Thesis

On the constraints in off-shell formulations of $D = N = 2$ and $D = N = 4$ super Yang-Mills theories with a gauged central charge

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February 10, 2014

Abstract

We formulate off-shell invariant two-dimensional $N = 2$ and four-dimensional $N = 4$ super Yang-Mills theories with a gauged central charge, and investigate whether constraints inevitably appear in these theories. We deal with Dirac-Kähler twisted version of supersymmetry and adopt the superconnection formalism. In the two-dimensional theory, we show that one can formulate two new models without constraints by introducing two central charges which are related to each other. This is in contrast with the already known super Yang-Mills theory with central charges, especially, the off-shell invariant $D = N = 4$ super Yang-Mills theory. We then apply this technique of dependent-central-charges to the formulation of the off-shell invariant $D = N = 4$ super Yang-Mills theory. In this model, there are four possible central charges. We found that there are no four-dimensional models without constraints, as far as central charges, which are dependent, are included in the algebra. It is also shown that a constraint is inevitable if only one of these central charges is included in the algebra.

Contents

1 Introduction

1.1 Motivations

Supersymmetry (SUSY) is one of the most important guiding principles in contemporary particle physics. In particular, it has been recognized that $D = N = 4$ super Yang-Mills (SYM) theory plays a crucial role in string-motivated gauge theory formulations [1]. It is known that $D = N = 4$ SYM theory with SU(4) R-symmetry can be formulated only at the on-shell level [2]. That is why there is a long-standing question as to whether one can find a superspace formulation for $D = N = 4$ SYM theory to obtain an off-shell invariant formulation. Although many efforts have been made so far and various formulations, e.g. Harmonic superspace formulation [3], are proposed and developed, there is no clear success for the problem in the present.

However, it is also known that one can find an off-shell invariant $D = N = 4$ SYM formulation by taking USp(4) as the R-symmetry instead of SU(4) and by introducing a central charge [4, 5]. Here we give a summary of this formulation proposed in Ref [5] because we use the method given there as a crucial reference. This method begins with an on-shell invariant higher dimensional theory. We now consider on-shell invariant five-dimensional SYM theory. And then dimensional reduction which uses the idea of Legendre transformation is performed.

First of all, Legendre transformation in the canonical formalism is performed as follows: The Hamiltonian density $\mathcal{H}$ is derived from the Lagrangian density $\mathcal{L}$ as

$$\mathcal{H} = -\mathcal{L} + \sum_{\varphi} \dot{\varphi} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \,. \tag{1.1}$$

It can be shown from the equations of motion that the Hamiltonian is time-independent

$$\dot{H} = \frac{d}{dt} \int d^3x \mathcal{H} = 0 \,. \tag{1.2}$$

Note that $\mathcal{H}$ need not be time-independent to make H time-independent. By the fact, one can obtain four-dimensional SYM theory from five-dimensional SYM theory with the following identifications:

$$\mathcal{L} \to \mathcal{L}_5 \,, \quad -\mathcal{H} \to \mathcal{L}_4 \,, \quad t \to x_5 \tag{1.3}$$

where $\mathcal{L}_5$ and $\mathcal{L}_4$ are Lagrangian densities in five and four dimension, respectively, and x_5 is the fifth coordinate in five-dimensional spacetime. One can realize that the four-dimensional theory is in fact "four -dimensional" from (1.1) and (1.2) which are now rewritten as

$$\mathcal{L}_4 = \mathcal{L}_5 - \sum_{\varphi} \partial_5 \varphi \frac{\partial \mathcal{L}_5}{\partial(\partial_5 \varphi)} \,, \tag{1.4}$$

$$\frac{d}{dx_5} \int d^4x \mathcal{L}_4 = 0 \,. \tag{1.5}$$

For (1.5) to be satisfied, five-dimensional equations of motion are necessary. The derivative of x_5-direction is then regarded as central charge transformation. The five-dimensional canonical equations of motion thus readily determine the laws of central charge transformations after the dimensional reduction. The canonical momentum which is conjugate to bosonic fields is defined as auxiliary fields. Especially, the canonical momentum of gauge field A_μ is defined as g_μ which is originally the curvature $F_{5\mu}$.

In the original on-shell invariant theory in five dimensions, the superalgebra closes in the following form with the equations of motion:

$$[\,\delta_\xi,\,\delta_\eta\,] \sim \partial_a + (\text{gauge transformations}) + (\text{equations of motion})\,, \tag{1.6}$$

where δ stands for a supertransformation, ξ and η are Grassmann parameters, and a runs 0$\sim$ 3, 5. After the dimensional reduction, (1.6) is altered to

$$[\,\delta_\xi,\,\delta_\eta\,] \sim \partial_\mu + \delta_z + (\text{gauge transformations})\,, \tag{1.7}$$

where δ_z is a central charge transformation and is originally ∂_5. The term of five-dimensional equations of motion in (1.6) is automatically satisfied due to the definitions of central charge transformations. The off-shell invariance of the four-dimensional theory is thus concluded from (1.7).

However, one of the five-dimensional equations of motion remains as a constraint while the others become the definitions of central charge transformations. The equations of motion for A_μ in five dimensions is the form of

$$D^b F_{ab} + \cdots = 0\,. \tag{1.8}$$

After the dimensional reduction, the equation with $a = 5$ becomes a relation of the form

$$D^\mu g_\mu + \cdots = 0\,, \tag{1.9}$$

which is regarded as a constraint relation among the component fields. We focus on this kind of constraint in the thesis.

As can be seen above, the dimensional reduction by Legendre transformation is a method to obtain an off-shell invariant supersymmetric theory with central charge from higher dimensional on-shell invariant theory. In the case that the method is performed to SYM theory, the appearance of such a constraint like (1.9) is unavoidable. In the case of an Abelian gauge group, i.e. free theory, the constraint takes the form of

$$\partial^\mu g_\mu = 0\,, \tag{1.10}$$

which can be easily solved as

$$g_\mu = \partial_\nu B_{\mu\nu}\,, \tag{1.11}$$

where $B_{\mu\nu}$ is an antisymmetric tensor field. It is, however, impossible to solve the constraint locally in the case of a non-Abelian gauge group. For this reason, the existence of such a constraint implies some difficulties in the stage of investigating the quantized theory. Calculation of path integral would demand to solve the constraint and deal with non-locality [6].

Since the theory after the reduction includes central charge as seen above, the R-symmetry is USp(4) so that the superalgebra is algebraically consistent. Note that the central charge is not included as a number but a transformation, i.e. central charge transformation. This is a characteristic feature of the theory.

Such a multiplet of the theory with central charge has been called vector-tensor multiplet and the properties have been investigated intensively [7]. The superspace formulation of $D = N = 4$ SYM theory with central charge derived in Ref. [5] is also developed in Ref. [8] for the first time. And the corresponding harmonic superspace formulation is also investigated [9].

One may now ask whether or not such a constraint is unavoidable for off-shell invariant SYM formulation with central charge. If one can formulate such a theory without any constraints, one can expect to avoid the difficulty of non-locality. Motivated by the question, we first investigate a $D = N = 2$ twisted SYM theory with a central charge for simplicity [10]. Note that we investigate the A model and the B model ansatz in the two-dimensional theory (see later), and it is revealed that these models have different property about the constraint as can be seen in section 2*. We then investigate $D = N = 4$ off-shell invariant twisted SYM with a central charge with the results and experiences of two-dimensional cases.

1.2 The Dirac-Kähler twisting procedure

In this thesis, we do not deal with ordinary supersymmetry but Dirac-Kähler twisted supersymmetry [11]. It has been recognized that the Dirac-Kähler twisting procedure plays a special role for $D = N = 2$ and $D = N = 4$ SYM formulation because the procedure gives a link between quantization and supersymmetry [11, 12] and gives an approach for lattice supersymmetry [13]. The origin of difficulties of lattice supersymmetry is the momentum operator $P_\mu = -i\partial_\mu$ in superalgebra. On a lattice, ∂_μ becomes a difference operator Δ_μ and the operator does not satisfy the Leibniz rule. Generally speaking, validity of Leibniz rule is necessary for an action to be superinvariant. The twisting procedure make some supercharges be nilpotent, in other words, P_μ is not appear in superalgebra of some supercharges, which means that one can realize partial supersymmetry on a lattice by the twisting procedure. The Dirac-Kähler twisting procedure and the approach for lattice supersymmetry are intensively studied in [13]. Though, in fact, it is not necessary for the investigations in this thesis to use the procedure, it should be noted that the representation of twisted superalgebra give some help for the investigations as can be seen later.

Although The Dirac-Kähler procedure has its own feature, the procedure can be regarded as just a redefinition of supercharges. Thus, as far as flat spacetime is considered, the twisted supersymmetry is expected to be equivalent to ordinary supersymmetry. In other words, results from twisted supersymmetric theory can be expected to also hold in the corresponding ordinary supersymmetric theory. However, one should realize that the property of spinor is different between each spacetime where twisted and ordinary supersymmetry are considered. As can be seen later, the twisting procedure is considered in Euclidean spacetime, which results in the difference of possibilities of Majorana condition and so on which are determined by the number of dimensions and metric. In two dimensions, the Euclidean spacetime allow only ordinary Majorana condition while Lorentzian spacetime allow Majorana-Weyl condition. In four dimensions, the Euclidean spacetime do not allow ordinary Majorana condition but USp(4)-Majorana condition as we will see later while Lorentzian spacetime allow ordinary Majorana condition. We expect, however, that such a difference would not yield an essential difference about the results in the thesis.

The Dirac-Kähler twisting procedure can be performed independently from the number of dimensions. On the other hand, it is known that there are three types of twisting procedure in four dimensions [14, 15, 16]. Among them, the Dirac-Kähler twisting procedure can be regarded as Marcus's type of twisting procedure which first pointed out in [14] and considered in [16]. It is noted that twisted SYM theory with central charge has not been formulated so far†.

*In the author's master thesis, the A model formulation is investigated. But the B model is not.

†Some investigations about the theory have ever been done by J. Saito. The author uses the results as a reference.

1.3 The superconnection formalism

In the thesis, we use the superconnection formalism [8, 17, 18, 19]. This method is based on a superspace. The usual supercovariant derivative anti-commutes with the supercharge differential operator whice defines the supertransformations of component fields, but it is not gauge covariant. To obtain a gauge covariance with keeping the property of supercovariance at the same time, the superconnection and the gauge-supercovariant derivative are introduced. It can be shown that the gauge-supercovariant derivative produces gauge covariant superfield as the supercurvature. We can restrict the superfield from Jacobi identities among the gauge-supercovariant derivatives by constraints on the supercurvature. Furthermore, we can reflect the form of superalgebra on the constraints on the supercurvature. We can then obtain irreducible supermultiplet based on the desired superalgebra. It is a remarkable feature of the method that the original structure of superalgebra is manifestly realized. The supertransformation of component fields can be also calculated by Jacobi identities. However, because it is not obvious what is an appropriate ansatz for constraints on the supercurvature, we have to investigate and find the appropriate ansatz for constraints. This point is thus the main problem of the method. It is noted that, in the case of SYM with central charge, whether a constraint is inevitable is understood not by calculating Jacobi identities but by the check of closure of superalgebra with concrete calculation after obtaining the supertransformations of component fields.

This paper is organized as follows: In Section 2, we formulate $D = N = 2$ twisted SYM theory with a central charge. At first the most general superalgebra with possible central charges is derived. Then the Dirac-Kähler twisting procedure is performed. The possible models are then formulated by the superconnection formalism and we obtain the supermultiplet and the action of each model. We summarize the results in the section and give some discussions. In Section 3, we formulate $D = N = 4$ twisted SYM theory with a central charge in the same manner as the two-dimensional case after formulating the corresponding model without central charge. We then summarize the result and give some discussions in the last section.

2 $D = N = 2$ twisted superalgebra with central charge

2.1 Twisted superalgebra with central charge

In this subsection, we introduce central charge to $N = 2$ superalgebra in two dimensions and perform the Dirac Kähler twisting procedure. We concentrate on Euclidean spacetime in this paper according to the twisting procedure [11]. The reason why this procedure is performed in Euclidean spacetime is given in the last part in the subsection.

2.1.1 Superalgebra with central charge

In two-dimensional Euclidean spacetime, the γ-matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ and the charge conjugation matrix C can be chosen to satisfy [20]

$$C\gamma^{\mu T}C^{-1} = \gamma^\mu, \quad C^T = C. \tag{2.1}$$

We can thus choose the representation of these matrices explicitly as

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = 1, \quad \gamma^5 \equiv i\gamma^1\gamma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{2.2}$$

where it is noted that γ^μ and γ^5 are symmetric and antisymmetric matrices, respectively.

The general extended superalgebra without central charge is given by

$$\begin{aligned}
\{Q_{\alpha i}, Q_{\beta j}\} =& 2\delta_{ij}\gamma^{\mu}_{\alpha\beta}P_{\mu}\,, \\
[Q_{\alpha i}, R_1] =& iS_{ij}Q_{\alpha j}\,, \\
[Q_{\alpha i}, R_2] =& S'_{ij}\gamma^{5}_{\alpha\beta}Q_{\beta j}\,, \\
[Q_{\alpha i}, P_{\mu}] = [R_i, P_{\mu}] =& [P_{\mu}, P_{\nu}] = [R_i, R_j] = 0\,,
\end{aligned} \tag{2.3}$$

where $Q_{\alpha i}$ is the supercharge and R_1, R_2 are the generators of the R-symmetry. The Majorana condition is given by $Q_{\alpha i} = Q_{\alpha i}{}^{*}$. Here we concentrate on the $N = 2$ case. The Jacobi identities w.r.t. Q, Q, R_1 and Q, Q, R_2 lead, respectively, to $S_{ij} = -S_{ji}$ and $S'_{ij} = -S'_{ji}$, which means R_1 and R_2 generate $U(1)$ symmetry.

We now introduce possible extra terms as follows:

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\delta_{ij}\gamma^{\mu}_{\alpha\beta}P_{\mu} + 2\delta_{\alpha\beta}U_{ij} + 2\gamma^{5}_{\alpha\beta}V_{ij}\,, \tag{2.4}$$

where $U_{ij} = U_{ji}$, $V_{ij} = -V_{ji}$ to be consistent with simultaneous replacements of $\alpha \leftrightarrow \beta$ and $i \leftrightarrow j$. Note that the right hand side in (2.12) is a form of expansion by the complete set of γ-matrices, and this is thus the most general form. U_{ij} and V_{ij} get the following restrictions according to the Jacobi identity w.r.t. Q, Q, R_1:

$$U_{ik}S_{kj} - S_{ik}U_{kj} = 0\,, \quad V_{ik}S_{kj} + S_{ik}V_{kj} = 0\,. \tag{2.5}$$

We can then solve the constraints up to an overall constant as

$$U_{ij} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_{ij} \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.6}$$

On the other hand, the Jacobi identity w.r.t. Q, Q, R_2 leads to the relation:

$$U_{ik}S'_{kj} + S'_{ik}U_{kj} = 0\,, \quad V_{ik}S'_{kj} - S'_{ik}V_{kj} = 0\,, \tag{2.7}$$

which can be solved as

$$U_{ij} \sim \begin{pmatrix} u' & u \\ u & -u' \end{pmatrix}, \quad V_{ij} = 0\,, \tag{2.8}$$

where u, u' are real parameters. The solutions (2.6) and (2.8) are not compatible. In other words, we cannot keep both of the R_1 and R_2 symmetries in the cases including central charges. But in either case, it is concluded that the $D = N = 2$ theory has two central charges.

