E^* \mid 1 \le i \le \ell, 1 \le p < q \le \ell\},\\
\Phi_C^+ &:= \{2x_i, x_p \pm x_q \in \Phi_B \mid 1 \le i \le \ell, 1 \le p < q \le \ell\}.
\end{aligned}$$

We express the cones over the Shi arrangements Shi[1] of the type C_ℓ by $\mathcal{S}(C_\ell)$. Then the defining polynomial of $\mathcal{S}(C_\ell)$ is

$$\begin{aligned}
Q(\mathcal{S}(C_\ell)) = z \prod_{i=1}^{\ell} 2x_i(2x_i - z) \prod_{1 \le p < q \le \ell} \{&(x_p + x_q)(x_p - x_q)\\
&(x_p + x_q - z)(x_p - x_q - z)\}.
\end{aligned}$$

It follows from Yoshinaga's Theorem 1.2.3 that $\mathcal{S}(C_\ell)$ is free with

$$\exp(\mathcal{S}(C_\ell)) = (1, 2\ell, 2\ell, \ldots, 2\ell).$$

4.2 A basis construction for the type C_ℓ

Definition 4.2.1. *For $(r,s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}$, define a polynomial $B^C_{r,s}(x)$ in x satisfying the following two conditions:*

(i) $B^C_{r,s}(x+1) - B^C_{r,s}(x) = \{(x+1)^{r-1} + (-x)^{r-1}\}(x+1)^s(-x)^s$,

(ii) $B^C_{r,s}(0) = 0$.

It is easy to see that $B^C_{r,s}(x)$ uniquely exists and

$$\deg B^C_{r,s}(x) = \begin{cases} r + 2s & \text{if } r \text{ is odd,} \\ r + 2s - 1 & \text{if } r \text{ is even.} \end{cases}$$

The following lemma can be proved by a smilar argument to the proof of Lemma 3.2.2:

Lemma 4.2.2. *$B^C_{r,s}(x)$ is an odd function.*

Definition 4.2.3. *The homogenization $\overline{B}^C_{r,s}(x,z)$ of $B^C_{r,s}(x)$ is defined by*

$$\overline{B}^C_{r,s}(x,z) := z^{r+2s} B^C_{r,s}(x/z).$$

Definition 4.2.4. *Define homogeneous derivations*

$$\varphi^C_j := (-1)^j \sum_{i=1}^{\ell} \Bigg\{ \sum_{\substack{N_1, N_2 \subset I_1^{(j)} \\ N_1 \cap N_2 = \emptyset}} \Bigg(\prod_{x_t \in N_1} x_t^2 \Bigg) \Bigg(\prod_{x_t \in N_2} (-x_t z) \Bigg) \sum_{\substack{0 \leq k_2 \leq 1 \\ 0 \leq k_3 \leq \ell - j}} (-1)^{k_2 + k_3} \sigma^{(2,j)}_{k_2} \tau^{(3,j)}_{k_3} \; \overline{B}^C_{r,s}(x_i, z) \Bigg\} \partial_i$$

where

$$r := 2\ell - 2j - k_2 - 2k_3 + 2 \geq 1, \quad s := |I_1^{(j)} \setminus (N_1 \cup N_2)| = (j-1) - |N_1| - |N_2| \geq 0$$

for $1 \leq j \leq \ell$.

Note that φ^C_j is exactly the same as φ^B_j with only one exception: the use of $\overline{B}^C_{r,s}(x_i,z)$ instead of $\overline{B}^B_{r,s}(x_i,z)$. Thus each φ^B_j is a homogeneous derivation of degree 2ℓ which is equal to the Coxeter number for C_ℓ. We will prove that the derivations θ_E and $\varphi^C_1, \ldots, \varphi^C_\ell$ form a basis for $D(\mathcal{S}(C_\ell))$. We first have the following propositions:

Proposition 4.2.5. *Let $\varepsilon \in \{-1, 0, 1\}$. Then we have the following congruence relations:*

$$\overline{B}^{C}_{r,s}(x_p, z) + \varepsilon \overline{B}^{C}_{r,s}(x_q, z) \equiv 0 \mod (x_p + \varepsilon x_q),$$

$$\overline{B}^{C}_{r,s}(x_p, z) + \varepsilon \overline{B}^{C}_{r,s}(x_q, z) \equiv (x_p + \varepsilon x_q)\{x_p^{r-1} + (\varepsilon x_q)^{r-1}\}(x_p \cdot \varepsilon x_q)^s \mod (x_p + \varepsilon x_q - z).$$

Proof. Imitate the proof of Proposition 3.2.5. □

Proposition 4.2.6. *The derivations φ^C_j $(1 \leq j \leq \ell)$ belong to the module $D(\mathcal{S}(C_\ell))$.*

Proof. This proof is very similar to the proof of Proposition 3.2.6. However, in this proof, we have to verify that $\varphi^C_j(2x_p - z)$ is divisible by $2x_p - z$ while we verified that $\varphi^B_j(x_p - z)$ is divisible by $x_p - z$ in the proof of Proposition 3.2.6. By Proposition 4.2.5, we first have

$$\begin{aligned}
&\varphi^C_j(x_p + \varepsilon x_q) \\
&= (-1)^j \sum_{\substack{N_1, N_2 \subset I_1^{(j)} \\ N_1 \cap N_2 = \emptyset}} \left(\prod_{x_t \in N_1} x_t^2\right)\left(\prod_{x_t \in N_2} (-x_t z)\right) \\
&\qquad \sum_{\substack{0 \leq k_2 \leq 1 \\ 0 \leq k_3 \leq \ell - j}} (-1)^{k_2 + k_3} \sigma^{(2,j)}_{k_2} \tau^{(3,j)}_{k_3} (\overline{B}^{C}_{r,s}(x_p, z) + \varepsilon \overline{B}^{C}_{r,s}(x_q, z)) \\
&\equiv 0 \qquad (\mathrm{mod}\ (x_p + \varepsilon x_q))
\end{aligned}$$

for $1 \leq j \leq \ell$. Thus we conclude that $\varphi^C_j(2x_p), \varphi^C_j(x_p \pm x_q)$ are divisible by $2x_p, x_p \pm x_q$ for $1 \leq p \leq \ell, 1 \leq p < q \leq \ell$ respectively.

Let the congruent notation $\equiv$ in the following calculation be modulo the ideal $(x_p + \varepsilon x_q - z)$. By Proposition 4.2.5, for $1 \leq j \leq \ell$, we also have

$$\begin{aligned}
&\varphi^C_j(x_p + \varepsilon x_q - z) = \varphi^C_j(x_p + \varepsilon x_q) \\
&= (-1)^j \sum_{\substack{N_1, N_2 \subset I_1^{(j)} \\ N_1 \cap N_2 = \emptyset}} \left(\prod_{x_t \in N_1} x_t^2\right)\left(\prod_{x_t \in N_2} (-x_t z)\right) \\
&\qquad \sum_{\substack{0 \leq k_2 \leq 1 \\ 0 \leq k_3 \leq \ell - j}} (-1)^{k_2 + k_3} \sigma^{(2,j)}_{k_2} \tau^{(3,j)}_{k_3} (\overline{B}^{C}_{r,s}(x_p, z) + \varepsilon \overline{B}^{C}_{r,s}(x_q, z))
\end{aligned}$$

$$\equiv (-1)^j \sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}} \left(\prod_{x_t\in N_1} x_t^2\right)\left(\prod_{x_t\in N_2}(-x_t(x_p+\varepsilon x_q))\right)$$

$$\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}} (-1)^{k_2+k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}(x_p+\varepsilon x_q)\{x_p^{r-1}+(\varepsilon x_q)^{r-1}\}(x_p\cdot\varepsilon x_q)^s$$

$$= (x_p+\varepsilon x_q) \sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}} \left(\prod_{x_t\in N_1} x_t^2\right)\left(\prod_{x_t\in N_2}(-x_t(x_p+\varepsilon x_q))\right)(x_p\cdot\varepsilon x_q)^s$$

$$(-1)^{\ell+1}\left\{\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}} (-1)^{\ell-j+1-k_2-k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}x_p^{r-1}\right.$$

$$\left. + \sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}} (-1)^{\ell-j+1-k_2-k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}(\varepsilon x_q)^{r-1}\right\}.$$

