Geometric study of Lauricella's hypergeometric function F_C

(Lauricella の超幾何関数 F_C に関する幾何学的研究)

by

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Abstract

We study Lauricella's hypergeometric function F_C of m-variables by using twisted (co)homology groups. We construct twisted cycles with respect to an integral representation of Euler type of F_C. These cycles correspond to 2^m linearly independent solutions to the system E_C of differential equations annihilating F_C. Using intersection forms of twisted (co)homology groups, we obtain twisted period relations which give quadratic relations for Lauricella's F_C.

We provide the monodromy representation of the system E_C. We give generators of the fundamental group of the complement of the singular locus of E_C. We represent the circuit transformations along these generators by the intersection form on twisted homology groups.

key words and phrases: Hypergeometric function, Lauricella's F_C, Twisted (co)homology groups, Twisted cycles, Twisted period relations, Monodromy representations.

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Contents

Introduction

Lauricella's hypergeometric series F_C of m-variables $x_1, \ldots, x_m$ with complex parameters $a, b, c_1, \ldots, c_m$ is defined by

$$F_C(a, b, c; x) = \sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{(a, n_1 + \cdots + n_m)(b, n_1 + \cdots + n_m)}{(c_1, n_1) \cdots (c_m, n_m) n_1! \cdots n_m!} x_1^{n_1} \cdots x_m^{n_m},$$

where $x = (x_1, \ldots, x_m)$, $c = (c_1, \ldots, c_m)$, $c_1, \ldots, c_m \notin \{0, -1, -2, \ldots\}$ and $(c_1, n_1) = \Gamma(c_1 + n_1)/\Gamma(c_1)$. This series converges in the domain

$$D_C := \left\{ (x_1, \ldots, x_m) \in \mathbb{C}^m \;\middle|\; \sum_{k=1}^{m} \sqrt{|x_k|} < 1 \right\},$$

and admits the integral representation (0.4). The system $E_C(a, b, c)$ of differential equations (0.1) annihilating $F_C(a, b, c; x)$ is a holonomic system of rank 2^m with the singular locus S given in (0.2). There is a fundamental system of solutions to $E_C(a, b, c)$ in a simply connected domain in $D_C - S$, which is given in terms of Lauricella's hypergeometric series F_C with different parameters, see (0.3) for their expressions.

In the case of $m = 2$, the series $F_C(a, b, c; x)$ and the system $E_C(a, b, c)$ are called Appell's hypergeometric series $F_4(a, b, c; x)$ and system $E_4(a, b, c)$ of differential equations. The monodromy representation of $E_4(a, b, c)$ is studied in [11] and [18]. In [11], an explicit expression of the fundamental group of the complement of the singular locus

$$S = \left(x_1 x_2 (x_1^2 + x_2^2 - 2x_1 x_2 - 2x_1 - 2x_2 + 1) = 0 \right) \subset \mathbb{C}^2$$

is also given. In [9], we study F_4 and E_4 by twisted (co)homology groups concerning with the integral representation (0.4) for $m = 2$. We construct twisted cycles corresponding to four solutions to $E_4(a, b, c)$ expressed by the series F_4, and evaluate their intersection numbers. We also evaluate the intersection matrix for a basis of the twisted cohomology group. By using these results, we determine the monodromy representation of $E_4(a, b, c)$ and give twisted period relations, which are quadratic relations between two fundamental systems of E_4 with different parameters.

In this paper, we study Lauricella's F_C of m-variables geometrically, by twisted (co)homology groups and intersection forms on these groups. First, we construct 2^m twisted cycles corresponding to the solutions to $E_C(a, b, c)$ represented by the series F_C, and evaluate the intersection matrix for them. We also evaluate the intersection numbers of some elements of the twisted cohomology groups. These intersection numbers give twisted period relations which are quadratic relations for Lauricella's F_C. Next, we investigate the monodromy representation of $E_C(a, b, c)$. We give generators of the fundamental group of the complement of the singular locus S, and obtain some relations among these generators. By using the intersection form on twisted homology groups, we provide the circuit transformations along these generators. We explain these results in four chapters.

In Chapter 1, we explain the construction of the twisted cycles which represent elements in the m-th twisted homology group concerning with the integral

representation (0.4). In the study of twisted homology groups, twisted cycles given by bounded chambers are useful. However there are few bounded chambers in our case. By avoiding this difficulty, we succeed in constructing 2^m twisted cycles. We explain our idea in the construction. Our twisted homology group is defined by the multi-valued function

$$u(t) = \prod_{k=1}^{m} t_k^{1-c_k+b} \cdot v^{\sum_{k=1}^{m} c_k - a - m + 1} \cdot w^{-b},$$
$$v = 1 - \sum_{k=1}^{m} t_k,\ w = \prod_{k=1}^{m} t_k \cdot \left(1 - \sum_{k=1}^{m} \frac{x_k}{t_k}\right),$$

for fixed small positive real numbers $x_1, \ldots, x_m$. Let $\{i_1, \ldots, i_r\}$ be a subset of $\{1, \ldots, m\}$ of cardinality r. We embed the direct product of an r-simplex and an $(m-r)$-simplex into the bounded chamber

$$\{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid t_1, \ldots, t_m,\ v,\ w > 0\}.$$

As is in Section 3.2.4 of [1], we can eliminate the boundary, and obtain the twisted cycle $\Delta_{i_1 \cdots i_r}$. In this elimination, we use a different way from the usual regularization. We realize it by using the twisted homology group defined by another multi-valued function, see Section 1.2 for details. Our first main theorem (Theorem 1.4) states that the twisted cycle $\Delta_{i_1 \cdots i_r}$ corresponds to the solution to $E_C(a, b, c)$ with the power function $\prod_{p=1}^{r} x_{i_p}^{1-c_{i_p}}$. By our first main theorem and the proof of Lemma 4.1 in [9], it turns out that the intersection matrix becomes diagonal. Moreover, our construction and results in [13] enable us to evaluate the diagonal entries of the intersection matrix.

In Chapter 2, we give twisted period relations. We evaluate the intersection numbers of some elements of twisted cohomology groups. These numbers and the intersection matrix for bases of twisted homology groups imply twisted period relations which are originally identities among the integrals given by the pairings of elements in twisted homology and cohomology groups. Our first main theorem transforms these identities into quadratic relations among hypergeometric series F_C's. Our second main theorem (Theorem 2.5) states these formulas in Section 2.2.

In Chapter 3, we study the fundamental group of the complement X of the singular locus S. Our third main theorem (Theorem 3.2) states that the fundamental group of X is generated by $m+1$ loops. We denote them by $\rho_0,\ \rho_1, \ldots,\ \rho_m$. Here ρ_k $(1 \leq k \leq m)$ turns the divisor $(x_k = 0)$, and ρ_0 turns the divisor

$$\prod_{\varepsilon_1, \ldots, \varepsilon_m = \pm 1} \left(1 + \sum_k \varepsilon_k \sqrt{x_k}\right) = 0$$

around the point $(\frac{1}{m^2}, \ldots, \frac{1}{m^2})$, positively. We show this claim by applying the Zariski theorem of Lefschetz type. Moreover, the proof of this theorem gives some relations among these generators.

In Chapter 4, we study the monodromy representation of $E_C(a, b, c)$ by using twisted homology groups and the intersection form. By the third main theorem, it is sufficient to investigate the circuit transformations $\mathcal{M}_i$ along ρ_i, where $0 \leq i \leq m$. By using twisted cycles constructed in Chapter 1, we obtain

the representation matrices of $\mathcal{M}_1, \ldots, \mathcal{M}_m$ easily. However, it is difficult to represent $\mathcal{M}_0$ with respect to these twisted cycles. Our fourth main theorem (Theorem 4.6) states that $\mathcal{M}_0$ has the following properties:

- The eigenvalues of $\mathcal{M}_0$ are $(-1)^{m-1}\prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}$ and 1.
- The eigenspace W_0 of $\mathcal{M}_0$ of eigenvalue $(-1)^{m-1}\prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}$ is one-dimensional, and spanned by the twisted cycle $D_{1\cdots m}$ defined by the bounded chamber

$$\{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid t_1, \ldots, \ t_m, \ v, \ w > 0\}.$$

- The eigenspace W_1 of $\mathcal{M}_0$ of eigenvalue 1 is $(2^m - 1)$-dimensional, and expressed as

$$W_1 = \{\delta \in H_m(T_{\dot{x}}, u_{\dot{x}}) \mid I_h(\delta, D^{\vee}_{1\cdots m}) = 0\},$$

  where I_h is the intersection form on twisted homology groups.

As a corollary, we can express the linear map $\mathcal{M}_0$ by using the intersection form. To represent $\mathcal{M}_0$ by a matrix with respect to a given basis, it is sufficient to evaluate some intersection numbers. We also make use of the intersection form to prove the fourth main theorem. The twisted cycles Δ_I's constructed in Chapter 1 play a key role, in the proof. Further, we write down the representation matrices of $\mathcal{M}_0, \ldots, \mathcal{M}_m$, for $m = 2, 3$.

Preliminaries

We collect some facts about Lauricella's F_C and the system E_C of differential equations annihilating it.

Notation 0.1. (i) Throughout this paper, the letter k always stands for an index running from 1 to m. If no confusion is possible, $\sum_{k=1}^m$ and $\prod_{k=1}^m$ are often simply denoted by $\sum$ (or $\sum_k$) and $\prod$ (or $\prod_k$), respectively. For example, under this convention $F_C(a, b, c; x)$ is expressed as

$$F_C(a, b, c; x) = \sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{(a, \sum n_k)(b, \sum n_k)}{\prod (c_k, n_k) \cdot \prod n_k!} \prod x_k^{n_k}.$$

(ii) For a subset I of $\{1, \ldots, m\}$, we denote the cardinality of I by $|I|$.

Let ∂_k $(1 \leq k \leq m)$ be the partial differential operator with respect to x_k. We set $\theta_k := x_k \partial_k$, $\theta := \sum_k \theta_k$. Lauricella's $F_C(a, b, c; x)$ satisfies differential equations

$$[\theta_k(\theta_k + c_k - 1) - x_k(\theta + a)(\theta + b)]\, f(x) = 0, \quad 1 \leq k \leq m. \tag{0.1}$$

The system generated by them is called Lauricella's hypergeometric system $E_C(a, b, c)$ of differential equations.

Fact 0.2 ([10], [14])**.** *The system $E_C(a,b,c)$ is a holonomic system of rank 2^m with the singular locus*

$$S := \left(\prod_k x_k \cdot \prod_{\varepsilon_1,\dots,\varepsilon_m=\pm 1} \left(1 + \sum_k \varepsilon_k \sqrt{x_k} \right) = 0 \right) \subset \mathbb{C}^m. \tag{0.2}$$

If $c_1,\dots,c_m \notin \mathbb{Z}$, then the vector space of solutions to $E_C(a,b,c)$ in a simply connected domain in $D_C - S$ is spanned by the following 2^m elements:

$$f_{i_1\cdots i_r} := \prod_{p=1}^{r} x_{i_p}^{1-c_{i_p}} \cdot F_C\left(a + r - \sum_{p=1}^{r} c_{i_p}, b + r - \sum_{p=1}^{r} c_{i_p}, c^{i_1\cdots i_r}; x \right). \tag{0.3}$$

Here r runs from 0 to m, indices $i_1,\dots,i_r$ satisfy $1 \le i_1 < \cdots < i_r \le m$, and the row vector $c^{i_1\cdots i_r}$ is defined by

$$c^{i_1\cdots i_r} := c + 2\sum_{p=1}^{r} (1 - c_{i_p}) e_{i_p},$$

where e_i is the i-th unit row vector of $\mathbb{C}^m$.

For the above $i_1,\dots,i_r$, we take $j_1,\dots,j_{m-r}$ so that $1 \le j_1 < \cdots < j_{m-r} \le m$ and $\{i_1,\dots,i_r,j_1,\dots,j_{m-r}\} = \{1,\dots,m\}$. It is easy to see that the i_p-th entry of $c^{i_1\cdots i_r}$ is $2 - c_{i_p}$ $(1 \le p \le r)$ and the j_q-th entry is c_{j_q} $(1 \le q \le m-r)$.

We denote the multi-index " $i_1 \cdots i_r$" by a letter I expressing the set $\{i_1,\dots,i_r\}$. For example, we also write

$$f_{i_1\cdots i_r} = f_I = \prod_{i\in I} x_i^{1-c_i} \cdot F_C\left(a + |I| - \sum_{i\in I} c_i, b + |I| - \sum_{i\in I} c_i, c^I; x \right).$$

Note that the solution (0.3) for $r = 0$ is $f(= f_\emptyset) = F_C(a,b,c;x)$.

Fact 0.3 (Integral representation of Euler type, Example 3.1 in [1])**.** *For sufficiently small positive real numbers $x_1,\dots,x_m$, if $c_1,\dots,c_m, a - \sum c_k \notin \mathbb{Z}$, then $F_C(a,b,c;x)$ admits the following integral representation:*

$$\begin{aligned} &F_C(a,b,c;x) \\ &= \frac{\Gamma(1-a)}{\prod \Gamma(1-c_k) \cdot \Gamma(\sum c_k - a - m - 1)} \\ &\quad \cdot \int_\Delta \prod t_k^{-c_k} \cdot (1 - \sum t_k)^{\sum c_k - a - m} \cdot \left(1 - \sum \frac{x_k}{t_k} \right)^{-b} dt_1 \wedge \cdots \wedge dt_m, \end{aligned} \tag{0.4}$$

where Δ is the twisted cycle made by an m-simplex in Sections 3.2 and 3.3 of [1].

In fact, this cycle is one of twisted cycles constructed in Section 1.2.

Chapter 1

Twisted cycles

In this chapter, we construct 2^m twisted cycles which represent elements of the m-th twisted homology group concerning with the integral representation (0.4). They imply integral representations of the solutions (0.3) expressed by the series F_C. We evaluate the intersection numbers of these 2^m twisted cycles. Further, we give relations between our twisted cycles and those defined in [9].

As is in [10], the irreducibility condition of the system $E_C(a,b,c)$ is known to be

$$a - \sum_{p=1}^{r} c_{i_r}, \quad b - \sum_{p=1}^{r} c_{i_r} \notin \mathbb{Z}$$

for any subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, m\}$. Since our interest is in the property of solutions to $E_C(a,b,c)$ expressed in terms of the hypergeometric series F_C, we assume throughout this and the next chapters that the parameters a, b, and $c = (c_1, \ldots, c_m)$ satisfy the conditions above and $c_1, \ldots, c_m \notin \mathbb{Z}$.

1.1 Twisted homology groups

We review twisted homology groups and the intersection form between twisted homology groups in general situations, by referring to Chapter 2 of [1] and Chapters IV, VIII of [19].

For polynomials $P_j(t) = P_j(t_1, \ldots, t_m)$ $(1 \leq j \leq n)$, we set $D_j := \{t \mid P_j(t) = 0\} \subset \mathbb{C}^m$ and $M := \mathbb{C}^m - (D_1 \cup \cdots \cup D_n)$. We consider a multi-valued function $u(t)$ on M defined as

$$u(t) := \prod_{j=1}^{n} P_j(t)^{\lambda_j}, \ \lambda_j \in \mathbb{C} - \mathbb{Z} \ (1 \leq j \leq n).$$

Let σ be a k-simplex in M, we define a loaded k-simplex $\sigma \otimes u$ by σ loading a branch of u on it. We denote the $\mathbb{C}$-vector space of finite sums of loaded k-simplexes by $\mathcal{C}_k(M,u)$, called the k-th twisted chain group. An element in $\mathcal{C}_k(M,u)$ is called a twisted k-chain. For a loaded k-simplex $\sigma \otimes u$ and a smooth k-form φ on M, the integral $\int_{\sigma \otimes u} u \cdot \varphi$ is defined by

$$\int_{\sigma \otimes u} u \cdot \varphi := \int_{\sigma} [\text{the fixed branch of } u \text{ on } \sigma] \cdot \varphi.$$

By the linear extension of this, we define the integral on a twisted k-chain.

We define the boundary operator $\partial^u : \mathcal{C}_k(M,u) \to \mathcal{C}_{k-1}(M,u)$ by

$$\partial^u(\sigma \otimes u) := \partial(\sigma) \otimes u|_{\partial(\sigma)},$$

where ∂ is the usual boundary operator and $u|_{\partial(\sigma)}$ is the restriction of u to $\partial(\sigma)$. It is easy to see that $\partial^u \circ \partial^u = 0$. Thus we have a complex

$$\mathcal{C}_\bullet(M,u) : \cdots \xrightarrow{\partial^u} \mathcal{C}_k(M,u) \xrightarrow{\partial^u} \mathcal{C}_{k-1}(M,u) \xrightarrow{\partial^u} \cdots ,$$

and its k-th homology group $H_k(M,u)$. It is called the k-th twisted homology group. An element in $\ker \partial^u$ is called a twisted cycle. By replacing $\mathcal{C}_k(M,u)$ with the $\mathbb{C}$-vector space $\mathcal{C}_k^{lf}(M,u)$ of locally finite sums of loaded k-simplexes, we obtain the locally finite twisted homology group $H_k^{lf}(M,u)$.

By considering $u^{-1} = 1/u$ instead of u, we have $H_k(M,u^{-1})$. There is the intersection form I_h between $H_m(M,u)$ and $H_m(M,u^{-1})$ (in fact, the intersection form is defined between $H_k(M,u)$ and $H_{2m-k}(M,u^{-1})$, however we do not consider the cases $k \neq m$). Let δ and δ' be elements in $H_m(M,u)$ and $H_m(M,u^{-1})$ given by twisted cycles $\sum_i \alpha_i \cdot \sigma_i \otimes u_i$ and $\sum_j \alpha'_j \cdot \sigma'_j \otimes u_j^{-1}$ respectively, where u_i (resp. u_j^{-1}) is a branch of u (resp. u^{-1}) on σ_i (resp. σ'_j). Then their intersection number is defined by

$$I_h(\delta,\delta') := \sum_{i,j} \sum_{s \in \sigma_i \cap \sigma'_j} \alpha_i \alpha'_j \cdot (\sigma_i \cdot \sigma'_j)_s \cdot \frac{u_i(s)}{u_j(s)},$$

where $(\sigma_i \cdot \sigma'_j)_s$ is the topological intersection number of m-simplexes σ_i and σ'_j at s.

In this paper, we mainly consider

$$M := \mathbb{C}^m - (H_1 \cup \cdots \cup H_m \cup H \cup D),$$

where

$$H_k := (t_k = 0)\ (1 \le k \le m),\ H := (v = 0),\ D := (w = 0),$$
$$v := 1 - \sum t_k,\ w := \prod t_k \cdot \left(1 - \sum \frac{x_k}{t_k}\right).$$

Note that w is a polynomial in $t_1, \ldots, t_m$. We consider the twisted homology group on M with respect to the multi-valued function

$$\begin{aligned} u &:= \prod t_k^{1-c_k+b} \cdot v^{\sum c_k - a - m + 1} w^{-b} \\ &= \prod t_k^{1-c_k} \cdot (1 - \sum t_k)^{\sum c_k - a - m + 1} \cdot \left(1 - \sum \frac{x_k}{t_k}\right)^{-b} \end{aligned}$$

(the second equality holds under the coordination of branches). Fact 0.3 means that the integral

$$\int_\Delta u\varphi, \quad \varphi := \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod t_k \cdot (1 - \sum t_k)}$$

represents $F_C(a,b,c;x)$ modulo Gamma factors.

1.2 Twisted cycles corresponding to local solutions $f_{i_1\cdots i_r}$

In this section, we construct 2^m twisted cycles in M corresponding to the solutions (0.3) to $E_C(a,b,c)$.

Let $0 \le r \le m$ and subsets $\{i_1,\dots,i_r\}$ and $\{j_1,\dots,j_{m-r}\}$ of $\{1,\dots,m\}$ satisfy $i_1 < \cdots < i_r,\ j_1 < \cdots < j_{m-r}$ and $\{i_1,\dots,i_r,j_1,\dots,j_{m-r}\} = \{1,\dots,m\}$.

Notation 1.1. From now on, the letter p (resp. q) is always stands for an index running from 1 to r (resp. from 1 to $m-r$). We use the abbreviations $\sum$, $\prod$ for the indices p,q as are mentioned in Notation 0.1(i).

We set

$$M_{i_1\cdots i_r} := \mathbb{C}^m - \left(\bigcup_k (s_k = 0) \cup (v_{i_1\cdots i_r} = 0) \cup (w_{i_1\cdots i_r} = 0)\right),$$

where

$$v_{i_1\cdots i_r} := \prod_p s_{i_p} \cdot \left(1 - \sum_p \frac{x_{i_p}}{s_{i_p}} - \sum_q s_{j_q}\right),\ w_{i_1\cdots i_r} := \prod_q s_{j_q} \cdot \left(1 - \sum_p s_{i_p} - \sum_q \frac{x_{j_q}}{s_{j_q}}\right)$$

are polynomials in $s_1,\dots,s_m$. Let $u_{i_1\cdots i_r}$ and $\varphi_{i_1\cdots i_r}$ be a multi-valued function and an m-form on $M_{i_1\cdots i_r}$ defined as

$$u_{i_1\cdots i_r} := \prod_k s_k^{C_k} \cdot v_{i_1\cdots i_r}^A \cdot w_{i_1\cdots i_r}^B, \quad \varphi_{i_1\cdots i_r} := \frac{ds_1 \wedge \cdots \wedge ds_m}{s_1 \cdots s_m v_{i_1\cdots i_r}},$$

respectively, where

$$A := \sum c_k - a - m + 1,\ B := -b,\ C_{i_p} := c_{i_p} - 1 - A,\ C_{j_q} := 1 - c_{j_q} - B.$$

We construct a twisted cycle $\tilde{\Delta}_{i_1\cdots i_r}$ in $M_{i_1\cdots i_r}$ with respect to $u_{i_1\cdots i_r}$. Note that if $\{i_1,\dots,i_r\} = \emptyset$, then these settings coincide with those in the end of Section 1.1. We choose positive real numbers $\varepsilon_1,\dots,\varepsilon_m$ and ε so that $\varepsilon < 1 - \sum_k \varepsilon_k$. And let $x_1,\dots,x_m$ be small positive real numbers satisfying

$$x_k < \frac{\varepsilon_k}{m}\varepsilon$$

(for example, if

$$\varepsilon_k = \varepsilon = \frac{1}{3m},\ 0 < x_k < \frac{1}{9m^3},$$

these conditions hold). Thus the closed subset

$$\sigma_{i_1\cdots i_r} := \left\{ (s_1,\dots,s_m) \in \mathbb{R}^m \;\middle|\; \begin{array}{l} s_{i_p} \ge \varepsilon_{i_p},\ 1 - \sum s_{i_p} \ge \varepsilon, \\ s_{j_q} \ge \varepsilon_{j_q},\ 1 - \sum s_{j_q} \ge \varepsilon \end{array} \right\}$$

is nonempty, since we have $(\varepsilon_1 + \frac{\delta}{2m},\dots,\varepsilon_m + \frac{\delta}{2m}) \in \sigma_{i_1\cdots i_r}$, where $\delta := 1 - \sum \varepsilon_k - \varepsilon > 0$. Further, $\sigma_{i_1\cdots i_r}$ is contained in the bounded domain

$$\left\{ (s_1,\dots,s_m) \in \mathbb{R}^m \;\middle|\; s_k > 0,\ \begin{array}{l} 1 - \sum \frac{x_{i_p}}{s_{i_p}} - \sum s_{j_q} > 0, \\ 1 - \sum s_{i_p} - \sum \frac{x_{j_q}}{s_{j_q}} > 0 \end{array} \right\} \subset (0,1)^m,$$

and is a direct product of an r-simplex and an $(m-r)$-simplex. Indeed, $(s_1, \ldots, s_m) \in \sigma_{i_1 \cdots i_r}$ satisfies

$$1 - \sum \frac{x_{i_p}}{s_{i_p}} - \sum s_{j_q} > 1 - \frac{r}{m}\varepsilon - \sum s_{j_q} > 1 - \sum s_{j_q} - \varepsilon \geq 0,$$
$$1 - \sum s_{i_p} - \sum \frac{x_{j_q}}{s_{j_q}} > 1 - \sum s_{i_p} - \frac{m-r}{m}\varepsilon > 1 - \sum s_{i_p} - \varepsilon \geq 0.$$

The orientation of $\sigma_{i_1 \cdots i_r}$ is induced from the natural embedding $\mathbb{R}^m \subset \mathbb{C}^m$. We construct a twisted cycle from $\sigma_{i_1 \cdots i_r}$. We may assume that $\varepsilon_k = \varepsilon$ (the above example satisfies this condition), and denote them by ε. Set $L_1 := (s_1 = 0)$, $\ldots$, $L_m := (s_m = 0)$, $L_{m+1} := (1 - \sum s_{i_p} = 0)$, $L_{m+2} := (1 - \sum s_{j_q} = 0)$, and let $U \subset \mathbb{R}^m$ be the bounded chamber surrounded by L_1, $\ldots$, L_m, L_{m+1}, L_{m+2}, then $\sigma_{i_1 \cdots i_r}$ is contained in U. Note that we do not consider the hyperplane L_{m+1} (resp. L_{m+2}), when $r = 0$ (resp. $r = m$). For $J \subset \{1, \ldots, m+2\}$, we consider $L_J := \cap_{j \in J} L_j$, $U_J := \overline{U} \cap L_J$ and $T_J :=$ ε-neighborhood of U_J. Then we have

$$\sigma_{i_1 \cdots i_r} = U - \bigcup_J T_J.$$

Using these neighborhoods T_J, we can construct a twisted cycle $\tilde{\Delta}_{i_1 \cdots i_r}$ in the same manner as Section 3.2.4 of [1] (notations L and U correspond to H and Δ in [1], respectively). Note that we have to consider contribution of branches of $s_{i_p}^{C_{i_p}} \cdot v_{i_1 \cdots i_r}^A$, when we deal with the circle associated to L_{i_p} $(p = 1, \ldots, r)$. Indeed, for fixed positive real numbers s_k $(k \neq i_p)$, s_{i_p} satisfying $1 - \sum \frac{x_{i_p}}{s_{i_p}} - \sum s_{j_q} = 0$ belongs to $\mathbb{R}$ and we have

$$s_{i_p} = \frac{x_{i_p}}{1 - \sum_q s_{j_q} - \sum_{p' \neq p} \frac{x_{i_{p'}}}{s_{i_{p'}}}} < \frac{\frac{\varepsilon}{m}\varepsilon}{\varepsilon - (r-1)\frac{\varepsilon}{m}} = \frac{\varepsilon}{m - r + 1} < \varepsilon.$$

Thus the exponent about this contribution is

$$C_{i_p} + A = c_{i_p} - 1.$$

The exponents about the contributions of the circles associated to L_{j_q}, L_{m+1}, L_{m+2} are also evaluated as

$$C_{j_q} + B = 1 - c_{j_q},\ B = -b,\ A = \sum c_k - a - m + 1,$$

respectively. We briefly explain the expression of $\tilde{\Delta}_{i_1 \cdots i_r}$. For $j = 1, \ldots, m+2$, let l_j be the $(m-1)$-face of $\sigma_{i_1 \cdots i_r}$ given by $\sigma_{i_1 \cdots i_r} \cap \overline{T_j}$, and let S_j be a positively oriented circle with radius ε in the orthogonal complement of L_j starting from the projection of l_j to this space and surrounding L_j. Then $\tilde{\Delta}_{i_1 \cdots i_r}$ is written as

$$\sigma_{i_1 \cdots i_r} + \sum_{\emptyset \neq J \subset \{1, \ldots, m+2\}} \prod_{j \in J} \frac{1}{d_j} \cdot \left(\left(\bigcap_{j \in J} l_j \right) \times \prod_{j \in J} S_j \right),$$

where

$$d_{i_p} := \gamma_{i_p} - 1,\ d_{j_q} := \gamma_{j_q}^{-1} - 1,\ d_{m+1} := \beta^{-1} - 1,\ d_{m+2} := \alpha^{-1} \prod \gamma_k - 1,$$

and $\alpha := e^{2\pi\sqrt{-1}a}$, $\beta := e^{2\pi\sqrt{-1}b}$, $\gamma_k := e^{2\pi\sqrt{-1}c_k}$. Note that we define an appropriate orientation for each $(\cap_{j\in J} l_j) \times \prod_{j\in J} S_j$, see Section 3.2.4 of [1] for details.