In the case where we choose the $R_1(\equiv R)$ symmetry, we obtain the following algebra:

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\delta_{ij}\gamma^{\mu}_{\alpha\beta}P_{\mu} + 2\delta_{\alpha\beta}\delta_{ij}U_0 + 2\gamma^{5}_{\alpha\beta}\gamma^{5}_{ij}V_5\,, \tag{2.9}$$

$$[Q_{\alpha i}, R] = iS_{ij}Q_{\alpha j}\,, \tag{2.10}$$

$$[U_0, \text{any}] = [V_5, \text{any}] = 0\,, \tag{2.11}$$

where U_0 and V_0 are the central charges. On the other hand, if we choose the R_2 symmetry, we cannot carry out the Dirac-Kähler twisting procedure. We thus not choose this case.

2.1.2 Twisted superalgebra

One can state that, in a simple term, the Dirac-Kähler twisting procedure includes two steps: expansion of the supercharge by the complete set of γ-matrices and redefinition of the Lorentz rotation generator [11, 8, 19].

We identify the representations of R-symmetry as that of spinor, and treat the extended SUSY suffix and spinor suffix of supercharge $Q_{\alpha i}$ in the same manner. We can thus expand the charge as

$$Q_{\alpha i} = (1s + \gamma^\mu s_\mu - i\gamma^5 \tilde{s})_{\alpha i}\,, \tag{2.12}$$

where s, s_μ, and $\tilde{s}$ are called twisted supercharges. Note that these supercharges can be expressed by the original charge $Q_{\alpha i}$ as

$$s = \frac{1}{2}\mathrm{tr}\, Q\,, \quad s_\mu = \frac{1}{2}\mathrm{tr}\, \gamma^\mu Q\,, \quad \tilde{s} = -\frac{1}{2}\mathrm{tr}\, \gamma^5 Q\,, \tag{2.13}$$

where the traces are calculated on the components of $Q_{\alpha i}$ and the γ-matrices. The charges may look strange because s_μ has, for example, a vector suffix even though it is a fermionic charge. This feature of the Dirac-Kähler mechanism can be understood in the next step as following.

In two dimensions, the Lorentz generator is represented by one component generator J satisfying

$$[P_\mu, J] = -i\epsilon_{\mu\nu} P_\nu\,, \tag{2.14}$$

$$[Q_{\alpha i}, J] = -\frac{1}{2}\gamma^5_{\alpha\beta} Q_{\beta i}\,. \tag{2.15}$$

On the other hand, we can rewrite (2.10) in the same form as (2.15) because S_{ij} and γ^5_{ij} are both antisymmetric and thus can be chosen to be proportional to each other,

$$[Q_{\alpha i}, R] = -\frac{1}{2}\gamma^5_{ij} Q_{\alpha j}\,. \tag{2.16}$$

Thus we can define $J' = J + R$ and obtain

$$[s, J'] = 0\,, \quad [s_\mu, J'] = -i\epsilon_{\mu\nu} s_\nu\,, \quad [\tilde{s}, J'] = 0\,, \tag{2.17}$$

from (2.13), (2.15), and (2.16). These relations mean that twisted supercharges s, s_μ, and $\tilde{s}$ transform as scalar, vector, and (pseudo-)scalar under J', respectively.

As can be seen above, the equivalence between the Lorentz group and the R-symmetry group is required to realize the Dirac-Kähler twist. The R-symmetry group is inevitably a compact group and thus the Lorentz group also needs to be compact. There is a natural reason that Euclidean spacetime is chosen. This is the Dirac-Kähler twisting procedure. By performing the procedure, such a fermionic ingredient which is scaler or vector in the twisted space appear. Similarly, a scaler field in original untwisted spacetime can have a vector suffix in the twisted spacetime as can be seen later.

The algebra among the twisted supercharges is derived from (2.9) and (2.13) as follows:

$$\begin{gathered}\{s, s_\mu\} = P_\mu\,, \quad \{\tilde{s}, s_\mu\} = -\epsilon_{\mu\nu} P_\nu\,, \quad \{s, \tilde{s}\} = 0\,, \\ s^2 = \tilde{s}^2 = \frac{1}{2}(U_0 - V_5)\,, \quad \{s_\mu, s_\nu\} = \delta_{\mu\nu}(U_0 + V_5)\,. \end{gathered} \tag{2.18}$$

This is the $N = 2$ twisted superalgebra with central charges in two dimensions.

In case where the central charges are not included, s and $\tilde{s}$ become nilpotent and covariant supercharges. This feature is useful to consider lattice SUSY theory as explained in Section 1.

2.2 Ansatz on supercurvature

In this subsection, we give a explanation of the basic mechanism of so-called superconnection formalism and find appropriate ansatz on supercurvatures based on the algebra derived in the previous subsection [8, 19]. The concrete forms of the ingredients are considered in the next subsection.

We introduce superfields in the superspace parametrized by (x_μ, θ_A, z)

$$\Phi(x_\mu, \theta_A, z) = \phi(x_\mu, z) + \theta_A \phi_A(x_\mu, z) + \frac{1}{2}\theta_A\theta_B\phi_{AB}(x_\mu, z) + \cdots , \tag{2.19}$$

where θ_A represents θ, θ_μ, and $\tilde{\theta}$ which are Grassmann coordinates, and z is a real parameter associated with a central charge. Using the supercharge differential operator $\mathcal{Q}_A$ generating a parameter shift in the superspace, we define the supertransformations of the component fields ϕ, ϕ_A, ... as follows:

$$\delta_\xi\Phi(x, \theta_A, z) = \delta_\xi\phi(x, z) + \theta_A\delta_\xi\phi_A(x, z) + \frac{1}{2}\theta_A\theta_B\delta_\xi\phi_{AB}(x, z) + \cdots \equiv \xi_B\mathcal{Q}_B\Phi(x, \theta_A, z) , \tag{2.20}$$

where ξ_A is a Grassmann parameter. It is well known that the resultant supermultiplet from the superfield above is reducible. It is thus necessary to reduce the degrees of freedom with gauge covariance for obtaining an irreducible supermultiplet of SYM theory. To implement that, one can first find the supercovariant derivative $\mathcal{D}_A$ which anticommutes with $\mathcal{Q}_A$, i.e. if Φ is a superfield, $\mathcal{D}_A\Phi$ is also a superfield. And then we introduce the (fermionic) gauge-supercovariant derivative as

$$\nabla_A = \mathcal{D}_A - i\Gamma_A , \tag{2.21}$$

where Γ_A is a fermionic superfield called a superconnection.

The bosonic gauge-supercovariant derivatives are also introduced as

$$\nabla_{\underline{\mu}} = \mathcal{D}_{\underline{\mu}} - i\Gamma_{\underline{\mu}} , \quad \nabla_z = \mathcal{D}_z - i\Gamma_z , \tag{2.22}$$

where $\mathcal{D}_{\underline{\mu}} \equiv \partial_\mu$, $\mathcal{D}_z \equiv \partial_z$, and $\Gamma_{\underline{\mu}}$ and Γ_z are bosonic superfields.

The gauge transformation of $\nabla_I \equiv \{\nabla_{\underline{\mu}}, \nabla_A, \nabla_z\}$ is defined as

$$\nabla_I' = e^K\nabla_I e^{-K} , \quad \text{or} \quad \delta_K\nabla_I = [\nabla_I, K] , \tag{2.23}$$

where K is an gauge parameter superfield. If Φ is a gauge covariant superfield, $[\nabla_A, \Phi] \equiv \nabla_A\Phi$ is also gauge covariant clearly. Furthermore, $\nabla_A\Phi$ is also a superfield:

$$\begin{aligned}
\delta_\xi(\nabla_I\Phi) &= (\delta_\xi\nabla_I)\Phi + \nabla_I(\delta_\xi\Phi) = (-i\delta_\xi\Gamma_I)\Phi + \nabla_I(\delta_\xi\Phi) \\
&= -i(\xi_A\mathcal{Q}_A\Gamma_I)\Phi + (\mathcal{D}_I - i\Gamma_I)(\xi_A\mathcal{Q}_A\Phi) \\
&= -i(\xi_A\mathcal{Q}_A\Gamma_I)\Phi + \xi_A\mathcal{Q}_A\mathcal{D}_I\Phi - i\Gamma_I(\xi_A\mathcal{Q}_A\Phi) \\
&= \xi_A\mathcal{Q}_A\mathcal{D}_I\Phi - i\xi_A\mathcal{Q}_A[\Gamma_I, \Phi] \\
&= \xi_A\mathcal{Q}_A(\nabla_I\Phi) .
\end{aligned} \tag{2.24}$$

That is why ∇_I is called gauge-supercovariant derivative.

The zeroth-order terms of $\Gamma_{\underline{\mu}}$ w.r.t. θ_A is defined as an usual gauge connection

$$\Gamma_{\underline{\mu}}| = A_\mu , \tag{2.25}$$

where | represents the zeroth-order term w.r.t. θ_A. We can thus define the standard gauge-covariant derivative as

$$\nabla_{\underline{\mu}}| \equiv D_\mu = \partial_\mu - iA_\mu \,. \tag{2.26}$$

In contrast that, we can take the following conditions

$$\Gamma_A| = \Gamma_z| = 0 \,. \tag{2.27}$$

These conditions are derived by implementing the following gauge transformation with

$$K = k(x,z) + \theta_A i(\Gamma_A - [k(x,z), \Gamma_A]) + K'(x,z,\theta_A) \,, \tag{2.28}$$

where a summation w.r.t. A is understood, K' contains no linear and zeroth order terms w.r.t. θ or $\tilde{\theta}$, and the lowest parameter field k is determined by the following differential equation:

$$\partial_z k(x,z) = i\Gamma_z| + i[\Gamma_z|, k(x,z)] \,. \tag{2.29}$$

This equation surely has a solution due to a form of first order equation. In fact,

$$\begin{aligned}
\Gamma'_A| &= \Gamma_A| + \delta_K \Gamma_A| \\
&= \Gamma_A| + i[\nabla_A, K]| \\
&= \Gamma_A| + i[\mathcal{D}_A - i\Gamma_A, K]| \\
&= \Gamma_A| - (\Gamma_A - [k(x,z), \Gamma_A])| + [\Gamma_A, K]| \\
&= 0 \,,
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
\Gamma'_z| &= \Gamma_z| + \delta_K \Gamma_z| \\
&= \Gamma_z| + i[\nabla_z, K]| \\
&= \Gamma_z| + i[\partial_z - i\Gamma_z, K]| \\
&= \Gamma_z| + i\partial_z k(x,z) + [\Gamma_z|, k(x,z)] \\
&= 0 \,,
\end{aligned} \tag{2.31}$$

where the last equality is from (2.29). The special gauge is chosen here and called Wess-Zumino gauge.

We can then define the supercurvatures by (anti-)commutation relations of all pairs of ∇_I. The supercurvatures transform gauge-covariantly under (2.23) as

$$\begin{aligned}
[\nabla_I, \nabla_J]_\pm &\equiv -iF_{IJ} \,, \\
\delta_K F_{IJ} &= i[\delta_K \nabla_I, \nabla_J]_\pm + i[\nabla_I, \delta_K \nabla_J]_\pm \\
&= i[[\nabla_I, K], \nabla_J]_\pm + i[\nabla_I, [\nabla_J, K]]_\pm \\
&= i[[\nabla_I, \nabla_J]_\pm, K] \\
&= [F_{IJ}, K] \,,
\end{aligned} \tag{2.32}$$

where [,]$_\pm$ represents commutation and anticommutation relation in case of $-$ sign and $+$ sign,

	∇	$\tilde{\nabla}$	∇_ν	$\nabla_{\underline{\nu}}$	∇_z
∇	$X - X'$	0	$-i(\nabla_{\underline{\nu}} + iX_\nu)$	$-iF_{\underline{\nu}}$	iG
$\tilde{\nabla}$		$X - X'$	$i\epsilon_{\nu\rho}(\nabla_{\underline{\rho}} - iX_\rho)$	$-i\tilde{F}_{\underline{\nu}}$	$i\tilde{G}$
∇_μ			$\delta_{\mu\nu}(X + X')$	$-iF_{\mu\underline{\nu}}$	iG_μ
$\nabla_{\underline{\mu}}$				$-iF_{\underline{\mu\nu}}$	$iG_{\underline{\mu}}$
∇_z					0

Table 1: Twisted version of supercurvature ansatz of (2.33). For example, $\{\nabla, \nabla\} = X - X'$, $[\nabla, \nabla_{\underline{\mu}}] = -iF_{\underline{\mu}}$. The positions of $\nabla_{\underline{\mu}}$ reflect those of P_μ in (2.18).

	∇	$\tilde{\nabla}$	∇_ν	$\nabla_{\underline{\nu}}$	∇_z
∇	$-iW + \nabla_z$	0	$-i\nabla_{\underline{\nu}}$	$-iF_{\underline{\nu}}$	iG
$\tilde{\nabla}$		$-iW + \nabla_z$	$i\epsilon_{\nu\rho}\nabla_{\underline{\rho}}$	$-i\tilde{F}_{\underline{\nu}}$	$i\tilde{G}$
∇_μ			$\pm\delta_{\mu\nu}(iW + \nabla_z)$	$-iF_{\mu\underline{\nu}}$	iG_μ
$\nabla_{\underline{\mu}}$				$-iF_{\underline{\mu\nu}}$	$iG_{\underline{\mu}}$
∇_z					0

Table 2: Supercurvature ansatz of the A model. Corresponding to the $\pm$ sign choice of algebra $\{\nabla_\mu, \nabla_\nu\} = \pm\delta_{\mu\nu}(iW + \nabla_z)$, we take $X = \nabla_z$ for $+$ and $X' = -\nabla_z$ for $-$.

respectively. Furthermore, the supercurvature is a superfield (or supercovariant) as

$$\begin{aligned}
\delta_\xi F_{IJ} &= i[\delta_\xi \nabla_I, \nabla_J]_\pm + i[\nabla_I, \delta_\xi \nabla_J]_\pm \\
&= [\xi_A \mathcal{Q}_A \Gamma_I, \nabla_J]_\pm + [\nabla_I, \xi_A \mathcal{Q}_A \Gamma_J]_\pm \\
&= [\xi_A \mathcal{Q}_A \Gamma_I, \mathcal{D}_J]_\pm + [\mathcal{D}_I, \xi_A \mathcal{Q}_A \Gamma_J]_\pm - i\xi_A \mathcal{Q}_A [\Gamma_I, \Gamma_J]_\pm \\
&= \xi_A \mathcal{Q}_A [\Gamma_I, \mathcal{D}_J]_\pm + \xi_A \mathcal{Q}_A [\mathcal{D}_I, \Gamma_J]_\pm - i\xi_A \mathcal{Q}_A [\Gamma_I, \Gamma_J]_\pm \\
&= \xi_A \mathcal{Q}_A [\Gamma_I, \nabla_J]_\pm + i\xi_A \mathcal{Q}_A [\mathcal{D}_I, \mathcal{D}_J - i\Gamma_J]_\pm \\
&= \xi_A \mathcal{Q}_A i[\nabla_I, \nabla_J]_\pm \\
&= \xi_A \mathcal{Q}_A F_{IJ}\,.
\end{aligned}$$

Thus one can obtain gauge covariant superfield as the supercurvature.

Once appropriate ansatz are taken on the supercurvature, one can obtain suitable constraints on the superfield and can obtain irreducible supermultiplet . To find such ansatz, it is useful to introduce supercurvatures X, X', and X_μ defined as

$$\{\nabla_{\alpha i}, \nabla_{\beta j}\} = -2i\delta_{ij}\gamma^\mu_{\alpha\beta}\nabla_{\underline{\mu}} + 2\delta_{\alpha\beta}\delta_{ij}X + 2\gamma^5_{\alpha\beta}\gamma^5_{ij}X' + 2\delta_{\alpha\beta}\gamma^\mu_{ij}X_\mu\,, \qquad (2.33)$$

where $\nabla_{\alpha i}$ are gauge-supercovariant derivative corresponding to $Q_{\alpha i}$ in (2.9). One can restrict the form of supercurvature by finding the possible form before the twisting procedure is performed. The right hand side in (2.33) are the most general terms for consistency with simultaneous replacement of $\alpha \leftrightarrow \beta$ and $i \leftrightarrow j$. As can be seen from (2.9), X and X' can be identified as including gauged central charge of U_0 and V_5, respectively, i.e. can be identified as including ∇_z . Table 1 shows the relations between the gauge-supercovariant derivative and supercurvatures given in (2.33) in the twisted space. It is, in principle, possible to find different types of supercurvature ansatz.