Here,

$$\sum_{\substack{N_1,N_2\subset I_1^{(j)}\\ N_1\cap N_2=\emptyset}} \left(\prod_{x_t\in N_1} x_t^2\right)\left(\prod_{x_t\in N_2}(-x_t(x_p+\varepsilon x_q))\right)(x_p\cdot\varepsilon x_q)^s$$

$$= \prod_{t=1}^{j-1}(x_t^2-(x_p+\varepsilon x_q)x_t+x_p\cdot\varepsilon x_q) = \prod_{t=1}^{j-1}(x_t-x_p)(x_t-\varepsilon x_q),$$

and

$$\sum_{\substack{0\le k_2\le 1\\ 0\le k_3\le \ell-j}} (-1)^{\ell-j+1-k_2-k_3}\sigma_{k_2}^{(2,j)}\tau_{k_3}^{(3,j)}x_p^{r-1}$$

$$= \sum_{k_2=0}^{1}\sigma_{k_2}^{(2,j)}(-x_p)^{1-k_2}\sum_{k_3=0}^{\ell-j}\tau_{k_3}^{(3,j)}(-x_p^2)^{\ell-j-k_3} = (x_j-x_p)\prod_{t=j+1}^{\ell}(x_t^2-x_p^2).$$

If $1\le p\le j-1$, then

$$\prod_{t=1}^{j-1}(x_t-x_p)(x_t-\varepsilon x_q)=0.$$

If $j\le p<q\le \ell$ and $\varepsilon\in\{-1,1\}$, then

$$(x_j-x_p)\prod_{t=j+1}^{\ell}(x_t^2-x_p^2) = (x_j-\varepsilon x_q)\prod_{t=j+1}^{\ell}(x_t^2-(\varepsilon x_q)^2)=0.$$

Therefore

$$\begin{aligned}
&\varphi_j^C(x_p+\varepsilon x_q - z)\\
&\equiv (-1)^{\ell-j+1}(x_p+\varepsilon x_q)\prod_{t=1}^{j-1}(x_t-x_p)(x_t-\varepsilon x_q)\\
&\qquad\left\{(x_j-x_p)\left(\prod_{t=j+1}^{\ell}(x_t^2-x_p^2)\right)+(x_j-\varepsilon x_q)\left(\prod_{t=j+1}^{\ell}(x_t^2-(\varepsilon x_q)^2)\right)\right\}\\
&=0
\end{aligned}$$

for all pairs (p,q) with $1\le p<q\le \ell$ where $\varepsilon\in\{-1,1\}$. When $p=q,\varepsilon=1$,

$$\begin{aligned}
&\varphi_j^C(x_p+\varepsilon x_q - z)=\varphi_j^C(2x_p-z)\\
&\equiv (-1)^{\ell-j+1}(2x_p)\prod_{t=1}^{j-1}(x_t-x_p)^2\left\{2(x_j-x_p)\prod_{t=j+1}^{\ell}(x_t^2-x_p^2)\right\}\\
&=0
\end{aligned}$$

for $1\le p\le\ell$. Hence $\varphi_j\in D(\mathcal{S}(C_\ell))$ for $1\le j\le \ell$. □

We may define an explicit $\mathbb{R}$-linear map

$$\Psi : E^* \to D_0(\mathcal{S}(C_\ell))_h$$

by

$$\Psi(x_j)=\varphi_j^C\quad (1\le j\le \ell)$$

using the derivations $\varphi_1^C,\ldots,\varphi_\ell^C$ in Definition 4.2.4.

Theorem 4.2.7. *Let Φ be a root system of the type C_ℓ.*

(1) The map

$$\Xi : E^*\to D(\mathcal{A}(C_\ell),2)_h$$

defined by $\Xi=\rho\circ\Psi$ is a W-equivariant isomorphism.

(2) The map

$$\Psi : E^* \to D_0(\mathcal{S}(C_\ell))_h$$

is a linear isomorphism.

Proof. Since

$$\overline{B}^C_{r,s}(x_i,0)=2\overline{B}^B_{r,s}(x_i,0)=\begin{cases}(-1)^s2x_i^{r+2s}/(r+2s) & (r:\text{odd number})\\ 0 & (r:\text{even number})\end{cases},$$

we may prove this theorem in the same way as Theorem 3.3.2. □

Theorem 4.2.8. *The derivations* $\theta_E, \varphi_1^C, \ldots, \varphi_\ell^C$ *form a basis for* $D(\mathcal{S}(C_\ell))$.

Proof. Apply Theorems 4.2.7 (2) and 3.3.1 (2) in the same way as the proof of Theorem 3.3.3. $\square$

Remark 4.2.9. *Remark 3.3.4 is also true for the type* C_ℓ*, that is, our construction of* $\varphi_1^C, \ldots, \varphi_\ell^C$ *can be regarded as an explicit realization of the basis existence theorem by M. Yoshinaga in [21].*

Chapter 5

Extended Shi and Catalan arrangements of the type A_2

In this chapter, we give the first explicit construction of a series of bases for the extended Shi and Catalan arrangements when the corresponding root system is of the type A_2. This chapter is based on [1].

5.1 Introduction

Let E be a 2-dimensional Euclidean space and $\Phi \subset E^*$ the root system of type A_2. Let W be the Weyl group of Φ and W_z the group generated by W and the reflection τ_z with respect to z. In this chapter we choose a simple system Δ and a positive system Φ^+ of Φ as follows:

$$\Delta = \{\alpha_1, \alpha_2\}, \quad \Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}.$$

Then, for $k \in \mathbb{Z}_{\geq 0}$, the cones over the extended Shi arrangement $\mathbf{c}\mathrm{Shi}^k$ of the type A_2 and the extended Catalan arrangement $\mathbf{c}\mathrm{Cat}^k$ of the type A_2 are defined by

$$Q(\mathbf{c}\mathrm{Shi}^k) = z \prod_{-k+1 \leq i \leq k} (\alpha_1 - iz)(\alpha_2 - iz)(\alpha_1 + \alpha_2 - iz),$$
$$Q(\mathbf{c}\mathrm{Cat}^k) = z \prod_{-k \leq i \leq k} (\alpha_1 - iz)(\alpha_2 - iz)(\alpha_1 + \alpha_2 - iz).$$