Example 1.2. We give explicit forms of $\tilde{\Delta}$ and $\tilde{\Delta}_1$, for $m = 2$.

(i) In the case of $I = \emptyset$ $(r = 0)$, we have

$$\begin{aligned}\tilde{\Delta} = \sigma &+ \frac{S_1 \times l_1}{1-\gamma_1^{-1}} + \frac{S_2 \times l_2}{1-\gamma_2^{-1}} + \frac{S_4 \times l_4}{1-\alpha^{-1}\gamma_1\gamma_2} \\ &+ \frac{S_1 \times S_2}{(1-\gamma_1^{-1})(1-\gamma_2^{-1})} + \frac{S_2 \times S_4}{(1-\gamma_2^{-1})(1-\alpha^{-1}\gamma_1\gamma_2)} + \frac{S_4 \times S_1}{(1-\alpha^{-1}\gamma_1\gamma_2)(1-\gamma_1^{-1})},\end{aligned}$$

where the 1-chains l_j satisfy $\partial\sigma = l_1 + l_2 + l_4$ (see Figure 1.1), and the orientation of each direct product is induced from those of its components. Note that the face l_3 does not appear in this case.

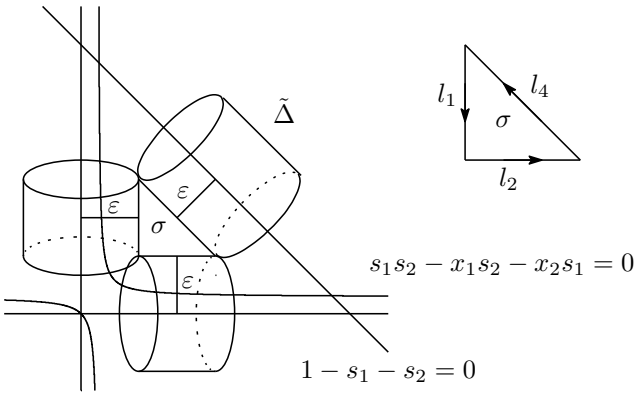

Figure 1.1: $\tilde{\Delta}$ for $m = 2$.

(ii) In the case of $I = \{1\}$, we have

$$\begin{aligned}\tilde{\Delta}_1 = \sigma_1 &+ \frac{S_1 \times l_1}{1-\gamma_1} + \frac{S_2 \times l_2}{1-\gamma_2^{-1}} + \frac{S_3 \times l_3}{1-\beta^{-1}} + \frac{S_4 \times l_4}{1-\alpha^{-1}\gamma_1\gamma_2} \\ &+ \frac{S_1 \times S_2}{(1-\gamma_1)(1-\gamma_2^{-1})} + \frac{S_2 \times S_3}{(1-\gamma_2^{-1})(1-\beta^{-1})} \\ &+ \frac{S_3 \times S_4}{(1-\beta^{-1})(1-\alpha^{-1}\gamma_1\gamma_2)} + \frac{S_4 \times S_1}{(1-\alpha^{-1}\gamma_1\gamma_2)(1-\gamma_1)},\end{aligned}$$

where the 1-chains l_j satisfy $\partial\sigma = \sum_{j=1}^{4} l_j$ (see Figure 1.2), and the orientation of each direct product is induced from those of its components.

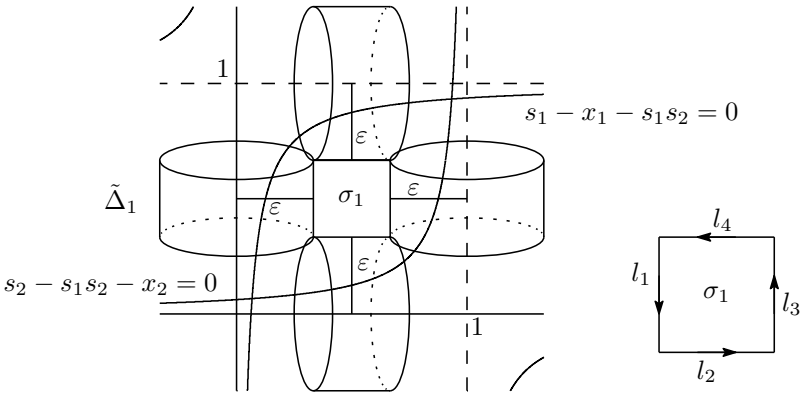

Figure 1.2: $\tilde{\Delta}_1$ for $m = 2$.

We consider the following integrals:

$$F_{i_1\cdots i_r} := \int_{\tilde{\Delta}_{i_1\cdots i_r}} u_{i_1\cdots i_r}\varphi_{i_1\cdots i_r}$$
$$= \int_{\tilde{\Delta}_{i_1\cdots i_r}} \prod_{p=1}^{r} s_{i_p}^{c_{i_p}-2} \cdot \prod_{q=1}^{m-r} s_{j_q}^{-c_{j_q}} \cdot \left(1 - \sum_{p=1}^{r} \frac{x_{i_p}}{s_{i_p}} - \sum_{q=1}^{m-r} s_{j_q}\right)^{\sum c_k - a - m}$$
$$\cdot \left(1 - \sum_{p=1}^{r} s_{i_p} - \sum_{q=1}^{m-r} \frac{x_{j_q}}{s_{j_q}}\right)^{-b} ds_1 \wedge \cdots \wedge ds_m.$$

Proposition 1.3.

$$F_{i_1\cdots i_r} = \prod_p \Gamma(c_{i_p} - 1) \cdot \prod_q \Gamma(1 - c_{j_q}) \cdot \frac{\Gamma(\sum c_k - a - m + 1)\Gamma(1-b)}{\Gamma(\sum c_{i_p} - a - r + 1)\Gamma(\sum c_{i_p} - b - r + 1)}$$
$$\cdot F_C\left(a + r - \sum_{p=1}^{r} c_{i_p}, b + r - \sum_{p=1}^{r} c_{i_p}, c^{i_1\cdots i_r}; x\right).$$

Proof. We compare the power series expansions of the both sides. Note that the coefficient of $x_1^{n_1}\cdots x_m^{n_m}$ in the series expression of $F_C(a + r - \sum_p c_{i_p}, b + r - \sum_p c_{i_p}, c^{i_1\cdots i_r}; x)$ is

$$A_{n_1\ldots n_m} := \frac{\Gamma(a + r - \sum_p c_{i_p} + \sum_k n_k)}{\Gamma(a + r - \sum_p c_{i_p})} \cdot \frac{\Gamma(b + r - \sum_p c_{i_p} + \sum_k n_k)}{\Gamma(b + r - \sum_p c_{i_p})}$$
$$\cdot \prod_p \frac{\Gamma(2 - c_{i_p})}{\Gamma(2 - c_{i_p} + n_{i_p})} \cdot \prod_q \frac{\Gamma(c_{j_q})}{\Gamma(c_{j_q} + n_{j_q})} \cdot \prod_k \frac{1}{n_k!}.$$

On the other hand, we have

$$\left(1 - \sum_p \frac{x_{i_p}}{s_{i_p}} - \sum_q s_{j_q}\right)^{\sum c_k - a - m}$$
$$= \sum_{n_{i_1},\ldots,n_{i_r}} \frac{\Gamma(a - \sum_k c_k + m + \sum_p n_{i_p})}{\Gamma(a - \sum_k c_k + m) \cdot \prod_p n_{i_p}!} \prod_p s_{i_p}^{-n_{i_p}} \cdot (1 - \sum_q s_{j_q})^{\sum c_k - a - m - \sum n_{i_p}} \cdot \prod_p x_{i_p}^{n_{i_p}}$$

and

$$\left(1-\sum_p s_{i_p}-\sum_q \frac{x_{j_q}}{s_{j_q}}\right)^{-b}$$
$$=\sum_{n_{j_1},\dots,n_{j_{m-r}}} \frac{\Gamma(b+\sum_q n_{j_q})}{\Gamma(b)\cdot\prod_q n_{j_q}!}\prod_q s_{j_q}^{-n_{j_q}}\cdot(1-\sum_p s_{i_p})^{-b-\sum n_{j_q}}\cdot\prod_q x_{j_q}^{n_{j_q}}.$$

When $r=0$ (resp. $r=m$), we do not need the first (resp. second) expansion. The convergences of these power series expansions are verified as follows. We explain only the first one. By the construction of $\tilde{\Delta}_{i_1\cdots i_r}$, we have

$$x_k<\frac{\varepsilon_k}{m}\varepsilon,\ |s_{i_p}|\geq\varepsilon_{i_p},\ \left|1-\sum_q s_{j_q}\right|\geq\varepsilon.$$

Thus the uniform convergence on $\tilde{\Delta}_{i_1\cdots i_r}$ follows from

$$\left|\frac{1}{1-\sum_q s_{j_q}}\cdot\sum_p\frac{x_{i_p}}{s_{i_p}}\right|\leq\frac{1}{|1-\sum_q s_{j_q}|}\cdot\sum_p\frac{|x_{i_p}|}{|s_{i_p}|}<\frac{1}{\varepsilon}\cdot\sum_p\frac{1}{\varepsilon_{i_p}}\frac{\varepsilon_{i_p}}{m}\varepsilon=\frac{r}{m}\leq 1.$$

Since $\tilde{\Delta}_{i_1\cdots i_r}$ is constructed as a finite sum of loaded (compact) simplexes, we can exchange the sum and the integral in the expression of $F_{i_1\cdots i_r}$. Then the coefficient of $x_1^{n_1}\cdots x_m^{n_m}$ in the series expansion of $F_{i_1\cdots i_r}$ is

$$\begin{aligned}B_{n_1\dots n_m}:=&\frac{\Gamma(a-\sum c_k+m+\sum n_{i_p})}{\Gamma(a-\sum c_k+m)}\cdot\frac{\Gamma(b+\sum n_{j_q})}{\Gamma(b)}\cdot\prod_k\frac{1}{n_k!}\\&\cdot\int_{\tilde{\Delta}_{i_1\cdots i_r}}\prod_p s_{i_p}^{c_{i_p}-n_{i_p}-2}\cdot\left(1-\sum_p s_{i_p}\right)^{-b-\sum n_{j_q}}\\&\cdot\prod_q s_{j_q}^{-c_{j_q}-n_{j_q}}\cdot\left(1-\sum_q s_{j_q}\right)^{\sum c_k-a-m-\sum n_{i_p}}ds.\end{aligned}\tag{1.1}$$

By the construction, the twisted cycle $\tilde{\Delta}_{i_1\cdots i_r}$ of this integral is identified with the usual regularization of the domain

$$\left\{(s_1,\dots,s_m)\in\mathbb{R}^m\ \middle|\ \begin{array}{l}s_{i_p}>0,\ 1-\sum s_{i_p}>0,\\ s_{j_q}>0,\ 1-\sum s_{j_q}>0\end{array}\right\}$$

for the multi-valued function

$$\prod_p s_{i_p}^{c_{i_p}-n_{i_p}-1}(1-\sum_p s_{i_p})^{-b-\sum n_{j_q}+1}\cdot\prod_q s_{j_q}^{-c_{j_q}-n_{j_q}+1}\cdot(1-\sum_q s_{j_q})^{\sum c_k-a-m-\sum n_{i_p}+1}$$

on $\mathbb{C}^m-\left(\bigcup_k(s_k=0)\cup(1-\sum s_{i_p}=0)\cup(1-\sum s_{j_q}=0)\right)$. Hence the integral in (1.1) is equal to

$$\begin{aligned}&\frac{\prod\Gamma(c_{i_p}-n_{i_p}-1)\cdot\Gamma(-b-\sum n_{j_q}+1)}{\Gamma(-b+\sum c_{i_p}-\sum n_k-r+1)}\\&\cdot\frac{\prod\Gamma(-c_{j_q}-n_{j_q}+1)\cdot\Gamma(\sum c_k-a-m-\sum n_{i_p}+1)}{\Gamma(\sum c_{i_p}-a-\sum n_k-r+1)}.\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{B_{n_1\dots n_m}}{A_{n_1\dots n_m}} =& \frac{\Gamma(a-\sum c_k+m+\sum n_{i_p})}{\Gamma(a-\sum c_k+m)}\cdot\frac{\Gamma(b+\sum n_{j_q})}{\Gamma(b)}\\ &\cdot\frac{\prod\Gamma(c_{i_p}-n_{i_p}-1)\cdot\Gamma(-b-\sum n_{j_q}+1)}{\Gamma(-b+\sum c_{i_p}-\sum n_k-r+1)}\\ &\cdot\frac{\prod\Gamma(-c_{j_q}-n_{j_q}+1)\cdot\Gamma(\sum c_k-a-m-\sum n_{i_p}+1)}{\Gamma(\sum c_{i_p}-a-\sum n_k-r+1)}\\ &\cdot\frac{\Gamma(a+r-\sum c_{i_p})}{\Gamma(a+r-\sum c_{i_p}+\sum n_k)}\cdot\frac{\Gamma(b+r-\sum c_{i_p})}{\Gamma(b+r-\sum c_{i_p}+\sum n_k)}\\ &\cdot\prod\frac{\Gamma(2-c_{i_p}+n_{i_p})}{\Gamma(2-c_{i_p})}\cdot\prod\frac{\Gamma(c_{j_q}+n_{j_q})}{\Gamma(c_{j_q})}.\end{aligned}$$

Using the formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}, \tag{1.2}$$

we obtain

$$\begin{aligned}&\frac{\Gamma(a-\sum c_k+m+\sum n_{i_p})\Gamma(\sum c_k-a-m-\sum n_{i_p}+1)}{\Gamma(a-\sum c_k+m)}\\ &\quad=\frac{\pi}{\Gamma(a-\sum c_k+m)\sin\pi(a-\sum c_k+m+\sum n_{i_p})}\\ &\quad=(-1)^{\sum n_{i_p}}\Gamma(1-a+\sum c_k-m),\\ &\frac{\Gamma(a+r-\sum c_{i_p})}{\Gamma(\sum c_{i_p}-a-\sum n_k-r+1)\Gamma(a+r-\sum c_{i_p}+\sum n_k)}\\ &\quad=\frac{\Gamma(a+r-\sum c_{i_p})\sin\pi(a+r-\sum c_{i_p}+\sum n_k)}{\pi}=\frac{(-1)^{\sum n_k}}{\Gamma(1-a-r+\sum c_{i_p})},\\ &\frac{\Gamma(b+\sum n_{j_q})\Gamma(-b-\sum n_{j_q}+1)}{\Gamma(b)}\\ &\quad=\frac{\pi}{\Gamma(b)\sin\pi(b+\sum n_{j_q})}=(-1)^{\sum n_{j_q}}\Gamma(1-b),\\ &\frac{\Gamma(b+r-\sum c_{i_p})}{\Gamma(-b+\sum c_{i_p}-\sum n_k-r+1)\Gamma(b+r-\sum c_{i_p}+\sum n_k)}\\ &\quad=\frac{\Gamma(b+r-\sum c_{i_p})\sin\pi(b+r-\sum c_{i_p}+\sum n_k)}{\pi}=\frac{(-1)^{\sum n_k}}{\Gamma(1-b-r+\sum c_{i_p})},\\ &\frac{\Gamma(c_{i_p}-n_{i_p}-1)\Gamma(2-c_{i_p}+n_{i_p})}{\Gamma(2-c_{i_p})}\\ &\quad=\frac{\pi}{\Gamma(1-(c_{i_p}-1))\sin\pi(c_{i_p}-n_{i_p}-1)}=(-1)^{n_{i_p}}\Gamma(c_{i_p}-1),\\ &\frac{\Gamma(-c_{j_q}-n_{j_q}+1)\Gamma(c_{j_q}+n_{j_q})}{\Gamma(c_{j_q})}=\frac{\pi}{\Gamma(c_{j_q})\sin\pi(c_{j_q}+n_{j_q})}=(-1)^{n_{j_q}}\Gamma(1-c_{j_q}).\end{aligned}$$

Therefore, we have

$$\frac{B_{n_1\dots n_m}}{A_{n_1\dots n_m}} = \prod_p \Gamma(c_{i_p}-1)\cdot\prod_q \Gamma(1-c_{j_q}) \cdot \frac{\Gamma(\sum c_k - a - m + 1)\Gamma(1-b)}{\Gamma(\sum c_{i_p} - a - r + 1)\Gamma(\sum c_{i_p} - b - r + 1)},$$

which implies the proposition. □

We define a bijection $\iota_{i_1\cdots i_r} : M_{i_1\cdots i_r} \to M$ by

$$\iota_{i_1\cdots i_r}(s_1,\dots,s_m) := (t_1,\dots,t_m);\ t_{i_p} = \frac{x_{i_p}}{s_{i_p}},\ t_{j_q} = s_{j_q}.$$

For example, $\iota(=\iota_\emptyset)$ is the identity map on $M = M_\emptyset$, and $\iota_{1\cdots m}$ defines an involution on $M = M_{1\cdots m}$. We state our first main theorem.

Theorem 1.4. *We define a twisted cycle $\Delta_{i_1\cdots i_r}$ in M by*

$$\Delta_{i_1\cdots i_r} := (-1)^r(\iota_{i_1\cdots i_r})_*(\tilde{\Delta}_{i_1\cdots i_r}). \tag{1.3}$$

Then we have

$$\int_{\Delta_{i_1\cdots i_r}} \prod t_k^{-c_k}\cdot\left(1-\sum t_k\right)^{\sum c_k - a - m}\cdot\left(1-\sum\frac{x_k}{t_k}\right)^{-b} dt_1\wedge\cdots\wedge dt_m$$
$$\left(=\int_{\Delta_{i_1\cdots i_r}} u\varphi\right) = \prod_p x_{i_p}^{1-c_{i_p}}\cdot F_{i_1\cdots i_r},$$

and hence this integral corresponds to the local solution $f_{i_1\cdots i_r}$ to $E_C(a,b,c)$ given in Fact 0.2.

Proof. Pull back the integral of the left hand side by $\iota_{i_1\cdots i_r}$. Indeed, we have

$$t_{i_p} = \frac{x_{i_p}}{s_{i_p}},\ dt_{i_p} = -\frac{x_{i_p}}{s_{i_p}^2}ds_{i_p}$$

by the definition of $\iota_{i_1\cdots i_r}$. Note that the sign $(-1)^r$ arising from the pull-back of $dt_1\wedge\cdots\wedge dt_m$ is canceled by that in (1.3). The second claim follows from the first equality and Proposition 1.3. □

Remark 1.5. (i) The sign $(-1)^r$ in (1.3) implies that the orientation of the direct product $\iota_{i_1\cdots i_r}(\sigma_{i_1\cdots i_r})$ of two simplexes in $\Delta_{i_1\cdots i_r}$ is coincide with that of $\mathbb{R}^m$ induced from the natural embedding $\mathbb{R}^m \subset \mathbb{C}^m$.

(ii) The twisted cycle Δ (for $r = 0$) equals to that mentioned in Fact 0.3.

The replacement $u \mapsto u^{-1} = 1/u$ and the construction same as $\Delta_{i_1\cdots i_r}$ give the twisted cycle $\Delta^\vee_{i_1\cdots i_r}$ which represents an element in $H_m(M, u^{-1})$. We obtain the intersection numbers of the twisted cycles $\{\Delta_{i_1\cdots i_r}\}$ and $\{\Delta^\vee_{i_1\cdots i_r}\}$.

Theorem 1.6. (i) *If $I \neq I'$, then we have $I_h(\Delta_I, \Delta_{I'}^\vee) = 0$.*

(ii) *The self-intersection number of $\Delta_{i_1 \cdots i_r}$ is*

$$I_h(\Delta_{i_1 \cdots i_r}, \Delta_{i_1 \cdots i_r}^\vee) = (-1)^r \cdot \frac{\prod_q \gamma_{j_q} \cdot \left(\alpha - \prod_p \gamma_{i_p}\right)\left(\beta - \prod_p \gamma_{i_p}\right)}{\prod_k (\gamma_k - 1) \cdot \left(\alpha - \prod_k \gamma_k\right)(\beta - 1)}.$$

Proof. (i) Since $\Delta_{i_1 \cdots i_r}$'s represent local solutions (0.3) to $E_C(a, b, c)$ by Theorem 1.4, this claim follows from similar arguments to the proof of Lemma 4.1 in [9].
(ii) By using $\iota_{i_1 \cdots i_r}$, the self-intersection number of $\Delta_{i_1 \cdots i_r}$ is equal to that of $\tilde{\Delta}_{i_1 \cdots i_r}$ with respect to the multi-valued function $u_{i_1 \cdots i_r}$. To calculate this, we apply results in [13]. Since we construct the twisted cycle $\tilde{\Delta}_{i_1 \cdots i_r}$ from the direct product $\sigma_{i_1 \cdots i_r}$ of two simplexes, the self-intersection number of $\tilde{\Delta}_{i_1 \cdots i_r}$ is obtained as the product of those of simplexes. Thus we have

$$I_h(\Delta_{i_1 \cdots i_r}, \Delta_{i_1 \cdots i_r}^\vee) = \frac{1 - \prod_p \gamma_{i_p} \cdot \beta^{-1}}{\prod_p (1 - \gamma_{i_p}) \cdot (1 - \beta^{-1})} \cdot \frac{1 - \prod_q \gamma_{j_q}^{-1} \cdot \prod_k \gamma_k \cdot \alpha^{-1}}{\prod_q (1 - \gamma_{j_q}^{-1}) \cdot (1 - \prod_k \gamma_k \cdot \alpha^{-1})}.$$

□

1.3 Comparison of twisted cycles Δ_I with those in [9]

Lauricella's F_C of two-variables is called Appell's F_4. We construct $2^2 = 4$ twisted cycles $\Delta,\ \Delta_1,\ \Delta_2,\ \Delta_{12}$ in Section 1.2. On the other hand, four twisted cycles corresponding to the local solutions (0.3) of E_4 are also given in [9].

Let D_i be the twisted cycle Δ_i $(i = 1, 2, 3, 4)$ in [9]. Then we have the following correspondence.