We eventually find three types of ansatz. The first ansatz is shown in Table 2. Here, one of X and X' is identified as ∇_z. We name it an A model ansatz when the bosonic scalar

	∇	$\tilde{\nabla}$	∇_ν	$\nabla_{\underline{\nu}}$	∇_z
∇	∇_z	0	$-i(\nabla_{\underline{\nu}} + F_\nu)$	$-iF_{\underline{\nu}}$	iG
$\tilde{\nabla}$		∇_z	$i\epsilon_{\nu\rho}(\nabla_{\underline{\rho}} - F_\rho)$	$-i\tilde{F}_{\underline{\nu}}$	$i\tilde{G}$
∇_μ			$\pm\delta_{\mu\nu}\nabla_z$	$-iF_{\mu\underline{\nu}}$	iG_μ
$\nabla_{\underline{\mu}}$				$-iF_{\underline{\mu}\underline{\nu}}$	$iG_{\underline{\mu}}$
∇_z					0

Table 3: Naive candidate for B model ansatz.

	∇	$\tilde{\nabla}$	∇_ν	$\nabla_{\underline{\nu}}$	∇_z
∇	0	0	$-i(\nabla_{\underline{\nu}} + F_\nu)$	$-iF_{\underline{\nu}}$	iG
$\tilde{\nabla}$		0	$i\epsilon_{\nu\rho}(\nabla_{\underline{\rho}} - F_\rho)$	$-i\tilde{F}_{\underline{\nu}}$	$i\tilde{G}$
∇_μ			$\delta_{\mu\nu}\nabla_z$	$-iF_{\mu\underline{\nu}}$	iG_μ
$\nabla_{\underline{\mu}}$				$-iF_{\underline{\mu}\underline{\nu}}$	$iG_{\underline{\mu}}$
∇_z					0

Table 4: Supercurvature ansatz of the B (0,0,Z) model.

	∇	$\tilde{\nabla}$	∇_ν	$\nabla_{\underline{\nu}}$	∇_z
∇	∇_z	0	$-i(\nabla_{\underline{\nu}} + F_\nu)$	$-iF_{\underline{\nu}}$	iG
$\tilde{\nabla}$		∇_z	$i\epsilon_{\nu\rho}(\nabla_{\underline{\rho}} - F_\rho)$	$-i\tilde{F}_{\underline{\nu}}$	$i\tilde{G}$
∇_μ			0	$-iF_{\mu\underline{\nu}}$	iG_μ
$\nabla_{\underline{\mu}}$				$-iF_{\underline{\mu}\underline{\nu}}$	$iG_{\underline{\mu}}$
∇_z					0

Table 5: Supercurvature ansatz of the B (Z,Z,0) model.

supercurvature (W in the case) is placed in diagonal positions. It is also possible to include X_μ as supercurvatures. In this case, the bosonic vector supercurvatures are placed in off-diagonal positions, which we call a B model ansatz. One naive candidate for a B model ansatz is given in Table 3. It is shown that, however, the naive ansatz for a B model does not work as can be seen later. On the other hand, it is possible to formulate two kinds of B model ansatz for one central charge by imposing $U_0 = \pm V_5$. In Table 4 and Table 5, we show the two kinds of ansatz which we name the B (0,0,Z) model ansatz and B (Z,Z,0) model ansatz.

Once suitable ansatz on the supercurvatures are taken, a set of relations between the supercurvatures can be derived from the Jacobi identities w.r.t. ∇_I, by which the degrees of freedom of the component fields are reduced. We use the notation $\nabla W \equiv [\nabla, W]$. For example, the component fields in an irreducible supermultiplet can be defined as

$$W| = A\,, \quad \nabla W| = \rho\,, \quad \ldots\,, \quad \text{if} \quad \nabla W \neq 0\,, \tag{2.34}$$

where W represents a supercurvature while A and ρ are the bosonic and fermionic component fields, respectively. One of the remarkable features of superconnection formalism is that the supertransformations and central charge transformations of component fields are obtained by

$$sA = s(W|) \equiv \mathcal{Q}W| = \mathcal{D}W| = \mathcal{D}W| - i[\Gamma, W]| = \nabla W| = \rho\,. \tag{2.35}$$

The third equality holds at the zeroth-order of θ_A while the fourth equality holds due to the first relation of (2.27). More-complicated supertransformations can be defined by more sophisticated Jacobi identities. One thus obtains all supertransformations of each component field in an irreducible supermultiplet.

Finally, we give the reason why naive ansatz for a B model in Table 3 does not work. The Jacobi identity w.r.t. ∇, ∇, ∇ leads to $G = 0$. Similarly the Jacobi identities w.r.t. $\tilde{\nabla}$, $\tilde{\nabla}$, $\tilde{\nabla}$ and w.r.t. ∇_μ, ∇_ν, ∇_ρ lead to $\tilde{G} = 0$ and $G_\mu = 0$, respectively. Using these relations, the Jacobi identities w.r.t. ∇, ∇_μ, ∇_z and w.r.t. $\tilde{\nabla}$, ∇_μ, ∇_z lead to $\nabla_z F_\mu = iG_{\underline{\mu}}$ and $\nabla_z F_\mu = -iG_{\underline{\mu}}$, respectively. Eventually the Jacobi identities based on the ansatz lead to

$$G_A = G_{\underline{\mu}} = \nabla_z F_\mu = 0\,. \tag{2.36}$$

On the other hand, the component fields of the model can be written in general by

$$\phi = \nabla_I \nabla_J \cdots F_\mu | \,, \tag{2.37}$$

where ϕ represents a component field symbolically. The central charge transformation of ϕ is then calculated with (2.36) as

$$Z\phi = \nabla_z \nabla_I \nabla_J \cdots F_\mu | = \nabla_I \nabla_z \nabla_J \cdots F_\mu | = \cdots = \nabla_I \nabla_J \cdots \nabla_z F_\mu | = 0 \,. \tag{2.38}$$

By the facts, it is resulted that the central charge transformations of all component fields are zero and the ansatz formulates a model without central charge, i.e. not a model which we require in this paper.

2.3 Supermultiplets and actions

In the previous subsection, we have given the basis of the superconnection formalism and some appropriate ansatz on the supercurvatures. We formulate models i.e. derive the supermultiplets and the actions with each ansatz in this subsection. The A model, B (0,0,Z) model and B (Z,Z,0) model are considered in section 2.3.1, 2.3.2, and 2.3.3, respectively. Note that $N = 2$ supersymmetric theory in two-dimensional Euclidean spacetime generally has four bosonic and four fermionic degrees of freedom at the off-shell level.

2.3.1 The A model

We now consider the following algebra:

$$\begin{aligned} &\{s, s_\mu\} = P_\mu \,, \quad \{\tilde{s}, s_\mu\} = -\epsilon_{\mu\nu} P_\nu \,, \quad \{s, \tilde{s}\} = 0 \,, \\ &s^2 = \tilde{s}^2 = \frac{1}{2} Z \,, \quad \{s_\mu, s_\nu\} = \pm \delta_{\mu\nu} Z \,, \quad [s_I, Z] = 0 \,. \end{aligned} \tag{2.39}$$

where $+$ represents the case $Z = U_0$ and $-$ represents the case $Z = -V_5$ of (2.18) in the double sign.

The corresponding supercharge differential operators for the superspace are given by

$$\begin{aligned} \mathcal{Q} &= \frac{\partial}{\partial \theta} + \frac{i}{2} \theta_\mu \partial_\mu + \frac{i}{2} \theta \partial_z \,, \\ \mathcal{Q}_\mu &= \frac{\partial}{\partial \theta^\mu} + \frac{i}{2} \theta \partial_\mu - \frac{i}{2} \tilde{\theta} \epsilon_{\mu\nu} \partial_\nu \pm \frac{i}{2} \theta_\mu \partial_z \,, \\ \tilde{\mathcal{Q}} &= \frac{\partial}{\partial \tilde{\theta}} - \frac{i}{2} \theta_\mu \epsilon_{\mu\nu} \partial_\nu + \frac{i}{2} \tilde{\theta} \partial_z \,, \end{aligned} \tag{2.40}$$

where P_μ and Z are represented by $-i\partial_\mu$ and $-i\partial_z$, respectively. The supercovariant derivatives are then found as

$$\begin{aligned} \mathcal{D} &= \frac{\partial}{\partial \theta} - \frac{i}{2} \theta_\mu \partial_\mu - \frac{i}{2} \theta \partial_z \,, \\ \mathcal{D}_\mu &= \frac{\partial}{\partial \theta^\mu} - \frac{i}{2} \theta \partial_\mu + \frac{i}{2} \tilde{\theta} \epsilon_{\mu\nu} \partial_\nu \mp \frac{i}{2} \theta_\mu \partial_z \,, \\ \tilde{\mathcal{D}} &= \frac{\partial}{\partial \tilde{\theta}} + \frac{i}{2} \theta_\mu \epsilon_{\mu\nu} \partial_\nu - \frac{i}{2} \tilde{\theta} \partial_z \,, \end{aligned} \tag{2.41}$$

	s	s_μ	$\tilde{s}$	Z
ϕ	ρ	λ_μ	$\tilde{\rho}$	D
A_ν	$-i\lambda_\nu$	$\pm i\delta_{\mu\nu}\rho \mp i\epsilon_{\mu\nu}\tilde{\rho}$	$-i\epsilon_{\nu\rho}\lambda_\rho$	g_ν
λ_ν	$\frac{i}{2}(g_\nu - D_\nu\phi)$	$\pm\frac{1}{2}\delta_{\mu\nu}D + \frac{1}{2}F_{\mu\nu}$	$-\frac{i}{2}\partial_{\nu\rho}(g_\rho - D_\rho\phi)$	$-iD_\nu\rho + i\epsilon_{\nu\rho}D_\rho\tilde{\rho} - i[\phi, \lambda_\nu]$
ρ	$\frac{1}{2}D$	$-\frac{i}{2}(g_\mu + D_\mu\phi)$	$\mp\frac{1}{4}\epsilon_{\mu\nu}F_{\mu\nu}$	$\mp iD_\mu\lambda_\mu + i[\phi, \rho]$
$\tilde{\rho}$	$\pm\frac{1}{4}\epsilon_{\mu\nu}F_{\mu\nu}$	$\frac{i}{2}\epsilon_{\mu\nu}(g_\nu + D_\nu\phi)$	$\frac{1}{2}D$	$\mp i\epsilon_{\mu\nu}D_\mu\lambda_\nu + i[\phi, \tilde{\rho}]$
D	$\mp iD_\mu\lambda_\mu$	$i\epsilon_{\mu\nu}D_\nu\tilde{\rho} - iD_\mu\rho$	$\mp i\epsilon_{\mu\nu}D_\mu\lambda_\nu$	$\pm D_\mu g_\mu \mp D_\mu D_\mu\phi$ $\pm 2i\{\lambda_\mu, \lambda_\mu\} + i[\phi, D]$
g_ν	$\epsilon_{\nu\rho}D_\rho\tilde{\rho} - [\phi, \lambda_\nu]$	$\epsilon_{\mu\sigma}\epsilon_{\nu\rho}D_\rho\lambda_\sigma$ $\mp\delta_{\mu\nu}[\phi, \rho] \pm \epsilon_{\mu\nu}[\phi, \tilde{\rho}]$	$-\epsilon_{\nu\rho}(D_\rho\rho + [\phi, \lambda_\rho])$	$\pm D_\rho F_{\nu\rho} - 2\epsilon_{\nu\rho}\{\lambda_\rho, \tilde{\rho}\}$ $-2\{\lambda_\nu, \rho\} + i[\phi, D_\nu\phi]$

Table 6: Supertransformations of the A model

where $\{\mathcal{Q}_A, \mathcal{D}_B\} = 0$. It should be noted that $\mathcal{D}_A$ satisfies the same algebraic relations as (2.39) with the identification of $\mathcal{D}_A \to s_A$, while $\mathcal{Q}_A$ satisfies the similar relations with the replacements, $\mathcal{Q}_A \to s_A$, $P_\mu \to -P_\mu$, and $Z \to -Z$ in (2.39).

We now consider the supercurvature ansatz in Table 2. The following relations can be derived by the Jacobi identities:

$$\begin{gathered} \nabla_\mu \nabla W = \epsilon_{\mu\nu}\nabla_\nu\tilde{\nabla}W\,, \\ F_{\underline{\mu}} = -i\nabla_\mu W\,, \quad \tilde{F}_{\underline{\mu}} = -i\epsilon_{\mu\nu}\nabla_\nu W\,, \\ F_{\mu\underline{\nu}} = \pm i\delta_{\mu\nu}\nabla W \mp i\epsilon_{\mu\nu}\tilde{\nabla}W\,, \quad F_{\underline{\mu\nu}} = \pm\epsilon_{\mu\nu}\tilde{\nabla}\nabla W + \frac{1}{2}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\nabla_\rho\nabla_\sigma W\,, \\ G = \nabla W\,, \quad \tilde{G} = \tilde{\nabla}W\,, \quad G_\mu = -\nabla_\mu W\,, \quad G_{\underline{\mu}} = 2i\nabla_\mu\nabla W - \nabla_{\underline{\mu}}W\,. \end{gathered} \tag{2.42}$$

In addition to these, we need to impose the following relation:

$$\nabla\tilde{\nabla}W = \frac{1}{2}\epsilon_{\mu\nu}\nabla_\mu\nabla_\nu W\,. \tag{2.43}$$

This relation is not derived by Jacobi identities but can be imposed consistently. The relation is interpreted as another constraint on supercurvatures to kill the reducibility of the representation.

The component fields are then defined as

$$W| = \phi\,, \quad \nabla W| = \rho\,, \quad \tilde{\nabla}W| = \tilde{\rho}\,, \quad \nabla_\mu W| = \lambda_\mu\,, \quad \nabla_z W| = D\,, \quad G_{\underline{\mu}}| = g_\mu\,, \tag{2.44}$$

where ϕ, D, and g_μ are bosonic fields, and ρ, $\tilde{\rho}$, and λ_μ are fermionic fields. Table 6 shows the supertransformations of each component field. Note that $F_{\underline{\mu\nu}}| \equiv F_{\mu\nu} = i[D_\mu, D_\nu]$ is a curvature in the usual gauge theory.

Off-shell closure of the supertransformations up to gauge transformations is shown with the following constraint on the component fields:

$$iD_\mu g_\mu \mp [\phi, D] - \{\lambda_\mu, \lambda_\mu\} \mp \{\rho, \rho\} \mp \{\tilde{\rho}, \tilde{\rho}\} = 0\,. \tag{2.45}$$

This constraint relation is not derived Jacobi identities. The concrete calculation for the check of off-shell closeness of the superalgebra. One can regard this constraint as the same type of constraint found for $D = N = 4$ SYM case in Ref [5].

Because of the constraint, the degrees of freedom of g_μ can be regarded as one. The bosonic degrees of freedom at the off-shell level is thus four (ϕ, A_μ, D, g_μ). Note that the gauge field A_μ has one bosonic degree of freedom at the off-shell level.