It follows from Yoshinaga's Theorem 1.2.3 that $\mathbf{c}\mathrm{Shi}^k$ is a free arrangement with $\exp(\mathbf{c}\mathrm{Shi}^k) = (1, 3k, 3k)$, and $\mathbf{c}\mathrm{Cat}^k$ is a free arrangement with $\exp(\mathbf{c}\mathrm{Cat}^k) = (1, 3k+1, 3k+2)$. We give bases for the logarithmic modules of these arrangements as follows:

Theorem 5.1.1. *Let $\Delta = \{\alpha_1, \alpha_2\}$ be a simple system and $\{\partial_1, \partial_2\}$ its dual basis for* $\mathrm{Der}(S)$*. For $k \in \mathbb{Z}_{\geq 0}$, define*

$$M_k = \begin{pmatrix} \alpha_1 + kz & (2\alpha_1 + 4\alpha_2 + 3kz)(\alpha_1 + kz) \\ \alpha_2 + kz & -(4\alpha_1 + 2\alpha_2 + 3kz)(\alpha_2 + kz) \end{pmatrix},$$

$$\begin{aligned} N_k &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t M_k|_{z \to -z} \\ &= \begin{pmatrix} (2\alpha_1 + 4\alpha_2 - 3kz)(\alpha_1 - kz) & -(4\alpha_1 + 2\alpha_2 - 3kz)(\alpha_2 - kz) \\ \alpha_1 - kz & \alpha_2 - kz \end{pmatrix}, \end{aligned}$$

$$T_k = \begin{pmatrix} \dfrac{1}{3k+1} & 0 \\ 0 & \dfrac{1}{3k+2} \end{pmatrix},$$

$$A = [I^*(\alpha_i, \alpha_j)]_{1 \leq i,j \leq 2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

where I^* is the natural inner product on E^* induced from the inner product I on E. Then the Euler derivation and

$$[\partial_1, \partial_2] \prod_{i=0}^{k-1} (M_i T_i N_{i+1} A^{-1})$$

form a basis for $D(\mathbf{c}\mathrm{Shi}^k)$, and

$$[\partial_1, \partial_2] (\prod_{i=0}^{k-1} (M_i T_i N_{i+1} A^{-1})) M_k$$

a W_z-invariant basis for $D(\mathbf{c}\mathrm{Cat}^k)$.

The idea to prove Theorem 5.1.1 is to use the simple-root bases ([3]) and Terao's matrix $B^{(k)}$ ([2], [19]) with the invariant theory. Namely, if we fix a simple system and a primitive derivation, then we obtain a family of nice bases (simple-root basis plus/minus) for the logarithmic modules of $\mathbf{c}\mathrm{Shi}^k$ for all $k \in \mathbb{Z}_{\geq 0}$. By computations based on invariant theory and Weyl group actions, we can find a way to construct the bases for that of $\mathbf{c}\mathrm{Cat}^k$ from these bases. Hence the rest problem is to connect these new bases, which is achieved by restricting them onto the infinite hyperplane and applying the invariant theoretic method. In that invariant theory, Terao's matrix $B^{(k)}$ plays the essential role.

The organization of this chapter is as follows: In section 5.2, we review the simple-root bases for extended Shi arrangements introduced in [3], which play key roles in our construction of bases. In section 5.3, we give an explicit construction of bases for the extended Shi and Catalan arrangements of the type A_2 in Theorem 5.3.1.

5.2 The simple-root basis

In this section we review the definition and properties of multiarrangemetns and the simple-root bases for the extended Shi arrangements.

First, let $\mathcal{A}$ be a central arrangement and fix $H \in \mathcal{A}$. Then define

$$D_0(\mathcal{A}) := \{\theta \in D(\mathcal{A}) \mid \theta(\alpha_H) = 0\}.$$

Let $\mathcal{A}^H := \{K \cap H \mid K \in \mathcal{A} \setminus \{H\}\}$ and define a map $m_H : \mathcal{A}^H \to \mathbb{Z}_{>0}$ by

$$m_H(K \cap H) := |\{L \in \mathcal{A} \setminus \{H\} \mid L \cap H = K \cap H\}|.$$

Then for a logarithmic module

$$\begin{aligned} D(\mathcal{A}^H, m_H) := \{\theta \in \mathrm{Der}(S/(\alpha_H)) \mid \\ \theta(\alpha_K) \in (S/(\alpha_H))(\alpha_K)^{m_H(K)} \ (\forall K \in \mathcal{A}^H)\}, \end{aligned}$$

the Ziegler restriction map $\pi : D_0(\mathcal{A}) \to D(\mathcal{A}^H, m_H)$ is defined by $\pi(\theta) := \theta|_{\alpha_H = 0}$.

Proposition 5.2.1. *([23]) Assume that $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, d_2, \ldots, d_\ell)$. Then $D_0(\mathcal{A}^H, m_H)$ is also free with basis $\varphi_2, \ldots, \varphi_\ell$ such that $\deg(\varphi_i) = d_i$ $(i = 2, \ldots, \ell)$. Moreover, the Ziegler restriction map is surjective.*

For the rest of this section, let $V = \mathbb{R}^\ell$, and we recall the simple-root bases introduced in [3]. Let W be a finite irreducible reflection group corresponding to an irreducible root system Φ. Then by the famous theorem of Chevalley, there are homogeneos basic invariants $P_1, \ldots, P_\ell$ generating the W-invariant ring S^W of S as $\mathbb{R}$-algebra such that

$$\deg P_1 < \deg P_2 \leq \cdots \leq \deg P_{\ell-1} < \deg P_\ell.$$

Let F be the quotient field of S. Then the **primitive derivation** $D = \frac{\partial}{\partial P_\ell} \in \mathrm{Der}(F)$ is characterized by

$$D(P_i) = \begin{cases} c \in \mathbb{R}^\times & (i = \ell) \\ 0 & (1 \leq i \leq \ell - 1) \end{cases}.$$

The primitive derivation D is uniquely determined up to nonzero constant multiple c independent of the choice of the basic invariants. We define an affine connection $\nabla : \mathrm{Der}(F) \times \mathrm{Der}(F) \to \mathrm{Der}(F)$ by

$$\nabla_{\theta_1}\theta_2 = \sum_{i=1}^{\ell} \theta_1(f_i)\frac{\partial}{\partial x_i}$$

for $\theta_1, \theta_2 \in \mathrm{Der}(F)$ with $\theta_2 = \sum_{i=1}^{\ell} f_i \frac{\partial}{\partial x_i}$. For $m \in \mathbb{Z}_{>0}$, we define an S-module $D(\mathcal{A}(\Phi), m)$ by

$$D(\mathcal{A}(\Phi), m) = \{\theta \in \mathrm{Der}(S) \mid \theta(\alpha_H) \in \alpha_H^m S \text{ for any } H \in \mathcal{A}(\Phi)\}.$$

Note that the action of W onto V canonically extends to those onto V^*, S, $\mathrm{Der}(S)$ and $D(\mathcal{A}(\Phi), m)$. Let $D(\mathcal{A}(\Phi), m)^W$ denote the W-invariant set of $D(\mathcal{A}(\Phi), m)$.