Proposition 1.7. *We have*

$$\begin{aligned}
D_1 &= \Delta, & D_1^\vee &= \Delta^\vee, \\
D_2 &= -\frac{(\gamma_1 - 1)(\gamma_2 - 1)\alpha\beta}{(\gamma_1 - \alpha)(\gamma_1 - \beta)\gamma_2}\Delta_1, & D_2^\vee &= -\frac{(\gamma_1 - 1)(\gamma_2 - 1)\gamma_1}{(\gamma_1 - \alpha)(\gamma_1 - \beta)}\Delta_1^\vee, \\
D_3 &= -\frac{(\gamma_1 - 1)(\gamma_2 - 1)\alpha\beta}{(\gamma_2 - \alpha)(\gamma_2 - \beta)\gamma_1}\Delta_2, & D_3^\vee &= -\frac{(\gamma_1 - 1)(\gamma_2 - 1)\gamma_2}{(\gamma_2 - \alpha)(\gamma_2 - \beta)}\Delta_2^\vee, \\
D_4 &= \Delta_{12}, & D_4^\vee &= \Delta_{12}^\vee.
\end{aligned}$$

Proof. By the construction, D_1 and D_4 are the same cycles as Δ and Δ_{12}, respectively. Since D_2 and Δ_1 correspond to constant multiples of the solution f_1, there exists K such that $D_2 = K\Delta_1$. We have

$$\begin{aligned}
K = &\frac{\Gamma(b - c_1 + 1)\Gamma(a - c_1 + 1)\Gamma(1 - b)\Gamma(c_1 + c_2 - a - 1)}{\Gamma(2 - c_1)\Gamma(c_2)} e^{-\pi\sqrt{-1}(c_1 + c_2 - a - b)} \\
&\cdot \frac{\Gamma(c_1 - a)\Gamma(c_1 - b)}{\Gamma(c_1 - 1)\Gamma(1 - c_2)\Gamma(c_1 + c_2 - a - 1)\Gamma(1 - b)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(c_1-a)\Gamma(1-(c_1-a))\Gamma(c_1-b)\Gamma(1-(c_1-b))}{\Gamma(c_1-1)\Gamma(1-(c_1-1))\Gamma(c_2)\Gamma(1-c_2)} e^{-\pi\sqrt{-1}(c_1+c_2-a-b)} \\
&= \frac{(e^{\pi\sqrt{-1}(c_1-1)} - e^{-\pi\sqrt{-1}(c_1-1)})(e^{\pi\sqrt{-1}c_2} - e^{-\pi\sqrt{-1}c_2})e^{-\pi\sqrt{-1}(c_1+c_2-a-b)}}{(e^{\pi\sqrt{-1}(c_1-a)} - e^{-\pi\sqrt{-1}(c_1-a)})(e^{\pi\sqrt{-1}(c_1-b)} - e^{-\pi\sqrt{-1}(c_1-b)})} \\
&= \frac{(\gamma_1-1)(\gamma_2-1)}{(\gamma_1\alpha^{-1}-1)(\gamma_1\beta^{-1}-1)} e^{\pi\sqrt{-1}(-c_1+1-c_2+c_1-a+c_1-b-(c_1+c_2-a-b))} \\
&= -\frac{(\gamma_1-1)(\gamma_2-1)\alpha\beta}{(\gamma_1-\alpha)(\gamma_1-\beta)\gamma_2}.
\end{aligned}
$$

The other equalities are proved similarly. $\square$

Chapter 2

Twisted period relations

In this chapter, we give twisted period relations for Lauricella's F_C. We evaluate the intersection numbers of some elements of the twisted cohomology group. The intersection matrix of twisted homology group and some intersection numbers of twisted cohomology group imply twisted period relations for two fundamental systems of E_C with different parameters.

2.1 Intersection numbers of twisted cohomology groups

In this section, we review twisted cohomology groups and the intersection form between twisted cohomology groups in our situation, and evaluate some intersection numbers of twisted cocycles.

Recall that

$$M = \mathbb{C}^m - \left(\bigcup_k (t_k = 0) \cup (v = 0) \cup (w = 0) \right),$$
$$u = \prod t_k^{1-c_k+b} \cdot v^{\sum c_k - a - m + 1} w^{-b}.$$

We consider the logarithmic 1-form

$$\omega := d \log u = \frac{du}{u}.$$

We denote the $\mathbb{C}$-vector space of smooth k-forms on M by $\mathcal{E}^k(M)$. We define the covariant differential operator $\nabla_\omega : \mathcal{E}^k(M) \to \mathcal{E}^{k+1}(M)$ by

$$\nabla_\omega(\psi) := d\psi + \omega \wedge \psi, \quad \psi \in \mathcal{E}^k(M).$$

Because of $\nabla_\omega \circ \nabla_\omega = 0$, we have a complex

$$\mathcal{E}^\bullet(M) : \cdots \xrightarrow{\nabla_\omega} \mathcal{E}^k(M) \xrightarrow{\nabla_\omega} \mathcal{E}^{k+1}(M) \xrightarrow{\nabla_\omega} \cdots,$$

and its k-th cohomology group $H^k(M, \nabla_\omega)$. It is called the k-th twisted de Rham cohomology group. An element in $\ker \nabla_\omega$ is called a twisted cocycle. By replacing $\mathcal{E}^k(M)$ with the $\mathbb{C}$-vector space $\mathcal{E}_c^k(M)$ of smooth k-forms on M with

compact support, we obtain the twisted de Rham cohomology group $H_c^k(M, \nabla_\omega)$ with compact support. By [2], we have $H^k(M, \nabla_\omega) = 0$ for all $k \neq m$. Further, by Lemma 2.9 in [1], there is a canonical isomorphism

$$\jmath : H^m(M, \nabla_\omega) \to H_c^m(M, \nabla_\omega).$$

By considering $u^{-1} = 1/u$ instead of u, we have the covariant differential operator $\nabla_{-\omega}$ and the twisted de Rham cohomology group $H^k(M, \nabla_{-\omega})$. The intersection form I_c between $H^m(M, \nabla_\omega)$ and $H^m(M, \nabla_{-\omega})$ is defined by

$$I_c(\psi, \psi') := \int_M \jmath(\psi) \wedge \psi', \quad \psi \in H^m(M, \nabla_\omega),\ \psi' \in H^m(M, \nabla_{-\omega}),$$

which converges because of the compactness of the support of $\jmath(\psi)$.

By the Poincaré duality (see Lemma 2.8 in [1]), we have

$$\begin{aligned} &\dim H_k(M, u) = 0 \quad (k \neq m), \\ &\dim H_m(M, u) = \dim H^m(M, \nabla_\omega). \end{aligned} \tag{2.1}$$

Proposition 2.1. *Let $x = (x_1, \ldots, x_m) \in D_C - S$.*

(i) *We have $\dim H_m(M, u) = 2^m$.*

(ii) *The twisted cycles $\{\Delta_I\}_I$ form a basis of $H_m(M, u)$.*

(iii) *The integrations of $u\varphi$ on twisted cycles give an isomorphism between $H_m(M, u)$ and the space of local solutions to $E_C(a, b, c)$.*

Proof. We prove (i). By (2.1) and Theorem 2.2 in [1], we have

$$\dim H_m(M, u) = (-1)^m \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. Let $h \in \mathbb{C}[T_0, \ldots, T_m]$ be a homogeneous polynomial defined by

$$h(T) := \prod_{i=0}^m T_i \cdot \left(\sum_{i=0}^m \prod_{j \neq i} T_j \right),$$

and let $D(h) = \{T = [T_0 : \cdots : T_m] \in \mathbb{P}^m \mid h(T) \neq 0\}$. Then we have $\chi(M) = \chi(D(h) - L)$ for some generic hyperplane L in $\mathbb{P}^m$. We consider the gradient map

$$\mathrm{grad}(h) : D(h) \to \mathbb{P}^m; \quad [T_0 : \cdots : T_m] \mapsto \left[\frac{\partial h}{\partial T_0}(T) : \cdots : \frac{\partial h}{\partial T_m}(T) \right].$$

It is easy to see that the degree of $\mathrm{grad}(h)$ is less than or equal to 2^m. By Theorem 1 in [5], we obtain

$$\dim H_m(M, u) = (-1)^m \chi(D(h) - L) = \deg(\mathrm{grad}(h)) \leq 2^m$$

(The author thanks to J. Kaneko for pointing out this fact). On the other hand, the determinant of the intersection matrix $(I_h(\Delta_I, \Delta_{I'}^\vee))$ is not zero by Theorem

1.6. This implies that Δ_I's are linearly independent, and hence $\dim H_m(M,u) \geq 2^m$. Therefore, we obtain (i).

As mentioned above, Δ_I's are linearly independent. Thus the claim (ii) follows from (i).

We show (iii). Let Sol_x be the space of local solutions to $E_C(a,b,c)$ around x. By Theorem 1.4, integrals of $u\varphi$ on linear combinations of $\Delta_{i_1\cdots i_r}$'s are in Sol_x. Then (ii) implies that the linear map

$$\Phi : H_m(M,u) \to Sol_x \ ; \quad C \mapsto \int_C u\varphi$$

is defined. Fact 0.2 and Theorem 1.4 imply that Φ is surjective. Therefore Φ is isomorphic because of $\dim H_m(M,u) = \dim Sol_x = 2^m$. □

We evaluate some intersection numbers of the twisted cocycles

$$\varphi = \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod t_k \cdot (1 - \sum t_k)},$$
$$\varphi' := \frac{dt_1 \wedge \cdots \wedge dt_m}{vw} = \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod t_k \cdot (1 - \sum t_k) \cdot (1 - \sum \frac{x_k}{t_k})}.$$

Theorem 2.2. (i) *The self-intersection number of φ is*

$$I_c(\varphi,\varphi)$$
$$= (2\pi\sqrt{-1})^m \left(\frac{1}{\sum c_k - a - m + 1} + \frac{1}{b + m - \sum c_k} \right) \cdot \sum_{\{I^{(r)}\}} \prod_{r=1}^{m-1} \frac{1}{b + r - \sum c_{i_p^{(r)}}},$$

where $\{I^{(r)}\}$ runs sequences of subsets of $\{1,\ldots,m\}$, which satisfy

$$\{1,\ldots,m\} \supsetneq I^{(m-1)} \supsetneq \cdots \supsetneq I^{(2)} \supsetneq I^{(1)} \neq \emptyset,$$

and we write $I^{(r)} = \{i_1^{(r)},\ldots,i_r^{(r)}\}$.

(ii) *We have $I_c(\varphi,\varphi') = 0$.*

Proof. (i) The hypersurfaces $H_1,\ldots,H_m,H,D$ do not form a normal crossing divisor, because of $H_i \cap H_j \subset D$ for $i \neq j$. Thus we blow up $\mathbb{C}^m$ along some intersections of hyperplanes so that the pole divisor of φ is normally crossing. Firstly we consider the blow-up at the origin ($= H_1 \cap \cdots \cap H_m$). Secondly we blow up this along $H_1 \cap \cdots \cap \check{H}_k \cap \cdots \cap H_m$ ($k = 1,\ldots,m$). Repeat the blowing-up process, lastly we blow up along $H_i \cap H_j$ ($1 \leq i < j \leq m$). For $i_1 < \cdots < i_r$, $r \geq 2$, the exceptional divisor $E_{i_1\cdots i_r}$ arising from the blow-up along $H_{i_1} \cap \cdots \cap H_{i_r}$ has the exponent

$$\sum_{p=1}^{r} (1 - c_{i_p} + b) - (r-1)b = b + r - \sum_{p=1}^{r} c_{i_p}.$$

By expressing φ in each coordinates system, results in [15] give the self-intersection number of φ.

(ii) By the definition, the pole divisor of φ' does not contain the exceptional divisors. Hence the pole divisors of φ and φ' do not have m or more common factors, which implies $I_c(\varphi,\varphi') = 0$. □

Remark 2.3. Precisely speaking, to evaluate intersection numbers of twisted cocycles, we should blow up a compactification of $\mathbb{C}^m$ ($\mathbb{P}^m$ or $(\mathbb{P}^1)^m$) so that the pole divisor of $\omega = d\log u$ is normally crossing. Though the blowing-up process in the above proof is not enough, it is shown that the exceptional divisors arising from the other blowing-up processes do not appear as the components of the pole divisor of φ. Thus the proof is completed.

Remark 2.4. It seems difficult to evaluate the self-intersection number $I_c(\varphi', \varphi')$. For $m = 2$ (i.e., Appell's F_4), this number ($= \mathcal{I}_c(\varphi_{x,4}, \varphi_{x,4})$ in [9]) is expressed by the parameters and the factor of the defining equation of the singular locus (see [9]). Furthermore, the author does not find m-forms which form a basis of $H^m(M, \nabla_\omega)$.

2.2 Twisted period relations

By Theorem 1.6 and 2.2, we state our second main theorem.

Theorem 2.5 (Twisted period relations)**.** *We have*

$$I_c(\varphi, \varphi) = \sum_I \frac{1}{I_h(\Delta_I, \Delta_I^\vee)} \cdot g_I \cdot g_I^\vee, \tag{2.2}$$

$$I_c(\varphi, \varphi') = \sum_I \frac{1}{I_h(\Delta_I, \Delta_I^\vee)} \cdot g_I \cdot h_I^\vee, \tag{2.3}$$

where

$$g_I = \int_{\Delta_I} u\varphi, \quad g_I^\vee = \int_{\Delta_I^\vee} u^{-1}\varphi, \quad h_I^\vee = \int_{\Delta_I^\vee} u^{-1}\varphi'.$$

Further, under the notations

$$a_I := a - \sum_{i \in I} c_i + |I|, \quad b_I := b - \sum_{i \in I} c_i + |I|, \quad \check{c}^I := (2, \ldots, 2) - c^I,$$

the equalities (2.2) and (2.3) are reduced to

$$\begin{aligned} &\sum_I (-1)^{|I|} \frac{1 - a_I}{b_I} \cdot F_C(a_I, b_I, c^I; x) \cdot F_C(2 - a_I, -b_I, \check{c}^I; x) \\ &\quad = \frac{(1 - a + b) \cdot \prod (1 - c_k)}{b b_{1 \cdots m}} \cdot \sum_{\{I^{(r)}\}} \prod_{r=1}^{m-1} \frac{1}{b_{I^{(r)}}}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\sum_I (-1)^{|I|} (a_I - 1) \cdot F_C(a_I, b_I, c^I; x) \cdot F_C(2 - a_I, 1 - b_I, \check{c}^I; x) \\ &\quad = 0, \end{aligned} \tag{2.5}$$

respectively.

Proof. Because of the compatibility of intersection forms and pairings obtained by integrations (see [3]), we obtain the equalities (2.2) and (2.3). We show that

(2.2) is reduced to (2.4). By Proposition 1.3 and Theorem 1.4, we have

$$\begin{aligned} g_{i_1\cdots i_r} =& \prod \Gamma(c_{i_p}-1)\cdot\prod\Gamma(1-c_{j_q})\cdot\frac{\Gamma(\sum c_k-a-m+1)\Gamma(1-b)}{\Gamma(\sum c_{i_p}-a-r+1)\Gamma(\sum c_{i_p}-b-r+1)} \\ &\cdot\prod x_{i_p}^{1-c_{i_p}}\cdot F_C\left(a+r-\sum c_{i_p}, b+r-\sum c_{i_p}, c^{i_1\cdots i_r}; x\right). \end{aligned}$$

On the other hand, we can express $g^{\vee}_{i_1\cdots i_r}$ like this by the replacement

$$(a,b,c)\longmapsto(2-a,-b,(2,\ldots,2)-c),$$

since $u^{-1}\varphi$ is written as

$$u^{-1}\varphi=\prod t_k^{c_k-2}\cdot(1-\sum t_k)^{-\sum c_k+a+m-2}\cdot\left(1-\sum\frac{x_k}{t_k}\right)^b dt_1\wedge\cdots\wedge dt_m.$$

Thus we obtain

$$\begin{aligned} g^{\vee}_{i_1\cdots i_r} =& \prod\Gamma(1-c_{i_p})\cdot\prod\Gamma(c_{j_q}-1) \\ &\cdot\frac{\Gamma(-\sum c_k+a+m-1)\Gamma(1+b)}{\Gamma(-\sum c_{i_p}+a+r-1)\Gamma(-\sum c_{i_p}+b+r+1)} \\ &\cdot\prod x_{i_p}^{c_{i_p}-1}\cdot F_C\left(2-a-r+\sum c_{i_p}, -b-r+\sum c_{i_p}, (2,\ldots,2)-c^{i_1\cdots i_r}; x\right). \end{aligned}$$

By the formula (1.2), we have

$$\begin{aligned} &\prod\Gamma(c_k-1)\Gamma(1-c_k)\cdot\frac{\Gamma(\sum c_k-a-m+1)\Gamma(-\sum c_k+a+m-1)}{\Gamma(\sum c_{i_p}-a-r+1)\Gamma(-\sum c_{i_p}+a+r-1)} \\ &\quad\cdot\frac{\Gamma(1-b)\Gamma(1+b)}{\Gamma(\sum c_{i_p}-b-r+1)\Gamma(-\sum c_{i_p}+b+r+1)} \\ &=\prod\Gamma(c_k-1)\Gamma(1-c_k)\cdot\frac{\Gamma(\sum c_k-a-m+1)\Gamma(-\sum c_k+a+m-1)}{\Gamma(\sum c_{i_p}-a-r+1)\Gamma(-\sum c_{i_p}+a+r-1)} \\ &\quad\cdot\frac{\Gamma(1-b)\Gamma(1+b)}{\Gamma(\sum c_{i_p}-b-r+1)\Gamma(-\sum c_{i_p}+b+r+1)} \\ &=\prod\frac{\pi}{-(c_k-1)\sin\pi(c_k-1)} \\ &\quad\cdot\frac{-(\sum c_{i_p}-a-r+1)\sin\pi(\sum c_{i_p}-a-r+1)}{-(\sum c_k-a-m+1)\sin\pi(\sum c_k-a-m+1)}\cdot\frac{b\cdot\sin\pi(\sum c_{i_p}-b-r)}{\sin\pi b\cdot(\sum c_{i_p}-b-r)} \\ &=-\frac{b}{\prod(1-c_k)\cdot(a+m-1-\sum c_k)}\cdot\frac{a+r-1-\sum c_{i_p}}{b+r-\sum c_{i_p}} \\ &\quad\cdot\pi^m(-1)^{-r+1-r-m+m-1}\cdot\frac{\sin\pi(\sum c_{i_p}-a)\cdot\sin\pi(\sum c_{i_p}-b)}{\prod\sin\pi c_k\cdot\sin\pi(\sum c_k-a)\cdot\sin\pi b} \\ &=-\frac{b}{\prod(1-c_k)\cdot(a+m-1-\sum c_k)}\cdot\frac{a+r-1-\sum c_{i_p}}{b+r-\sum c_{i_p}}\cdot(2\pi\sqrt{-1})^m \\ &\quad\cdot\frac{(e^{\pi\sqrt{-1}(\sum c_{i_p}-a)}-e^{-\pi\sqrt{-1}(\sum c_{i_p}-a)})(e^{\pi\sqrt{-1}(\sum c_{i_p}-b)}-e^{-\pi\sqrt{-1}(\sum c_{i_p}-b)})}{\prod(e^{\pi\sqrt{-1}c_k}-e^{-\pi\sqrt{-1}c_k})\cdot(e^{\pi\sqrt{-1}(\sum c_k-a)}-e^{-\pi\sqrt{-1}(\sum c_k-a)})(e^{\pi\sqrt{-1}b}-e^{-\pi\sqrt{-1}b})} \end{aligned}$$

$$
\begin{aligned}
&= -(2\pi\sqrt{-1})^m \frac{b}{\prod(1-c_k)\cdot(a+m-1-\sum c_k)} \cdot \frac{a+r-1-\sum c_{i_p}}{b+r-\sum c_{i_p}} \\
&\quad \cdot \frac{(\alpha^{-1}\prod\gamma_{i_p}-1)(\beta^{-1}\prod\gamma_{i_p}-1)}{\prod(\gamma_k-1)\cdot(\alpha^{-1}\prod\gamma_k-1)(\beta-1)} e^{-\pi\sqrt{-1}(\sum c_{i_p}-a+\sum c_{i_p}-b-\sum c_k-\sum c_k+a-b)} \\
&= -(2\pi\sqrt{-1})^m \frac{b}{\prod(1-c_k)\cdot(a+m-1-\sum c_k)} \cdot \frac{a+r-1-\sum c_{i_p}}{b+r-\sum c_{i_p}} \\
&\quad \cdot \frac{(\alpha^{-1}\prod\gamma_{i_p}-1)(\beta^{-1}\prod\gamma_{i_p}-1)}{\prod(\gamma_k-1)\cdot(\alpha^{-1}\prod\gamma_k-1)(\beta-1)} \beta\prod\gamma_k\cdot\prod\gamma_{i_p}^{-1} \\
&= (2\pi\sqrt{-1})^m \frac{b}{\prod(1-c_k)\cdot(a+m-1-\sum c_k)} \cdot \frac{a+r-1-\sum c_{i_p}}{b+r-\sum c_{i_p}} \\
&\quad \cdot \frac{\prod\gamma_{j_q}\cdot(\prod\gamma_{i_p}-\alpha)(\prod\gamma_{i_p}-\beta)}{\prod(\gamma_k-1)\cdot(\alpha-\prod\gamma_k)(\beta-1)} \\
&= (2\pi\sqrt{-1})^m \cdot \frac{b}{\prod(1-c_k)\cdot(a+m-1-\sum c_k)} \\
&\quad \cdot (-1)^r \cdot \frac{a+r-1-\sum c_{i_p}}{b+r-\sum c_{i_p}} \cdot I_h(\Delta_{i_1\cdots i_r}, \Delta^\vee_{i_1\cdots i_r}).
\end{aligned}
$$

Hence, under the notations $a_I,\ b_I,\ \check{c}^I$, and $|I|=r$, the equality (2.2) is reduced to

$$
\begin{aligned}
&\left(\frac{1}{1-a_{1\cdots m}}+\frac{1}{b_{1\cdots m}}\right)\cdot\sum_{\{I^{(r)}\}}\prod_{r=1}^{m-1}\frac{1}{b_{I^{(r)}}} \\
&= \frac{b}{\prod(1-c_k)\cdot(a_{1\cdots m}-1)} \\
&\quad \cdot \sum_I (-1)^{|I|}\frac{a_I-1}{b_I}\cdot F_C(a_I,b_I,c^I;x)\cdot F_C(2-a_I,-b_I,\check{c}^I;x).
\end{aligned}
$$

By multiplying $\dfrac{(1-a_{1\cdots m})\cdot\prod(1-c_k)}{b}$, we obtain (2.4).

By similar arguments, we obtain

$$
\begin{aligned}
h^\vee_{i_1\cdots i_r} = &\prod\Gamma(1-c_{i_p})\cdot\prod\Gamma(c_{j_q}-1) \\
&\cdot\frac{\Gamma(-\sum c_k+a+m-1)\Gamma(b)}{\Gamma(-\sum c_{i_p}+a+r-1)\Gamma(-\sum c_{i_p}+b+r)} \\
&\cdot\prod x_{i_p}^{c_{i_p}-1}\cdot F_C(2-a_{i_1\cdots i_r},1-b_{i_1\cdots i_r},\check{c}^{i_1\cdots i_r};x),
\end{aligned}
$$

and

$$
\begin{aligned}
&\prod\Gamma(c_k-1)\Gamma(1-c_k)\cdot\frac{\Gamma(\sum c_k-a-m+1)\Gamma(-\sum c_k+a+m-1)}{\Gamma(\sum c_{i_p}-a-r+1)\Gamma(-\sum c_{i_p}+a+r-1)} \\
&\quad\cdot\frac{\Gamma(1-b)\Gamma(b)}{\Gamma(\sum c_{i_p}-b-r+1)\Gamma(-\sum c_{i_p}+b+r)} \\
&= \prod\frac{\pi}{-(c_k-1)\sin\pi(c_k-1)} \\
&\quad\cdot\frac{-(\sum c_{i_p}-a-r+1)\sin\pi(\sum c_{i_p}-a-r+1)}{-(\sum c_k-a-m+1)\sin\pi(\sum c_k-a-m+1)}\cdot\frac{\sin\pi(\sum c_{i_p}-b-r)}{\sin\pi b}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{a+r-1-\sum c_{i_p}}{\prod(1-c_k)\cdot(a+m-1-\sum c_k)} \\
&\quad \cdot \pi^m(-1)^{-r+1-r-m+m-1}\cdot\frac{\sin\pi(\sum c_{i_p}-a)\cdot\sin\pi(\sum c_{i_p}-b)}{\prod\sin\pi c_k\cdot\sin\pi(\sum c_k-a)\cdot\sin\pi b} \\
&= (2\pi\sqrt{-1})^m\cdot(-1)^r\frac{a+r-1-\sum c_{i_p}}{\prod(1-c_k)\cdot(a+m-1-\sum c_k)}I_h(\Delta_{i_1\cdots i_r},\Delta^\vee_{i_1\cdots i_r}).
\end{aligned}
$$

By multiplying $\dfrac{\prod(1-c_k)\cdot(a+m-1-\sum c_k)}{(2\pi\sqrt{-1})^m}$, we reduce (2.3) to (2.5). $\square$

Note that (2.4) and (2.5) are generalizations of some equalities in Corollary 6.1 of [9].

Chapter 3

Fundamental group

In this chapter, we study the fundamental group of the complement X of the singular locus S. In [11], the fundamental group of X for $m = 2$ is characterized by generators and their relations. However, we do not have a such presentation of the fundamental group of X for $m \geq 3$. We show that it is generated by $m + 1$ loops. Further, we give some relations among these generators.

3.1 Generators of the fundamental group

Recall that the singular locus S is given by

$$S = \left(\prod_k x_k \cdot R(x) = 0 \right), \quad R(x_1, \ldots, x_m) := \prod_{\varepsilon_1, \ldots, \varepsilon_m = \pm 1} \left(1 + \sum_k \varepsilon_k \sqrt{x_k} \right).$$

Note that $R(x)$ is an irreducible polynomial of degree 2^{m-1} in $x_1, \ldots, x_m$.