For an Abelian gauge group the constraint (2.45) becomes simply

$$\partial_\mu g_\mu = 0\,, \tag{2.46}$$

which can be solved as

$$g_\mu = \epsilon_{\mu\nu}\partial_\nu B\,. \tag{2.47}$$

The degrees of freedom is, in fact, one. The explicit form of action which includes field B is

$$\begin{aligned} S = \int d^2x \mathrm{Tr}\Big(&\pm \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}D^2 \pm \frac{1}{2}(\partial_\mu B)^2 \mp 2i\lambda_\mu(\partial_\mu\rho - \epsilon_{\mu\nu}\partial_\nu\tilde{\rho}) \\ &+ e(\frac{1}{2}\phi\epsilon_{\mu\nu}F_{\mu\nu} + 2\rho\tilde{\rho} + BD + \epsilon_{\mu\nu}\lambda_\mu\lambda_\nu)\Big)\,, \end{aligned} \tag{2.48}$$

where e is a parameter with mass dimension 1. In this case, invariance of the action and closure of the superalgebra are satisfied without constraints. For a non-Abelian gauge group the constraint cannot be solved locally [6].

Finally, one can find an action for a non-Abelian gauge group as

$$\begin{aligned} S = \int d^2x \mathrm{Tr}\Big(&\pm \frac{1}{2}(D_\mu\phi)^2 - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}D^2 \pm \frac{1}{2}g_\mu^2 \mp 2i\lambda_\mu(D_\mu\rho - \epsilon_{\mu\nu}D_\nu\tilde{\rho}) \\ &- i\phi\{\rho,\rho\} - i\phi\{\tilde{\rho},\tilde{\rho}\} \pm i\phi\{\lambda_\mu,\lambda_\mu\}\Big)\,. \end{aligned} \tag{2.49}$$

It is worth mentioning that this action cannot be derived by superspace.

2.3.2 The B (0,0,Z) model

The following algebra is considered:

$$\begin{aligned} &\{s, s_\mu\} = P_\mu\,, \quad \{\tilde{s}, s_\mu\} = -\epsilon_{\mu\nu}P_\nu\,, \quad \{s,\tilde{s}\} = 0\,, \\ &s^2 = \tilde{s}^2 = 0\,, \quad \{s_\mu, s_\nu\} = \delta_{\mu\nu}Z\,, \quad [s_I, Z] = 0\,, \end{aligned} \tag{2.50}$$

where $Z = 2U_0 = 2V_5$ in (2.18). The corresponding supercharge differential operators are given by

$$\begin{aligned} \mathcal{Q} &= \frac{\partial}{\partial\theta} + \frac{i}{2}\theta_\mu\partial_\mu\,, \\ \mathcal{Q}_\mu &= \frac{\partial}{\partial\theta^\mu} + \frac{i}{2}\theta\partial_\mu - \frac{i}{2}\tilde{\theta}\epsilon_{\mu\nu}\partial_\nu + \frac{i}{2}\theta_\mu\partial_z\,, \\ \tilde{\mathcal{Q}} &= \frac{\partial}{\partial\tilde{\theta}} - \frac{i}{2}\theta_\mu\epsilon_{\mu\nu}\partial_\nu\,. \end{aligned} \tag{2.51}$$

And the supercovariant derivatives are found as

$$\begin{aligned} \mathcal{D} &= \frac{\partial}{\partial\theta} - \frac{i}{2}\theta_\mu\partial_\mu\,, \\ \mathcal{D}_\mu &= \frac{\partial}{\partial\theta^\mu} - \frac{i}{2}\theta\partial_\mu + \frac{i}{2}\tilde{\theta}\epsilon_{\mu\nu}\partial_\nu - \frac{i}{2}\theta_\mu\partial_z\,, \\ \tilde{\mathcal{D}} &= \frac{\partial}{\partial\tilde{\theta}} + \frac{i}{2}\theta_\mu\epsilon_{\mu\nu}\partial_\nu\,. \end{aligned} \tag{2.52}$$

The B (0,0,Z) model ansatz is shown in Table 4. The following relations are derived by Jacobi identities:

$$\nabla F_\mu = \epsilon_{\mu\nu}\tilde{\nabla} F_\nu\,, \quad \nabla_\mu F_\nu + \nabla_\nu F_\mu = \delta_{\mu\nu}\nabla_\rho F_\rho\,, \quad G_\mu = 0\,, \tag{2.53}$$

$$F_{\underline{\mu}} = -i\nabla F_\mu\,, \quad \tilde{F}_{\underline{\mu}} = i\tilde{\nabla} F_\mu\,, \tag{2.54}$$

$$F_{\mu\underline{\nu}} = -\frac{i}{2}\delta_{\mu\nu}(\nabla_\rho F_\rho - G) + \frac{i}{2}\epsilon_{\mu\nu}(\epsilon_{\rho\sigma}\nabla_\rho F_\sigma - \tilde{G})\,, \tag{2.55}$$

$$F_{\underline{\mu}\underline{\nu}} = \nabla_\mu \nabla F_\nu - \nabla_\nu \nabla F_\mu + i[F_\mu, F_\nu] + \frac{1}{2}\epsilon_{\mu\nu}\nabla\tilde{G}\,, \tag{2.56}$$

$$\nabla G = \tilde{\nabla}\tilde{G} = \nabla\tilde{G} + \tilde{\nabla} G = 0\,, \tag{2.57}$$

$$\nabla_z F_\mu = \frac{1}{2}(\nabla_\mu G - \epsilon_{\mu\nu}\nabla_\nu \tilde{G})\,, \quad G_{\underline{\mu}} = \frac{i}{2}(\nabla_\mu G + \epsilon_{\mu\nu}\nabla_\nu\tilde{G})\,. \tag{2.58}$$

The component fields are defined as

$$F_\mu| = \phi_\mu\,, \quad \nabla F_\mu| = \lambda_\mu\,, \quad \nabla_\mu F_\nu| = \frac{1}{2}(\delta_{\mu\nu}\rho + \epsilon_{\mu\nu}\tilde{\rho})\,, \quad \nabla_\mu \nabla F_\mu| = D\,, \tag{2.59}$$

where ϕ_μ and D are bosonic fields and ρ, $\tilde{\rho}$, and λ_μ are fermionic fields.

The supertransformations of each component field are derived straightforwardly. In contrast with the previous model where G_A is related to W, G and $\tilde{G}$ should satisfy (2.57) and seem to be independent of F_μ as far as Jacobi identities are concerned. It is a remarkable feature that the superalgebra is closed up to gauge transformations without constraints at the off-shell level as far as (2.57) is satisfied.

To obtain the action, we introduce a linear combination of s_μ,

$$s_\pm \equiv s_1 \pm i s_2\,, \tag{2.60}$$

which satisfies

$$s_\pm^2 = 0\,, \quad \{s_+, s_-\} = 2Z\,. \tag{2.61}$$

We define $\lambda_\pm \equiv \lambda_1 \pm i\lambda_2$ similarly, and introduce the notation $\nabla^\pm_{\underline{\mu}} \equiv \nabla_{\underline{\mu}} \pm F_\mu$ and $D^\pm_\mu \equiv D_\mu \pm \phi_\mu$ for convenience. Then one can derive action by using the nilpotency of s, $\tilde{s}$ and $s_\pm$. In fact we can find S_1, S_2, and S_3 satisfying $sS_1 = \tilde{s}S_1 = s_+S_2 = s_-S_3 = 0$ where S_1, S_2, and S_3 are not generally identical. In facts it is shown that

$$S_1 = \frac{1}{2}\int d^2x \mathrm{Tr} s\tilde{s}\tilde{\rho}\rho = S_0 + \int d^2x \mathrm{Tr}\{-\frac{1}{4}\epsilon_{\mu\nu}(F_{\mu\nu} + i[\phi_\mu, \phi_\nu])\nabla\tilde{G}| + \frac{1}{8}(\nabla\tilde{G})^2\}\,, \tag{2.62}$$

$$\begin{aligned} S_2 &= \frac{1}{2}\int d^2x \mathrm{Tr} s_+ s_- \lambda_+ \lambda_- \\ &= S_0 + \int d^2x \mathrm{Tr}\{-\frac{1}{2}(\epsilon_{\mu\nu}D^-_\mu\lambda_\nu - i(-D^+_\mu\lambda_\mu))(G| + i\tilde{G}|) + \frac{i}{2}(\lambda_\mu\nabla_\mu\nabla\tilde{G}| + i\epsilon_{\mu\nu}\lambda_\mu\nabla_\nu\nabla\tilde{G}|) \\ &\qquad - \frac{1}{4}\epsilon_{\mu\nu}(F_{\mu\nu} - i[\phi_\mu, \phi_\nu])\nabla\tilde{G}| + \frac{1}{8}(\nabla\tilde{G}|)^2\}\,, \end{aligned} \tag{2.63}$$

$$\begin{aligned} S_3 &= \frac{1}{2}\int d^2x \mathrm{Tr} s_- s_+ \lambda_- \lambda_+ \\ &= S_0 + \int d^2x \mathrm{Tr}\{\frac{1}{2}(\epsilon_{\mu\nu}D^-_\mu\lambda_\nu + i(-D^+_\mu\lambda_\mu))(G| - i\tilde{G}|) - \frac{i}{2}(\lambda_\mu\nabla_\mu\nabla\tilde{G}| - i\epsilon_{\mu\nu}\lambda_\mu\nabla_\nu\nabla\tilde{G}|) \\ &\qquad - \frac{1}{4}\epsilon_{\mu\nu}(F_{\mu\nu} - i[\phi_\mu, \phi_\nu])\nabla\tilde{G}| + \frac{1}{8}(\nabla\tilde{G}|)^2\}\,, \end{aligned} \tag{2.64}$$

	s	s_μ	$\tilde{s}$	Z
ϕ_ν	λ_ν	$\frac{1}{2}(\delta_{\mu\nu}\rho + \epsilon_{\mu\nu}\tilde{\rho})$	$-\epsilon_{\nu\rho}\lambda_\rho$	$\frac{1}{2}(\nabla_\nu G\vert - \epsilon_{\nu\rho}\nabla_\rho \tilde{G}\vert)$
A_ν	$-i\lambda_\nu$	$-\frac{i}{2}\delta_{\mu\nu}\rho + \frac{i}{2}\epsilon_{\mu\nu}\tilde{\rho}$ $+\frac{i}{2}\delta_{\mu\nu}G\vert - \frac{i}{2}\epsilon_{\mu\nu}\tilde{G}\vert$	$-i\epsilon_{\nu\rho}\lambda_\rho$	$\frac{i}{2}(\nabla_\nu G\vert + \epsilon_{\nu\rho}\nabla_\rho \tilde{G}\vert)$
λ_ν	0	$A_{\mu\nu}$	0	$-\frac{i}{2}(D^-_\nu G\vert - \epsilon_{\nu\rho}D^+_\rho \tilde{G}\vert)$
ρ	$\frac{i}{2}[D^+_\rho, D^-_\rho] - D$	$\frac{1}{2}(\nabla_\mu G\vert - \epsilon_{\mu\nu}\nabla_\nu \tilde{G}\vert)$	$\frac{i}{2}\epsilon_{\rho\sigma}[D^-_\rho, D^-_\sigma]$	$\frac{1}{2}(\nabla_z G\vert - \epsilon_{\rho\sigma}\nabla_\rho\nabla_\sigma \tilde{G}\vert)$
$\tilde{\rho}$	$-\frac{i}{2}\epsilon_{\rho\sigma}[D^+_\rho, D^+_\sigma]$	$\frac{1}{2}(\nabla_\mu \tilde{G}\vert + \epsilon_{\mu\nu}\nabla_\nu G\vert)$	$-\frac{i}{2}[D^+_\rho, D^-_\rho] - D$	$\frac{1}{2}(\nabla_z \tilde{G}\vert + \epsilon_{\rho\sigma}\nabla_\rho\nabla_\sigma G\vert)$
D	$-iD^+_\rho\lambda_\rho$	$\frac{i}{2}(D^+_\mu\rho - \epsilon_{\mu\nu}D^-_\nu\tilde{\rho})$ $-\frac{i}{2}(D_\mu G\vert - \epsilon_{\mu\nu}D_\nu\tilde{G}\vert)$	$-i\epsilon_{\rho\sigma}D^-_\rho\lambda_\sigma$	$-\frac{i}{2}(D^-_\rho\nabla_\rho G\vert + \epsilon_{\rho\sigma}D^+_\rho\nabla_\sigma\tilde{G}\vert)$ $i\{\rho, G\vert\} + i\{\tilde{\rho}, \tilde{G}\vert\}$ $-\frac{i}{2}(\{G\vert, G\vert\} + \{\tilde{G}\vert, \tilde{G}\vert\})$

Table 7: Supertransformations of the B (0,0,Z) model. $A_{\mu\nu} = \frac{1}{2}\delta_{\mu\nu}D - \frac{i}{2}(D_\mu\phi_\nu + D_\nu\phi_\mu - \delta_{\mu\nu}D_\rho\phi_\rho) + \frac{1}{2}(F_{\mu\nu} - i[\phi_\mu, \phi_\nu])$.

where

$$S_0 = \int d^2x \mathrm{Tr}\{\frac{1}{2}(D_\mu\phi_\nu)^2 + \frac{1}{4}F^2_{\mu\nu} + \frac{1}{2}D^2 - i\rho D^+_\mu\lambda_\mu - i\tilde{\rho}\epsilon_{\mu\nu}D^-_\mu\lambda_\nu - \frac{1}{4}[\phi_\mu, \phi_\nu]^2\}\,, \qquad (2.65)$$

which corresponds to the action of the model without central charge and to the twisted versions of the action in [21]. However, in the case of $\nabla\tilde{G} = \tilde{\nabla}G = 0$ together with (2.57), we find

$$S_1 = S_0\,, \qquad (2.66)$$

$$S_2 = S_0 + \int d^2x \mathrm{Tr}\{-\frac{1}{2}(\epsilon_{\mu\nu}D^-_\mu\lambda_\nu - i(-D^+_\mu\lambda_\mu))(G| + i\tilde{G}|)\}\,, \qquad (2.67)$$

$$S_3 = S_0 + \int d^2x \mathrm{Tr}\{\frac{1}{2}(\epsilon_{\mu\nu}D^-_\mu\lambda_\nu + i(-D^+_\mu\lambda_\mu))(G| - i\tilde{G}|)\}\,. \qquad (2.68)$$

Here it is important to recognize that we can find solutions satisfying $\nabla\tilde{G} = \tilde{\nabla}G = 0$ and (2.57),

$$G = a\epsilon_{\mu\nu}\nabla^-_{\underline{\mu}}\nabla F_\nu\,, \quad \tilde{G} = -a\nabla^+_{\underline{\mu}}\nabla F_\mu\,, \qquad (2.69)$$

where a is a parameter with mass dimension -1. Moreover the above choice of G and $\tilde{G}$ makes S_2 and S_3 identical

$$S_2 = S_3 = S_0 - ia^{-1}\int d^2x \mathrm{Tr}\, G|\tilde{G}|\,. \qquad (2.70)$$

We can then find the following action satisfying $sS = \tilde{s}S = s_\mu S = 0$, where the supertransformations are given in Table 7,

$$S = S_0 - ia^{-1}\int d^2x \mathrm{Tr}\, G|\tilde{G}|\,. \qquad (2.71)$$

In Table 7 the following expressions are used:

$$G| = a\epsilon_{\mu\nu}D^-_\mu\lambda_\nu\,, \quad \tilde{G}| = -aD^+_\mu\lambda_\mu\,,$$

$$\nabla_\mu G| = a(-\epsilon_{\mu\nu}\{\lambda_\nu, \bar{\rho}\} + \frac{1}{2}\{\lambda_\mu, \tilde{G}|\} + \epsilon_{\rho\sigma}D^-_\rho A_{\mu\sigma})\,,$$

$$\nabla_\mu \tilde{G}| = a(-\epsilon_{\mu\nu}\{\lambda_\nu, \bar{\tilde{\rho}}\} - \frac{1}{2}\{\lambda_\mu, G|\} - D^+_\nu A_{\mu\nu}) ,$$

$$\begin{aligned}
Z\rho &= \frac{1}{2}(\nabla_z G| - \epsilon_{\rho\sigma}\nabla_\rho\nabla_\sigma\tilde{G}|) \\
&= \frac{a}{2}([\nabla_\mu\tilde{G}|, \lambda_\mu] + \epsilon_{\mu\nu}[\nabla_\mu G|, \lambda_\nu] - 2[D, \bar{\tilde{\rho}}] + \epsilon_{\mu\nu}[A_{\mu\nu}, G|] \\
&- i\epsilon_{\mu\nu}D^+_\mu D^+_\nu\bar{\rho} - iD^+_\mu D^-_\mu\bar{\tilde{\rho}} - \frac{i}{2}\epsilon_{\mu\nu}D^-_\mu D^-_\nu G| - \frac{i}{2}D^-_\mu D^+_\mu\tilde{G}|) , \\
Z\tilde{\rho} &= \frac{1}{2}(\nabla_z\tilde{G}| + \epsilon_{\rho\sigma}\nabla_\rho\nabla_\sigma G|) \\
&= \frac{a}{2}(-[\nabla_\mu G|, \lambda_\mu] + \epsilon_{\mu\nu}[\nabla_\mu\tilde{G}|, \lambda_\nu] + 2[D, \bar{\rho}] + \epsilon_{\mu\nu}[A_{\mu\nu}, \tilde{G}|] \\
&+ iD^-_\mu D^+_\mu\bar{\rho} - i\epsilon_{\mu\nu}D^-_\mu D^-_\nu\bar{\tilde{\rho}} - \frac{i}{2}D^-_\mu D^+_\mu G| - \frac{i}{2}\epsilon_{\mu\nu}D^+_\mu D^+_\nu\tilde{G}|) ,
\end{aligned} \tag{2.72}$$

where $\bar{\rho} \equiv \rho - \frac{1}{2}G|$ and $\bar{\tilde{\rho}} \equiv \tilde{\rho} - \frac{1}{2}\tilde{G}|$.