Lemma 5.2.2. *([20]) For the derivations $\frac{\partial}{\partial P_i} \in \mathrm{Der}(S^W)$ $(1 \le i \le \ell)$,*

$$\nabla_{\frac{\partial}{\partial P_i}} D(\mathcal{A}(\Phi), 2k+1)^W \subset D(\mathcal{A}(\Phi), 2k-1)^W \quad (k > 0).$$

In particular, as mentioned in [2], the connection ∇_D induces an $\mathbb{R}[P_1, \dots, P_{\ell-1}]$-isomorphism

$$\nabla_D : D(\mathcal{A}(\Phi), 2k+1)^W \xrightarrow{\sim} D(\mathcal{A}(\Phi), 2k-1)^W \quad (k > 0).$$

So we can consider the inverse map

$$\nabla_D^{-1} : D(\mathcal{A}(\Phi), 2k-1)^W \xrightarrow{\sim} D(\mathcal{A}(\Phi), 2k+1)^W.$$

Proposition 5.2.3. *([3],[20]) Let $\theta_E = \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i}$ be the Euler derivation and define ∂_v $(v \in E)$ by $\partial_v(\alpha) := \langle v, \alpha \rangle$ for $\alpha \in E^*$. We define $\Xi : E \to D(\mathcal{A}_\Phi, 2k)$ by $\Xi(v) = \nabla_{\partial_v} \nabla_D^{-k} \theta_E$. Then Ξ is a W-isomorphism.*

Proposition 5.2.4. *([21]) Let $D_0(\mathbf{c}\mathrm{Shi}^k) = \{\theta \in D(\mathbf{c}\mathrm{Shi}^k) \mid \theta(z) = 0\}$. Then the Ziegler restriction map* $\mathbf{res}$ *: $D_0(\mathbf{c}\mathrm{Shi}^k) \to D(\mathcal{A}_\Phi, 2k)$ defined by* $\mathbf{res}(\theta) = \theta|_{z=0}$ *is surjective. In particular,* $\mathbf{res} : D_0(\mathbf{c}\mathrm{Shi}^k)_{kh} \to D(\mathcal{A}_\Phi, 2k)_{kh}$ *is $\mathbb{R}$-linear isomorphism where $D_0(\mathbf{c}\mathrm{Shi}^k)_{kh}$ and $D(\mathcal{A}_\Phi, 2k)_{kh}$ are the homogeneous parts of degree kh of $D_0(\mathbf{c}\mathrm{Shi}^k)$ and $D(\mathcal{A}_\Phi, 2k)$ respectively and h is the Coxeter number.*

Definition 5.2.5. *([3]) Fix $k \in \mathbb{Z}_{\ge 0}$. Define a linear isomorphism $\Theta : E \to D_0(\mathbf{c}\mathrm{Shi}^k)$ by $\Theta = \mathbf{res}^{-1} \circ \Xi$. Let $\{\alpha_1, \dots, \alpha_\ell\} \subset E^*$ be a simple system of Φ^+ and $\{\alpha_1^*, \dots, \alpha_\ell^*\} \subset E$ be its dual basis. Then the derivations*

$$\varphi_i^{(k)} = \Theta(\alpha_i^*) \quad (1 \le i \le \ell)$$

are called a ***simple-root basis plus*** ($\mathbf{SRB}_+$) *of* $D_0(\mathbf{c}\mathrm{Shi}^k)$ *and the derivations*

$$\psi_i^{(k)} = \sum_{p=1}^{\ell} I^*(\alpha_i, \alpha_p)\varphi_p^{(k)} \quad (1 \le i \le \ell)$$

are called a ***simple-root basis minus*** ($\mathbf{SRB}_-$) *of* $D_0(\mathbf{c}\mathrm{Shi}^k)$. *Here* I^* *is the natural inner product on* E^* *induced from the inner product* I *on* E.

Remark 5.2.6. *Let* $A = [I^*(\alpha_i, \alpha_j)]_{1 \le i,j \le \ell}$ *be the inner product matrix. Then we have the following relation between an* SRB_+ $\{\varphi_1^{(k)}, \dots, \varphi_\ell^{(k)}\}$ *and an* SRB_- $\{\psi_1^{(k)}, \dots, \psi_\ell^{(k)}\}$ *by the definitions:*

$$[\varphi_1^{(k)}, \dots, \varphi_\ell^{(k)}] = [\psi_1^{(k)}, \dots, \psi_\ell^{(k)}]A^{-1}.$$

It follows from Schur's lemma that these bases are uniquely determined if we fix a simple system and a primitive derivation D. These bases can be characterized by the following conditions:

Proposition 5.2.7. *([3])*

(1) *Let* $\varphi_1^{(k)}, \dots, \varphi_\ell^{(k)}$ *be an* SRB_+ *of* $D_0(\mathbf{c}\mathrm{Shi}^k)$. *Then* $\varphi_1^{(k)}, \dots, \varphi_\ell^{(k)}$ *satisfy*

$$\varphi_i^{(k)}(\alpha_j + kz) \in (\alpha_j + kz)S_z \quad (i \ne j).$$

(2) *Let* $\psi_1^{(k)}, \dots, \psi_\ell^{(k)}$ *be an* SRB_- *of* $D_0(\mathbf{c}\mathrm{Shi}^k)$. *Then* $\psi_1^{(k)}, \dots, \psi_\ell^{(k)}$ *satisfy*

$$\psi_i^{(k)} \in (\alpha_i - kz)\,\mathrm{Der}(S_z) \quad (1 \le i \le \ell).$$

Remark 5.2.8. *For an arbitrary root system, we do not know an explicit expression of the simple-root basis because the inverse mapping of Ziegler restriction* $\mathbf{res}^{-1}$ *is impossible to describe at this writing.*

Now we introduce some propositions concerning the action of W to these bases.

Proposition 5.2.9. *([3]) The derivation*

$$\sum_{i=1}^{\ell}(\alpha_i + kz)\varphi_i^{(k)}$$

is called the ***k-Euler derivation***. *The* k*-Euler derivation is* W*-invariant and belongs to* $D_0(\mathbf{c}\mathrm{Cat}^k)_{kh+1}$.

Proposition 5.2.10. *([3]) Let* $s_i \in W$ *be the reflection with respect to* α_i *for* $1 \le i \le \ell$. *Then*

(1) $s_i\varphi_j^{(k)} = \varphi_j^{(k)}$ *whenever* $i \ne j$, *and*

(2) $s_i\left(\dfrac{\psi_i^{(k)}}{(\alpha_i - kz)}\right) = \dfrac{\psi_i^{(k)}}{(\alpha_i - kz)}$ *for* $1 \le i \le \ell$.

5.3 Construction of bases of the type A_2

For the rest of this paper, we assume that the root system Φ is of the type A_2. Let $\{\alpha_1, \alpha_2\} \subset E^*$ be a simple system. For $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, let $H_{\alpha-kz} := \{\alpha - kz = 0\}$. Then the results in [3] shows that $\mathbf{c}\mathrm{Shi}^k \setminus \{H_{\alpha_i-kz}\}$ and $\mathbf{c}\mathrm{Shi}^k \setminus \{H_{\alpha_1-kz}, H_{\alpha_2-kz}\}$ are also both free with exponents

$$\begin{aligned}\exp(\mathbf{c}\mathrm{Shi}^k \setminus \{H_{\alpha_i-kz}\}) &= (1, 3k-1, 3k),\\ \exp(\mathbf{c}\mathrm{Shi}^k \setminus \{H_{\alpha_1-kz}, H_{\alpha_2-kz}\}) &= (1, 3k-1, 3k-1)\end{aligned}$$

for $i = 1, 2$.