We put $X := \mathbb{C}^m - S$. Let $\dot{x} := (\frac{1}{2m^2}, \ldots, \frac{1}{2m^2}) \in X$ be a base point of X. For $1 \leq k \leq m$, let ρ_k be the loop in X defined by

$$\rho_k : [0,1] \ni \theta \mapsto \left(\frac{1}{2m^2}, \ldots, \frac{e^{2\pi\sqrt{-1}\theta}}{2m^2}, \ldots, \frac{1}{2m^2} \right) \in X,$$

where $\frac{e^{2\pi\sqrt{-1}\theta}}{2m^2}$ is the k-th entry of $\rho_k(\theta)$. We take a positive real number ε_0 so that $\varepsilon_0 < \min\{\frac{1}{2m^2}, \frac{1}{(m-2)^2} - \frac{1}{m^2}\}$, and we define the loop ρ_0 in X as $\rho_0 := \tau_0 \rho_0' \overline{\tau_0}$, where

$$\tau_0 : [0,1] \ni \theta \mapsto \left((1-\theta)\frac{1}{2m^2} + \theta\Big(\frac{1}{m^2} - \varepsilon_0\Big) \right) (1, \ldots, 1) \in X,$$

$$\rho_0' : [0,1] \ni \theta \mapsto \left(\frac{1}{m^2} - \varepsilon_0 e^{2\pi\sqrt{-1}\theta} \right) (1, \ldots, 1) \in X,$$

and $\overline{\tau_0}$ is the reverse path of τ_0. Here, the composition $\rho \cdot \rho'$ of loops ρ and ρ' is the loop defined as the loop going first along ρ, and then along ρ'.

Remark 3.1. The loop ρ_k $(1 \leq k \leq m)$ turns the divisor $(x_k = 0)$, and ρ_0 turns the divisor $(R(x) = 0)$ around the point $(\frac{1}{m^2}, \ldots, \frac{1}{m^2})$, positively. Note that $(\frac{1}{m^2}, \ldots, \frac{1}{m^2})$ is the nearest to the origin in $(R(x) = 0) \cap (x_1 = x_2 = \cdots = x_m)$ $= \{\frac{1}{m^2}(1, \ldots, 1), \frac{1}{(m-2)^2}(1, \ldots, 1), \ldots\}$.

Theorem 3.2. *The loops $\rho_0, \rho_1, \ldots, \rho_m$ generate the the fundamental group $\pi_1(X, \dot{x})$. Moreover, they satisfy the following relations:*

$$\rho_i \rho_j = \rho_j \rho_i \quad (1 \leq i, j \leq m), \quad (\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2 \quad (1 \leq k \leq m).$$

Remark 3.3. We conjecture that $\pi_1(X, \dot{x})$ is the group generated by ρ_0, ρ_1, $\ldots$, ρ_m with the relations in Theorem 3.2. To prove this conjecture, we need to show that there are no extra relations among ρ_0, ρ_1, $\ldots$, ρ_m. It is shown in [11] that this conjecture is true for $m = 2$.

3.2 Proof of Theorem 3.2

In this section, we prove Theorem 3.2.

We regard $\mathbb{C}^m$ as a subset of $\mathbb{P}^m$ and put $L_\infty := \mathbb{P}^m - \mathbb{C}^m$. Then we can consider that $S \cup L_\infty$ is a hypersurface in $\mathbb{P}^m$, and

$$X = \mathbb{C}^m - S = \mathbb{P}^m - (S \cup L_\infty).$$

By a special case of the Zariski theorem of Lefschetz type (refer to Proposition 3.3.1 in [4]), the inclusion $L - (L \cap (S \cup L_\infty)) \hookrightarrow X$ induces an surjection

$$\eta : \pi_1(L - (L \cap (S \cup L_\infty))) \to \pi_1(X),$$

for a generic line L in $\mathbb{P}^m$. Note that generators of $\pi_1(L - (L \cap (S \cup L_\infty)))$ are given by $m + 2^{m-1}$ loops going once around each of the intersection points in $L \cap S (\subset \mathbb{C}^m)$. Then we consider a generic line L in $\mathbb{C}^m$. Let $r_1, \ldots, r_{m-1}$ be positive real numbers satisfying

$$r_1 < \frac{1}{4}, \quad r_k < \frac{r_{k-1}}{4} \ (2 \leq k \leq m-1),$$

and let ε be a sufficiently small positive real number. We consider lines

$$\begin{aligned} L_0 :& (x_1, \ldots, x_{m-1}, x_m) = (r_1, \ldots, r_{m-1}, 0) + t(0, \ldots, 0, 1) \quad (t \in \mathbb{C}), \\ L_\varepsilon :& (x_1, \ldots, x_{m-1}, x_m) = (r_1, \ldots, r_{m-1}, 0) + t(\varepsilon, \ldots, \varepsilon, 1) \quad (t \in \mathbb{C}) \end{aligned}$$

in $\mathbb{C}^m$. Though L_0 is not a generic line (for example, $L_0 \cap (x_1 \cdots x_{m-1} = 0) = \emptyset$), L_ε is one of generic lines. We identify L_ε with t-space $\mathbb{C}$. The intersection point $L_\varepsilon \cap (x_k = 0)$ is coordinated by $t = -\frac{r_k}{\varepsilon} < 0$, for $1 \leq k \leq m-1$. The intersection point $L_\varepsilon \cap (x_m = 0)$ is coordinated by $t = 0$. There are 2^{m-1} intersection points of L_ε and $(R(x) = 0)$. We coordinate the intersection points $L_\varepsilon \cap (R(x) = 0)$ by $t = t_{a_1 \cdots a_{m-1}}$, $(a_1, \ldots, a_{m-1}) \in \{0, 1\}^{m-1}$. The correspondence is as follows. We denote the coordinates of the intersection points $L_0 \cap (R(x) = 0)$ by

$$t^{(0)}_{a_1 \cdots a_{m-1}} := \left(1 + \sum_{k=1}^{m-1} (-1)^{a_k} \sqrt{r_k}\right)^2.$$

By this definition, we have

$$\begin{aligned} & t^{(0)}_{a_1 \cdots a_{m-1}} < t^{(0)}_{a'_1 \cdots a'_{m-1}} \\ & \Longleftrightarrow a_1 - a'_1 = \cdots = a_{r-1} - a'_{r-1} = 0, \ a_r = 1, \ a'_r = 0 \\ & \Longleftrightarrow a_1 \cdots a_{m-1} > a'_1 \cdots a'_{m-1}, \end{aligned}$$

where $a_1 \cdots a_{m-1}$ is regarded as a binary number. For example, if $m = 4$, then

$$t^{(0)}_{111} < t^{(0)}_{110} < t^{(0)}_{101} < t^{(0)}_{100} < t^{(0)}_{011} < t^{(0)}_{010} < t^{(0)}_{001} < t^{(0)}_{000}.$$

Since L_ε is sufficiently close to L_0, $t_{a_1 \cdots a_{m-1}}$ are supposed to be arranged near to $t^{(0)}_{a_1 \cdots a_{m-1}}$.

Let ℓ_k be the loop going once around the intersection point $L_\varepsilon \cap (x_k = 0)$, and let $\ell_{a_1 \cdots a_{m-1}}$ be the loop going once around $t_{a_1 \cdots a_{m-1}}$. Each loop approaches the intersection point through the upper half plane of t-space, see Figure 3.1.

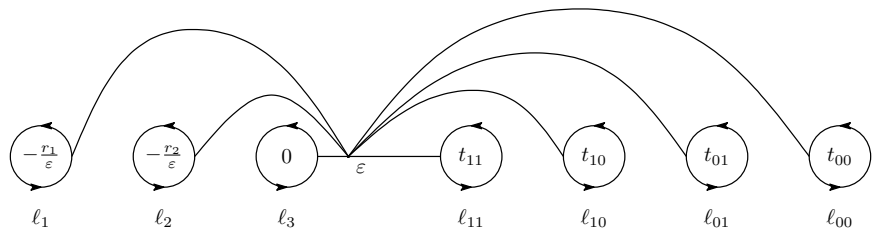

Figure 3.1: ℓ_* for $m = 3$.

It is easy to see that

$$\eta(\ell_k) = \rho_k \ (1 \le k \le m), \quad \eta(\ell_{1 \cdots 1}) = \rho_0.$$

Further, we have

$$\rho_i \rho_j = \rho_j \rho_i \quad (1 \le i, j \le m),$$

since the fundamental group of $(\mathbb{C}^\times)^m$ is abelian. To investigate relations among $\eta(\ell_{a_1 \cdots a_{m-1}})$'s, we may consider these loops are in L_0, since $\ell_{a_1 \cdots a_{m-1}}$'s can be defined by a same way as in L_ε. For simplicity, we denote $t^{(0)}_{a_1 \cdots a_{m-1}}$ by $t_{a_1 \cdots a_{m-1}}$.

Lemma 3.4. (i) $\eta(\ell_{a_1 \cdots a_{k-1} 0 a_{k+1} \cdots a_{m-1}}) = \rho_k \eta(\ell_{a_1 \cdots a_{k-1} 1 a_{k+1} \cdots a_{m-1}}) \rho_k^{-1}$.

(ii) $\eta(\ell_{1 \cdots 1}) = \rho_{m-1} \eta(\ell_{1 \cdots 1} \ell_{1 \cdots 10} \ell^{-1}_{1 \cdots 1}) \rho^{-1}_{m-1}$.

Temporarily, we admit this lemma. By (i), we have

$$\begin{aligned} \eta(\ell_{a_1 \cdots a_{m-1}}) &= (\rho_1^{b_1} \cdots \rho_{m-1}^{b_{m-1}}) \cdot \eta(\ell_{1 \cdots 1}) \cdot (\rho_1^{b_1} \cdots \rho_{m-1}^{b_{m-1}})^{-1} \\ &= (\rho_1^{b_1} \cdots \rho_{m-1}^{b_{m-1}}) \cdot \rho_0 \cdot (\rho_1^{b_1} \cdots \rho_{m-1}^{b_{m-1}})^{-1}, \end{aligned}$$

where $(b_1, \ldots, b_{m-1}) := (1 - a_1, \ldots, 1 - a_{m-1})$. This implies that the loops $\rho_0, \ldots, \rho_m$ generate $\pi_1(X)$, since the images of ℓ_k's and $\ell_{a_1 \cdots a_{m-1}}$'s by η generate $\pi_1(X)$. By (ii) and the above argument, we obtain

$$\begin{aligned} \rho_0 = \eta(\ell_{1 \cdots 1}) &= \rho_{m-1} \eta(\ell_{1 \cdots 1} \ell_{1 \cdots 10} \ell^{-1}_{1 \cdots 1}) \rho^{-1}_{m-1} \\ &= \rho_{m-1} \cdot \rho_0 \cdot \rho_{m-1} \rho_0 \rho^{-1}_{m-1} \cdot \rho_0^{-1} \cdot \rho^{-1}_{m-1}, \end{aligned}$$

that is, $(\rho_0 \rho_{m-1})^2 = (\rho_{m-1} \rho_0)^2$. Changing the definitions of L_0 and L_ε, we obtain the relations

$$(\rho_0 \rho_k)^2 = (\rho_k \rho_0)^2 \quad (1 \le k \le m).$$

For example, if we put

$$L_\varepsilon : (x_1, x_2, \ldots, x_m) = (0, r_1, \ldots, r_{m-1}) + t(1, \varepsilon, \ldots, \varepsilon) \quad (t \in \mathbb{C}),$$

then a similar argument shows $(\rho_0\rho_m)^2 = (\rho_m\rho_0)^2$. Therefore, the proof of Theorem 3.2 is completed.

Proof of Lemma 3.4. For $\theta \in [0,1]$, let $L(\theta)$ be a line defined by

$$\begin{aligned} L(\theta) :&(x_1, \ldots, x_k, \ldots, x_{m-1}, x_m) \\ &= (r_1, \ldots, e^{2\pi\sqrt{-1}\theta} r_k, \ldots, r_{m-1}, 0) + t(0, \ldots, 0, 1) \quad (t \in \mathbb{C}). \end{aligned}$$

Note that $L(0) = L(1) = L_0$. We identify $L(\theta)$ with $\mathbb{C}$ by the coordinate t. It is easy to see that the intersection points of $L(\theta)$ and $(R(x) = 0)$ are the following 2^{m-1} elements:

$$t^{(\theta)}_{a_1 \cdots a_{m-1}} := \left(1 + \sum_{\substack{j=1 \\ j \neq k}}^{m-1} (-1)^{a_j} \sqrt{r_j} + (-1)^{a_k} \sqrt{r_k} e^{\pi\sqrt{-1}\theta} \right)^2 .$$

The points $1 + \sum_{j \neq k} (-1)^{a_j} \sqrt{r_j} + (-1)^{a_k} \sqrt{r_k} e^{\pi\sqrt{-1}\theta}$ are in the right half plane for any $\theta \in [0,1]$, since $\sum_{j=1}^{m-1} \sqrt{r_j} < \sum_{j=1}^{m-1} 2^{-j} < 1$. Let θ move from 0 to 1, then

(a) $t_{a_1 \cdots a_{k-1} 0 a_{k+1} \cdots a_{m-1}}$ and $t_{a_1 \cdots a_{k-1} 1 a_{k+1} \cdots a_{m-1}}$ interchange,

(b) $t^{(\theta)}_{a_1 \cdots a_{k-1} 0 a_{k+1} \cdots a_{m-1}}$ moves in the upper half plane,

(c) $t^{(\theta)}_{a_1 \cdots a_{k-1} 1 a_{k+1} \cdots a_{m-1}}$ moves in the lower half plane.

For example, $t_{a_1 a_2 a_3}$'s move as Figure 3.2, for $m = 4$ and $k = 2$.

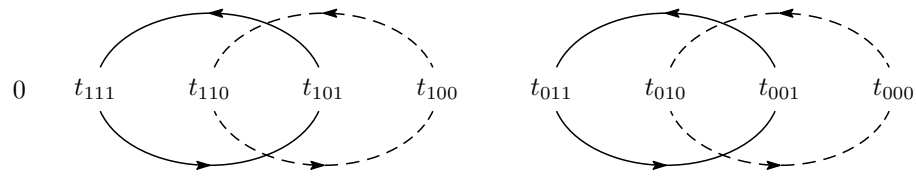

Figure 3.2: $t_{a_1 a_2 a_3}$ for $m = 4,\ k = 2$.

Put $P(\theta) := \mathbb{C} - \{t^{(\theta)}_{a_1 \cdots a_{m-1}} \mid a_j \in \{0,1\}\}$. Let ε' be a sufficiently small positive real number, and we consider the fundamental group $\pi_1(P(\theta), \varepsilon')$. As mentioned above, $\ell_{a_1 \cdots a_{m-1}}$'s are defined as elements in $\pi_1(P(0), \varepsilon') = \pi_1(P(1), \varepsilon')$. Let θ move from 0 to 1, then $\ell_{a_1 \cdots a_{m-1}}$'s vary naturally and define elements in each $\pi_1(P(\theta), \varepsilon')$. The properties (a), (b), (c) imply that $\ell_{a_1 \cdots a_{k-1} 0 a_{k+1} \cdots a_{m-1}}$ in $\pi_1(P(0), \varepsilon')$ changes into $\ell_{a_1 \cdots a_{k-1} 1 a_{k+1} \cdots a_{m-1}}$ in $\pi_1(P(1), \varepsilon')$. By this variation, the base point moves around the divisor $(x_k = 0)$, since $\varepsilon' \in P(\theta)$ corresponds

to the point $(r_1, \ldots, e^{2\pi\sqrt{-1}\theta} r_k, \ldots, r_{m-1}, \varepsilon') \in L(\theta)$. It implies the conjugation by ρ_k in $\pi_1(X)$. Hence we obtain the relation (i):

$$\eta(\ell_{a_1 \cdots a_{k-1} 0 a_{k+1} \cdots a_{m-1}}) = \rho_k \eta(\ell_{a_1 \cdots a_{k-1} 1 a_{k+1} \cdots a_{m-1}}) \rho_k^{-1}.$$

To prove (ii), we use a similar argument for $k = m-1$ and $\ell_{1\cdots 1} \in \pi_1(P(\theta), \varepsilon')$. Let θ move from 0 to 1, then $\ell_{1\cdots 1}$ changes into a loop in $P(1)$, which goes once around $t_{1\cdots 10} (= t^{(1)}_{1\cdots 1})$ and approaches this point through the lower half plane (see Figure 3.3). Since such a loop is homotopic to $\ell_{1\cdots 1}\ell_{1\cdots 10}\ell^{-1}_{1\cdots 1}$, we obtain (ii). $\square$

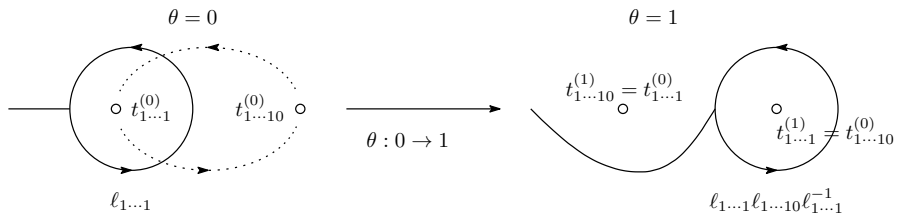

Figure 3.3: the variation of $\ell_{1\cdots 1}$.

Chapter 4

Monodromy representation

In this chapter, we study the monodromy representation of $E_C(a,b,c)$ for general m, by using twisted homology groups and the intersection form. By Theorem 3.2, it is sufficient to study $\mathcal{M}_0,\dots,\mathcal{M}_m$, which are the circuit transformations along the generators $\rho_0,\dots,\rho_m$, respectively. With respect to the basis $\{\Delta_I\}_I$ in Chapter 1, $\mathcal{M}_1,\dots,\mathcal{M}_m$ are represented by diagonal matrices. It is difficult to represent $\mathcal{M}_0$ with respect to our basis. We give some properties of $\mathcal{M}_0$, which enable us to express the linear map $\mathcal{M}_0$ by the intersection form. In this consideration, the basis $\{\Delta_I\}_I$ plays an important role. Moreover, we take a basis of the twisted homology group so that $\mathcal{M}_0,\dots,\mathcal{M}_m$ are represented by triangular matrices.

Hereafter, we assume that the parameters a, b, and $c=(c_1,\dots,c_m)$ are generic, that is, we add other conditions in Remark 4.20 to those mentioned in Chapter 1:

$$a-\sum_{i\in I}c_i,\quad b-\sum_{i\in I}c_i\notin\mathbb{Z}\quad (I\subset\{1,\dots,m\}),\quad c_k\notin\mathbb{Z}\quad (1\le k\le m).$$

4.1 Local system

In this chapter, we need to vary the variable $x=(x_1,\dots,x_m)$. Then we change some notations. Further, for a subset $I=\{i_1,\dots,i_r\}$ of $\{1,\dots,m\}$, we use the suffix "I" more frequently than "$i_1\cdots i_r$".

We denote

$$v(t):=1-\sum_k t_k,\quad w(t,x):=\prod_k t_k\cdot\left(1-\sum_k\frac{x_k}{t_k}\right),$$

and put

$$\mathfrak{X}:=\left\{(t,x)\in\mathbb{C}^m\times X\;\middle|\;\prod_k t_k\cdot v(t)\cdot w(t,x)\neq 0\right\}.$$

There is a natural projection

$$pr:\mathfrak{X}\to X;\ (t,x)\mapsto x,$$

and we define $T_x := pr^{-1}(x)$ for any $x \in X$. We consider the twisted homology groups on T_x with respect to the multi-valued function

$$u_x(t) := \prod t_k^{1-c_k+b} \cdot v(t)^{\sum c_k - a - m + 1} w(t,x)^{-b}.$$

We denote the k-th twisted homology group by $H_k(T_x, u_x)$, and locally finite one by $H_k^{lf}(T_x, u_x)$. Note that $H_m(T_x, u_x)$ (resp. $H_k^{lf}(T_x, u_x)$) is equal to $H_m(M, u)$ (resp. $H_m^{lf}(M, u)$) in Chapter 1.

For $x, x' \in X$, let τ be a path in X from x to x'. There is a canonical isomorphism

$$\tau_* : H_m(T_x, u_x) \to H_m(T_{x'}, u_{x'}),$$

and hence the family

$$\mathcal{H} := \bigcup_{x \in X} H_m(T_x, u_x)$$

forms a local system on X.

Fact 4.1 ([1]). (i) *$H_k(T_x, u_x) = 0$, $H_k^{lf}(T_x, u_x) = 0$, for $k \neq m$.*

(ii) *The natural map $H_m(T_x, u_x) \to H_m^{lf}(T_x, u_x)$ is isomorphic (the inverse map is called the regularization).*

Hereafter, we identify $H_k^{lf}(T_x, u_x)$ with $H_k(T_x, u_x)$. Recall that the intersection form I_h is defined between $H_m(T_x, u_x)$ and $H_m(T_x, u_x^{-1})$.

Let δ be a twisted cycle in T_x for a fixed x. If x' is a sufficiently close point to x, there is a unique cycle δ' such that $\int_{\delta'} u_{x'}\varphi$ is obtained by analytic continuation of $\int_\delta u_x \varphi$. Thus we regard

$$\int_\delta u_x \varphi$$

as a holomorphic function in x. Let Sol be the sheaf on X whose sections are holomorphic solutions to $E_C(a, b, c)$. The stalk Sol_x of $x \in X$ is the space of local holomorphic solutions near x. Let U be a sufficiently small simply connected domain in $D_C - S$. Fact 0.2 implies that Sol_x is a $\mathbb{C}$-vector space of dimension 2^m and spanned by f_I's, for $x \in U$. Proposition 2.1 (i) and (iii) are written as follows.

Fact 4.2. *For any $x \in X$, we have $\dim H_m(T_x, u_x) = 2^m$, and*

$$\Phi_x : H_m(T_x, u_x) \to Sol_x;\ \delta \mapsto \int_\delta u_x \varphi$$

is isomorphic.

In Chapter 1, we construct twisted cycles Δ_I for all subsets I of $\{1, \ldots, m\}$. We consider the basis $\{\Delta_I\}_I$ arranged as $(\Delta, \Delta_1, \Delta_2, \ldots, \Delta_m, \Delta_{12}, \Delta_{13}, \ldots, \Delta_{1\cdots m})$. We summarize Theorem 1.4 and 1.6.

Fact 4.3. *We have*

$$\Phi_x(\Delta_I) = \frac{\prod_{i \in I} \Gamma(c_i - 1) \cdot \prod_{j \notin I} \Gamma(1 - c_j) \cdot \Gamma(\sum_k c_k - a - m + 1)\Gamma(1 - b)}{\Gamma(\sum_{i \in I} c_i - a - r + 1)\Gamma(\sum_{i \in I} c_i - b - r + 1)} \cdot f_I.$$

The intersection matrix $H := (I_h(\Delta_I, \Delta_{I'}^\vee))_{I,I'}$ *is diagonal. Further, the* (I, I) *entry* $H_{I,I}$ *of* H *is*

$$H_{I,I} = (-1)^{|I|} \cdot \frac{\prod_{j \notin I} \gamma_j \cdot (\alpha - \prod_{i \in I} \gamma_i)(\beta - \prod_{i \in I} \gamma_i)}{\prod_k (\gamma_k - 1) \cdot (\alpha - \prod_k \gamma_k)(\beta - 1)},$$

where $\alpha = e^{2\pi\sqrt{-1}a}$, $\beta = e^{2\pi\sqrt{-1}b}$, $\gamma_k = e^{2\pi\sqrt{-1}c_k}$.

4.2 Monodromy representation

For $\rho \in \pi_1(X, \dot{x})$ and $g \in Sol_{\dot{x}}$, let $\rho_* g$ be the analytic continuation of g along ρ. Since $\rho_* g$ is also a solution to $E_C(a, b, c)$, the map $\rho_* : Sol_{\dot{x}} \to Sol_{\dot{x}};\ g \mapsto \rho_* g$ is a $\mathbb{C}$-linear automorphism, and we have $(\rho \cdot \rho')_* = \rho'_* \circ \rho_*$ for $\rho, \rho' \in \pi_1(X, \dot{x})$. We thus obtain a representation

$$\mathcal{M}' : \pi_1(X, \dot{x}) \to GL(Sol_{\dot{x}})$$

of $\pi_1(X, \dot{x})$, where $GL(V)$ is the general linear group on a $\mathbb{C}$-vector space V. Since we can identify $Sol_{\dot{x}}$ with $H_m(T_{\dot{x}}, u_{\dot{x}})$ by Fact 4.2, $\mathcal{M}'$ is equivalent to

$$\mathcal{M} : \pi_1(X, \dot{x}) \to GL(H_m(T_{\dot{x}}, u_{\dot{x}})).$$

Note that for $\rho \in \pi_1(X, \dot{x})$, the map $\mathcal{M}(\rho) : H_m(T_{\dot{x}}, u_{\dot{x}}) \to H_m(T_{\dot{x}}, u_{\dot{x}})$ means the canonical isomorphism $\rho_* : H_m(T_{\dot{x}}, u_{\dot{x}}) \to H_m(T_{\dot{x}}, u_{\dot{x}})$ in the local system $\mathcal{H}$. $\mathcal{M}$ (and $\mathcal{M}'$) is called the monodromy representation. In this paper, we study $\mathcal{M}$ mainly.