2.3.3 The B (Z,Z,0) model

We consider the following algebra:

$$\begin{aligned}
&\{s, s_\mu\} = P_\mu , \quad \{\tilde{s}, s_\mu\} = -\epsilon_{\mu\nu}P_\nu , \quad \{s, \tilde{s}\} = 0 , \\
&s^2 = \tilde{s}^2 = Z , \quad \{s_\mu, s_\nu\} = 0 , \quad [s_I, Z] = 0 ,
\end{aligned} \tag{2.73}$$

where $Z = 2U_0 = -2V_5$. This model can be constructed similarly to the previous model. We thus follow the steps of the construction.

In this case, the corresponding supercharge differential operators are given by

$$\begin{aligned}
\mathcal{Q} &= \frac{\partial}{\partial\theta} + \frac{i}{2}\theta_\mu\partial_\mu + \frac{i}{2}\theta\partial_z , \\
\mathcal{Q}_\mu &= \frac{\partial}{\partial\theta^\mu} + \frac{i}{2}\theta\partial_\mu - \frac{i}{2}\tilde{\theta}\epsilon_{\mu\nu}\partial_\nu , \\
\tilde{\mathcal{Q}} &= \frac{\partial}{\partial\tilde{\theta}} - \frac{i}{2}\theta_\mu\epsilon_{\mu\nu}\partial_\nu + \frac{i}{2}\tilde{\theta}\partial_z .
\end{aligned} \tag{2.74}$$

And the supercovariant derivatives are found as

$$\begin{aligned}
\mathcal{D} &= \frac{\partial}{\partial\theta} - \frac{i}{2}\theta_\mu\partial_\mu - \frac{i}{2}\partial_z , \\
\mathcal{D}_\mu &= \frac{\partial}{\partial\theta^\mu} - \frac{i}{2}\theta\partial_\mu + \frac{i}{2}\tilde{\theta}\epsilon_{\mu\nu}\partial_\nu , \\
\tilde{\mathcal{D}} &= \frac{\partial}{\partial\tilde{\theta}} + \frac{i}{2}\theta_\mu\epsilon_{\mu\nu}\partial_\nu - \frac{i}{2}\tilde{\theta}\partial_z .
\end{aligned} \tag{2.75}$$

The B-(Z,Z,0) supercurvature ansatz is shown in Table 5. With the ansatz, the following relations are derived by Jacobi identities:

$$\nabla F_\mu = \epsilon_{\mu\nu}\tilde{\nabla} F_\nu \,, \quad \nabla_\mu F_\nu + \nabla_\nu F_\mu = \delta_{\mu\nu}\nabla_\rho F_\rho \,, \quad G = \tilde{G} = 0 \,, \tag{2.76}$$

$$F_{\underline{\mu}} = -i\nabla F_\mu + \frac{i}{2}G_\mu \,, \quad \tilde{F}_{\underline{\mu}} = i\tilde{\nabla} F_\mu + \frac{i}{2}\epsilon_{\mu\nu}G_\nu \,, \tag{2.77}$$

$$F_{\mu\underline{\nu}} = -\frac{i}{2}\delta_{\mu\nu}\nabla_\rho F_\rho + \frac{i}{2}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\nabla_\rho F_\sigma \,, \tag{2.78}$$

$$F_{\underline{\mu}\underline{\nu}} = \nabla_\mu\nabla F_\nu - \nabla_\nu\nabla F_\mu + i[F_\mu, F_\nu] - \frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\nabla_\rho G_\sigma \,, \tag{2.79}$$

$$\nabla_\mu G_\nu + \nabla_\nu G_\mu = 0 \,, \tag{2.80}$$

$$\nabla_z F_\mu = \frac{1}{2}(\nabla G_\mu - \epsilon_{\mu\nu}\tilde{\nabla} G_\nu) \,, \quad G_{\underline{\mu}} = \frac{i}{2}(\nabla G_\mu + \epsilon_{\mu\nu}\tilde{\nabla} G_\nu) \,. \tag{2.81}$$

The component fields are defined in the same way as the B (0,0,Z) model, i.e. defined as in (2.59). We show it again:

$$F_\mu| = \phi_\mu \,, \quad \nabla F_\mu| = \lambda_\mu \,, \quad \nabla_\mu F_\nu| = \frac{1}{2}(\delta_{\mu\nu}\rho + \epsilon_{\mu\nu}\tilde{\rho}) \,, \quad \nabla_\mu\nabla F_\mu| = D \,, \tag{2.82}$$

where ϕ_μ and D are bosonic fields and ρ, $\tilde{\rho}$, and λ_μ are fermionic fields.

The key relation of the model derived by Jacobi identity is (2.80). Similarly to the previous model, G_μ is not directly related to F_μ by Jacobi identity, it is then necessary to solve (2.80). As far as the above relation holds, one can derive the supertransformations of each component field and show off-shell closure up to gauge transformations without constraints.

To obtain the action, we introduce a linear combination of s and $\tilde{s}$ as

$$s'_\pm \equiv s \pm i\tilde{s} \,, \tag{2.83}$$

which satisfies

$$s'^2_\pm = 0 \,, \quad \{s'_+, s'_-\} = 2Z \,. \tag{2.84}$$

We also define $\rho_\pm \equiv \rho \pm i\tilde{\rho}$. Then one can derive action by using the nilpotency of s, $\tilde{s}$ and $s_\pm$. We can find S'_1, S'_2, and S'_3 satisfying $s_\mu S'_1 = s'_+ S'_2 = s'_- S'_3 = 0$ where S_1, S_2, and S_3 are not generally identical. In facts it is shown that

$$
\begin{aligned}
S_1 &= -\int d^2x \mathrm{Tr} s_\mu s_\nu \lambda_\mu \lambda_\nu \\
&= S_0 + \int d^2x \mathrm{Tr}\{\frac{1}{8}\epsilon_{\mu\nu}(F_{\mu\nu} - i[\phi_\mu, \phi_\nu])\epsilon_{\rho\sigma}\nabla_\rho G_\sigma| + \frac{1}{32}(\epsilon_{\mu\nu}\nabla_\mu G_\nu|)^2\}\,, \qquad (2.85) \\
S_2 &= \frac{1}{8i}\int d^2x \mathrm{Tr} s_+ s_- \rho\tilde{\rho} \\
&= S_0 + \int d^2x \mathrm{Tr}\{-\frac{i}{16}\nabla_+\epsilon_{\mu\nu}\nabla_\mu G_\nu|\rho_- + \frac{1}{8}(F_{\mu\nu} + i[\phi_\mu, \phi_\nu] + \frac{1}{4}\nabla_\mu G_\nu|)\nabla_\mu G_\nu| \\
&\qquad - \frac{i}{8}(D_\mu\rho_- \cdot G_\mu| + i\epsilon_{\mu\nu}D_\mu\rho_- \cdot G_\nu| + [\phi_\mu, \rho_+]G_\mu| + i\epsilon_{\mu\nu}[\phi_\mu, \rho_+]G_\nu|)\}\,, \qquad (2.86) \\
S_3 &= \frac{1}{8i}\int d^2x \mathrm{Tr} s_- s_+ \tilde{\rho}\rho \\
&= S_0 + \int d^2x \mathrm{Tr}\{\frac{i}{16}\nabla_-\epsilon_{\mu\nu}\nabla_\mu G_\nu|\rho_+ + \frac{1}{8}(F_{\mu\nu} + i[\phi_\mu, \phi_\nu] + \frac{1}{4}\nabla_\mu G_\nu|)\nabla_\mu G_\nu| \\
&\qquad - \frac{i}{8}(D_\mu\rho_+ \cdot G_\mu| - i\epsilon_{\mu\nu}D_\mu\rho_+ \cdot G_\nu| + [\phi_\mu, \rho_-]G_\mu| - i\epsilon_{\mu\nu}[\phi_\mu, \rho_-]G_\nu|)\}\,, \qquad (2.87)
\end{aligned}
$$

where S_0 is defined as in (2.65). In the case of $\nabla_\mu G_\nu = 0$ together with (2.80), we then find

$$
\begin{aligned}
S_1 &= S_0\,, \qquad (2.88) \\
S_2 &= S_0 + \int d^2x \mathrm{Tr}\{-\frac{i}{8}(D_\mu\rho_- \cdot G_\mu| + i\epsilon_{\mu\nu}D_\mu\rho_- \cdot G_\nu| + [\phi_\mu, \rho_+]G_\mu| + i\epsilon_{\mu\nu}[\phi_\mu, \rho_+]G_\nu|)\}\,, \qquad (2.89) \\
S_3 &= S_0 + \int d^2x \mathrm{Tr}\{-\frac{i}{8}(D_\mu\rho_+ \cdot G_\mu| - i\epsilon_{\mu\nu}D_\mu\rho_+ \cdot G_\nu| + [\phi_\mu, \rho_-]G_\mu| - i\epsilon_{\mu\nu}[\phi_\mu, \rho_-]G_\nu|)\}\,. \qquad (2.90)
\end{aligned}
$$

Furthermore, we can find a solution satisfying $\nabla_\mu G_\nu = 0$ and (2.80),

$$
G_\mu = -\frac{a}{2}\epsilon_{\rho\sigma}(\nabla_{\underline{\rho}}\nabla_\sigma F_\mu + [F_\rho, \nabla_\mu F_\sigma])\,. \qquad (2.91)
$$

where a is a parameter with mass dimension -1. The above choice of G_μ makes S_2 and S_3 identical as

$$
S_2 = S_3 = S_0 - ia^{-1}\int d^2x \mathrm{Tr}\,\frac{1}{2}\epsilon_{\mu\nu}G_\mu|G_\nu|\,. \qquad (2.92)
$$

One can then realize that the following action S has full supersymmetry for the supertransformations given in Table 8:

$$
S = S_0 - ia^{-1}\int d^2x \mathrm{Tr}\,\frac{1}{2}\epsilon_{\mu\nu}G_\mu|G_\nu|\,. \qquad (2.93)
$$

In Table 8 the following expressions are used:

$$
G_\mu| = -\frac{a}{4}(D^-_\mu\tilde{\rho} + \epsilon_{\mu\nu}D^+_\nu\rho)\,,
$$

	s	s_μ	$\tilde{s}$	Z
ϕ_ν	λ_ν	$\frac{1}{2}(\delta_{\mu\nu}\rho+\epsilon_{\mu\nu}\tilde{\rho})$	$-\epsilon_{\nu\rho}\lambda_\rho$	$\frac{1}{2}(\nabla G_\nu\| - \epsilon_{\nu\rho}\tilde{\nabla}G_\rho\|)$
A_ν	$-i\lambda_\nu+\frac{i}{2}G_\nu\|$	$\frac{i}{2}(-\delta_{\mu\nu}\rho+\epsilon_{\mu\nu}\tilde{\rho})$	$-i\epsilon_{\nu\rho}\lambda_\rho+\frac{i}{2}\epsilon_{\nu\rho}G_\rho\|$	$\frac{i}{2}(\nabla G_\nu\| + \epsilon_{\nu\rho}\tilde{\nabla}G_\rho\|)$
λ_ν	$\frac{1}{4}(\nabla G_\nu\| - \epsilon_{\nu\rho}\tilde{\nabla}G_\rho\|)$	$A_{\mu\nu}$	$\frac{1}{4}(\tilde{\nabla}G_\mu\| + \epsilon_{\mu\nu}\nabla G_\nu\|)$	$\frac{1}{4}\nabla_z G_\nu\| - \frac{1}{2}\epsilon_{\nu\rho}\nabla\tilde{\nabla}G_\rho\|$
ρ	$-iD_\rho\phi_\rho - D$	0	$\frac{i}{2}\epsilon_{\rho\sigma}[D_\rho, D_\sigma]$	$-iD^-_\rho G_\rho\|$
$\tilde{\rho}$	$-\frac{i}{2}\epsilon_{\rho\sigma}[D^+_\rho, D^+_\sigma]$	0	$iD_\rho\phi_\rho - D$	$-i\epsilon_{\rho\sigma}D^+_\rho G_\sigma\|$
D	$-iD^+_\rho\lambda_\rho+\frac{i}{2}D_\rho G_\rho\|$	$\frac{i}{2}(D^+_\mu\rho - \epsilon_{\mu\nu}D^-_\nu\tilde{\rho})$	$-i\epsilon_{\rho\sigma}D^-_\rho\lambda_\sigma+\frac{i}{2}\epsilon_{\rho\sigma}D_\rho G_\sigma\|$	$\frac{i}{2}(D^-_\rho\nabla G_\rho\| + \epsilon_{\rho\sigma}D^+_\rho\tilde{\nabla}G_\sigma\|)$ $-2i\{\lambda_\rho, G_\rho\|\}+\frac{i}{2}\{G_\rho\|, G_\rho\|\}$

Table 8: Supertransformations of the B (Z,Z,0) model. $A_{\mu\nu}$ is defined in the same way as that in Table 8.

$$\nabla G_\mu| = \frac{a}{2}(-\frac{i}{2}\epsilon_{\rho\sigma}D^-_\mu[D^+_\rho, D^+_\sigma] - i\epsilon_{\mu\nu}D^+_\nu D_\rho\phi_\rho - \epsilon_{\mu\nu}D^+_\nu D - 2\{\tilde{\rho}, \bar{\lambda}_\mu\} + \frac{1}{2}\epsilon_{\mu\nu}\{\rho, G_\nu|\}),$$

$$\tilde{\nabla}G_\mu| = \frac{a}{2}(-iD^+_\nu[D^-_\mu, D^-_\nu] + iD^-_\mu D_\nu\phi_\nu - D^- D + 2\{\rho, \bar{\lambda}_\mu\} + \frac{1}{2}\epsilon_{\mu\nu}\{\tilde{\rho}, G_\nu|\}),$$

$$\begin{aligned} Z\lambda_\mu &= \frac{1}{4}\nabla_z G_\nu| - \frac{1}{2}\epsilon_{\nu\rho}\nabla\tilde{\nabla}G_\rho| \\ &= \frac{a}{4}(\frac{1}{2}[\nabla G_\mu| + \epsilon_{\mu\nu}\tilde{\nabla}G_\nu|, \tilde{\rho}] - \frac{1}{2}[\tilde{\nabla}G_\mu| - \epsilon_{\mu\nu}\nabla G_\nu|, \rho] + 4\epsilon_{\mu\nu}[K, \bar{\lambda}_\nu] \\ &\quad - 2i\epsilon_{\mu\nu}D^-_\nu D^+_\rho\bar{\lambda}_\rho + 2i\epsilon_{\mu\nu}D^+_\rho D^-_{[\nu}\bar{\lambda}_{\rho]} + \frac{i}{2}\epsilon_{\mu\nu}D^-_{[\nu}D^-_{\rho]}G_\rho| - \frac{i}{2}\epsilon_{\rho\sigma}D^-_\mu D^+_\rho G_\sigma| \\ &\quad - \frac{i}{2}\epsilon_{\mu\nu}D^+_\nu D^-_\rho G_\rho| - \frac{i}{2}\epsilon_{\rho\sigma}D^+_\rho D^+_\sigma G_\mu|), \end{aligned} \tag{2.94}$$

where [,] denotes the antisymmetrization of suffixes and $\bar{\lambda}_\mu \equiv \lambda_\mu - \frac{1}{4}G_\mu|$.