Theorem 5.3.1. *Let us fix basic invariants*

$$P_1 := \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2,\ P_2 := \frac{2}{27}(\alpha_1 - \alpha_2)(\alpha_1 + 2\alpha_2)(2\alpha_1 + \alpha_2)$$

of the Weyl group W and choose the primitive derivation D in such a way that $D(P_2) = 1/3$. For $k \in \mathbb{Z}_{\geq 0}$, let M_k, N_k, T_k and A be the same as in Theorem 5.1.1.

Let $\varphi_1^{(k)}, \varphi_2^{(k)}$ be an SRB$_+$ *of $D_0(\mathbf{c}\mathrm{Shi}^k)$. Then*

$$[\varphi_1^{(k)}, \varphi_2^{(k)}]M_k$$

form a W_z-invariant basis for $D_0(\mathbf{c}\mathrm{Cat}^k)$, and

$$[\varphi_1^{(k)}, \varphi_2^{(k)}]M_k T_k N_{k+1}$$

is an SRB$_-$ *of $D_0(\mathbf{c}\mathrm{Shi}^{k+1})$.*

We prove Theorem 5.3.1 by using following propositions.

Proposition 5.3.2. *Let $\varphi_1^{(k)}, \varphi_2^{(k)}$ be an* SRB$_+$ *of $D_0(\mathbf{c}\mathrm{Shi}^k)$ and $[\theta_1^{(k)}, \theta_2^{(k)}] := [\varphi_1^{(k)}, \varphi_2^{(k)}]M_k$. Then $\theta_1^{(k)}, \theta_2^{(k)}$ form a W-invariant basis for $D_0(\mathbf{c}\mathrm{Cat}^k)$.*

Proof. Since $\theta_1^{(k)} = (\alpha_1 + kz)\varphi_1^{(k)} + (\alpha_2 + kz)\varphi_2^{(k)}$ is the k-Euler derivation, it follows from Proposition 5.2.9 that $\theta_1^{(k)} \in D_0(\mathbf{c}\mathrm{Cat}^k)^W$. Let us show $\theta_2^{(k)} \in D_0(\mathbf{c}\mathrm{Cat}^k)^W$. By Proposition 5.2.7 (1), it is clear that $\theta_2^{(k)}(\alpha_i + kz) \in (\alpha_i + kz)S_z$ $(i = 1, 2)$. Since

$$\begin{aligned}\theta_2^{(k)} &= (2\alpha_1 + 4\alpha_2 + 3kz)(\alpha_1 + kz)\varphi_1^{(k)} - (4\alpha_1 + 2\alpha_2 + 3kz)(\alpha_2 + kz)\varphi_2^{(k)}\\ &= (2\alpha_1 + 4\alpha_2 + 3kz)\{\theta_1^{(k)} - (\alpha_2 + kz)\varphi_2^{(k)}\}\\ &\qquad\qquad - (4\alpha_1 + 2\alpha_2 + 3kz)(\alpha_2 + kz)\varphi_2^{(k)}\\ &= (2\alpha_1 + 4\alpha_2 + 3kz)\theta_1^{(k)} - 6(\alpha_1 + \alpha_2 + kz)(\alpha_2 + kz)\varphi_2^{(k)},\end{aligned}$$

it holds that $\theta_2^{(k)}(\alpha_1+\alpha_2+kz) \in (\alpha_1+\alpha_2+kz)S_z$. So $\theta_2^{(k)} \in D_0(\mathbf{c}\mathrm{Cat}^k)$. Moreover, since $s_i\varphi_j^{(k)} = \varphi_j^{(k)}$ $(i \neq j)$ for the reflection s_i with respect to α_i because of Proposition 5.2.10 (1),

$$\begin{aligned} s_1\theta_2^{(k)} &= (2\alpha_1+4\alpha_2+3kz)s_1\theta_1^{(k)} - 6(\alpha_2+kz)(\alpha_1+\alpha_2+kz)s_1\varphi_2^{(k)} \\ &= \theta_2^{(k)}. \end{aligned}$$

Similarly, we can express $\theta_2^{(k)}$ in terms of $\theta_1^{(k)}$ and $\varphi_1^{(k)}$. Then the same argument as the above shows that $s_2\theta_2^{(k)} = \theta_2^{(k)}$. Hence $\theta_2^{(k)}$ is W-invariant. Finally, since

$$\det(M_k) = -6(\alpha_1+kz)(\alpha_2+kz)(\alpha_1+\alpha_2+kz),$$

and $\varphi_1^{(k)}, \varphi_2^{(k)}$ form a basis for $D_0(\mathbf{c}\mathrm{Shi}^k)$, it follows that $\theta_1^{(k)}, \theta_2^{(k)}$ form a basis for $D_0(\mathbf{c}\mathrm{Cat}^k)$. $\square$

Lemma 5.3.3. *Let $\Omega^1(\mathcal{A}_\Phi)$ denote the module of logarithmic differential forms of $\mathcal{A}_\Phi$ (i.e., the dual S-module of $D(\mathcal{A}_\Phi)$). If $\omega \in \Omega^1(\mathcal{A}_\Phi)$, then $\nabla_{I^*(\omega)}\nabla_D^{-k}\theta_E \in D(\mathcal{A}_\Phi, 2k-1)$.*

Proof. By [2], it follows that

$$I^*(\Omega^1(\mathcal{A}_\Phi)) \subset \bigoplus_{i=1}^{\ell} S\frac{\partial}{\partial P_i}.$$

Since $\nabla_{\frac{\partial}{\partial P_i}}\nabla_D^{-k}\theta_E \in D(\mathcal{A}_\Phi, 2k-1)$ by Lemma 5.2.2, we conclude that $\nabla_{I^*(\omega)}\nabla_D^{-k}\theta_E \in D(\mathcal{A}_\Phi, 2k-1)$. $\square$

Proposition 5.3.4. *Let $\psi_1^{(k)}, \psi_2^{(k)}$ be an* SRB$_-$ *of $D_0(\mathbf{c}\mathrm{Shi}^k)$. Then $[\eta_1^{(k-1)}, \eta_2^{(k-1)}] := [\psi_1^{(k)}, \psi_2^{(k)}]N_k^{-1}$ form a W-invariant basis for $D_0(\mathbf{c}\mathrm{Cat}^{k-1})$.*

Proof. First we will show that $\eta_1^{k-1} \in D_0(\mathbf{c}\mathrm{Cat}^{k-1})^W$. Since

$$N_k^{-1} = \begin{pmatrix} \dfrac{1}{6(\alpha_1-kz)(\alpha_1+\alpha_2-kz)} & \dfrac{4\alpha_1+2\alpha_2-3kz}{6(\alpha_1-kz)(\alpha_1+\alpha_2-kz)} \\ -\dfrac{1}{6(\alpha_2-kz)(\alpha_1+\alpha_2-kz)} & \dfrac{2\alpha_1+4\alpha_2-3kz}{6(\alpha_2-kz)(\alpha_1+\alpha_2-kz)} \end{pmatrix},$$

we have

$$\eta_1^{(k-1)} = \frac{1}{6(\alpha_1+\alpha_2-kz)}\left(\frac{\psi_1^{(k)}}{\alpha_1-kz} - \frac{\psi_2^{(k)}}{\alpha_2-kz}\right).$$