By Theorem 3.2, for the study of $\mathcal{M}$, it is sufficient to investigate $m + 1$ linear maps

$$\mathcal{M}_i := \mathcal{M}(\rho_i) \quad (0 \le i \le m).$$

Proposition 4.4. *For* $1 \le k \le m$*, the eigenvalues of* $\mathcal{M}_k$ *are* γ_k^{-1} *and* 1*. The eigenspace of* $\mathcal{M}_k$ *of eigenvalue* γ_k^{-1} *is spanned by the twisted cycles*

$$\Delta_I, \quad k \in I \subset \{1, \ldots, m\}.$$

That of eigenvalue 1 *is spanned by*

$$\Delta_I, \quad k \notin I \subset \{1, \ldots, m\}.$$

In particular, both eigenspaces are of dimension 2^{m-1}.

Proof. By Fact 4.3, the twisted cycle Δ_I corresponds to the solution

$$f_I = \prod_{i \in I} x_i^{1-c_i} \cdot F_C\left(a + |I| - \sum_{i \in I} c_i, b + |I| - \sum_{i \in I} c_i, c^I; x\right)$$

to $E_C(a, b, c)$. Since the series F_C defines a single-valued function around the origin, we have

$$\mathcal{M}'(\rho_k)(f_I) = \begin{cases} \gamma_k^{-1} f_I & (k \in I), \\ f_I & (k \notin I). \end{cases}$$

Therefore we obtain this proposition. □

Corollary 4.5. *For $1 \le k \le m$, the linear map $\mathcal{M}_k : H_m(T_{\dot{x}}, u_{\dot{x}}) \to H_m(T_{\dot{x}}, u_{\dot{x}})$ is expressed as*

$$\mathcal{M}_k : \delta \mapsto \delta - (1 - \gamma_k^{-1}) \sum_{I \ni k} \frac{I_h(\delta, \Delta_I^\vee)}{I_h(\Delta_I, \Delta_I^\vee)} \Delta_I.$$

Further, the representation matrix M_k of $\mathcal{M}_k$ with respect to the basis $\{\Delta_I\}_I$ is the diagonal matrix whose (I, I) entry is

$$\begin{cases} \gamma_k^{-1} & (I \ni k), \\ 1 & (I \not\ni k). \end{cases}$$

Proof. By Proposition 4.4, $H_m(T_{\dot{x}}, u_{\dot{x}})$ is decomposed into the direct product of the eigenspaces: $H_m(T_{\dot{x}}, u_{\dot{x}}) = (\bigoplus_{I \ni k} \mathbb{C}\Delta_I) \oplus (\bigoplus_{I \not\ni k} \mathbb{C}\Delta_I)$. Then it is sufficient to show that the claim holds for $\delta = \Delta_I$. This is clear by Fact 4.3 and Proposition 4.4. The second claim is obvious. □

For each subset $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$, we define a chamber $D_I = D_{i_1 \cdots i_r}$ which gives an element in $H_m(T_{\dot{x}}, u_{\dot{x}})$. For $I = \{1, \ldots, m\}$, we put

$$D_{1 \cdots m} := \{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid t_k > 0 \ (1 \le k \le m),\ v(t) > 0,\ w(t, \dot{x}) > 0\}.$$

For $I = \emptyset$, we put

$$D_\emptyset = D := \{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid t_k < 0 \ (1 \le k \le m)\}.$$

For $I \ne \emptyset, \{1, \ldots, m\}$, we put

$$D_I = D_{i_1 \cdots i_r} := \left\{ (t_1, \ldots, t_m) \in \mathbb{R}^m \;\middle|\; \begin{array}{l} t_i > 0 \ (i \in I),\ t_j < 0 \ (j \notin I), \\ v(t) > 0,\ (-1)^{m - |I| + 1} w(t, \dot{x}) > 0 \end{array} \right\}.$$

The arguments of the factors of $u_{\dot{x}}(t)$ are defined as follows:

	$t_i (i \in I)$	$t_j (j \notin I)$	$v(t)$	$w(t, \dot{x})$
$D_{1 \cdots m}$	0	$-$	0	0
D	$-$	$-\pi$	0	$-m\pi$
otherwise	0	$-\pi$	0	$-(m - \lvert I \rvert + 1)\pi$

Note that if $m = 2$, then $D,\ D_1,\ D_2$, and D_{12} are equal to $\Delta_6,\ \Delta_7,\ \Delta_8$, and Δ_5 in [9], respectively.

Theorem 4.6. *The eigenvalues of $\mathcal{M}_0$ are $(-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}$ and 1. The eigenspace W_0 of $\mathcal{M}_0$ of eigenvalue $(-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}$ is spanned by $D_{1 \cdots m}$, and hence is one dimensional. The eigenspace W_1 of $\mathcal{M}_0$ of eigenvalue 1 is spanned by*

$$D_I, \quad I \subsetneq \{1, \ldots, m\},$$

and expressed as

$$W_1 = \{\delta \in H_m(T_{\dot{x}}, u_{\dot{x}}) \mid I_h(\delta, D_{1 \cdots m}^\vee) = 0\}.$$

In particular, this space is $(2^m - 1)$-dimensional.

The proof of this theorem is given in the next section.

Corollary 4.7. *The linear map $\mathcal{M}_0 : H_m(T_{\dot{x}}, u_{\dot{x}}) \to H_m(T_{\dot{x}}, u_{\dot{x}})$ is expressed as*

$$\mathcal{M}_0 : \delta \mapsto \delta - (1 + (-1)^m \prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}) \frac{I_h(\delta, D^\vee_{1\cdots m})}{I_h(D_{1\cdots m}, D^\vee_{1\cdots m})} D_{1\cdots m}.$$

Proof. By Theorem 4.6, we have $H_m(T_{\dot{x}}, u_{\dot{x}}) = W_0 \oplus W_1 = \mathbb{C}D_{1\cdots m} \oplus W_1$. Then it is sufficient to show that the claim holds for $\delta = D_{1\cdots m}$ and $\delta \in W_1$. This is clear by Theorem 4.6. □

Proposition 4.8. *We have*

$$I_h(D_{1\cdots m}, \Delta^\vee_I) = I_h(\Delta_I, \Delta^\vee_I) = I_h(\Delta_I, D^\vee_{1\cdots m}). \tag{4.1}$$

Thus we obtain

$$D_{1\cdots m} = \sum_{I \subset \{1,\ldots,m\}} \Delta_I, \tag{4.2}$$

$$I_h(D_{1\cdots m}, D^\vee_{1\cdots m}) = \frac{\alpha\beta + (-1)^m \prod_k \gamma_k}{(\beta - 1)(\alpha - \prod_k \gamma_k)}. \tag{4.3}$$

This proposition is also proved in the next section. By this proposition, we obtain the following corollary.

Corollary 4.9. *The linear map $\mathcal{M}_0$ is expressed as*

$$\mathcal{M}_0 : \delta \mapsto \delta - \frac{(\beta - 1)(\alpha - \prod_k \gamma_k)}{\alpha\beta} I_h(\delta, D^\vee_{1\cdots m}) D_{1\cdots m}.$$

Let M_0 be the representation matrix M_0 of $\mathcal{M}_0$ with respect to the basis $\{\Delta_I\}_I$. Then we have

$$M_0 = E_{2^m} - \frac{(\beta - 1)(\alpha - \prod_k \gamma_k)}{\alpha\beta} NH,$$

where E_{2^m} is the unit matrix of size 2^m, N is the $2^m \times 2^m$ matrix with all entries 1, and $H = (I_h(\Delta_I, \Delta^\vee_{I'}))_{I,I'}$ is the intersection matrix given in Fact 4.3.

Proof. The expression of $\mathcal{M}_0$ follows from Corollary 4.7 and (4.3) immediately. To obtain the representation matrix, we have to show that the representation matrix of $\delta \mapsto I_h(\delta, D^\vee_{1\cdots m}) D_{1\cdots m}$ is given by NH. By Proposition 4.8, we have

$$\begin{aligned} I_h(\Delta_I, D^\vee_{1\cdots m}) D_{1\cdots m} &= I_h(\Delta_I, \Delta^\vee_I) D_{1\cdots m} = \sum_{I'} I_h(\Delta_I, \Delta^\vee_I) \Delta_{I'} \\ &= (\Delta, \Delta_1, \Delta_2, \ldots, \Delta_m, \Delta_{12}, \Delta_{13}, \ldots, \Delta_{1\cdots m}) \begin{pmatrix} I_h(\Delta_I, \Delta^\vee_I) \\ I_h(\Delta_I, \Delta^\vee_I) \\ \vdots \\ I_h(\Delta_I, \Delta^\vee_I) \end{pmatrix}, \end{aligned}$$

and hence the claim is proved. □

Proposition 4.10. *Let ρ_∞ be a loop in X turning the divisor $L_\infty \subset \mathbb{P}^m$. $x_m^{-a} f(\frac{x_1}{x_m}, \ldots, \frac{x_{m-1}}{x_m}, \frac{1}{x_m})$ is a solution to $E_C(a, b, c)$ if and only if $f(\xi_1, \ldots, \xi_m)$ is a solution to $E_C(a, a-c_m+1, (c_1, \ldots, c_{m-1}, a-b+1))$ with variables $\xi_1, \ldots, \xi_m$. Hence, the eigenvalues of $\mathcal{M}(\rho_\infty)$ are α and β. Moreover, both eigenspaces are 2^{m-1}-dimensional.*

Remark 4.11. Because of $\rho_\infty = \eta(\ell_1 \cdots \ell_m \ell_{1\cdots 1} \ell_{1\cdots 10} \cdots \ell_{0\cdots 0})^{-1}$, we have $\mathcal{M}(\rho_\infty)$ by Proposition 4.4 and Theorem 4.6. However, it is too complicated to write down.

Proof of Proposition 4.10. We prove this proposition similarly to Section 2.3 of [12]. Recall that $E_C(a,b,c)$ is generated by

$$P_k : \theta_k(\theta_k + c_k - 1) - x_k(\theta + a)(\theta + b) \quad (1 \le k \le m),$$

where $\theta_k = x_k \partial_k$, $\theta = \sum_k \theta_k$. We consider the variable changes

$$x_k = \frac{\xi_k}{\xi_m} \ (1 \le k \le m-1), \quad x_m = \frac{1}{\xi_m} \quad \left(\Leftrightarrow \xi_k = \frac{x_k}{x_m},\ \xi_m = \frac{1}{x_m} \right).$$

Let δ_k $(k = 1, \ldots, m)$ be the partial differential operator with respect to ξ_k, and put $\vartheta_k := \xi_k \delta_k$, $\vartheta := \sum_k \vartheta_k$. We have

$$\begin{aligned}
\partial_k &= \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial x_k} \delta_i = \frac{1}{x_m} \delta_k = \xi_m \delta_k \ (1 \le k \le m-1), \\
\partial_m &= \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial x_m} \delta_i = \sum_{i=1}^{m-1} \left(-\frac{x_i}{x_m^2} \right) \delta_i - \frac{1}{x_m^2} \delta_m = -\sum_{i=1}^{m} \xi_i \xi_m \delta_i,
\end{aligned}$$

and hence

$$\begin{aligned}
\theta_k &= \frac{\xi_k}{\xi_m} \cdot \xi_m \delta_k = \vartheta_k, \quad \theta_m = -\frac{1}{\xi_m} \sum_{i=1}^{m} \xi_i \xi_m \delta_i = -\sum_{i=1}^{m} \vartheta_i = -\vartheta, \\
\theta &= \sum_{i=1}^{m-1} \vartheta_i - \sum_{i=1}^{m} \vartheta_i = -\vartheta_m.
\end{aligned}$$

Thus we have

$$\begin{aligned}
P_k &= \vartheta_k(\vartheta_k + c_k - 1) - \xi_k \xi_m^{-1}(-\vartheta_m + a)(-\vartheta_m + b) \\
&= \vartheta_k(\vartheta_k + c_k - 1) - \xi_k \xi_m^{-1}(\vartheta_m - a)(\vartheta_m - b)
\end{aligned}$$

for $1 \le k \le m-1$, and

$$\begin{aligned}
P_m &= -\vartheta(-\vartheta + c_m - 1) - \xi_m^{-1}(-\vartheta_m + a)(-\vartheta_m + b) \\
&= \vartheta(\vartheta - c_m + 1) - \xi_m^{-1}(\vartheta_m - a)(\vartheta_m - b).
\end{aligned}$$

By the equalities

$$\vartheta_k \xi_m^a = \xi_m^a \vartheta_k \ (1 \le k \le m-1), \quad \vartheta_m \xi_m^a = \xi_m^a(\vartheta_m + a), \quad \vartheta \xi_m^a = \xi_m^a(\vartheta + a),$$

we obtain

$$\begin{aligned}
P_k \xi_m^a &= \xi_m^a \vartheta_k(\vartheta_k + c_k - 1) - \xi_k \xi_m^{a-1} \vartheta_m(\vartheta_m + a - b) \\
&= \xi_m^{a-1} \left[\xi_m \vartheta_k(\vartheta_k + c_k - 1) - \xi_k \vartheta_m(\vartheta_m + a - b) \right] \quad (1 \le k \le m-1), \\
P_m \xi_m^a &= \xi_m^a(\vartheta + a)(\vartheta + a - c_m + 1) - \xi_m^{a-1} \vartheta_m(\vartheta_m + a - b) \\
&= -\xi_m^{a-1} \left[\vartheta_m(\vartheta_m + a - b) - \xi_m(\vartheta + a)(\vartheta + a - c_m + 1) \right].
\end{aligned}$$

Since

$$(P_k - \xi_k P_m)\xi_m^a = \xi_m^a \left[\vartheta_k(\vartheta_k + c_k - 1) - \xi_k(\vartheta + a)(\vartheta + a - c_m + 1)\right]$$

for $1 \leq k \leq m-1$, and

$$P_m \xi_m^a = -\xi_m^{a-1} \left[\vartheta_m(\vartheta_m + (a-b+1) - 1) - \xi_m(\vartheta + a)(\vartheta + a - c_m + 1)\right],$$

$f(\xi_1, \ldots, \xi_m)$ is a solution to $E_C(a, a - c_m + 1, (c_1, \ldots, c_{m-1}, a - b + 1))$ with variables $\xi_1, \ldots, \xi_m$ if and only if $\xi_m^a f(\xi_1, \ldots, \xi_m)$ is annihilated by $P_1, \ldots, P_m$, i.e., $x_m^{-a} f(\frac{x_1}{x_m}, \ldots, \frac{x_{m-1}}{x_m}, \frac{1}{x_m})$ is a solution to $E_C(a, b, c)$. To study $\mathcal{M}(\rho_\infty)$, it is sufficient to investigate the local monodromy of the system $E_C(a, b, c)$ with variables $\xi_1 = \frac{x_1}{x_m}, \ldots, \xi_{m-1} = \frac{x_{m-1}}{x_m}, \xi_m = \frac{1}{x_m}$ along a loop turning the divisor $(\xi_m = 0)$. The space S_∞ of the solutions consists of the elements $\xi_m^a \cdot f$, where f runs elements in the space of solutions to $E_C(a, a-c_m+1, (c_1, \ldots, c_{m-1}, a-b+1))$ with variables $\xi_1, \ldots, \xi_m$. Thus S_∞ is spanned by the following elements: for any subsets $N \subset \{1, \ldots, m-1\}$,

$$f_N^\infty(\xi) := \prod_{i \in N} \xi_i^{1-c_i} \cdot \xi_m^a \cdot F_N^\infty(\xi),$$
$$g_N^\infty(\xi) := \prod_{i \in N} \xi_i^{1-c_i} \cdot \xi_m^{1-(a-b+1)+a} \cdot G_N^\infty(\xi) = \prod_{i \in N} \xi_i^{1-c_i} \cdot \xi_m^b \cdot G_N^\infty(\xi),$$

where $F_N^\infty(\xi)$ and $G_N^\infty(\xi)$ are Lauricella's hypergeometric series F_C of variables $\xi_1, \ldots, \xi_m$ with some parameters. Then the eigenvalues of $\mathcal{M}(\rho_\infty)$ are α and β. Further, the eigenspace of α (resp. β) is 2^{m-1}-dimensional and spanned by $\{f_N^\infty\}_N$ (resp. $\{g_N^\infty\}_N$). □

4.3 Proof of Theorem 4.6

In this section, we prove Theorem 4.6. Since $\dim H_m(T_{\dot{x}}, u_{\dot{x}}) = 2^m$, it is sufficient to show that D_I's are eigenvectors and linearly independent. First, we evaluate the intersection numbers $I_h(\Delta_I, D_{I'}^\vee)$. Second, we show the linear independence by evaluating the determinant of the matrix $(I_h(\Delta_I, D_{I'}^\vee))_{I,I'}$. Third, we prove the properties of the eigenspace of $\mathcal{M}_0$ of eigenvalue 1. Finally, we show that $D_{1\cdots m}$ is an eigenvector of $\mathcal{M}_0$ of eigenvalue $(-1)^{m-1} \prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}$.

Notation 4.12. We use same notations as in Section 1.2. For fixed sufficiently small positive real numbers $x_1, \ldots, x_m$, we denote $M = T_x$, where $x = (x_1, \ldots, x_m)$.

We review the construction of $\Delta_{i_1 \cdots i_r} = \Delta_I$ briefly. We set $J := I^c = \{1, \ldots, m\} - I$. We consider

$$M_I = \mathbb{C}^m - \left(\bigcup_k (s_k = 0) \cup (v_I = 0) \cup (w_I = 0) \right),$$

where v_I and w_I are polynomials in $s_1, \ldots, s_m$ defined by

$$v_I := \prod_{i \in I} s_i \cdot \left(1 - \sum_{i \in I} \frac{x_i}{s_i} - \sum_{j \in J} s_j\right), \; w_I := \prod_{j \in J} s_j \cdot \left(1 - \sum_{i \in I} s_i - \sum_{j \in J} \frac{x_j}{s_j}\right).$$

The multi-valued function u_I on M_I is defined by

$$u_I := \prod_k s_k^{C_k} \cdot v_I^A \cdot w_I^B,$$

where

$$A := \sum c_k - a - m + 1, \quad B := -b,$$
$$C_i := c_i - 1 - A \ (i \in I), \quad C_j := 1 - c_j - B \ (j \in J).$$

We construct the twisted cycle $\tilde{\Delta}_I$ in M_I with respect to u_I from the direct product $\sigma_I \subset \mathbb{R}^m$ of an r-simplex and an $(m-r)$-simplex. The orientation of σ_I is induced from the natural embedding $\mathbb{R}^m \subset \mathbb{C}^m$. Note that when we eliminate the boundary of σ_I, we regulate the difference of branches of u_I by a different way from the usual regularization. We consider the ε-neighborhood of the divisors $(s_1 = 0), \ldots, (s_m = 0)$, $(1 - \sum_{i \in I} s_i = 0)$, $(1 - \sum_{j \in J} s_j = 0)$, where ε is a positive real number satisfying $\varepsilon < \frac{1}{m+1}$ and $x_k < \frac{\varepsilon^2}{m}$ (we use the assumption $\varepsilon_1 = \cdots = \varepsilon_m = \varepsilon$ in Section 1.2). By using the bijection

$$\iota_I : M_I \to M; \quad \iota_I(s_1, \ldots, s_m) := (t_1, \ldots, t_m),$$
$$t_i = \frac{x_i}{s_i} \ (i \in I), \ t_j = s_j \ (j \in J),$$

we define Δ_I as the following:

$$\Delta_I := (-1)^{|I|} (\iota_I)_* (\tilde{\Delta}_I).$$

Note that σ_I is contained in the bounded domain $\iota_I^{-1}(D_{1\cdots m})$.

4.3.1 An expression of $D_{1\cdots m}$

We prove Proposition 4.8 by using imaginary cycles and the construction of Δ_I's.

Fix any $t_0 \in \sigma_I$, and set

$$\sqrt{-1}\mathbb{R}_I^m := \{t_0 + \sqrt{-1}(\eta_1, \ldots, \eta_m) \mid (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m\} \subset M_I,$$

which is called an imaginary cycle. By arguments similar to those in the proofs of Proposition 1.3 and Theorem 1.4, we can prove that the integration of $u\varphi$ on $(\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)$ also gives the solution f_I to $E_C(a, b, c)$, under some conditions for the parameters a, b, c. Therefore $(\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee$ is orthogonal to the cycles $\Delta_{I'}$ $(I' \neq I)$ with respect to I_h, and hence $(\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee$ is a constant multiple of $\Delta_I^\vee$. Since $D_{1\cdots m}$ and the simplex $\iota_I(\sigma_I)$ in Δ_I have a same orientation (c.f. Remark 1.5 (i)), we have

$$I_h(D_{1\cdots m}, (\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee) = I_h(\Delta_I, (\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee).$$

Thus we obtain

$$\Delta_I^\vee = \frac{I_h(\Delta_I, \Delta_I^\vee)}{I_h(D_{1\cdots m}, (\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee)} \cdot (\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee,$$

which implies the first equality of (4.1) because of

$$\begin{aligned} I_h(D_{1\cdots m}, \Delta_I^\vee) =& \frac{I_h(\Delta_I, \Delta_I^\vee)}{I_h(D_{1\cdots m}, (\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee)} \cdot I_h(D_{1\cdots m}, (\iota_I)_*(\sqrt{-1}\mathbb{R}_I^m)^\vee) \\ =& I_h(\Delta_I, \Delta_I^\vee). \end{aligned}$$

The second equality of (4.1) is followed as

$$I_h(\Delta_I, D_{1\cdots m}^\vee) = (-1)^m I_h(D_{1\cdots m}, \Delta_I^\vee)^\vee = (-1)^m I_h(\Delta_I, \Delta_I^\vee)^\vee = I_h(\Delta_I, \Delta_I^\vee),$$

where $g(\alpha, \beta, \gamma_1, \ldots, \gamma_m)^\vee = g(\alpha^{-1}, \beta^{-1}, \gamma_1^{-1}, \ldots, \gamma_m^{-1})$ for $g(\alpha, \beta, \gamma_1, \ldots, \gamma_m) \in \mathbb{C}(\alpha, \beta, \gamma_1, \ldots, \gamma_m)$. The orthogonality of Δ_I's implies

$$D_{1\cdots m} = \sum_I \frac{I_h(D_{1\cdots m}, \Delta_I^\vee)}{I_h(\Delta_I, \Delta_I^\vee)} \Delta_I = \sum_I \Delta_I,$$

which is the equality (4.2). Hence the self-intersection number of $D_{1\cdots m}$ is

$$\begin{aligned} & I_h(D_{1\cdots m}, D_{1\cdots m}^\vee) = \sum_I I_h(\Delta_I, \Delta_I^\vee) \\ & = \sum_I (-1)^{|I|} \frac{\prod_{j\notin I} \gamma_j \cdot \left(\alpha - \prod_{i\in I} \gamma_i\right) \left(\beta - \prod_{i\in I} \gamma_i\right)}{\prod_k (\gamma_k - 1) \cdot (\alpha - \prod_k \gamma_k) (\beta - 1)} \\ & = \frac{1}{\prod_k (\gamma_k - 1) \cdot (\alpha - \prod_k \gamma_k) (\beta - 1)} \\ & \quad \cdot \left(-(\alpha+\beta) \prod_k \gamma_k \cdot \sum_I (-1)^{|I|} + \alpha\beta \sum_I (-1)^{|I|} \prod_{j\notin I} \gamma_j + \prod_k \gamma_k \cdot \sum_I (-1)^{|I|} \prod_{i\in I} \gamma_i \right). \end{aligned}$$

Since

$$\begin{aligned} & \sum_I (-1)^{|I|} = \sum_{r=0}^m \binom{m}{r} (-1)^r = (1-1)^m = 0, \quad \sum_I (-1)^{|I|} \prod_{j\notin I} \gamma_j = \prod_k (\gamma_k - 1), \\ & \sum_I (-1)^{|I|} \prod_{i\in I} \gamma_i = (-1)^m \sum_J (-1)^{|J|} \prod_{i\notin J} \gamma_i = (-1)^m \prod_k (\gamma_k - 1), \end{aligned}$$

we have the equality (4.3):

$$I_h(D_{1\cdots m}, D_{1\cdots m}^\vee) = \frac{\alpha\beta + (-1)^m \prod_k \gamma_k}{(\beta - 1)(\alpha - \prod_k \gamma_k)}.$$

4.3.2 Intersection numbers

For $I, I' \subset \{1, \ldots, m\}$, we evaluate the intersection number $I_h(\Delta_I, D_{I'}^\vee)$. By Proposition 4.8, we may assume $I' \neq \{1, \ldots, m\}$. We set

$$\begin{aligned} & J := \{1, \ldots, m\} - I, \quad J' := \{1, \ldots, m\} - I', \\ & I_0 := I \cap I', \quad I_1 := I \cap J', \quad J_0 := J \cap I', \quad J_1 := J \cap J'. \end{aligned}$$

By using ι_I, we have $I_h(\Delta_I, D_{I'}^\vee) = I_h(\tilde{\Delta}_I, \tilde{D}_{I'}^\vee)$, where $\tilde{D}_{I'} := (-1)^{|I|} \cdot (\iota_I)_*^{-1}(D_{I'})$. Note that the orientation of $\tilde{D}_{I'}$ is also induced from the natural embedding

$\mathbb{R}^m \subset \mathbb{C}^m$. Thus σ_I and $\tilde{D}_{I'}$ have the same orientation. For $I' \neq \emptyset$, $\tilde{D}_{I'}$ is a chamber

$$\left\{ (s_1, \ldots, s_m) \in \mathbb{R}^m \;\middle|\; \begin{array}{l} s_i > 0 \ (i \in I'),\ s_j < 0 \ (j \notin I'), \\ (-1)^{|I_1|} v_I(s) > 0,\ (-1)^{|I_1|+|J'|+1} w_I(s) > 0 \end{array} \right\}$$

loaded the branch of u_I by the assignment of arguments:

	$s_i(i \in I')$	$s_i(i \in I_1)$	$s_i(i \in J_1)$	$v_I(s)$	$w_I(s)$
argument	0	π	$-\pi$	$\lvert I_1\rvert\pi$	$(\lvert I_1\rvert - (\lvert J'\rvert + 1))\pi$

In fact, the conditions for v_I and w_I are simply given by

$$1 - \sum_{i \in I} \frac{x_i}{s_i} - \sum_{j \in J} s_j > 0, \quad 1 - \sum_{i \in I} s_i - \sum_{j \in J} \frac{x_j}{s_j} < 0,$$

because of $|J'| = |I_1| + |J_1|$. In the case of $I' = \emptyset$ (then $I_0 = J_0 = \emptyset$), $\tilde{D}_\emptyset$ is a chamber

$$\{(s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_k < 0 \ (1 \leq k \leq m)\}$$

loaded the branch of u_I by the assignment of arguments:

	$s_i(i \in I_1)$	$s_i(i \in J_1)$	$v_I(s)$	$w_I(s)$
argument	π	$-\pi$	$\lvert I_1\rvert\pi$	$(\lvert I_1\rvert - m)\pi$

Lemma 4.13. *If $I' \neq \emptyset$ and $I \subset J'$, we have $I_h(\tilde{\Delta}_I, \tilde{D}^\vee_{I'}) = 0$.*

Proof. By the assumption, we have $J_0 = J \cap I' = I' \neq \emptyset$. For $(s_1, \ldots, s_m) \in \tilde{D}_{I'}$, we show that at least one of s_j's ($j \in J_0$) satisfies $0 < s_j < mx_j$. Because of $mx_j < m \cdot \frac{\varepsilon^2}{m} < \varepsilon$, it implies that the chamber $\tilde{D}_{I'}$ is included in the ε-neighborhood of ($s_j = 0$), and hence $\tilde{D}_{I'}$ does not intersect $\tilde{\Delta}_I$. We assume that all of s_j's ($j \in J_0$) satisfy $s_j \geq mx_j$. By

$$0 > 1 - \sum_{i \in I} s_i - \sum_{j \in J} \frac{x_j}{s_j} = 1 - \sum_{i \in I_1} s_i - \sum_{j \in J_0} \frac{x_j}{s_j} - \sum_{j \in J_1} \frac{x_j}{s_j},$$

$s_i < 0$ ($i \in I_1$) and $s_j < 0$ ($j \in J_1$), we have

$$1 < 1 - \sum_{i \in I_1} s_i - \sum_{j \in J_1} \frac{x_j}{s_j} < \sum_{j \in J_0} \frac{x_j}{s_j}.$$

However, the inequalities

$$\sum_{j \in J_0} \frac{x_j}{s_j} \leq \sum_{j \in J_0} \frac{x_j}{mx_j} \leq \sum_{j \in J_0} \frac{1}{m} \leq 1$$

lead to a contradiction to $1 < \sum_{j \in J_0} \frac{x_j}{s_j}$. □

We consider in the case of $I' \neq \emptyset$. By Lemma 4.13, we may assume that $I \not\subset J'$. If we consider $x_1, \ldots, x_m \to 0$, the condition $(-1)^{|I_1|} v_I(s) > 0$ may be replaced with $1 - \sum_{j \in J} s_j > 0$, and $(-1)^{|I_1|+|J'|+1} w(s) > 0$ may be replaced with $1 - \sum_{i \in I} s_i < 0$ to judge if s belongs to a central area of $\tilde{D}_{I'}$. This observation means that we can evaluate the intersection number $I_h(\tilde{\Delta}_I, \tilde{D}^\vee_{I'})$

like that of the regularization of V_I and ${V'_{I'}}^{\vee}$ by omitting the difference of the branch of u_I, where

$$V_I := \{(s_1,\dots,s_m)\in\mathbb{R}^m \mid s_k>0,\ 1-\sum_{i\in I}s_i>0,\ 1-\sum_{j\in J}s_j>0\},$$

$$V'_{I'} := \left\{(s_1,\dots,s_m)\in\mathbb{R}^m \;\middle|\; \begin{array}{c} s_k>0\ (k\in I'),\ s_k<0\ (k\in J'),\\ 1-\sum_{i\in I}s_i<0,\ 1-\sum_{j\in J}s_j>0 \end{array}\right\}. \tag{4.4}$$

Note that this chamber is not empty, because of $I\not\subset J'$. In the case of $I'=\emptyset$, we can see that the above claim is valid, by replacing (4.4) with

$$V' := \{(s_1,\dots,s_m)\in\mathbb{R}^m \mid s_k<0\ (1\le k\le m)\}$$

(note that $1-\sum_{i\in I}s_i>0,\ 1-\sum_{j\in J}s_j>0$ hold clearly). Recall that when we construct the twisted cycle $\tilde{\Delta}_I$, the exponents of $(s_i=0)$, $(s_j=0)$, $(1-\sum_{i\in I}s_i=0)$ and $(1-\sum_{j\in J}s_j=0)$ are

$$c_i-1,\quad 1-c_j,\quad -b,\quad \sum_{k=1}^{m}c_k-a-m+1,$$

respectively, where $i\in I$ and $j\in J$.

Let t_0 be an intersection point of $\tilde{\Delta}_I$ and $\tilde{D}_{I'}$. We denote the differences of the branches of u_I at t_0 by $\chi_{I,I'}$, namely,

$$\chi_{I,I'} := \frac{\text{the value } u_I(t_0) \text{ with respect to the branch defined on } \tilde{\Delta}_I}{\text{the value } u_I(t_0) \text{ with respect to the branch defined on } \tilde{D}_{I'}}.$$

Note that $\chi_{I,I'}$ is independent of the choice of the intersection point t_0.

Lemma 4.14. *For $I'\neq\emptyset$, we have*

$$\begin{aligned} I_h(\tilde{\Delta}_I,\tilde{D}^{\vee}_{I'}) = \chi_{I,I'}\cdot(-1)^{m-(|J'|+1)}\cdot\prod_{i\in I_1}\frac{1}{\gamma_i-1}\cdot\prod_{j\in J_1}\frac{1}{\gamma_j^{-1}-1}\cdot\frac{1}{\beta^{-1}-1}\\ \cdot\left[1+\sum_{\substack{K_I\subsetneq I_0\\ K_J\subsetneq J_0}}\left(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{1}{\gamma_j^{-1}-1}\right)\right.\\ \left.+\frac{1}{\alpha^{-1}\prod_k\gamma_k-1}\sum_{\substack{K_I\subsetneq I_0\\ K_J\subsetneq J_0}}\left(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{1}{\gamma_j^{-1}-1}\right)\right]. \end{aligned} \tag{4.5}$$

For $I'=\emptyset$, we have

$$I_h(\tilde{\Delta}_I,\tilde{D}^{\vee}) = \chi_{I,\emptyset}\cdot(-1)^{m-m}\cdot\prod_{i\in I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in J}\frac{1}{\gamma_j^{-1}-1}. \tag{4.6}$$

Proof. We show (4.5), by using results in [13]. Obviously we have

$$\overline{V_I}\cap\overline{V'_{I'}} = \left\{(s_1,\dots,s_m)\in\mathbb{R}^m \;\middle|\; \begin{array}{c} s_j=0\ (j\in J'),\ 1-\sum_{i\in I}s_i=0,\\ s_i\ge 0\ (i\in I'),\ 1-\sum_{j\in J}s_j\ge 0 \end{array}\right\},$$

which implies that the intersection number $I_h(\tilde{\Delta}_I,\tilde{D}^{\vee}_{I'})$ is equal to the product of

$$\chi_{I,I'}\cdot\prod_{i\in I\cap J'}\frac{1}{\gamma_i-1}\cdot\prod_{j\in J\cap J'}\frac{1}{\gamma_j^{-1}-1}\cdot\frac{1}{\beta^{-1}-1}$$

and the self-intersection number of the twisted cycle determined by the chamber

$$\left\{(s_1,\dots,s_m)\in\mathbb{R}^m \;\middle|\; \begin{array}{l} s_j=0\ (j\in J'),\ 1-\sum_{i\in I}s_i=0,\\ s_i>0\ (i\in I'),\ 1-\sum_{j\in J}s_j>0\end{array}\right\}$$

in the $(m-(|J'|+1))$-dimensional space $L:=\bigcap_{j\in J'}(s_j=0)\cap(1-\sum_{i\in I}s_i=0)$. Thus we investigate non-empty intersections of $(s_i=0)\ (i\in I'),\ (1-\sum_{j\in J}s_j=0)$ with L.

(i) Without $(1-\sum_{j\in J}s_j=0)$: we choose subsets K of I' such that $\bigcap_{k\in K}(s_k=0)\cap L\neq\emptyset$. By the condition $1-\sum_{i\in I}s_i=0$, we have

$$\bigcap_{k\in K}(s_k=0)\cap L\neq\emptyset \Leftrightarrow K\cap I\subsetneq I \Leftrightarrow K=K_I\cup K_J\ (K_I\subsetneq I,\ K_J\subset J).$$

(ii) With $(1-\sum_{j\in J}s_j=0)$: we choose subsets K of I' such that $\bigcap_{k\in K}(s_k=0)\cap(1-\sum_{j\in J}s_j=0)\cap L\neq\emptyset$. By the conditions $1-\sum_{i\in I}s_i=0,\ 1-\sum_{j\in J}s_j=0$, we have

$$\begin{aligned}&\bigcap_{k\in K}(s_k=0)\cap(1-\sum_{j\in J}s_j=0)\cap L\neq\emptyset\\ &\Leftrightarrow K\cap I\subsetneq I,\ K\cap J\subsetneq J\Leftrightarrow K=K_I\cup K_J\ (K_I\subsetneq I,\ K_J\subsetneq J).\end{aligned}$$

Therefore the self-intersection number is equal to

$$\begin{aligned}(-1)^{m-(|J'|+1)}\cdot\Bigg[1+&\sum_{\substack{K_I\subsetneq I_0\\ K_J\subset J_0}}\left(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{1}{\gamma_j^{-1}-1}\right)\\ &+\frac{1}{\alpha^{-1}\prod_k\gamma_k-1}\sum_{\substack{K_I\subsetneq I_0\\ K_J\subsetneq J_0}}\left(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{1}{\gamma_j^{-1}-1}\right)\Bigg],\end{aligned}$$

and hence (4.5) is proved. The equality (4.6) is shown similarly. □

Theorem 4.15. *For $I'\neq\emptyset$, we have*

$$\begin{aligned}I_h(\tilde\Delta_I,\tilde D^\vee_{I'})=&(-1)^{m-|J_1|-1}\cdot\prod_{k\in J'}\frac{1}{1-\gamma_k}\cdot\frac{1}{1-\beta}\\ &\cdot\Bigg[1+\sum_{\substack{K_I\subsetneq I_0\\ K_J\subset J_0}}\left(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{\gamma_j}{1-\gamma_j}\right)\\ &+\frac{\alpha}{\prod_k\gamma_k-\alpha}\sum_{\substack{K_I\subsetneq I_0\\ K_J\subsetneq J_0}}\left(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{\gamma_j}{1-\gamma_j}\right)\Bigg].\end{aligned}\tag{4.7}$$

For $I'=\emptyset$, we have

$$I_h(\tilde\Delta_I,\tilde D^\vee)=(-1)^{|I|}\cdot\prod_{k=1}^m\frac{1}{1-\gamma_k}.$$

Proof. By Lemma 4.14, it is sufficient to evaluate $\chi_{I,I'}$. We consider the differences of the branches of the factors of u_I at an intersection point of $\tilde{\Delta}_I$ and $\tilde{D}_{I'}$.

(i) The argument of s_k on $\tilde{\Delta}_I$ and $\tilde{D}_{I'}$ are as follows:

	$k \in I' = I_0 \cup J_0$	$k \in I_1$	$k \in J_1$
$\tilde{\Delta}_I$	0	π	π
$\tilde{D}_{I'}$	0	π	$-\pi$

Since the exponent of s_j $(j \in J)$ is $C_j = 1 - c_j + b$, the contribution by the branch of $\prod_{k=1}^{m} s_k^{C_k}$ is $\prod_{j \in J_1} (\gamma_j^{-1}\beta)$.

(ii) We have

$$v_I = \prod_{i \in I} s_i \cdot \left(1 - \sum_{j \in J} s_j - \sum_{i \in I} \frac{x_i}{s_i}\right),$$

and the term $\sum_{i \in I} \frac{x_i}{s_i}$ does not concern the difference of the branch. By (i) and the fact that $s \in V'_{I'}$ satisfies $1 - \sum_{j \in J} s_j > 0$, both the argument of v_I on $\tilde{\Delta}_I$ and that on $\tilde{D}_{I'}$ are $|I_1|\pi$, and hence the contribution by the branch of v_I^A is 1.

(iii) We have

$$w_I = \prod_{j \in J} s_j \cdot \left(1 - \sum_{i \in I} s_i - \sum_{j \in J} \frac{x_j}{s_j}\right),$$

and the term $\sum_{j \in J} \frac{x_j}{s_j}$ does not concern the difference of the branch. By (i) and the fact that $s \in V'_{I'}$ satisfies

$$\begin{cases} 1 - \sum_{i \in I} s_i < 0 & (I' \neq \emptyset), \\ 1 - \sum_{i \in I} s_i > 0 & (I' = \emptyset), \end{cases}$$

the arguments of w_I on $\tilde{\Delta}_I$ and $\tilde{D}_{I'}$ at the intersection points are as follows:

$$(\text{argument on } \tilde{\Delta}_I) = \begin{cases} (|J_1| + 1)\pi & (I' \neq \emptyset), \\ |J_1|\pi & (I' = \emptyset), \end{cases}$$

$$(\text{argument on } \tilde{D}_{I'}) = \begin{cases} (|I_1| - |J'| - 1)\pi & (I' \neq \emptyset), \\ (|I_1| - m)\pi = -|J_1|\pi & (I' = \emptyset) \end{cases}$$

(note that $m = |J'| = |I_1| + |J_1|$, if $I' = \emptyset$). Because of $|J'| = |I_1| + |J_1|$, we obtain

$$\begin{aligned} &(\text{difference of the argument of } w_I) \\ &= \begin{cases} (|J_1| + 1)\pi - (|I_1| - |J'| - 1)\pi = 2(|J_1| + 1)\pi & (I' \neq \emptyset), \\ |J_1|\pi - (-|J_1|)\pi = 2|J_1|\pi & (I' = \emptyset). \end{cases} \end{aligned}$$

Since the exponent of w_I is $B = -b$, the contribution by the branch of w_I^B is

$$\begin{cases} \beta^{-(|J_1|+1)} & (I' \neq \emptyset), \\ \beta^{-|J_1|} & (I' = \emptyset). \end{cases}$$

We thus have

$$\chi_{I,I'} = \prod_{j\in J_1} (\gamma_j^{-1}\beta)\cdot \beta^{-(|J_1|+1)} \quad (I'\neq\emptyset), \quad \chi_{I,\emptyset} = \prod_{j\in J_1} (\gamma_j^{-1}\beta)\cdot \beta^{-|J_1|}.$$

By Lemma 4.14, if $I'\neq\emptyset$, then we obtain

$$\begin{aligned}
I_h(\tilde{\Delta}_I, \tilde{D}^\vee_{I'}) = &\prod_{j\in J_1} (\gamma_j^{-1}\beta)\cdot \beta^{-(|J_1|+1)}\cdot (-1)^{m-(|J'|+1)} \\
&\cdot \prod_{i\in I_1}\frac{1}{\gamma_i-1}\cdot \prod_{j\in J_1}\frac{1}{\gamma_j^{-1}-1}\cdot \frac{1}{\beta^{-1}-1} \\
&\cdot \Bigg[1+\sum_{\substack{K_I\subsetneq I_0\\ K_J\subset J_0}}\left(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{1}{\gamma_j^{-1}-1}\right) \\
&\quad + \frac{1}{\alpha^{-1}\prod_k\gamma_k-1}\sum_{\substack{K_I\subsetneq I_0\\ K_J\subsetneq J_0}}\left(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{1}{\gamma_j^{-1}-1}\right)\Bigg] \\
=&(-1)^{m-|J_1|-1}\cdot\prod_{k\in J'}\frac{1}{1-\gamma_k}\cdot\frac{1}{1-\beta} \\
&\cdot\Bigg[1+\sum_{\substack{K_I\subsetneq I_0\\ K_J\subset J_0}}\left(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{\gamma_j}{1-\gamma_j}\right) \\
&\quad + \frac{\alpha}{\prod_k\gamma_k-\alpha}\sum_{\substack{K_I\subsetneq I_0\\ K_J\subsetneq J_0}}\left(\prod_{i\in K_I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in K_J}\frac{\gamma_j}{1-\gamma_j}\right)\Bigg].
\end{aligned}$$

If $I'=\emptyset$, then we have

$$I_h(\tilde{\Delta}_I, \tilde{D}^\vee) = \prod_{j\in J_1}(\gamma_j^{-1}\beta)\cdot\beta^{-|J_1|}\cdot\prod_{i\in I}\frac{1}{\gamma_i-1}\cdot\prod_{j\in J}\frac{1}{\gamma_j^{-1}-1} = (-1)^{|I|}\cdot\prod_{k=1}^m\frac{1}{1-\gamma_k}.$$

□

To simplify the equality (4.7), we use the following formulas.

Lemma 4.16. *For a positive integer n and complex numbers $\lambda_1,\dots,\lambda_n$, we have*

$$\sum_{N\subset\{1,\dots,n\}}\prod_{l\in N}\frac{\lambda_l}{1-\lambda_l} = \prod_{l=1}^n\frac{1}{1-\lambda_l}, \qquad \sum_{N\subset\{1,\dots,n\}}\prod_{l\in N}\frac{1}{\lambda_l-1} = \prod_{l=1}^n\frac{\lambda_l}{\lambda_l-1}.$$

Proof. Because of

$$1+\frac{\lambda_l}{1-\lambda_l} = \frac{1}{1-\lambda_l}, \quad 1+\frac{1}{\lambda_l-1} = \frac{\lambda_l}{\lambda_l-1},$$

this lemma is shown by the induction on n. □

We summarize the results in this subsection.

Corollary 4.17. *If $I' \neq \emptyset, \{1, \dots, m\}$, then we have*

$$I_h(\Delta_I, D_{I'}^\vee) = (-1)^{|I|+|I'|-1} \cdot \prod_{k=1}^{m} \frac{1}{1-\gamma_k} \cdot \frac{\prod_{i\in I_0}\gamma_i - 1}{1-\beta} \cdot \frac{\prod_k \gamma_k - \alpha \prod_{j\in J_0}\gamma_j}{\prod_k \gamma_k - \alpha}. \tag{4.8}$$

This equality holds, even if $I \subset J'$. For $I' = \emptyset$, we have

$$I_h(\tilde{\Delta}_I, \tilde{D}^\vee) = (-1)^{|I|} \cdot \prod_{k=1}^{m} \frac{1}{1-\gamma_k}. \tag{4.9}$$

Proof. Recall that $I_h(\Delta_I, D_{I'}^\vee) = I_h(\tilde{\Delta}_I, \tilde{D}_{I'}^\vee)$. The equality (4.9) is equal to that in Theorem 4.15. If $I \subset J'$, then we have $I_0 = I \cap I' = \emptyset$ and hence $\prod_{i\in I_0}\gamma_i - 1 = 0$. Thus the right hand side of (4.8) is 0, which is compatible with Lemma 4.13. Then we have to show that the right hand side of (4.7) is equal to that of (4.8). By Lemma 4.16, we have

$$\begin{aligned}
1 + \sum_{\substack{K_I \subsetneq I_0 \\ K_J \subset J_0}} \left(\prod_{i\in K_I} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in K_J} \frac{\gamma_j}{1-\gamma_j} \right) &= \left(\sum_{K_I \subsetneq I_0} \prod_{i\in K_I} \frac{1}{\gamma_i - 1} \right) \cdot \left(\sum_{K_J \subset J_0} \prod_{j\in K_J} \frac{\gamma_j}{1-\gamma_j} \right) \\
= \left(\prod_{i\in I_0} \frac{\gamma_i}{\gamma_i - 1} - \prod_{i\in I_0} \frac{1}{\gamma_i - 1} \right) \cdot \prod_{j\in J_0} \frac{1}{1-\gamma_j} &= (-1)^{|I_0|} \cdot \left(\prod_{i\in I_0} \gamma_i - 1 \right) \cdot \prod_{k\in I'} \frac{1}{1-\gamma_k},
\end{aligned}$$

$$\begin{aligned}
\sum_{\substack{K_I \subsetneq I_0 \\ K_J \subsetneq J_0}} \left(\prod_{i\in K_I} \frac{1}{\gamma_i - 1} \cdot \prod_{j\in K_J} \frac{\gamma_j}{1-\gamma_j} \right) &= \left(\sum_{K_I \subsetneq I_0} \prod_{i\in K_I} \frac{1}{\gamma_i - 1} \right) \cdot \left(\sum_{K_J \subsetneq J_0} \prod_{j\in K_J} \frac{\gamma_j}{1-\gamma_j} \right) \\
&= \left(\prod_{i\in I_0} \frac{\gamma_i}{\gamma_i - 1} - \prod_{i\in I_0} \frac{1}{\gamma_i - 1} \right) \cdot \left(\prod_{j\in J_0} \frac{1}{1-\gamma_j} - \prod_{j\in J_0} \frac{\gamma_j}{1-\gamma_j} \right) \\
&= (-1)^{|I_0|} \cdot \left(\prod_{i\in I_0} \gamma_i - 1 \right) \cdot \left(1 - \prod_{j\in J_0} \gamma_j \right) \cdot \prod_{k\in I'} \frac{1}{1-\gamma_k}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&I_h(\Delta_I, D_{I'}^\vee) = I_h(\tilde{\Delta}_I, \tilde{D}_{I'}^\vee) \\
&= (-1)^{m-|J_1|-1} \cdot \prod_{k\in J'} \frac{1}{1-\gamma_k} \cdot \frac{1}{1-\beta} \\
&\quad \cdot (-1)^{|I_0|} \cdot \left(\prod_{i\in I_0} \gamma_i - 1 \right) \cdot \prod_{k\in I'} \frac{1}{1-\gamma_k} \cdot \left(1 + \frac{\alpha}{\prod_k \gamma_k - \alpha} \cdot \left(1 - \prod_{j\in J_0} \gamma_j \right) \right) \\
&= (-1)^{|I_1|+|J_0|-1} \cdot \prod_{k=1}^{m} \frac{1}{1-\gamma_k} \cdot \frac{\prod_{i\in I_0}\gamma_i - 1}{1-\beta} \cdot \frac{\prod_k \gamma_k - \alpha \prod_{j\in J_0}\gamma_j}{\prod_k \gamma_k - \alpha}.
\end{aligned}$$

Here we use $m = |I_0| + |I_1| + |J_0| + |J_1|$. Further, since

$$|I_1| + |J_0| = |I \cap I'^c| + |I^c \cap I'| = |I \cup I'| - |I \cap I'| = |I| + |I'| - 2|I \cap I'|,$$

we have $(-1)^{|I_1|+|J_0|-1} = (-1)^{|I|+|I'|-1}$. □

Corollary 4.18. *We have $I_h(D_{1\cdots m}, D_{I'}^{\vee}) = 0$, if $I' \neq \{1, \ldots, m\}$.*

Proof. This is obvious, since

$$\overline{D_{1\cdots m}} \subset \{(s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_k > x_k \ (1 \leq k \leq m)\},$$
$$\overline{D_{I'}} \cap \{(s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_k \geq x_k \ (1 \leq k \leq m)\} = \emptyset.$$

We also show this corollary by straightforward calculation. By (4.2), we have

$$I_h(D_{1\cdots m}, D_{I'}^{\vee}) = \sum_{I \subset \{1,\ldots,m\}} I_h(\Delta_I, D_{I'}).$$

If $I' = \emptyset$, then we obtain

$$I_h(D_{1\cdots m}, D^{\vee}) = \prod_{k=1}^{m} \frac{1}{1-\gamma_k} \cdot \sum_{I \subset \{1,\ldots,m\}} (-1)^{|I|} = \prod_{k=1}^{m} \frac{1}{1-\gamma_k} \cdot \sum_{r=0}^{m} (-1)^r \binom{m}{r} = 0.$$

Thus we assume $I' \neq \emptyset$. By (4.8), it is sufficient to show that

$$\sum_{I \subset \{1,\ldots,m\}} (-1)^{|I|+|I'|-1} \cdot \left(\prod_{i \in I \cap I'} \gamma_i - 1 \right) \cdot \left(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I^c \cap I'} \gamma_j \right) = 0.$$

The left hand side is equal to

$$\sum_{\substack{I_0 \subset I' \\ I_1 \subset J'}} (-1)^{(|I_0|+|I_1|)+|I'|-1} \left(\prod_{i \in I_0} \gamma_i - 1 \right) \cdot \left(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I' - I_0} \gamma_j \right)$$
$$= \sum_{I_0 \subset I'} \left((-1)^{|I'|+|I_0|-1} \left(\prod_{i \in I_0} \gamma_i - 1 \right) \cdot \left(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I' - I_0} \gamma_j \right) \right) \cdot \sum_{I_1 \subset J'} (-1)^{|I_1|}.$$

Because of $J' \neq \emptyset$, we have

$$\sum_{I_1 \subset J'} (-1)^{|I_1|} = 0,$$

which shows the corollary. □

4.3.3 Linear independence

Let Λ_0 be the matrix $(I_h(\Delta_I, D_{I'}))_{I,I'}$ with arranging $I,\ I'$ in the same way to the basis $\{\Delta_I\}_I$ (see Section 4.1). In this subsection, we evaluate the determinant of Λ_0.