2.4 Conclusion and discussions

In this section, we have investigated $D = N = 2$ SYM theory with central charge and have found one A model and two B model formulations. In the A model, the superalgebra is closed at the off-shell level up to gauge transformation with an extra constraint (2.45). The model has a similarity to the already known D=N=4 SYM theory with a central charge with an unavoidable extra constraint in Ref. [5]. For an Abelian gauge group in the A model, we can explicitly solve the constraint (2.45), and we can thus construct the supertransformations and action without any constraints. We cannot, however, solve the constraint for the non-Abelian case in the local level. Thus the investigation of properties of the model for an non-Abelian gauge group seems to need a framework of non-local field theory [6].

On the other hand, we have found two types of B model whose superalgebra is closed at the off-shell level up to gauge transformation without any constraints. The way to formulate the B models is to include the two central charges U_0 and V_5 and to relate them in the following special ways

$$U_0 = \pm V_5\,. \tag{2.95}$$

We have included the dependent-central-charges in the algebra and found the two models without constraints. We have named the two models the B (0,0,Z) and B (Z,Z,0) model in the case of $U_0 = V_5$ and $U_0 = -V_5$, respectively. It should be noted that (2.95) is naturally found due to the twisted form of supersymmetry. There are no certain reason to find (2.95) in case of ordinary supersymetry.

One of the future works after the success in this section is to investigate the reason why the models without constraints are allowed. This question may relate whether it is possible to obtain the models by dimensional reduction from a theory in three or four dimensions. If the B models in this section is obtained by some sort of dimensional reduction, the meanings of (2.95) would be naturally understood. In analogy with $D = N = 4$ SYM theory which is obtained by dimensional reduction from $D = 10$ $N = 1$ SYM theory [5], $D = N = 2$ SYM theory is assumed to be obtained from $D = 4$ $N = 1$ SYM theory due to the number of fermionic degrees of freedom. There are two kinds of dimensional reduction, i.e. ordinary reduction and reduction by Legendre transformation. The former is independent from whether the theory before reduction is on-shell or off-shell invariant while the latter is the original theory has to be on-shell. Furthermore, the former unchanges the features whether central charges are included and whether constraints are inevitable, and the latter certainly results in a model with a constraint. That is why obtaining the B models in this section by dimensional reduction from $D = 4$ $N = 1$ SYM theory seems not to be easy.

Furthermore, as one of the future works, some sort of A model without constraints may be possible because we have not already investigated all possibilities of ansatz for A model. Additionally, the spinor representation (not twisted one) of the B model should be constructed.

And another work which should be investigated is to apply the technique of dependent-central-charges to four dimensional theory. We then deal with it in the next section.

3 $D = N = 4$ twisted superalgebra with central charge

3.1 Twisted superalgebra with central charge

In section 3.1, we introduce central charge to $N = 4$ superalgebra in four dimensions and perform the Dirac-Kähler twisting procedure.

3.1.1 Superalgebra with central charge

In four-dimensional Euclidean spacetime, γ-matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ and charge conjugation matrix C can be chosen to satisfy [20]

$$C\gamma^{\mu T}C^{-1} = \gamma^\mu, \quad C^T = -C. \tag{3.1}$$

In this section, we use the following notation:

$$\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 , \quad \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] , \quad \tilde{\gamma}^\mu = \gamma^\mu\gamma^5 . \tag{3.2}$$

We consider the superalgebra

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2C^{-1}_{ij}(\gamma^\mu C)_{\alpha\beta}P_\mu , \tag{3.3}$$

$$[Q_{\alpha i}, R_{\mu\nu}] = -\frac{i}{2}Q_{\alpha j}(\gamma^{\mu\nu})_{ji} , \tag{3.4}$$

$$[Q_{\alpha i}, P_\mu] = [R_{\mu\nu}, P_\rho] = [P_\mu, P_\nu] = 0 , \tag{3.5}$$

$$[R_{\mu\nu}, R_{\rho\sigma}] = -2(\delta_{\mu\rho}R_{\nu\sigma} - \delta_{\mu\sigma}R_{\nu\rho} - \delta_{\nu\rho}R_{\mu\sigma} + \delta_{\nu\sigma}R_{\mu\rho}) , \tag{3.6}$$

where $Q_{\alpha i}$ is supercharge and $R_{\mu\nu}$ is the generator of $SO(4)$ R-symmetry to be performed the Dirac-Kähler twisting procedure. Note that the spinor representation of $SO(4)$ covers that of $SO(5) \sim USp(4)$, and the symmetry allows to include central charge. The Majorana condition is given by $Q^c_{\alpha i} \equiv C^{-1}_{ij} C_{\alpha\beta} \bar{Q}^{\beta j} = Q_{\alpha i}$.

We now introduce possible extra terms as follows:

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2C^{-1}_{ij}(\gamma^\mu C)_{\alpha\beta} P_\mu + 2C_{\alpha\beta} U_{ij} + 2(\gamma^5 C)_{\alpha\beta} V_{ij}\,, \tag{3.7}$$

where $U_{ij} = -U_{ji}$ and $V_{ij} = -V_{ji}$ to be consistent with simultaneous replacements of $\alpha \leftrightarrow \beta$ and $i \leftrightarrow j$. U_{ij} and V_{ij} get the following restrictions according to the Jacobi identity w.r.t. Q, Q, R:

$$U_{ik}(\gamma^{\mu\nu})_{kj} = U_{jk}(\gamma^{\mu\nu})_{ki}\,, \quad V_{ik}(\gamma^{\mu\nu})_{kj} = V_{jk}(\gamma^{\mu\nu})_{ki}\,. \tag{3.8}$$

Since U_{ij} is antisymmetric, U_{ij} can be expanded as follows:

$$U_{ij} = U_0 C^{-1}_{ij} + U_\mu (C^{-1}\gamma^\mu)_{ij} + U_5 (C^{-1}\gamma^5)_{ij}\,. \tag{3.9}$$

In the three terms, it is shown that only the first and third terms satisfy (3.8). One thus obtain possible form of U_{ij} and similarly V_{ij} as

$$U_{ij} = U_0 C^{-1}_{ij} + U_5 (C^{-1}\gamma^5)_{ij}\,, \quad V_{ij} = V_0 C^{-1}_{ij} + V_5 (C^{-1}\gamma^5)_{ij}\,, \tag{3.10}$$

and obtain the following algebra:

$$\begin{aligned} \{Q_{\alpha i}, Q_{\beta j}\} =& 2C^{-1}_{ij}(\gamma^\mu C)_{\alpha\beta} P_\mu + 2C_{\alpha\beta}(C^{-1}_{ij} U_0 + (C^{-1}\gamma_5)_{ij} U_5) \\ &+ 2(\gamma_5 C)_{\alpha\beta}(C^{-1}_{ij} V_0 + (C^{-1}\gamma_5)_{ij} V_5) \end{aligned} \tag{3.11}$$

$$[Z, \text{any}] = 0 \quad \left[Z = U_0, U_5, V_0, V_5\right], \tag{3.12}$$

where U_0, U_5, V_0, and V_5 are identified as central charges. It is important to notice that V_0 can be regarded as five-dimensional momentum in the view point of dimensional reduction. The central charge included in [5] is thus V_0.

3.1.2 Twisted superalgebra

In four dimensions, one can expand the charge as

$$Q_{\alpha i} = \frac{i}{\sqrt{2}}(1s + \gamma^\mu s_\mu + \frac{1}{2}\gamma^{\mu\nu} s_{\mu\nu} + \tilde{\gamma}^\mu \tilde{s}_\mu + \gamma^5 \tilde{s})_{\alpha i}\,, \tag{3.13}$$

where s, s_μ, $s_{\mu\nu}$, $\tilde{s}_\mu$, and $\tilde{s}$ are $N = 4$ twisted supercharges in four dimensions. Note that these supercharges can be expressed by the original charge as

$$\begin{aligned} s = -\frac{i}{2\sqrt{2}}\mathrm{tr}\, Q\,, \quad s_\mu = -\frac{i}{2\sqrt{2}}\mathrm{tr}\, \gamma^\mu Q\,, \quad s_{\mu\nu} = \frac{i}{2\sqrt{2}}\mathrm{tr}\, \gamma^{\mu\nu} Q\,, \\ \tilde{s}_\mu = \frac{i}{2\sqrt{2}}\mathrm{tr}\, \tilde{\gamma}_\mu Q\,, \quad \tilde{s} = -\frac{i}{2\sqrt{2}}\mathrm{tr}\, \gamma^5 Q\,. \end{aligned} \tag{3.14}$$

The Lorentz generator $J_{\mu\nu}$ is satisfying

$$[P_\mu, J_{\rho\sigma}] = i\delta_{\mu\nu\rho\sigma} P_\nu\,, \tag{3.15}$$

$$[Q_{\alpha i}, J_{\mu\nu}] = \frac{i}{2}(\gamma_{\mu\nu})_{\alpha\beta} Q_{\beta i}\,, \tag{3.16}$$

where $\delta_{\mu\nu\rho\sigma} \equiv \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}$.

We can thus define $J'_{\mu\nu} = J_{\mu\nu} + R_{\mu\nu}$ and obtain

$$
\begin{aligned}
&[s, J'_{\mu\nu}] = 0\,, \quad [s_\mu, J'_{\rho\sigma}] = i\delta_{\mu\nu\rho\sigma} s_\nu\,, \\
&[s_{\mu\nu}, J'_{\rho\sigma}] = -i\delta_{\rho\sigma\mu\lambda} s_{\nu\lambda} + i\delta_{\rho\sigma\nu\lambda} s_{\mu\lambda},\,, \\
&[\tilde{s}_\mu, J'_{\rho\sigma}] = i\delta_{\mu\nu\rho\sigma}\tilde{s}_\nu\,, \quad [\tilde{s}, J'_{\mu\nu}] = 0\,,
\end{aligned}
\tag{3.17}
$$

from (3.4), (3.14), and (3.16). These relations mean that twisted supercharges s, s_μ, $s_{\mu\nu}$, $\tilde{s}_\mu$, and $\tilde{s}$ transform as scalar, vector, tensor, (pseudo-)vector, and (pseudo-)scalar under $J'_{\mu\nu}$, respectively.

The algebra among the twisted supercharges is then derived from (3.11) and (3.14) as follows:

$$
\begin{aligned}
&\{s, s_\mu\} = \{\tilde{s}, \tilde{s}_\mu\} = P_\mu\,, \quad \{s_\mu, s_{\rho\sigma}\} = -\delta_{\mu\nu\rho\sigma} P_\nu\,, \quad \{\tilde{s}_\mu, s_{\rho\sigma}\} = \epsilon_{\mu\nu\rho\sigma} P_\nu\,, \\
&\{s, \tilde{s}_\mu\} = \{\tilde{s}, s_\mu\} = \{s, s_{\mu\nu}\} = \{\tilde{s}, s_{\mu\nu}\} = 0\,, \\
&2s^2 = 2\tilde{s}^2 = U_0 + V_5\,, \quad \{s_\mu, s_\nu\} = \{\tilde{s}_\mu, \tilde{s}_\nu\} = \delta_{\mu\nu}(U_0 - V_5)\,, \quad \{s, \tilde{s}\} = U_5 + V_0\,, \\
&\{s_\mu, \tilde{s}_\nu\} = \delta_{\mu\nu}(U_5 - V_0)\,, \quad \{s_{\mu\nu}, s_{\rho\sigma}\} = \delta_{\mu\nu\rho\sigma}(U_0 + V_5) - \epsilon_{\mu\nu\rho\sigma}(U_5 + V_0)\,,
\end{aligned}
\tag{3.18}
$$

This is the $N = 4$ twisted superalgebra with central charge in four dimensions.

3.2 Formulation of a model without central charge

In this subsection, before including central charge, we formulate a model without central charge‡. The model is expected to have an on-shell invariance. Here we can thus understand some features of model with on-shell invariance.

We consider the following algebra:

$$
\begin{aligned}
&\{s, s_\mu\} = \{\tilde{s}, \tilde{s}_\mu\} = P_\mu\,, \\
&\{s_\mu, s_{\rho\sigma}\} = -\delta_{\mu\nu\rho\sigma} P_\nu\,, \quad \{\tilde{s}_\mu, s_{\rho\sigma}\} = \epsilon_{\mu\nu\rho\sigma} P_\nu\,, \\
&\{\,\text{others}\,\} = 0\,.
\end{aligned}
\tag{3.19}
$$

And we introduce superfields in the superspace parametrized by (x_μ, θ_A) where θ_A represents θ, θ_μ, $\theta_{\mu\nu}$, $\tilde{\theta}_\mu$, and $\tilde{\theta}$. The supercharge differential operator $\mathcal{Q}_A$, supercovariant derivative $\mathcal{D}_A$, superconnection Γ_I, gauge-supercovariant derivative ∇_I are introduced similarly to the last section except for absence of the ingredients due to central charge, for example, ∇_z.

Ansatz on supercurvatures in this case is shown in Table 9 [?]. With this ansatz, the following relations can be derived by the Jacobi identities:

$$
\begin{aligned}
&\nabla W = \tilde{\nabla} W = \nabla_{\mu\nu} W = 0\,, \quad \nabla_\mu F = \tilde{\nabla}_\mu F = 0\,, \\
&-\delta_{\mu\nu}\nabla F = \tilde{\nabla}_\mu F_\nu + \tilde{\nabla}_\nu F_\mu\,, \quad \delta_{\mu\nu}\tilde{\nabla} F = \nabla_\mu F_\nu + \nabla_\nu F_\mu\,, \\
&\nabla_{[\mu} F_{\nu]} = \epsilon_{\mu\nu\rho\sigma}\tilde{\nabla}_\rho F_\sigma\,, \\
&\nabla_{\mu\nu} F_\rho = -\delta_{\mu\nu\rho\sigma}\nabla F_\sigma + \epsilon_{\mu\nu\rho\sigma}\tilde{\nabla} F_\sigma\,, \\
&F_{\underline{\mu}} = -i\nabla F_\mu\,, \quad \tilde{F}_{\underline{\mu}} = i\tilde{\nabla} F_\mu\,, \\
&\nabla_\mu W = -2\tilde{\nabla} F_\mu\,, \quad \tilde{\nabla}_\mu W = 2\nabla F_\mu\,, \\
&F_{\mu\underline{\nu}} = -\frac{i}{2}(\delta_{\mu\nu}\tilde{\nabla} F - \delta_{\mu\nu\rho\sigma}\nabla_\rho F_\sigma)\,, \quad \tilde{F}_{\mu\underline{\nu}} = -\frac{i}{2}(\delta_{\mu\nu}\nabla F + \delta_{\mu\nu\rho\sigma}\tilde{\nabla}\rho F_\sigma)\,, \\
&F_{\mu\nu\underline{\rho}} = -i\delta_{\mu\nu\rho\sigma}\nabla F_\sigma - i\epsilon_{\mu\nu\rho\sigma}\tilde{\nabla} F_\sigma\,.
\end{aligned}
\tag{3.20}
$$

‡About the model, the author uses K. Nagata's doctor thesis as a reference [?].