Consider a commtative diagram

$$\begin{array}{ccc} D_0(\mathbf{c}\mathrm{Shi}^k \setminus \{H_{\alpha_1-kz}, H_{\alpha_2-kz}\})_{3k-1} & \xrightarrow[\sim]{\mathbf{res}} & D(\mathcal{A}_\Phi, 2k-\mathbf{m})_{3k-1} \\ \bigcup & & \bigcup \\ (\alpha_1+\alpha_2-kz)D_0(\mathbf{c}\mathrm{Cat}^{k-1})_{3k-2} & \xrightarrow[\sim]{\mathbf{res}} & (\alpha_1+\alpha_2)D(\mathcal{A}_\Phi, 2k-1)_{3k-2}, \end{array}$$

where $\mathbf{m} : \mathcal{A}_\Phi \to \{0,1\}$ is a multiplicity map defined by

$$\mathbf{m}(H) = \begin{cases} 1 & H \in \{H_{\alpha_1}, H_{\alpha_2}\} \\ 0 & H = H_{\alpha_1+\alpha_2} \end{cases} \quad (H \in \mathcal{A}_\Phi).$$

Let

$$\eta := 6(\alpha_1+\alpha_2-kz)\eta_1^{(k-1)} = \frac{\psi_1^{(k)}}{\alpha_1-kz} - \frac{\psi_2^{(k)}}{\alpha_2-kz}.$$

Then it follows from Proposition 5.2.7 (2) that η is a regular derivation and $\eta \in D_0(\mathbf{c}\mathrm{Shi}^k \setminus \{H_{\alpha_1-kz}, H_{\alpha_2-kz}\})_{3k-1}$. By the definition of SRB_-, we have

$$\begin{aligned} \frac{1}{\alpha_1+\alpha_2}\mathbf{res}(\eta) &= \frac{1}{\alpha_1+\alpha_2}\mathbf{res}\left(\frac{\psi_1^{(k)}}{\alpha_1-kz} - \frac{\psi_2^{(k)}}{\alpha_2-kz}\right) \\ &= \frac{1}{\alpha_1+\alpha_2}\left(\frac{\nabla_{I^*(d\alpha_1)}\nabla_D^{-k}\theta_E}{\alpha_1} - \frac{\nabla_{I^*(d\alpha_2)}\nabla_D^{-k}\theta_E}{\alpha_2}\right) \\ &= \nabla_{I^*\left(\frac{1}{\alpha_1+\alpha_2}(\frac{d\alpha_1}{\alpha_1}-\frac{d\alpha_2}{\alpha_2})\right)}\nabla_D^{-k}\theta_E. \end{aligned}$$

Since

$$\frac{1}{\alpha_1+\alpha_2}(\frac{d\alpha_1}{\alpha_1} - \frac{d\alpha_2}{\alpha_2}) \in \Omega^1(\mathcal{A}_\Phi),$$

Lemma 5.3.3 implies that

$$\frac{1}{\alpha_1+\alpha_2}\mathbf{res}(\eta) \in D(\mathcal{A}_\Phi, 2k-1)_{3k-2}.$$

Hence

$$\mathbf{res}(\eta) \in (\alpha_1+\alpha_2)D(\mathcal{A}_\Phi, 2k-1)_{3k-2}.$$

Then we can see that $\eta \in (\alpha_1+\alpha_2-kz)D_0(\mathbf{c}\mathrm{Cat}^{k-1})_{3k-2}$ by chasing the diagram above. Thus we may conclude that $\eta_1^{(k-1)} \in D_0(\mathbf{c}\mathrm{Cat}^{k-1})_{3k-2}$. Since $D_0(\mathbf{c}\mathrm{Cat}^{k-1})_{3k-2} = D_0(\mathbf{c}\mathrm{Cat}^{k-1})^W_{3k-2}$ is the one-dimensional $\mathbb{R}$-vector space generated by $(k-1)$-Euler derivation by Proposition 5.2.9, we obtain $\eta_1^{(k-1)} \in D_0(\mathbf{c}\mathrm{Cat}^{k-1})^W$. Next we will prove that $\eta_2^{(k-1)} \in D_0(\mathbf{c}\mathrm{Cat}^{k-1})^W$. We compute

$$\begin{aligned}
\eta_2^{(k-1)} &= \frac{4\alpha_1+2\alpha_2-3kz}{6(\alpha_1-kz)(\alpha_1+\alpha_2-kz)}\psi_1^{(k)} + \frac{2\alpha_1+4\alpha_2-3kz}{6(\alpha_2-kz)(\alpha_1+\alpha_2-kz)}\psi_2^{(k)} \\
&= (4\alpha_1+2\alpha_2-3kz)\left(\eta_1^{(k-1)} + \frac{\psi_2^{(k)}}{6(\alpha_2-kz)(\alpha_1+\alpha_2-kz)}\right) \\
&\qquad + \frac{2\alpha_1+4\alpha_2-3kz}{6(\alpha_2-kz)(\alpha_1+\alpha_2-kz)}\psi_2^{(k)} \\
&= (4\alpha_1+2\alpha_2-3kz)\eta_1^{(k-1)} + \frac{\psi_2^{(k)}}{\alpha_2-kz}.
\end{aligned}$$

Since $\psi_2^{(k)}/(\alpha_2-kz) \in D_0(\mathbf{c}\mathrm{Shi}^k \setminus \{H_{\alpha_2-kz}\}) \subset D_0(\mathbf{c}\mathrm{Cat}^{k-1})$, $\eta_2^{(k-1)}$ belongs to $D_0(\mathbf{c}\mathrm{Cat}^{k-1})$. Moreover, since $s_i(\psi_i^{(k)}/(\alpha_i-kz)) = (\psi_i^{(k)}/(\alpha_i-kz))$ for the reflection s_i with respect to α_i because of Proposition 5.2.10 (2),

$$\begin{aligned}
s_2\eta_2^{(k-1)} &= s_2(4\alpha_1+2\alpha_2-3kz)\cdot s_2\eta_1^{(k-1)} + s_2\left(\frac{\psi_2^{(k)}}{\alpha_2-kz}\right) \\
&= (4\alpha_1+2\alpha_2-3kz)\eta_1^{(k-1)} + \frac{\psi_2^{(k)}}{\alpha_2-kz} = \eta_2^{(k-1)}.
\end{aligned}$$

Similarly, we can express $\eta_2^{(k-1)}$ in terms of $\eta_1^{(k-1)}$ and $\psi_1^{(k)}$. Then the same argument as the above shows that $s_1\eta_2^{(k-1)} = \eta_2^{(k-1)}$. Hence $\eta_2^{(k-1)}$ is W-invariant. Finally, since

$$\det(N_k^{-1}) = \frac{1}{6(\alpha_1-kz)(\alpha_2-kz)(\alpha_1+\alpha_2-kz)},$$

and $\psi_1^{(k)}, \psi_2^{(k)}$ form a basis for $D_0(\mathbf{c}\mathrm{Shi}^k)$, it holds that $\eta_1^{(k-1)}, \eta_2^{(k-1)}$ form a basis for $D_0(\mathbf{c}\mathrm{Cat}^{k-1})$. □

Proposition 5.3.5. *Let $\theta_1^{(k)}, \eta_1^{(k-1)}$ be as in Proposition 5.3.2 and 5.3.4. Then $\theta_1^{(k)}, \eta_1^{(k-1)}$ are W_z-invariant.*