Theorem 4.19. *The determinant of* Λ_0 *is*

$$\begin{cases} -\left(\alpha\beta-\prod_{k=1}^{m}\gamma_k\right)\dfrac{(\prod_k\gamma_k+\alpha)^{2^{m-1}-1}}{(1-\beta)^{2^m-1}(\prod_k\gamma_k-\alpha)^{2^{m-1}}}\cdot\prod_{k=1}^{m}\dfrac{1}{(1-\gamma_k)^{2^{m-1}}} & (m:\text{odd}),\\ \left(\alpha\beta+\prod_{k=1}^{m}\gamma_k\right)\dfrac{(\prod_k\gamma_k+\alpha)^{2^{m-1}-2}}{(1-\beta)^{2^m-1}(\prod_k\gamma_k-\alpha)^{2^{m-1}-1}}\cdot\prod_{k=1}^{m}\dfrac{1}{(1-\gamma_k)^{2^{m-1}}} & (m:\text{even}). \end{cases}$$

In particular, we obtain $\det\Lambda_0\neq 0$*, and hence* $\{D_I\}_I$ *is linearly independent.*

Remark 4.20. In this chapter, we assume that the parameters a, b, and $c=(c_1,\dots,c_m)$ are generic. In fact, it is sufficient for our proof of Theorem 4.6 to assume the irreducibility condition of the system $E_C(a,b,c)$

$$a-\sum_{i\in I}c_i,\quad b-\sum_{i\in I}c_i\notin\mathbb{Z}\quad (I\subset\{1,\dots,m\}),$$

and the conditions

$$c_1,\dots,c_m\notin\mathbb{Z},\quad a-\sum_{k=1}^{m}c_k\notin\frac{1}{2}\mathbb{Z},\quad \begin{cases} a+b-\sum_{k=1}^{m}c_k\notin\mathbb{Z} & (m:\text{odd}),\\ a+b-\sum_{k=1}^{m}c_k\notin\mathbb{Z}+\frac{1}{2} & (m:\text{even}). \end{cases}$$

We change Λ_0 into Λ by elementary transformations with keeping the determinant as follows.

Add the first, second, …, (2^m-1)-th row of Λ_0 to the 2^m-th row of Λ_0, then 2^m-th row becomes

$$\begin{aligned}&\left(I_h(\sum_I\Delta_I,D^\vee),\dots,I_h(\sum_I\Delta_I,D^\vee_{2\cdots m}),I_h(\sum_I\Delta_I,D^\vee_{1\cdots m})\right)\\ &=(I_h(D_{1\cdots m},D^\vee),\dots,I_h(D_{1\cdots m},D^\vee_{2\cdots m}),I_h(D_{1\cdots m},D^\vee_{1\cdots m}))\\ &=(0,\dots,0,I_h(D_{1\cdots m},D^\vee_{1\cdots m}))\,,\end{aligned}$$

by Corollary 4.18. It means that

$$\det\Lambda_0=I_h(D_{1\cdots m},D^\vee_{1\cdots m})\cdot\det\Lambda',$$

where Λ' is the leading principal minor of Λ_0 of size 2^m-1. By Proposition 4.8 and Corollary 4.17, we have

$$\begin{aligned}\det\Lambda_0&=\frac{\alpha\beta+(-1)^m\prod_k\gamma_k}{(\beta-1)(\alpha-\prod_k\gamma_k)}\cdot\prod_{k=1}^{m}\frac{1}{(1-\gamma_k)^{2^m-1}}\cdot\frac{\det\Lambda}{(1-\beta)^{2^m-2}(\prod_k\gamma_k-\alpha)^{2^m-2}}\\ &=\frac{\alpha\beta+(-1)^m\prod_k\gamma_k}{(1-\beta)^{2^m-1}(\prod_k\gamma_k-\alpha)^{2^m-1}}\cdot\prod_{k=1}^{m}\frac{1}{(1-\gamma_k)^{2^m-1}}\cdot\det\Lambda,\end{aligned}$$

where Λ is a $(2^m-1)\times(2^m-1)$ matrix whose (I,I') entry is

$$\Lambda_{I,I'}:=(-1)^{|I|+|I'|-1}\cdot(\prod_{i\in I\cap I'}\gamma_i-1)\cdot(\prod_{k=1}^{m}\gamma_k-\alpha\prod_{j\in I^c\cap I'}\gamma_j)\quad (I'\neq\emptyset),\tag{4.10}$$

$$\Lambda_{I,\emptyset}:=(-1)^{|I|}.\tag{4.11}$$

We write

$$\Lambda = \begin{pmatrix} \Lambda(0,0) & \Lambda(0,1) & \cdots & \Lambda(0,m-1) \\ \Lambda(1,0) & \Lambda(1,1) & \cdots & \Lambda(1,m-1) \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda(m-1,0) & \Lambda(m-1,1) & \cdots & \Lambda(m-1,m-1) \end{pmatrix},$$

where $\Lambda(k,k')$ is the $\binom{m}{k} \times \binom{m}{k'}$ matrix. Note that the entries of $\Lambda(k,k')$ are the (I,I') entries of Λ with $|I| = k,\ |I'| = k'$. We compute $\det \Lambda$. Put $\Lambda^{(0)} := \Lambda$.

Proposition 4.21. *By the induction on $n \geq 1$, we can take $\Lambda^{(n)}$ from $\Lambda^{(n-1)}$ such that*

(i) $\det \Lambda^{(n)} = \det \Lambda$ *and* $\Lambda^{(n)}_{\emptyset,\emptyset} = 1$,

(ii) *if* $|I'| \geq n+1$, *then*

$$\begin{aligned} \Lambda^{(n)}_{I,I'} =& (-1)^{|I|+|I'|-1} \cdot \Bigg[\left(\prod_{i \in I \cap I'} \gamma_i - 1 \right) \cdot \left(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I^c \cap I'} \gamma_j \right) \\ & - \sum_{\substack{K \subset I \cap I' \\ 0 < |K| \leq n}} \left(\prod_{i \in K} (\gamma_i - 1) \cdot \left(\prod_{k=1}^{m} \gamma_k + (-1)^{|K|} \alpha \prod_{j \in K^c \cap I'} \gamma_j \right) \right) \Bigg], \end{aligned}$$

(iii) *for $k \leq n$, we have $\Lambda^{(n)}(k,k') = O\ (k' > k)$,*

(iv) $\Lambda^{(n)}(1,1), \ldots, \Lambda^{(n)}(n+1,n+1)$ *are diagonal,*

(v) $\Lambda^{(n)}_{I,I} = -\prod_{i \in I}(\gamma_i - 1) \cdot (\prod_k \gamma_k + (-1)^{|I|}\alpha)$ *for* $1 \leq |I| \leq n+1$.

We denote the conditions (i), …,(v) by $C(n)$. By admitting this proposition temporarily, we prove Theorem 4.19.

Proof of Theorem 4.19. By Proposition 4.21, $\Lambda^{(m-2)}$ is a lower triangular matrix, and hence we obtain

$$\begin{aligned} \det \Lambda_0 &= \frac{\alpha\beta + (-1)^m \prod_k \gamma_k}{(1-\beta)^{2^m-1}(\prod_k \gamma_k - \alpha)^{2^m-1}} \cdot \prod_{k=1}^{m} \frac{1}{(1-\gamma_k)^{2^m-1}} \cdot \det \Lambda^{(m-2)} \\ &= \frac{\alpha\beta + (-1)^m \prod_k \gamma_k}{(1-\beta)^{2^m-1}(\prod_k \gamma_k - \alpha)^{2^m-1}} \cdot \prod_{k=1}^{m} \frac{1}{(1-\gamma_k)^{2^m-1}} \\ &\quad \cdot \prod_{\emptyset \neq I \subsetneq \{1,\ldots,m\}} \left(-\prod_{i \in I}(\gamma_i - 1) \cdot \left(\prod_{k=1}^{m} \gamma_k + (-1)^{|I|}\alpha \right) \right) \\ &= (-1)^m \cdot \frac{\alpha\beta + (-1)^m \prod_k \gamma_k}{(1-\beta)^{2^m-1}(\prod_k \gamma_k - \alpha)^{2^m-1}} \cdot \prod_{k} \frac{1}{(1-\gamma_k)^{2^{m-1}}} \\ &\quad \cdot \prod_{\emptyset \neq I \subsetneq \{1,\ldots,m\}} \left(\prod_{k=1}^{m} \gamma_k + (-1)^{|I|}\alpha \right). \end{aligned}$$

If m is odd, we have

$$\prod_{\emptyset\neq I\subsetneq\{1,\dots,m\}}\left(\prod_{k=1}^{m}\gamma_k+(-1)^{|I|}\alpha\right)=\left(\prod_{k=1}^{m}\gamma_k-\alpha\right)^{2^{m-1}-1}\cdot\left(\prod_{k=1}^{m}\gamma_k+\alpha\right)^{2^{m-1}-1}.$$

If m is even, we have

$$\prod_{\emptyset\neq I\subsetneq\{1,\dots,m\}}\left(\prod_{k=1}^{m}\gamma_k+(-1)^{|I|}\alpha\right)=\left(\prod_{k=1}^{m}\gamma_k-\alpha\right)^{2^{m-1}}\cdot\left(\prod_{k=1}^{m}\gamma_k+\alpha\right)^{2^{m-1}-2}.$$

Therefore, the proof of Theorem 4.19 is completed. □

Proof of Proposition 4.21. We write down properties of $\Lambda^{(0)}=\Lambda$:

(a) $\Lambda^{(0)}_{\emptyset,\emptyset}=1$,

(b) $\Lambda^{(0)}(0,k')=O$ $(k'>0)$,

(c) $\Lambda^{(0)}(1,1)$ is diagonal,

(d) $\Lambda^{(0)}_{\{l\},\{l\}}=-(\gamma_l-1)\cdot(\prod_k\gamma_k-\alpha)$, for $1\le l\le m$.

The properties (a) and (b) follows from (4.11) and (4.10), respectively. Since $I\cap I'=\emptyset$ if $I=\{l\},I=\{l'\}$ with $l\neq l'$, (4.10) implies (c). The property (d) also follows from (4.10). By (4.10), for $|I'|\ge 2$, we have

$$\begin{aligned}
&l\notin I'\Rightarrow\Lambda^{(0)}_{\{l\},I'}=0,\\
&l\in I'\Rightarrow\Lambda^{(0)}_{\{l\},I'}=(-1)^{|I'|}(\gamma_l-1)\Big(\prod_{k=1}^{m}\gamma_k-\alpha\prod_{j\in I'-\{l\}}\gamma_j\Big).
\end{aligned}$$

We define $\Lambda^{(1)}$ by replacing the columns of $\Lambda^{(0)}$ of I' $(|I'|\ge 2)$ with

$$\Lambda^{(0)}_{*,I'}+\sum_{l\in I'}(-1)^{|I'|}\frac{\prod_k\gamma_k-\alpha\prod_{j\in I'-\{l\}}\gamma_j}{\prod_k\gamma_k-\alpha}\cdot\Lambda^{(0)}_{*,\{l\}},$$

where $\Lambda^{(0)}_{*,I'}$ is the column of I' of $\Lambda^{(0)}$. By this definition and (b), the columns of I' $(|I'|=0,1)$ and the row of $\emptyset$ do not change. We verify that $\Lambda^{(1)}$ satisfies the condition $C(1)$.

(i) It is obvious.

(ii) We assume $|I'|\ge 2$. By the definition of $\Lambda^{(1)}$, we have

$$\begin{aligned}
&\Lambda^{(1)}_{I,I'}\\
&=(-1)^{|I|+|I'|-1}\cdot\Big(\prod_{i\in I\cap I'}\gamma_i-1\Big)\cdot\Big(\prod_{k=1}^{m}\gamma_k-\alpha\prod_{j\in I^c\cap I'}\gamma_j\Big)\\
&\quad+\sum_{l\in I'}(-1)^{|I'|}\frac{\prod_k\gamma_k-\alpha\prod_{j\in I'-\{l\}}\gamma_j}{\prod_k\gamma_k-\alpha}\cdot\Lambda^{(0)}_{I,\{l\}}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{|I|+|I'|-1} \cdot \Big(\prod_{i \in I \cap I'} \gamma_i - 1 \Big) \cdot \Big(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I^c \cap I'} \gamma_j \Big) \\
&\quad + \sum_{l \in I \cap I'} (-1)^{|I'|} \frac{\prod_k \gamma_k - \alpha \prod_{j \in I' - \{l\}} \gamma_j}{\prod_k \gamma_k - \alpha} \cdot (-1)^{|I|} (\gamma_l - 1) \Big(\prod_{k=1}^{m} \gamma_k - \alpha \Big) \\
&= (-1)^{|I|+|I'|-1} \cdot \Bigg[\Bigg(\prod_{i \in I \cap I'} \gamma_i - 1 \Bigg) \cdot \Bigg(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I^c \cap I'} \gamma_j \Bigg) \\
&\qquad - \sum_{l \in I \cap I'} \Bigg((\gamma_l - 1) \Bigg(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I' - \{l\}} \gamma_j \Bigg) \Bigg) \Bigg],
\end{aligned}
$$

which is the equality in (ii) of $C(1)$. Here we use the equality $\Lambda^{(0)}_{I,\{l\}} = 0$ $(l \notin I)$.

(iii) By (b), we have to show that $\Lambda^{(1)}(1, k') = O$ $(k' > 1)$, i.e., $\Lambda^{(1)}_{\{i\},I'} = 0$ $(|I'| \geq 2)$. It is clear in the case of $i \notin I'$. Thus we assume that $i \in I'$. Because of $\{i\} \cap I' = \{i\}$ and $\{i\}^c \cap I' = I' - \{i\}$, we obtain $\Lambda^{(1)}_{\{i\},I'} = 0$.

(iv) By (c), we have to show that $\Lambda^{(1)}(2, 2)$ is diagonal. We assume that $|I| = |I'| = 2$ and $I \neq I'$. If $I \cap I' = \emptyset$, clearly we have $\Lambda^{(1)}_{I,I'} = 0$. We consider the cases $I \cap I' \neq \emptyset$. We may assume that $I = \{i_1, i\}$, $I' = \{i'_1, i\}$ with $i_1 \neq i'_1$. Then we have $I \cap I' = \{i\}$ and $I^c \cap I' = I' - \{i\} = \{i'_1\}$, which imply $\Lambda^{(1)}_{I,I'} = 0$.

(v) By (d), we have to show that

$$
\Lambda^{(1)}_{\{i_1,i_2\},\{i_1,i_2\}} = -(\gamma_{i_1} - 1)(\gamma_{i_2} - 1) \Big(\prod_{k=1}^{m} \gamma_k + \alpha \Big),
$$

for $i_1 \neq i_2$. By (ii), we obtain

$$
\begin{aligned}
&\Lambda^{(1)}_{\{i_1,i_2\},\{i_1,i_2\}} \\
&= - \Bigg((\gamma_{i_1}\gamma_{i_2} - 1)(\prod_{k=1}^{m} \gamma_k - \alpha) \\
&\qquad - (\gamma_{i_1} - 1)(\prod_{k=1}^{m} \gamma_k - \alpha\gamma_{i_2}) - (\gamma_{i_2} - 1)(\prod_{k=1}^{m} \gamma_k - \alpha\gamma_{i_1}) \Bigg) \\
&= - \Bigg(\prod_{k=1}^{m} \gamma_k \cdot (\gamma_{i_1}\gamma_{i_2} - \gamma_{i_1} - \gamma_{i_2} + 1) + \alpha(\gamma_{i_1}\gamma_{i_2} - \gamma_{i_1} - \gamma_{i_2} + 1) \Bigg) \\
&= -(\gamma_{i_1} - 1)(\gamma_{i_2} - 1)(\prod_{k=1}^{m} \gamma_k + \alpha).
\end{aligned}
$$

Therefore, $\Lambda^{(1)}$ satisfies the condition $C(1)$. We assume that we obtain $\Lambda^{(n-1)}$ satisfying the condition $C(n-1)$, for $n \geq 2$. We define $\Lambda^{(n)}$ by replacing the

columns of $\Lambda^{(n-1)}$ of I' ($|I'| \geq n+1$) with

$$\Lambda^{(n-1)}_{*,I'} + \sum_{\substack{K' \subset I' \\ |K'|=n}} (-1)^{|I'|+n+1} \frac{\prod_k \gamma_k + (-1)^n \alpha \prod_{j \in K'^c \cap I'} \gamma_j}{\prod_k \gamma_k + (-1)^n \alpha} \cdot \Lambda^{(n-1)}_{*,K'}.$$

Note that the columns of I' ($|I'| \leq n$) and the rows of I ($|I| \leq n-1$) are not changed from $\Lambda^{(n-1)}$. We verify that $\Lambda^{(n)}$ satisfies the condition $C(n)$.

(i) It is obvious.

(ii) We assume $|I'| \geq n+1$. For $K' \subset I'$ with $|K'| = n$, (ii) of $C(n-1)$ means

$$\begin{aligned}
&\Lambda^{(n-1)}_{I,K'} \\
&= (-1)^{|I|+|K'|-1} \cdot \Bigg[\Big(\prod_{i \in I \cap K'} \gamma_i - 1 \Big) \Big(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I^c \cap K'} \gamma_j \Big) \\
&\qquad - \sum_{\substack{K \subset I \cap K' \\ 0<|K| \leq n-1}} \prod_{i \in K} (\gamma_i - 1) \Big(\prod_{k=1}^{m} \gamma_k + (-1)^{|K|} \alpha \prod_{j \in K^c \cap K'} \gamma_j \Big) \Bigg] \\
&= (-1)^{|I|+n-1} \cdot \Bigg[\Bigg(\Big(\prod_{i \in I \cap K'} \gamma_i - 1 \Big) - \sum_{\substack{K \subset I \cap K' \\ 0<|K| \leq n-1}} \prod_{i \in K} (\gamma_i - 1) \Bigg) \prod_{k=1}^{m} \gamma_k \\
&\quad - \Bigg(\Big(\prod_{i \in I \cap K'} \gamma_i - 1 \Big) + \sum_{\substack{K \subset I \cap K' \\ 0<|K'| \leq n-1}} (-1)^{|K|} \prod_{i \in K} (\gamma_i - 1) \prod_{j \in I \cap K' - K} \gamma_j \Bigg) \cdot \alpha \prod_{j \in I^c \cap K'} \gamma_j \Bigg],
\end{aligned}$$

since $K^c \cap K' = (I^c \cap K') \cup (I \cap K' - K)$. As a corollary of Lemma 4.16, we obtain the formulas

$$\sum_{N \subset \{1,\ldots,k\}} \prod_{l \in N} (1 - \lambda_l) \prod_{l \notin N} \lambda_l = \sum_{N \subset \{1,\ldots,k\}} (-1)^{|N|} \prod_{l \in N} (\lambda_l - 1) \prod_{l \notin N} \lambda_l = 1, \tag{4.12}$$

$$\sum_{N \subset \{1,\ldots,k\}} \prod_{l \in N} (\lambda_l - 1) = \prod_{l=1}^{k} \lambda_l, \tag{4.13}$$

for a positive integer k and complex numbers $\lambda_1, \ldots \lambda_k$. The equality (4.13) implies

$$\prod_{i \in I \cap K'} \gamma_k - 1 = \sum_{\emptyset \neq K \subset I \cap K'} \prod_{i \in K} (\gamma_i - 1).$$

By (4.12), we have

$$\prod_{i \in I \cap K'} \gamma_i - 1 = - \sum_{\emptyset \neq K \subset I \cap K'} (-1)^{|K|} \prod_{i \in K} (\gamma_i - 1) \prod_{j \in I \cap K' - K} \gamma_j.$$

Therefore, if $|I \cap K'| \leq n-1$, then we have

$$\Lambda^{(n-1)}_{I,K'} = 0.$$

If $|I \cap K'| = n$ (i.e., $K' \subset I$), then we have

$$\Lambda^{(n-1)}_{I,K'} = (-1)^{|I|+n-1} \prod_{i \in K'} (\gamma_i - 1) \cdot \Big(\prod_{k=1}^{m} \gamma_k + (-1)^n \alpha \Big).$$

Hence we obtain

$$\begin{aligned}
\Lambda^{(n)}_{I,I'} =& (-1)^{|I|+|I'|-1} \cdot \Bigg[\Bigg(\prod_{i \in I \cap I'} \gamma_i - 1 \Bigg) \cdot \Bigg(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I^c \cap I'} \gamma_j \Bigg) \\
&- \sum_{\substack{K \subset I \cap I' \\ 0 < |K| \le n-1}} \Bigg(\prod_{i \in K} (\gamma_i - 1) \cdot \Bigg(\prod_{k=1}^{m} \gamma_k + (-1)^{|K|} \alpha \prod_{j \in K^c \cap I'} \gamma_j \Bigg) \Bigg) \Bigg] \\
&+ \sum_{\substack{K' \subset I \cap I' \\ |K'| = n}} \Bigg[(-1)^{|I'|+n+1} \frac{\prod_k \gamma_k + (-1)^n \alpha \prod_{j \in K'^c \cap I'} \gamma_j}{\prod_k \gamma_k + (-1)^n \alpha} \\
&\cdot (-1)^{|I|+n-1} \prod_{i \in K'} (\gamma_i - 1) \cdot \Big(\prod_{k=1}^{m} \gamma_k + (-1)^n \alpha \Big) \Bigg] \\
=& (-1)^{|I|+|I'|-1} \cdot \Bigg[\Bigg(\prod_{i \in I \cap I'} \gamma_i - 1 \Bigg) \cdot \Bigg(\prod_{k=1}^{m} \gamma_k - \alpha \prod_{j \in I^c \cap I'} \gamma_j \Bigg) \\
&- \sum_{\substack{K \subset I \cap I' \\ 0 < |K| \le n}} \Bigg(\prod_{i \in K} (\gamma_i - 1) \cdot \Bigg(\prod_{k=1}^{m} \gamma_k + (-1)^{|K|} \alpha \prod_{j \in K^c \cap I'} \gamma_j \Bigg) \Bigg) \Bigg],
\end{aligned}$$

which is (ii) of the condition $C(n)$.

(iii) By (iii) of $C(n-1)$, we have to show that $\Lambda^{(n)}(n, k') = O$ $(k' > n)$, i.e., $\Lambda^{(n)}_{I,I'} = 0$ $(|I| = n,\ |I'| \ge n+1)$. We assume that $|I| = n$ and $|I'| \ge n+1$. By a similar calculation to the proof of (ii), we obtain

$$\begin{aligned}
&\Lambda^{(n)}_{I,I'} \\
&= (-1)^{|I|+|I'|-1} \cdot \Bigg[\Bigg(\Big(\prod_{i \in I \cap I'} \gamma_i - 1 \Big) - \sum_{\substack{K \subset I \cap I' \\ 0 < |K| \le n}} \prod_{i \in K} (\gamma_i - 1) \Bigg) \prod_{k=1}^{m} \gamma_k \\
&\quad - \Bigg(\Big(\prod_{i \in I \cap I'} \gamma_i - 1 \Big) + \sum_{\substack{K \subset I \cap I' \\ 0 < |I'| \le n}} (-1)^{|K|} \prod_{i \in K} (\gamma_i - 1) \prod_{j \in I \cap I' - K} \gamma_j \Bigg) \cdot \alpha \prod_{j \in I^c \cap I'} \gamma_j \Bigg].
\end{aligned} \tag{4.14}$$

Because of $|I \cap I'| \le n$, we conclude that $\Lambda^{(n)}_{I,I'} = 0$.

(iv) By (iv) of $C(n-1)$, we have to show that $\Lambda^{(n)}(n+1, n+1)$ is diagonal. We assume that $|I| = |I'| = n+1$ and $I \ne I'$. Then we also have $|I \cap I'| \le n$, and hence $\Lambda^{(n)}_{I,I'} = 0$, by (4.14).

(v) By (v) of $C(n-1)$, we have to show that

$$\Lambda^{(n)}_{I,I} = -\prod_{i\in I}(\gamma_i - 1)\cdot\Big(\prod_{k=1}^{m}\gamma_k + (-1)^{|I|}\alpha\Big),$$

for $|I| = n+1$. By (4.14), $I \cap I^c = \emptyset$, and a similar argument to proof of (ii), we obtain

$$\begin{aligned}\Lambda^{(n)}_{I,I} &= (-1)^{|I|+|I|-1}\left[\prod_{i\in I}(\gamma_i - 1)\cdot\prod_{k=1}^{m}\gamma_i + (-1)^{|I|}\prod_{i\in I}(\gamma_i - 1)\cdot\alpha\right]\\ &= -\prod_{i\in I}(\gamma_i - 1)\cdot\Big(\prod_{k=1}^{m}\gamma_k + (-1)^{|I|}\alpha\Big).\end{aligned}$$

Consequently, we can construct $\Lambda^{(n)}$ inductively. □

4.3.4 The eigenspace of $\mathcal{M}_0$ of eigenvalue 1

By Corollary 4.18 and Theorem 4.19, we have to show that

- $\mathcal{M}_0(D_{1\cdots m}) = [(-1)^{m-1}\prod_k \gamma_k\cdot\alpha^{-1}\beta^{-1}]\cdot D_{1\cdots m}$,
- $\mathcal{M}_0(D_I) = D_I$ for $I \subsetneq \{1,\dots,m\}$,

to prove Theorem 4.6. In this subsection, we show the second claim. The first one is proved in the next subsection.