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$
∇	0	$-iW$	$-i(\nabla_{\underline{\rho}} + F_\rho)$	0	0	$-iF_{\underline{\rho}}$
$\tilde{\nabla}$		0	0	$-i(\nabla_{\underline{\rho}} - F_\rho)$	0	$-i\tilde{F}_{\underline{\rho}}$
∇_μ			0	$-i\delta_{\mu\rho}F$	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} - F_\nu)$	$-iF_{\mu\underline{\rho}}$
$\tilde{\nabla}_\mu$				0	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} + F_\nu)$	$-i\tilde{F}_{\mu\underline{\rho}}$
$\nabla_{\mu\nu}$					$i\epsilon_{\mu\nu\rho\sigma}W$	$-iF_{\mu\nu\underline{\rho}}$
$\nabla_{\underline{\mu}}$						$-iF_{\underline{\mu}\underline{\rho}}$

Table 9: Supercurvature ansatz of a model without central charge.

The component fields are then defined as

$$F_\mu| = \phi_\mu\,, \quad W| = A\,, \quad F| = B\,,$$
$$\nabla F_\mu| = \lambda_\mu\,, \quad \tilde{\nabla} F_\mu| = \tilde{\lambda}_\mu\,,$$
$$\nabla_\mu F_\nu| = \delta_{\mu\nu}\rho + \rho_{\mu\nu}\,, \quad \tilde{\nabla}_\mu F_\nu| = \delta_{\mu\nu}\tilde{\rho} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\rho_{\rho\sigma}\,, \tag{3.21}$$

where ϕ_μ, A, and B are bosonic fields, and λ_μ, $\tilde{\lambda}_\mu$, ρ, $\tilde{\rho}$, and $\rho_{\mu\nu}$ are fermionic fields. Table 10 shows the supertransformations of each component field. The superalgebra is not closed at the off-shell level but closed at the on-shell level up to gauge transformations as, for example,

$$\{s, s_\mu\}\lambda_\nu = -iD^+_\mu\lambda_\nu + \frac{i}{2}(D^+_\rho\lambda_\rho - [A, \tilde{\rho}])\,, \tag{3.22}$$

where $D^\pm_\mu \equiv D_\mu \pm \phi_\mu$. The second term in the right hand side of (3.22) is unnecessary for the off-shell invariance. After the check of the closeness of superalgebra, it is shown that the superalgebra is closed with the following equations:

$$D^+_\mu\lambda_\mu - [A, \tilde{\rho}] = 0\,, \quad D^-_\mu\tilde{\lambda}_\mu - [A, \rho] = 0\,,$$
$$D^+_\mu\rho - D^-_\nu\rho_{\mu\nu} - [B, \tilde{\lambda}] = 0\,, \quad D^-_\mu\tilde{\rho} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}D^+_\nu\rho_{\rho\sigma} - [B, \lambda] = 0\,,$$
$$\delta_{\mu\nu\rho\sigma}D^-_\rho\lambda_\sigma + \epsilon_{\mu\nu\rho\sigma}(D^+_\rho\tilde{\lambda}_\sigma - \frac{1}{2}[A, \rho_{\rho\sigma}]) = 0\,. \tag{3.23}$$

The action is found so that (3.23) is derived as equations of motion. By the restriction, the fermionic part of action is found as

$$S_F = \int d^4x \mathrm{Tr}\big\{ - i\lambda_\mu(D^+_\mu\rho - D^-_\nu\rho_{\mu\nu} - [B, \tilde{\lambda}_\mu]) - i\tilde{\lambda}_\mu(D^-_\mu\tilde{\rho} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}D^+_\nu\rho_{\rho\sigma})$$
$$+ i\tilde{\rho}[A, \rho] - \frac{i}{8}\epsilon_{\mu\nu\rho\sigma}A\{\rho_{\mu\nu}, \rho_{\rho\sigma}\}\big\}\,. \tag{3.24}$$

One can then find the bosonic part of action so that the s transformation of the bosonic part and that of (3.24) cancel out each other. Eventually we can obtain the following bosonic part of action which has full supersymmetry with (3.24) as

$$S_B = \int d^4x \mathrm{Tr}\big\{-\frac{1}{8}[D^+_\mu, D^+_\nu][D^-_\mu, D^-_\nu] + \frac{1}{16}[D^+_\mu, D^-_\mu]^2 + \frac{1}{8}D^+_\mu A D^-_\mu B + \frac{1}{8}D^-_\mu A D^+_\mu B + \frac{1}{16}[A, B]^2\big\}\,. \tag{3.25}$$

	s	$\tilde{s}$	$s_{\mu\nu}$
ϕ_ρ	λ_ρ	$\tilde{\lambda}_\rho$	$-\delta_{\mu\nu\rho\sigma}\lambda_\sigma + \epsilon_{\mu\nu\rho\sigma}\tilde{\lambda}_\sigma$
A_ρ	$-i\lambda_\rho$	$i\tilde{\lambda}_\rho$	$-i\delta_{\mu\nu\rho\sigma}\lambda_\sigma - i\epsilon_{\mu\nu\rho\sigma}\tilde{\lambda}_\sigma$
A	0	0	0
B	$-2\tilde{\rho}$	2ρ	$-\epsilon_{\mu\nu\rho\sigma}\rho_{\rho\sigma}$
λ_ρ	0	$\frac{i}{2}D^-_\mu A$	$-\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}D^+_\sigma A$
$\tilde{\lambda}_\rho$	$\frac{i}{2}D^+_\mu A$	0	$-\frac{i}{2}\delta_{\mu\nu\rho\sigma}D^-_\sigma A$
ρ	$-\frac{i}{2}(D_\alpha\phi_\alpha + \frac{1}{2}[A,B])$	0	$\frac{i}{2}[D^-_\mu, D^-_\nu]$
$\tilde{\rho}$	0	$-\frac{i}{2}(D_\alpha\phi_\alpha - \frac{1}{2}[A,B])$	$\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}[D^+_\mu, D^+_\nu]$
$\rho_{\rho\sigma}$	$-\frac{i}{2}[D^+_\rho, D^+_\sigma]$	$\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}[D^-_\mu, D^-_\nu]$	$\frac{i}{2}\delta_{\mu\nu\alpha\gamma}\delta_{\rho\sigma\beta\gamma}[D^-_\alpha, D^+_\beta] - \frac{i}{2}\delta_{\mu\nu\rho\sigma}(D_\alpha\phi_\alpha + \frac{1}{2}[A,B])$

	s_μ	$\tilde{s}_\mu$
ϕ_ρ	$\delta_{\mu\rho} + \rho_{\mu\rho}$	$\delta_{\mu\rho}\tilde{\rho} + \frac{1}{2}\epsilon_{\mu\rho\alpha\beta}\rho_{\alpha\beta}$
A_ρ	$-i\delta_{\mu\rho} + i\rho_{\mu\rho}$	$i\delta_{\mu\rho}\tilde{\rho} - \frac{i}{2}\epsilon_{\mu\rho\alpha\beta}\rho_{\alpha\beta}$
A	$-2\tilde{\lambda}_\mu$	$2\lambda_\mu$
B	0	0
λ_ρ	$\frac{i}{2}[D^+_\mu, D^-_\rho] + \frac{i}{2}\delta_{\mu\rho}(D_\alpha\phi_\alpha + \frac{1}{2}[A,B])$	$\frac{i}{4}\epsilon_{\mu\rho\alpha\beta}[D^+_\alpha, D^+_\beta]$
$\tilde{\lambda}$	$-\frac{i}{4}\epsilon_{\mu\rho\alpha\beta}[D^-_\alpha, D^-_\beta]$	$-\frac{i}{2}[D^-_\mu, D^+_\rho] + \frac{i}{2}\delta_{\mu\rho}(D_\alpha\phi_\alpha - \frac{1}{2}[A,B])$
ρ	0	$-\frac{i}{2}D^-_\mu B$
$\tilde{\rho}$	$\frac{i}{2}D^+_\mu B$	0
$\rho_{\rho\sigma}$	$-\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}D^-_\nu B$	$\frac{i}{2}\delta_{\mu\nu\rho\sigma}D^+_\nu B$

Table 10: Supertransformations of the model without central charge.

The action can be rewritten with the bosonic fields explicitly as

$$S = \int d^4x \mathrm{Tr}\Big\{\frac{1}{4}(D_\mu\phi_\nu)^2 + \frac{1}{8}F_{\mu\nu}^2 + \frac{1}{8}D_\mu^+ A D_\mu^- B + \frac{1}{8}D_\mu^- A D_\mu^+ B - \frac{1}{8}[\phi_\mu,\phi_\nu]^2 + \frac{1}{16}[A,B]^2$$
$$- i\lambda_\mu(D_\mu^+\rho - D_\nu^-\rho_{\mu\nu} - [B,\tilde{\lambda}_\mu]) - i\tilde{\lambda}_\mu(D_\mu^-\tilde{\rho} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}D_\nu^+\rho_{\rho\sigma})$$
$$+ i\tilde{\rho}[A,\rho] - \frac{i}{8}\epsilon_{\mu\nu\rho\sigma}A\{\rho_{\mu\nu},\rho_{\rho\sigma}\}\Big\}\,. \qquad (3.26)$$

As can be seen above, the $D = N = 4$ SYM theory without central charge is an on-shell invariant theory. The supermultiplet includes no auxiliary field as well as the equations of motion (3.23) are necessary for the supertransformations to be closed.

3.3 Ansatz on supercurvature for models with central charge

In this subsection, we find appropriate ansatz on supercurvatures for models with central charge based on the algebra (3.18) derived in section 3.1.

We introduce superfields in the superspace parametrized by (x_μ, θ_A, z) where θ_A represents θ, θ_μ, $\theta_{\mu\nu}$, $\tilde{\theta}_\mu$, and $\tilde{\theta}$. The supercharge differential operator $\mathcal{Q}_A$, supercovariant derivative $\mathcal{D}_A$, superconnection Γ_I, gauge-supercovariant derivative ∇_I are introduced similarly to the last section. To find some kinds of appropriate ansatz, the supercurvatures X_0, X_μ, X_5, X'_0, X'_μ, and X'_5 are introduced as

$$\{\nabla_{\alpha i},\nabla_{\beta j}\} = 2C_{ij}^{-1}(\gamma^\mu C)_{\alpha\beta}P_\mu$$
$$+ 2C_{\alpha\beta}(C_{ij}^{-1}X_0 + (C^{-1}\gamma_\mu)_{ij}X_\mu + (C^{-1}\gamma_5)_{ij}X_5)$$
$$+ 2(\gamma_5 C)_{\alpha\beta}(C_{ij}^{-1}X'_0 + (C^{-1}\gamma_\mu)_{ij}X'_\mu + (C^{-1}\gamma_5)_{ij}X'_5)\,, \qquad (3.27)$$

where $\nabla_{\alpha i}$ are gauge-supercovariant derivative corresponding to $Q_{\alpha i}$ in (3.11). Right hand side in (3.27) is the most general terms with consistency of simultaneous replacement of $\alpha \leftrightarrow \beta$ and $i \leftrightarrow j$. As can be seen from (3.11), a part of X_0, X_5, X'_0, and X'_5 can be identified as gauged central charge of U_0, U_5, V_0, and V_5, respectively, i.e. can be identified as including ∇_z. Table 11 shows the relations between the gauge-supercovariant derivative and supercurvatures given in (3.27) in the twisted space.

At first we show two types of ansatz which correspond to models including either U_0, U_5, V_0, or V_5 as a central charge, The first ansatz is shown in Table 12 namely (V_0 , U_5) model. Here either X'_0 or X_5 is identified as ∇_z, i.e. either V_0 or U_5 is included as a central charge. The second ansatz is shown in Table 13 namely (U_0 , V_5) model. Here one of X and X'_5 is identified as ∇_z, i.e. either U_0 or V_5 is included as a central charge. Totally four models including one central charge U_0, U_5, V_0, or V_5 can be formulated by these two types of ansatz.

As can be seen in section 3.3, one can obtain each set of supertransformations and each action of the two models. One may then ask whether or not these models are different in principle. The answer is that the two models are theoretically equivalent. There is a set of redefinitions of the gauge-supercovariant derivative and the supercurvature which interchanges one to the other model. In Table 11, the following redefinitions make no change of the form:

$$\nabla^{\mathrm{new}} = \frac{1}{\sqrt{2}}(-i\nabla + \tilde{\nabla})\,, \quad \tilde{\nabla}^{\mathrm{new}} = \frac{1}{\sqrt{2}}(\nabla - i\tilde{\nabla})\,,$$

$$\nabla_\mu^{\mathrm{new}} = \frac{1}{\sqrt{2}}(i\nabla_\mu + \tilde{\nabla}_\mu)\,, \quad \tilde{\nabla}_\mu^{\mathrm{new}} = \frac{1}{\sqrt{2}}(\nabla_\mu + i\tilde{\nabla}_\mu)\,,$$

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	$X_0 + X'_5$	$X_5 + X'_0$	$-i(\nabla_{\underline{\rho}} + iX_\rho)$	$-X'_\rho$	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$		$X_0 + X'_5$	X'_ρ	$-i(\nabla_{\underline{\rho}} - iX_\rho)$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ			$\delta_{\mu\rho}(X_0 - X'_5)$	$\delta_{\mu\rho}(X_5 - X'_0)$	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} - iX_\nu) - \epsilon_{\mu\nu\rho\sigma}X'_\nu$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$				$\delta_{\mu\rho}(X_0 - X'_5)$	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} + iX_\nu) - \delta_{\mu\nu\rho\sigma}X'_\nu$	$-i\tilde{F}_{\mu\underline{\rho}}$	$i\tilde{G}_\mu$
$\nabla_{\mu\nu}$					$\delta_{\mu\nu\rho\sigma}(X_0 + X'_5) - \epsilon_{\mu\nu\rho\sigma}(X_5 + X'_0)$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$						$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z							0

Table 11: Twisted version of supercurvature ansatz of (3.27). The positions of $\nabla_{\underline{\mu}}$ reflect those of P_μ in (3.11).

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	0	∇_z	$-i(\nabla_{\underline{\rho}} + F_\rho)$	0	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$		0	0	$-i(\nabla_{\underline{\rho}} - F_\rho)$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ			0	$\delta_{\mu\rho}(\mp\nabla_z - iW)$	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} - F_\nu)$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$				0	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} + F_\nu)$	$-i\tilde{F}_{\mu\underline{\rho}}$	iG_μ
$\nabla_{\mu\nu}$					$-\epsilon_{\mu\nu\rho\sigma}\nabla_z$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$						$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z							0

Table 12: Supercurvature ansatz of (V_0 , U_5) model. Corresponding to the $\mp$ sign choice of algebra $\{\nabla_\mu, \tilde{\nabla}_\nu\} = \delta_{\mu\nu}(\mp Z - iW)$, we take $X'_0 = \nabla_z$ for $-$ and $X_5 = \nabla_z$ for $+$.