Proof. First we note that the action of the reflection τ_z with respect to z preserves $\mathbf{c}\mathrm{Cat}^k$. Hence τ_z acts on $D_0(\mathbf{c}\mathrm{Cat}^k)$. Since $D_0(\mathbf{c}\mathrm{Cat}^k)_{3k+1} = D_0(\mathbf{c}\mathrm{Cat}^k)^W_{3k+1}$ is the one-dimensional $\mathbb{R}$-vector space generated by k-Euler derivation $\theta_1^{(k)}$, we can express $\tau_z\theta_1^{(k)} = c\theta_1^{(k)}$ for some $c \in \mathbb{R}^\times$. Then $\theta_1^{(k)}|_{z=0} = \tau_z\theta_1^{(k)}|_{z=0} = c\theta_1^{(k)}|_{z=0}$. Thus $c = 1$ and we conclude that $\theta_1^{(k)}$ is W_z-invariant. Similarly, we can check that $\eta_1^{(k-1)}$ is W_z-invariant. □

Proposition 5.3.6. *Let* $\varphi_1^{(k)}, \varphi_2^{(k)}$ *be an* SRB_+ *and* $\psi_1^{(k)}, \psi_2^{(k)}$ *an* SRB_- *of* $D_0(\mathbf{c}\mathrm{Shi}^k)$. *Then*

$$\tau_z\varphi_1^{(k)} = \frac{(\alpha_1+\alpha_2)(\alpha_1+kz)\varphi_1^{(k)} - kz(\alpha_2+kz)\varphi_2^{(k)}}{(\alpha_1-kz)(\alpha_1+\alpha_2-kz)}, \tag{5.1}$$

$$\tau_z\varphi_2^{(k)} = \frac{-kz(\alpha_1+kz)\varphi_1^{(k)} + (\alpha_1+\alpha_2)(\alpha_2+kz)\varphi_2^{(k)}}{(\alpha_2-kz)(\alpha_1+\alpha_2-kz)}, \tag{5.2}$$

and

$$\tau_z\psi_1^{(k)} = \frac{\alpha_1+kz}{\alpha_1+\alpha_2-kz}\left(\frac{\alpha_1+\alpha_2}{\alpha_1-kz}\psi_1^{(k)} - \frac{kz}{\alpha_2-kz}\psi_2^{(k)}\right), \tag{5.3}$$

$$\tau_z\psi_2^{(k)} = \frac{\alpha_2+kz}{\alpha_1+\alpha_2-kz}\left(-\frac{kz}{\alpha_1-kz}\psi_1^{(k)} + \frac{\alpha_1+\alpha_2}{\alpha_2-kz}\psi_2^{(k)}\right). \tag{5.4}$$

Proof. By Proposition 5.3.5, $\theta_1^{(k)}, \eta_1^{(k-1)}$ are W_z-invariant. Thus we have two equations:

$$(\alpha_1+kz)\varphi_1^{(k)} + (\alpha_2+kz)\varphi_2^{(k)} = (\alpha_1-kz)\tau_z\varphi_1^{(k)} + (\alpha_2-kz)\tau_z\varphi_2^{(k)}, \tag{5.5}$$

$$\begin{aligned}\frac{1}{6(\alpha_1+\alpha_2-kz)}&\left(\frac{\psi_1^{(k)}}{\alpha_1-kz} - \frac{\psi_2^{(k)}}{\alpha_2-kz}\right)\\ &= \frac{1}{6(\alpha_1+\alpha_2+kz)}\left(\frac{\tau_z\psi_1^{(k)}}{\alpha_1+kz} - \frac{\tau_z\psi_2^{(k)}}{\alpha_2+kz}\right).\end{aligned} \tag{5.6}$$

By Remark 5.2.6, we have

$$[\psi_1^{(k)}, \psi_2^{(k)}] = [\varphi_1^{(k)}, \varphi_2^{(k)}]A = [\varphi_1^{(k)}, \varphi_2^{(k)}]\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{5.7}$$

Therefore we can rewrite the equation (5.6) by using $\varphi_1^{(k)}, \varphi_2^{(k)}$ instead of $\psi_1^{(k)}, \psi_2^{(k)}$, and solving the equations (5.5) and (5.6), we get (5.1) and (5.2). Applying τ_z to the both sides of (5.7) we obtain

$$\begin{aligned}[\tau_z\psi_1^{(k)}, \tau_z\psi_2^{(k)}] &= [\tau_z\varphi_1^{(k)}, \tau_z\varphi_2^{(k)}]A\\ &= [\varphi_1^{(k)}, \varphi_2^{(k)}]\begin{pmatrix} \frac{(\alpha_1+\alpha_2)(\alpha_1+kz)}{(\alpha_1-kz)(\alpha_1+\alpha_2-kz)} & \frac{-kz(\alpha_1+kz)}{(\alpha_2-kz)(\alpha_1+\alpha_2-kz)} \\ \frac{-kz(\alpha_2+kz)}{(\alpha_1-kz)(\alpha_1+\alpha_2-kz)} & \frac{(\alpha_1+\alpha_2)(\alpha_2+kz)}{(\alpha_2-kz)(\alpha_1+\alpha_2-kz)} \end{pmatrix} A\end{aligned}$$

$$= [\psi_1^{(k)}, \psi_2^{(k)}]A^{-1}\begin{pmatrix} \dfrac{(\alpha_1+\alpha_2)(\alpha_1+kz)}{(\alpha_1-kz)(\alpha_1+\alpha_2-kz)} & \dfrac{-kz(\alpha_1+kz)}{(\alpha_2-kz)(\alpha_1+\alpha_2-kz)} \\ \dfrac{-kz(\alpha_2+kz)}{(\alpha_1-kz)(\alpha_1+\alpha_2-kz)} & \dfrac{(\alpha_1+\alpha_2)(\alpha_2+kz)}{(\alpha_2-kz)(\alpha_1+\alpha_2-kz)} \end{pmatrix} A$$

$$= [\psi_1^{(k)}, \psi_2^{(k)}]\begin{pmatrix} \dfrac{(\alpha_1+\alpha_2)(\alpha_1+kz)}{(\alpha_1-kz)(\alpha_1+\alpha_2-kz)} & \dfrac{-kz(\alpha_2+kz)}{(\alpha_1-kz)(\alpha_1+\alpha_2-kz)} \\ \dfrac{-kz(\alpha_1+kz)}{(\alpha_2-kz)(\alpha_1+\alpha_2-kz)} & \dfrac{(\alpha_1+\alpha_2)(\alpha_2+kz)}{(\alpha_2-kz)(\alpha_1+\alpha_2-kz)} \end{pmatrix},$$

thus we have (5.3) and (5.4). □

Proposition 5.3.7. *Let $\theta_2^{(k)}, \eta_2^{(k-1)}$ be as in Proposition 5.3.2 and 5.3.4. Then $\theta_2^{(k)}, \eta_2^{(k-1)}$ are W_z-invariant.*

Proof. The W-invariance is checked in Proposition 5.3.2 and 5.3.4. The τ_z-invariance follows by the direct computation combined with Proposition 5.3.6. □