Hereafter, we use coordinates $(s_1,\dots s_m) = \left(\frac{t_1}{x_1},\dots,\frac{t_m}{x_m}\right)$. $v(t)$ and $w(t,x)$ are expressed as

$$1 - \sum_{k=1}^{m} x_k s_k, \quad \prod_{k=1}^{m}(x_k s_k)\cdot\left(1 - \sum_{k=1}^{m}\frac{1}{s_k}\right),$$

respectively. Let

$$v'(s,x) := 1 - \sum_{k=1}^{m} x_k s_k, \quad w'(s) := \prod_{k=1}^{m} s_k\cdot\left(1 - \sum_{k=1}^{m}\frac{1}{s_k}\right).$$

If $x_1,\dots,x_m$ are positive real numbers, then we have

$$t_k \gtreqqless 0 \Leftrightarrow s_k \gtreqqless 0, \quad v(t) \gtreqqless 0 \Leftrightarrow v'(s,x) \gtreqqless 0, \quad w(t,x) \gtreqqless 0 \Leftrightarrow w'(s) \gtreqqless 0,$$

and hence the expressions of D_I's are as follows:

$$\begin{aligned} D_{1\cdots m} &\leftrightarrow s_k > 0\ (1\le k\le m),\ v'(s,x) > 0,\ w'(s) > 0,\\ D &\leftrightarrow s_k < 0\ (1\le k\le m),\\ D_I\ \text{(otherwise)} &\leftrightarrow \begin{cases} s_i > 0\ (i\in I),\ s_j < 0\ (j\notin I),\\ v'(s,x) > 0,\ (-1)^{m-|I|+1}w'(s) > 0.\end{cases}\end{aligned}$$

Note that if $x = (x_1,\dots,x_m)$ moves, then only the divisor $(v'(s,x) = 0)$ varies.

Recall that the loop ρ_0 is homotopic to the composition $\tau_0\rho_0'\overline{\tau_0}$, where

$$\tau_0 : [0,1] \ni \theta \mapsto \left((1-\theta)\frac{1}{2m^2} + \theta\Big(\frac{1}{m^2} - \varepsilon_0\Big)\right)(1,\dots,1),$$
$$\rho_0' : [0,1] \ni \theta \mapsto \left(\frac{1}{m^2} - \varepsilon_0 e^{2\pi\sqrt{-1}\theta}\right)(1,\dots,1),$$

for a sufficiently small positive real number ε_0. Since variations along the paths τ_0 and $\overline{\tau_0}$ give trivial transformations of the cycles D_I's, we have to consider the variation along ρ_0' for a sufficiently small ε_0. Let $\varepsilon_0 \to 0$, then $(v'(s,x) = 0)$ and $(w'(s) = 0)$ are tangent at $(s_1,\dots,s_m) = (m,\dots,m)$. Thus $D_{1\cdots m}$ is a vanishing cycle, and each D_I ($I \subsetneq \{1,\dots,m\}$) changes trivially as $\varepsilon_0 \to 0$. This implies that $\mathcal{M}_0(D_I) = D_I$ for $|I| < m$.

4.3.5 An eigenvector of $\mathcal{M}_0$ of eigenvalue $(-1)^{m-1}\prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}$

In this subsection, we show $\mathcal{M}_0(D_{1\cdots m}) = [(-1)^{m-1}\prod_k \gamma_k \cdot \alpha^{-1}\beta^{-1}] \cdot D_{1\cdots m}$. As mentioned in the previous subsection, it is sufficient to consider the variation of $D_{1\cdots m}$ along ρ_0' for a sufficiently small ε_0. Thus we consider that $D_{1\cdots m}$ is contained in a small neighborhood of $s = (m,\dots,m)$ in $\mathbb{R}^m$.

Put $x := \frac{1}{m^2} - \varepsilon_0$. We have

$$v'(s,\rho_0'(0)) = 1 - \left(\frac{1}{m^2} - \varepsilon_0\right)\sum_{k=1}^{m} s_k.$$

We use the coordinates system

$$(s_1',\dots,s_{m-1}',s_m') := \left(s_1 - m,\dots,s_{m-1} - m, \sum_{k=1}^{m} s_k - m^2\right).$$

Note that $s_l = s_l' + m$ $(1 \le l \le m-1)$ and $s_m = s_m' - \sum_{l=1}^{m-1} s_l' + m$. Then the origin $(s_1',\dots,s_m') = (0,\dots,0)$ corresponds to $(s_1,\dots,s_m) = (m\dots,m)$. Let U be a small neighborhood of $(s_1',\dots,s_m') = (0,\dots,0)$ so that $s_k > 0$ $(1 \le k \le m)$. In U, we have

$$v'(s,\rho_0'(0)) > 0 \Leftrightarrow 1 - \left(\frac{1}{m^2} - \varepsilon_0\right)(s_m' + m^2) > 0 \Leftrightarrow s_m' < \frac{m^2}{\frac{1}{m^2} - \varepsilon_0}\cdot\varepsilon_0,$$
$$w'(s) > 0 \Leftrightarrow 1 - \sum_{k=1}^{m}\frac{1}{s_k} > 0 \Leftrightarrow s_m' > \sum_{l=1}^{m-1} s_l' - m + \frac{1}{1 - \sum_{l=1}^{m-1}\frac{1}{s_l'+m}}.$$

Hence $D_{1\cdots m}$ is expressed as

$$\left\{(s_1',\dots,s_m') \in U \;\middle|\; \sum_{l=1}^{m-1} s_l' - m + \frac{1}{1 - \sum_{l=1}^{m-1}\frac{1}{s_l'+m}} < s_m' < \frac{m^2}{\frac{1}{m^2} - \varepsilon_0}\cdot\varepsilon_0\right\}.$$

Let θ move from 0 to 1, then the arguments of $\frac{1}{m^2} - \varepsilon_0 e^{2\pi\sqrt{-1}\theta}$ at the start point and the end point are equal. Thus the argument of $\frac{m^2}{\frac{1}{m^2} - \varepsilon_0 e^{2\pi\sqrt{-1}\theta}}\cdot\varepsilon_0 e^{2\pi\sqrt{-1}\theta}$

increases by 2π, when θ moves from 0 to 1. Put

$$f(s'_1,\dots,s'_{m-1}) := \sum_{l=1}^{m-1} s'_l - m + \frac{1}{1-\sum_{l=1}^{m-1}\frac{1}{s'_l+m}}.$$

For $i \neq j$, we have

$$\begin{aligned}
f(0,\dots,0) &= -m + \frac{1}{1-(m-1)\frac{1}{m}} = 0,\\
\frac{\partial f}{\partial s'_i}(s'_1,\dots,s'_{m-1}) &= 1 - \frac{1}{(1-\sum_{l=1}^{m-1}\frac{1}{s'_l+m})^2}\cdot\frac{1}{(s_i+m)^2},\\
\frac{\partial f}{\partial s'_i}(0,\dots,0) &= 1 - \frac{1}{(1-(m-1)\frac{1}{m})^2}\cdot\frac{1}{m^2} = 0,\\
\frac{\partial^2 f}{\partial s'_i\partial s'_j}(s'_1,\dots,s'_{m-1}) &= \frac{2}{(1-\sum_{l=1}^{m-1}\frac{1}{s'_l+m})^3}\cdot\frac{1}{(s_i+m)^2(s_j+m)^2},\\
\frac{\partial^2 f}{\partial s'_i\partial s'_j}(0,\dots,0) &= \frac{2}{(1-(m-1)\frac{1}{m})^3}\cdot\frac{1}{m^4} = \frac{2}{m},\\
\frac{\partial^2 f}{\partial {s'_i}^2}(s'_1,\dots,s'_{m-1}) &= \frac{2}{(1-\sum_{l=1}^{m-1}\frac{1}{s'_l+m})^3}\cdot\frac{1}{(s_i+m)^4}\\
&\quad + \frac{1}{(1-\sum_{l=1}^{m-1}\frac{1}{s'_l+m})^2}\cdot\frac{2}{(s_i+m)^3},\\
\frac{\partial^2 f}{\partial {s'_i}^2}(0,\dots,0) &= \frac{2}{(1-(m-1)\frac{1}{m})^3}\cdot\frac{1}{m^4} + \frac{2}{(1-(m-1)\frac{1}{m})^2}\cdot\frac{1}{m^3} = \frac{4}{m},
\end{aligned}$$

Thus $(s'_1,\dots,s'_{m-1}) = (0,\dots,0)$ is a critical point, and the Hessian matrix $H_f(0,\dots,0)$ of f at this point is

$$H_f(0,\dots,0) = \frac{2}{m}\begin{pmatrix} 2 & 1 & \cdots & 1\\ 1 & 2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 1 & \cdots & 1 & 2\end{pmatrix}.$$

Since $H_f(0,\dots,0)$ is clearly positive definite, the Morse lemma means that f is expressed as

$$\sum_{l=1}^{m-1} z_l^2,$$

with appropriate coordinates $(z_1,\dots,z_{m-1})$ around the origin. Therefore, the claim $\mathcal{M}_0(D_{1\cdots m}) = [(-1)^{m-1}\prod_k \gamma_k\cdot\alpha^{-1}\beta^{-1}]\cdot D_{1\cdots m}$ is obtained from the following lemma.

Lemma 4.22. *For $y,\lambda,\mu\in\mathbb{C}$, we put*

$$\begin{aligned}
Z_y &:= \mathbb{C}^m - \left(\left(z_m - \sum_{l=1}^{m-1} z_l^2 = 0\right)\cup(y - z_m = 0)\right)\subset\mathbb{C}^m,\\
\nu_y &:= \left(z_m - \sum_{l=1}^{m-1} z_l^2\right)^{\lambda}\cdot(y-z_m)^{\mu},
\end{aligned}$$

where $z_1, \ldots, z_m$ are coordinates of $\mathbb{C}^m$. We consider the twisted homology groups $H_m(Z_y, \nu_y)$ $(y \in \mathbb{C})$. Let $\delta_y \in H_m(Z_y, \nu_y)$ $(y > 0)$ be expressed by the twisted cycle

$$\left\{ (z_1, \ldots, z_m) \in \mathbb{R}^m \;\middle|\; \sum_{l=1}^{m-1} z_l^2 < z_m < y \right\},$$

and let δ' be the element in $H_m(Z_1, \nu_1)$, which is obtained by the deformation of δ_1 along $y = e^{2\pi\sqrt{-1}\theta}$ as $\theta : 0 \to 1$. Then we have

$$\delta' = (-1)^{m-1} e^{2\pi\sqrt{-1}(\lambda+\mu)} \cdot \delta_1.$$

Proof. It is easy to see that δ_y is expressed by $(\xi_1, \ldots, \xi_m) \in [0,1]^m$ as

$$\begin{aligned} z_l &= (2\xi_l - 1)\sqrt{y\xi_m \prod_{j=l+1}^{m-1} (1 - (2\xi_j - 1)^2)} \quad (1 \le l \le m-1), \\ z_m &= y\xi_m. \end{aligned}$$

The functions $z_m - \sum_{l=1}^{m-1} z_l^2$ and $y - z_m$ are expressed as

$$y\xi_m \left(1 - \sum_{l=1}^{m-1} (2\xi_l - 1)^2 \prod_{j=l+1}^{m-1} (1 - (2\xi_j - 1)^2) \right), \quad y(1 - \xi_m),$$

respectively. We consider the variation along $y = e^{2\pi\sqrt{-1}\theta}$ as $\theta : 0 \to 1$. Both arguments of $z_m - \sum_{l=1}^{m-1} z_l^2$ and $y - z_m$ increase by 2π. Further, the expression of the domain of δ by $(\xi_1, \ldots, \xi_m) \in [0,1]^m$ is also changed. However, by a bijection

$$\xi_l \mapsto 1 - \xi_l \ (1 \le l \le m-1), \quad \xi_m \mapsto \xi_m,$$

the expression coincides with the original one with contributions to orientation. Therefore we obtain

$$\delta' = (-1)^{m-1} e^{2\pi\sqrt{-1}(\lambda+\mu)} \cdot \delta_1.$$

□

4.4 Representation matrices

For $0 \le i \le m$, the matrix representation of $\mathcal{M}_i$ with respect to the basis $\{\Delta_I\}_I$ is given by M_i in Corollaries 4.5 and 4.9. However, M_0 is too complicated to write down. In this section, we give another basis $\{\Delta'_I\}_I$ of $H_m(T_{\dot{x}}, u_{\dot{x}})$, and write down the representation matrix of $\mathcal{M}_i$ with respect to this basis.

Let P be the $2^m \times 2^m$ matrix whose (N, I) entry is

$$\begin{cases} \alpha\beta \displaystyle\prod_{j \notin I} \frac{\gamma_j - 1}{\gamma_j} \cdot \frac{\prod_{n \in N} \gamma_n}{(\alpha - \prod_{n \in N} \gamma_n)(\beta - \prod_{n \in N} \gamma_n)} & (N \subset I), \\ 0 & (N \not\subset I), \end{cases}$$

and let $\{\Delta'_I\}_I$ be the basis of $H_m(T_{\dot{x}}, u_{\dot{x}})$ defined as

$$\begin{aligned}&(\Delta', \Delta'_1, \Delta'_2, \dots, \Delta'_m, \Delta'_{12}, \Delta'_{13}, \dots, \Delta'_{1\cdots m})\\ &= (\Delta, \Delta_1, \Delta_2, \dots, \Delta_m, \Delta_{12}, \Delta_{13}, \dots, \Delta_{1\cdots m})P.\end{aligned}$$

Namely, Δ'_I is defined by

$$\Delta'_I = \alpha\beta \prod_{j\notin I} \frac{\gamma_j - 1}{\gamma_j} \cdot \sum_{N\subset I} \frac{\prod_{n\in N}\gamma_n}{(\alpha - \prod_{n\in N}\gamma_n)(\beta - \prod_{n\in N}\gamma_n)} \Delta_N.$$

Note that P is an upper triangular matrix.

Lemma 4.23. *We have*

$$\begin{aligned}&\frac{(\alpha - \prod_k \gamma_k)(\beta - \prod_k \gamma_k)}{\alpha\beta \prod_k \gamma_k} \Delta'_{1\cdots m}\\ &+ \sum_{I\subsetneq\{1,\dots,m\}} \left(\frac{1}{\prod_{i\in I}\gamma_i} + (-1)^{m-|I|} \frac{\prod_k \gamma_k}{\alpha\beta} \right) \Delta'_I = D_{1\cdots m}.\end{aligned}$$

Proof. By the definition, the left hand side is equal to

$$\begin{aligned}&\frac{(\alpha - \prod_k \gamma_k)(\beta - \prod_k \gamma_k)}{\alpha\beta \prod_k \gamma_k} \cdot \alpha\beta \sum_{N\subset\{1,\dots,m\}} \frac{\prod_{n\in N}\gamma_n}{(\alpha - \prod_{n\in N}\gamma_n)(\beta - \prod_{n\in N}\gamma_n)} \Delta_N\\ &+ \sum_{I\subsetneq\{1,\dots,m\}} \Bigg[\prod_{j\notin I} (\gamma_j - 1) \Big(\frac{\alpha\beta}{\prod_k \gamma_k} + (-1)^{m-|I|} \prod_{i\in I} \gamma_i \Big)\\ &\qquad \cdot \sum_{N\subset I} \frac{\prod_{n\in N}\gamma_n}{(\alpha - \prod_{n\in N}\gamma_n)(\beta - \prod_{n\in N}\gamma_n)} \Delta_N \Bigg]. \qquad (4.15)\end{aligned}$$

Clearly the coefficient of $\Delta_{1\cdots m}$ in (4.15) is 1. The coefficient of Δ_N ($N \neq \{1, \dots, m\}$) is

$$\begin{aligned}&\frac{\prod_{n\in N}\gamma_n}{(\alpha - \prod_{n\in N}\gamma_n)(\beta - \prod_{n\in N}\gamma_n)} \cdot \Bigg[\frac{(\alpha - \prod_k \gamma_k)(\beta - \prod_k \gamma_k)}{\prod_k \gamma_k}\\ &\qquad + \sum_{\substack{I\supset N\\ I\neq\{1,\dots,m\}}} \prod_{j\notin I} (\gamma_j - 1) \Big(\frac{\alpha\beta}{\prod_k \gamma_k} + (-1)^{m-|I|} \prod_{i\in I} \gamma_i \Big) \Bigg]\\ &= \frac{\prod_{n\in N}\gamma_n}{(\alpha - \prod_{n\in N}\gamma_n)(\beta - \prod_{n\in N}\gamma_n)} \cdot \Bigg[\frac{\alpha\beta}{\prod_k \gamma_k} \Big(1 + \sum_{\substack{J\subset N^c\\ J\neq\emptyset}} \prod_{j\in J} (\gamma_j - 1) \Big)\\ &\qquad - \alpha - \beta + \prod_{k=1}^m \gamma_k + \prod_{n\in N} \gamma_n \sum_{\substack{J\subset N^c\\ J\neq\emptyset}} (-1)^{|J|} \prod_{j\in J} (\gamma_j - 1) \prod_{i\in N^c - J} \gamma_i \Bigg]\\ &= \frac{\prod_{n\in N}\gamma_n}{(\alpha - \prod_{n\in N}\gamma_n)(\beta - \prod_{n\in N}\gamma_n)} \cdot \Bigg[\frac{\alpha\beta}{\prod_k \gamma_k} \Big(\sum_{J\subset N^c} \prod_{j\in J} (\gamma_j - 1) \Big)\\ &\qquad - \alpha - \beta + \prod_{n\in N} \gamma_n \sum_{J\subset N^c} (-1)^{|J|} \prod_{j\in J} (\gamma_j - 1) \prod_{i\in N^c - J} \gamma_i \Bigg]\end{aligned}$$

$$= \frac{\prod_{n\in N}\gamma_n}{(\alpha-\prod_{n\in N}\gamma_n)(\beta-\prod_{n\in N}\gamma_n)}\cdot\left(\frac{\alpha\beta}{\prod_{n\in N}\gamma_n}-\alpha-\beta+\prod_{n\in N}\gamma_n\right)$$
$$= 1.$$

Here we use the equalities (4.12) and (4.13). Therefore, (4.15) is equal to

$$\sum_{I\subset\{1,\dots,m\}}\Delta_I = D_{1\cdots m}.$$

□

Corollary 4.24. *For $0\le i\le m$, let M'_i be the representation matrix of $\mathcal{M}_i$ with respect to the basis $\{\Delta'_I\}_I$. Then we have*

$$M'_0 = E_{2^m}-N_0,\quad M'_k = M_k+N_k\ (1\le k\le m),$$

where N_i is defined as follows. The (I,I') entry of N_0 (resp. N_k) is zero, except in the case of $I'=\emptyset$ (resp. $k\in I'$ and $I=I'-\{k\}$). The $(I,\emptyset)$ entry of N_0 is

$$\begin{cases}\dfrac{(\alpha-\prod_k\gamma_k)(\beta-\prod_k\gamma_k)}{\alpha\beta\prod_k\gamma_k} & (I=\{1,\dots,m\}),\\[2ex] \dfrac{1}{\prod_{i\in I}\gamma_i}+(-1)^{m-|I|}\dfrac{\prod_k\gamma_k}{\alpha\beta} & (\text{otherwise}).\end{cases}$$

The $(I-\{k\},I)$ entry of N'_k is 1.

In particular, M'_k $(1\le k\le m)$ is upper triangular, M'_0 is lower triangular, and the $(\emptyset,\emptyset)$ entry of M'_0 is

$$1-\left(1+(-1)^m\frac{\prod\gamma_k}{\alpha\beta}\right)=(-1)^{m-1}\prod\gamma_k\cdot\alpha^{-1}\beta^{-1}.$$

Proof. We evaluate M'_0. By Corollary 4.9, it is sufficient to show that the matrix representation of

$$\delta\mapsto\frac{(\beta-1)(\alpha-\prod_k\gamma_k)}{\alpha\beta}I_h(\delta,D^\vee_{1\cdots m})D_{1\cdots m}$$

is given by N_0. By Fact 4.3 and Proposition 4.8, we have

$$\begin{aligned}&\frac{(\beta-1)(\alpha-\prod_k\gamma_k)}{\alpha\beta}I_h(\Delta'_I,D^\vee_{1\cdots m})D_{1\cdots m}\\
&=\frac{(\beta-1)(\alpha-\prod_k\gamma_k)}{\alpha\beta}\cdot\alpha\beta\prod_{j\notin I}\frac{\gamma_j-1}{\gamma_j}\\
&\quad\cdot\sum_{N\subset I}\Bigg[\frac{\prod_{n\in N}\gamma_n}{(\alpha-\prod_{n\in N}\gamma_n)(\beta-\prod_{n\in N}\gamma_n)}\\
&\qquad\cdot(-1)^{|N|}\frac{\prod_{j\notin N}\gamma_j\cdot(\alpha-\prod_{n\in N}\gamma_n)(\beta-\prod_{n\in N}\gamma_n)}{\prod_k(\gamma_k-1)\cdot(\alpha-\prod_k\gamma_k)(\beta-1)}\Bigg]D_{1\cdots m}\\
&=\left(\sum_{N\subset I}(-1)^{|N|}\right)\prod_{i\in I}\frac{\gamma_i}{\gamma_i-1}\cdot D_{1\cdots m},\end{aligned}$$

and hence we obtain

$$\frac{(\beta-1)(\alpha-\prod_k \gamma_k)}{\alpha\beta} I_h(\Delta'_I, D^\vee_{1\cdots m}) D_{1\cdots m} = \begin{cases} D_{1\cdots m} & (I=\emptyset), \\ 0 & (\text{otherwise}). \end{cases}$$

Thus Lemma 4.23 shows the claim.

We evaluate M'_k $(1 \le k \le m)$. We have to show that

$$\mathcal{M}_k(\Delta'_I) = \begin{cases} \Delta'_I & (k \notin I), \\ \gamma_k^{-1}\Delta'_I + \Delta'_{I-\{k\}} & (k \in I). \end{cases}$$

If $k \notin I$, then the subsets N of I also satisfy $k \notin N$, and hence we have $\mathcal{M}_k(\Delta_N) = \Delta_N$ by Proposition 4.4. This implies that $\mathcal{M}_k(\Delta'_I) = \Delta'_I$, for $k \notin I$. We assume $k \in I$. For a subset N of $I - \{k\}$, we have

$$\mathcal{M}_k(\Delta_N) = \Delta_N = \left(\gamma_k^{-1} + \frac{\gamma_k - 1}{\gamma_k}\right)\Delta_N, \quad \mathcal{M}_k(\Delta_{N\cup\{k\}}) = \gamma_k^{-1}\Delta_{N\cup\{k\}}.$$

Then we obtain

$$\begin{aligned} &\mathcal{M}_k(\Delta'_I) \\ &= \gamma_k^{-1}\Delta'_I + \frac{\gamma_k-1}{\gamma_k}\cdot\alpha\beta\prod_{j\notin I}\frac{\gamma_j-1}{\gamma_j}\cdot\sum_{N\subset I-\{k\}}\frac{\prod_{n\in N}\gamma_n}{(\alpha-\prod_{n\in N}\gamma_n)(\beta-\prod_{n\in N}\gamma_n)}\Delta_N \\ &= \gamma_k^{-1}\Delta'_I + \alpha\beta\prod_{j\notin I-\{k\}}\frac{\gamma_j-1}{\gamma_j}\cdot\sum_{N\subset I-\{k\}}\frac{\prod_{n\in N}\gamma_n}{(\alpha-\prod_{n\in N}\gamma_n)(\beta-\prod_{n\in N}\gamma_n)}\Delta_N \\ &= \gamma_k^{-1}\Delta'_I + \Delta'_{I-\{k\}}. \end{aligned}$$

□

Example 4.25. We write down M'_i's for $m = 2, 3$.

(i) In the case of $m = 2$, the representation matrices M'_0, M'_1, M'_2 are as follows.

$$M'_0 = \begin{pmatrix} -\frac{\gamma_1\gamma_2}{\alpha\beta} & 0 & 0 & 0 \\ -\frac{1}{\gamma_1} + \frac{\gamma_1\gamma_2}{\alpha\beta} & 1 & 0 & 0 \\ -\frac{1}{\gamma_2} + \frac{\gamma_1\gamma_2}{\alpha\beta} & 0 & 1 & 0 \\ -\frac{(\alpha-\gamma_1\gamma_2)(\beta-\gamma_1\gamma_2)}{\alpha\beta\gamma_1\gamma_2} & 0 & 0 & 1 \end{pmatrix},$$

$$M'_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & \frac{1}{\gamma_1} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{\gamma_1} \end{pmatrix}, \quad M'_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{\gamma_2} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma_2} \end{pmatrix}.$$

These are equal to transpose matrices of those in Remark 4.3 of [9].

(ii) In the case of $m = 3$, the representation matrices M'_0, M'_1, M'_2, M'_3 are as follows.

$$M'_0 = \begin{pmatrix} \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\gamma_1} - \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\gamma_2} - \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\gamma_3} - \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{\gamma_1\gamma_2} + \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{\gamma_1\gamma_3} + \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{\gamma_2\gamma_3} + \frac{\gamma_1\gamma_2\gamma_3}{\alpha\beta} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{(\alpha-\gamma_1\gamma_2\gamma_3)(\beta-\gamma_1\gamma_2\gamma_3)}{\alpha\beta\gamma_1\gamma_2\gamma_3} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M'_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_1} \end{pmatrix},$$

$$M'_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\gamma_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_2} \end{pmatrix},$$

$$M'_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\gamma_3} \end{pmatrix}.$$

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