It follows from Proposition 5.3.2 and 5.3.4 that both $[\varphi_1^{(k)}, \varphi_2^{(k)}]M_k$ and $[\psi_1^{(k+1)}, \psi_2^{(k+1)}]N_{k+1}^{-1}$ are bases for $D_0(\mathbf{c}\mathrm{Cat}^k)^{W_z}$ and their exponents are equal to $(3k+1, 3k+2)$ as ordered sets. Therefore, there exists a matrix $T_k \in M_2(\mathbb{R}[\alpha_1, \alpha_2, z])$ such that

$$[\varphi_1^{(k)}, \varphi_2^{(k)}]M_k \cdot T_k = [\psi_1^{(k+1)}, \psi_2^{(k+1)}]N_{k+1}^{-1} = [\varphi_1^{(k+1)}, \varphi_2^{(k+1)}]AN_{k+1}^{-1}.$$

Note that every entry of T_k is W_z-invariant since T_k gives a transformation between the W_z-invariant bases in $D_0(\mathbf{c}\mathrm{Cat}^k)^W$. Comparing the degrees of both sides, we can see that the $(2,1)$-entry of T_k is 0, the $(1,1)$-entry and the $(2,2)$-entry of T_k are constants, and the $(1,2)$-entry of T_k is a polynomial of degree 1. Furthermore, the $(1,2)$ entry of T_k is 0 because it must be W_z-invariant but there is no polynomial of degree 1 in $\mathbb{R}[\alpha_1, \alpha_2, z]^{W_z}$. Hence we may assume that

$$T_k = \begin{pmatrix} a_k & 0 \\ 0 & b_k \end{pmatrix} \quad (a_k, b_k \in \mathbb{R}).$$

Hence $T_k|_{z=0} = T_k$ and

$$[\varphi_1^{(k)}, \varphi_2^{(k)}]|_{z=0}M_k|_{z=0} \cdot T_k = [\varphi_1^{(k+1)}, \varphi_2^{(k+1)}]|_{z=0}AN_{k+1}^{-1}|_{z=0}.$$

Now recall the following:

Theorem 5.3.8. *(T. Abe-H. Terao [2]) Define*

$$R_{2k} := (-1)^k J(D^k(\alpha_1), D^k(\alpha_2))^{-1},$$

where $J(f,g)$ denotes the Jacobian matrix of α,β with respect to the simple system α_1,α_2. Then

$$[\varphi_1^{(k)}|_{z=0},\varphi_2^{(k)}|_{z=0}] = [\nabla_{\partial_1}\nabla_D^{-k}\theta_E, \nabla_{\partial_2}\nabla_D^{-k}\theta_E] = [\partial_1,\partial_2]AR_{2k}A^{-1}.$$

By using these two, let us compute T_k directly in terms of $D(\mathcal{A}_\Phi, 2k+1)$. For that purpose, let us rewrite several polynomials and matrices in [3] in terms of α_1 and α_2. First, it is easy to check that

$$P_1 = \alpha_1^2+\alpha_1\alpha_2+\alpha_2^2,\ P_2 = \frac{2}{27}(\alpha_1-\alpha_2)(\alpha_1+2\alpha_2)(2\alpha_1+\alpha_2).$$

are basic invariants of the type A_2. Let ∂_1,∂_2 denote the dual basis of $\{\alpha_1,\alpha_2\}$ for $\mathrm{Der}(S)$. Then the Jacobian matrix $J := (\partial P_j/\partial\alpha_i)$ is

$$J = \begin{pmatrix} 2\alpha_1+\alpha_2 & \frac{2}{9}(2\alpha_1^2+2\alpha_1\alpha_2-\alpha_2^2) \\ \alpha_1+2\alpha_2 & \frac{2}{9}(\alpha_1^2-2\alpha_1\alpha_2-2\alpha_2^2) \end{pmatrix}.$$

Hence the primitive derivation D is expressed as

$$\begin{aligned} D &= \frac{1}{Q}\begin{vmatrix} \partial_1(P_1) & \partial_1 \\ \partial_2(P_1) & \partial_2 \end{vmatrix} \\ &\doteq \frac{1}{6\alpha_1\alpha_2(\alpha_1+\alpha_2)}[(\alpha_1+2\alpha_2)\partial_1-(2\alpha_1+\alpha_2)\partial_2], \end{aligned}$$

where $Q=\alpha_1\alpha_2(\alpha_1+\alpha_2)$ is the defining polynomial of the Weyl arrangement of the type A_2. Also in the above, we multiplied $-1/6$ to D to satisfy the condition $D(P_2) = 1/3$ in Theorem 5.3.1. For a matrix $M = (m_{ij})$, let $D[M] := (D(m_{ij}))$. Then we can compute

$$D[J] = \frac{1}{18\alpha_1\alpha_2(\alpha_1+\alpha_2)}\begin{pmatrix} 9\alpha_2 & 4\alpha_2(2\alpha_1+\alpha_2) \\ -9\alpha_1 & 4\alpha_1(\alpha_1+2\alpha_2) \end{pmatrix},$$

Moreover, the matrix $B := J^TAD[J]$ and $B^{(k)} := kB+(k-1)B^T$ are also computed as follows:

$$B = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},\ B^{(k)} = \begin{pmatrix} 0 & 3k-1 \\ 3k-2 & 0 \end{pmatrix}$$

Hence

$$(B^{(k)})^{-1} = \begin{pmatrix} 0 & \frac{1}{3k-2} \\ \frac{1}{3k-1} & 0 \end{pmatrix}.$$

Now by using Theorem 5.3.8, we can determine the matrix T_k.

Proposition 5.3.9.

$$T_k = \begin{pmatrix} \dfrac{1}{3k+1} & 0 \\ 0 & \dfrac{1}{3k+2} \end{pmatrix}.$$

Proof. First recall that

$$[\varphi_1^{(k)}, \varphi_2^{(k)}]M_k T_k = [\varphi_1^{(k+1)}, \varphi_2^{(k+1)}]AN_{k+1}^{-1}.$$

Restricting the equality above onto $z = 0$ and applying Theorem 5.3.8, we obtain

$$AR_{2k}A^{-1}(M_k|_{z=0})(T_k|_{z=0}) = AR_{2k+2}A^{-1}A(N_{k+1}|_{z=0})^{-1}.$$

Therefore,

$$T_k|_{z=0} = (M_k|_{z=0})^{-1}AR_{2k}^{-1}R_{2k+2}(N_{k+1}|_{z=0})^{-1}.$$

By Proposition 2.6 in [2],

$$R_{2k}^{-1}R_{2k+2} = J(B^{(k+1)})^{-1}J^T A.$$

Now we can compute $T_{k+1}|_{z=0}$ directly as follows:

$$\begin{aligned} T_k|_{z=0} &= (M_k|_{z=0})^{-1}AJ(B^{(k+1)})^{-1}J^T A(N_{k+1}|_{z=0})^{-1} \\ &= \begin{pmatrix} \dfrac{1}{3k+1} & 0 \\ 0 & \dfrac{1}{3k+2} \end{pmatrix}. \end{aligned}$$

□

Proof of Theorem 5.3.1. Combine Propositions 5.3.2, 5.3.4 and 5.3.9.

Proof of Theorem 5.1.1. First, note that P_1 and P_2 are unique up to nozero-constant when Φ is of the type A_2 since there is no invariant polynomial of degree one. Therefore the construction in Theorem 5.3.1 shows that for any choice of P_1, P_2 and D, the bases constructed by them are unique up to nonzero constants. Moreover, we can connect the SRB_+ and SRB_- using the inner product matrix A as Remark 5.2.6. Hence we may apply Theorem 5.3.1 starting from $[\partial_1, \partial_2]$ inductively to obtain the bases stated in Theorem 2.3.5, which completes the proof.

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