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	$\nabla_z - iW$	0	$-i\nabla_{\underline{\rho}}$	iF_ρ	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$		$\nabla_z - iW$	$-iF_\rho$	$-i\nabla_{\underline{\nu}}$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ			$\pm\delta_{\mu\nu}\nabla_z$	0	$i\delta_{\mu\nu\rho\sigma}\nabla_{\underline{\nu}} + i\epsilon_{\mu\nu\rho\sigma}F_\nu$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$				$\pm\delta_{\mu\rho}\nabla_z$	$-i\epsilon_{\mu\nu\rho\sigma}\nabla_{\underline{\nu}} + i\delta_{\mu\nu\rho\sigma}F_\nu$	$-i\tilde{F}_{\mu\underline{\rho}}$	iG_μ
$\nabla_{\mu\nu}$					$\delta_{\mu\nu\rho\sigma}(\nabla_z - iW)$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$						$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z							0

Table 13: Supercurvature ansatz of (U_0 , V_5) model. Corresponding to the $\pm$ sign choice of algebra $\{\nabla_\mu, \nabla_\nu\} = \pm\delta_{\mu\nu}(iW + \nabla_z)$, we take $X = \nabla_z$ for $+$ and $X' = -\nabla_z$ for $-$.

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$
∇	$(a+b)\nabla_z - iW$	$(c+d)\nabla_z - iW'$	$-i(\nabla_{\underline{\rho}} + F_\rho)$	0
$\tilde{\nabla}$		$(a+b)\nabla_z - iW$	0	$-i(\nabla_{\underline{\rho}} - F_\rho)$
∇_μ			$\delta_{\mu\rho}((a-b)\nabla_z - iF)$	$\delta_{\mu\rho}((c-d)\nabla_z - iF')$
$\tilde{\nabla}_\mu$				$\delta_{\mu\rho}((a-b)\nabla_z - iF)$

	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} - F_\nu)$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} + F_\nu)$	$-i\tilde{F}_{\mu\underline{\rho}}$	$i\tilde{G}_\mu$
$\nabla_{\mu\nu}$	$\delta_{\mu\nu\rho\sigma}((a+b)\nabla_z - iW) - \epsilon_{\mu\nu\rho\sigma}((c+d)\nabla_z - iW')$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$		$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z			0

Table 14: Trial 1.

$$\nabla^{\text{new}}_{\mu\nu} = \frac{1}{\sqrt{2}}(-i\nabla_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla_{\rho\sigma}),$$
$$X_0^{\text{new}} = -iX'_0, \quad X_\mu^{\text{new}} = iX'_\mu, \quad X_5^{\text{new}} = -iX'_5,$$
$$X_0'^{\text{new}} = -iX_0, \quad X_\mu'^{\text{new}} = iX_\mu, \quad X_5'^{\text{new}} = -iX_5. \tag{3.28}$$

One can thus show that the (V_0, U_5) and (U_0, V_5) models are equivalent in principle. The two models above, however, need a constraint for the superalgebra to be closed at the off-shell level as can be seen in section 3.3.

One of the main interests in the thesis is to investigate $D = N = 4$ models by applying the technique of dependent-central-charges used in the $D = N = 2$ B model and answer whether there are models which do not have any constraints. To find an appropriate ansatz for such a model, we now consider general form of dependent-central-charges as:

$$U_0 = aZ, \quad U_5 = cZ, \quad V_0 = dZ, \quad V_5 = bZ, \tag{3.29}$$

where a, b, c, and d are just parameters, and Z is a central charge. The most general ansatz with the dependent-central-charges is named Trial 1 ansatz and is shown in Table 14 If both of X_μ and X'_μ are included to the ansatz, it is clear that there are too many degrees of freedom, i.e. more than 16 bosonic and 16 fermionic degrees of freedom. Thus only X_μ is included and X'_μ and omitted. How is the ansatz where X'_μ is included and X_μ is omitted without any other changes in Trial 1 ansatz ? The answer is understood by (3.28). The transformations switches X_μ and X'_μ. That is why the ansatz where X'_μ is included and X_μ is omitted without any other changes is equivalent to Trial 1 ansatz itself, and Trial 1 ansatz is the most general ansatz which has one central charge.

Here we calculate the Jacobi identities and try to construct models with the Trial 1 ansatz. First of all, we consider the case except for $a = b = 0$. The case of $a = b = 0$ is considered later as Trial 2 ansatz. If $a + b = 0$ in Trial 1 ansatz, we can redefine the ingredients and can obtain Trial 1-1 ansatz shown in Table 15. In this paper, we use the notation of $(\nabla_I, \nabla_J, \nabla_K)$ to represent the Jacobi identity w.r.t. ∇_I, ∇_J, and ∇_K. From Trial 1-1 ansatz, the Jacobi identities $(\nabla, \tilde{\nabla}_\mu, \tilde{\nabla}_\nu)$, $(\tilde{\nabla}, \nabla_\mu, \nabla_\nu)$, $(\nabla_\mu, \nabla_\nu, \nabla_\rho)$, and $(\tilde{\nabla}_\mu, \tilde{\nabla}_\nu, \tilde{\nabla}_\rho)$ lead to

$$G = \tilde{G} = G_\mu = \tilde{G}_\mu = 0, \tag{3.30}$$

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$
∇	$-iW$	$(c+d)\nabla_z - iW'$	$-i(\nabla_{\underline{\rho}} + F_\rho)$	0
$\tilde{\nabla}$		$-iW$	0	$-i(\nabla_{\underline{\rho}} - F_\rho)$
∇_μ			$\delta_{\mu\rho}\nabla_z$	$\delta_{\mu\rho}((c-d)\nabla_z - iF')$
$\tilde{\nabla}_\mu$				$\delta_{\mu\rho}\nabla_z$

	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} - F_\nu)$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} + F_\nu)$	$-i\tilde{F}_{\mu\underline{\rho}}$	$i\tilde{G}_\mu$
$\nabla_{\mu\nu}$	$-i\delta_{\mu\nu\rho\sigma}W - \epsilon_{\mu\nu\rho\sigma}((c+d)\nabla_z - iW')$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$		$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z			0

Table 15: Trial 1-1.

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$
∇	∇_z	$(c+d)\nabla_z - iW'$	$-i(\nabla_{\underline{\rho}} + F_\rho)$	0
$\tilde{\nabla}$		∇_z	0	$-i(\nabla_{\underline{\rho}} - F_\rho)$
∇_μ			$\delta_{\mu\rho}(a'\nabla_z - iF)$	$\delta_{\mu\rho}((c-d)\nabla_z - iF')$
$\tilde{\nabla}_\mu$				$\delta_{\mu\rho}(a'\nabla_z - iF)$

	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} - F_\nu)$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} + F_\nu)$	$-i\tilde{F}_{\mu\underline{\rho}}$	$i\tilde{G}_\mu$
$\nabla_{\mu\nu}$	$\delta_{\mu\nu\rho\sigma}\nabla_z - \epsilon_{\mu\nu\rho\sigma}((c+d)\nabla_z - iW')$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$		$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z			0

Table 16: Trial 1-2.

and the Jacobi identities $(\nabla_\mu, \nabla_\nu, \nabla_{\rho\sigma})$ and $(\tilde{\nabla}, \nabla_\mu, \tilde{\nabla}_\nu)$ lead to

$$G_{\mu\nu} = 0\,. \tag{3.31}$$

Additionally, the Jacobi identities $(\nabla, \nabla_\mu, \nabla_z)$ and $(\tilde{\nabla}, \tilde{\nabla}_\mu, \nabla_z)$ lead to

$$G_{\underline{\mu}} = \nabla_z F_\mu = 0\,. \tag{3.32}$$

From (3.30), (3.31), and (3.32), the following relation is derived:

$$G_A = G_{\underline{\mu}} = \nabla_z F_\mu = 0\,, \tag{3.33}$$

which is the same as (2.36) in two-dimensional case, and results in a formulation of model without central charge. If $a+b \neq 0$ in Trial 1 ansatz, we can redefine the ingredients and obtain Trial 1-2 ansatz shown in Table 16. From Trial 1-2 ansatz, the Jacobi identities (∇, ∇, ∇),

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$
∇	$-iW$	$(c+d)\nabla_z - iW'$	$-i(\nabla_{\underline{\rho}} + F_\rho)$	0
$\tilde{\nabla}$		$-iW$	0	$-i(\nabla_{\underline{\rho}} - F_\rho)$
∇_μ			$-i\delta_{\mu\rho}F$	$\delta_{\mu\rho}((c-d)\nabla_z - iF')$
$\tilde{\nabla}_\mu$				$-i\delta_{\mu\rho}F$

	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} - F_\nu)$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} + F_\nu)$	$-i\tilde{F}_{\mu\underline{\rho}}$	$i\tilde{G}_\mu$
$\nabla_{\mu\nu}$	$-i\delta_{\mu\nu\rho\sigma}W - \epsilon_{\mu\nu\rho\sigma}((c+d)\nabla_z - iW')$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$		$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z			0

Table 17: Trial 2.

$(\tilde{\nabla}, \tilde{\nabla}, \tilde{\nabla})$, $(\nabla, \nabla, \tilde{\nabla}_\mu)$, $(\tilde{\nabla}, \tilde{\nabla}, \nabla_\mu)$, and $(\nabla, \nabla, \nabla_{\mu\nu})$ lead to

$$G_A = 0\,, \tag{3.34}$$

while the Jacobi identities $(\nabla, \nabla_\mu, \nabla_z)$ and $(\tilde{\nabla}, \tilde{\nabla}_\mu, \nabla_z)$ lead to

$$G_{\underline{\mu}} = \nabla_z F_\mu = 0\,. \tag{3.35}$$

Thus (3.34) and (3.35) result in (3.33) again. We can then conclude that the Trial 1 ansatz do not work to formulate a model with a central charge.

Next we try to construct models with the Trial 2 ansatz shown in Table 17, which is the case $a = b = 0$ in Trial 1 ansatz. From Trial 2 ansatz, the Jacobi identities (∇, ∇, ∇), $(\tilde{\nabla}, \tilde{\nabla}, \tilde{\nabla})$, $(\tilde{\nabla}, \tilde{\nabla}, \nabla_\mu)$, $(\nabla, \nabla, \tilde{\nabla}_\mu)$, and $(\nabla, \nabla, \nabla_{\mu\nu})$ readily lead to

$$\nabla_A W = 0\,, \tag{3.36}$$

which results $W = 0$ otherwise the component field in the zeroth order of W clearly does not satisfy the superalgebra. On the other hand, the Jacobi identities $(\nabla_\mu, \nabla_\nu, \nabla_\rho)$, $(\tilde{\nabla}_\mu, \tilde{\nabla}_\nu, \tilde{\nabla}_\rho)$, $(\nabla, \tilde{\nabla}_\mu, \tilde{\nabla}_\nu)$, and $(\tilde{\nabla}, \nabla_\mu, \nabla_\nu)$ readily lead to

$$\nabla F = \tilde{\nabla} F = \nabla_\mu F = \tilde{\nabla}_\mu F = 0\,, \tag{3.37}$$

which results $F = 0$ by the same reason. We can then regard the Trial 2 ansatz as the Trial $2'$ ansatz shown in Table 18.

Here we consider the three cases of $c + d = 0$, $c - d = 0$, and $c + d \neq 0$ and $c - d \neq 0$ in the Trial $2'$, namely the Trial 2-1, Trial 2-2, and Trial 2-3 ansatz, respectively. And as can be seen below, the ansatz also do not work to construct the desired model.

In contrast to the Trial 1 ansatz, the reason that the Trial 2-1 and 2-2 ansatz does not work is not derivation of (3.33) but too many degrees of freedom in the multiplet, in other words, not enough constraint on the supercurvatures. The Trial 2-1 ansatz is shown in Table 19. The

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	0	$(c+d)\nabla_z - iW'$	$-i(\nabla_{\underline{\rho}} + F_\mu)$	0	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$		0	0	$-i(\nabla_{\underline{\nu}} - F_\mu)$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ			0	$\delta_{\mu\rho}((c-d)\nabla_z - iF')$	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} - F_\nu)$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$				0	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} + F_\nu)$	$-i\tilde{F}_{\mu\underline{\rho}}$	$i\tilde{G}_\mu$
$\nabla_{\mu\nu}$					$-\epsilon_{\mu\nu\rho\sigma}((c+d)\nabla_z - iW')$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$						$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z							0

Table 18: Trial $2'$.

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	0	$-iW'$	$-i(\nabla_{\underline{\rho}} + F_\mu)$	0	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$		0	0	$-i(\nabla_{\underline{\nu}} - F_\mu)$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ			0	$\delta_{\mu\rho}\nabla_z$	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} - F_\nu)$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$				0	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} + F_\nu)$	$-i\tilde{F}_{\mu\underline{\rho}}$	$i\tilde{G}_\mu$
$\nabla_{\mu\nu}$					$i\epsilon_{\mu\nu\rho\sigma}W'$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$						$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z							0

Table 19: Trial 2-1.

Jacobi identities $(\nabla_A, \nabla_B, \nabla_C)$ finally leads to

$$\begin{aligned} &\nabla W = \tilde{\nabla} W = \nabla_{\mu\nu} W = 0\,, \quad \nabla_\mu W = -2\tilde{\nabla} F_\mu\,, \quad \tilde{\nabla}_\mu W = 2\nabla F_\mu\,, \\ &G = \frac{1}{2}\tilde{\nabla}_\mu F_\mu\,, \quad \tilde{G} = -\frac{1}{2}\nabla_\mu F_\mu\,, \quad G_\mu = \tilde{G}_\mu = 0\,, \quad G_{\mu\nu} = \frac{1}{2}(\epsilon_{\mu\nu\rho\sigma}\nabla_\rho F_\sigma + \delta_{\mu\nu\rho\sigma}\tilde{\nabla}_\rho F_\sigma)\,, \\ &F_{\underline{\mu}} = -i\nabla F_\mu\,, \quad \tilde{F}_{\underline{\mu}} = i\tilde{\nabla} F_\mu\,, \quad F_{\mu\underline{\nu}} = i\nabla_\mu F_\nu + i\delta_{\mu\nu}\tilde{G}\,, \quad \tilde{F}_{\mu\underline{\nu}} = -i\tilde{\nabla}_\mu F_\nu + i\delta_{\mu\nu} G\,, \\ &F_{\mu\nu\underline{\rho}} = -i\delta_{\mu\nu\rho\sigma}\nabla F_\sigma - i\epsilon_{\mu\nu\rho\sigma}\tilde{\nabla} F_\sigma\,, \quad \nabla_{\mu\nu} F_\rho = -\delta_{\mu\nu\rho\sigma}\nabla F_\sigma + \epsilon_{\mu\nu\rho\sigma}\tilde{\nabla} F_\sigma\,. \end{aligned} \tag{3.38}$$

One may expect that one can construct a model with the relations. However, this model has too many component fields due to lack of constraints. Especially, constraints on $\nabla_\mu F_\nu$ and $\tilde{\nabla}_\mu F_\nu$ are not enough to obtain a model with the suitable degrees of freedom. The fact results in a defect of the Trial 2-1 ansatz.

The Trial 2-2 ansatz is shown in Table 20. This ansatz also results in a defect by the same

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	0	∇_z	$-i(\nabla_{\underline{\rho}} + F_\mu)$	0	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$		0	0	$-i(\nabla_{\underline{\nu}} - F_\mu)$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ			0	$-i\delta_{\mu\rho}F'$	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} - F_\nu)$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$				0	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}} + F_\nu)$	$-i\tilde{F}_{\mu\underline{\rho}}$	$i\tilde{G}_\mu$
$\nabla_{\mu\nu}$					$-\epsilon_{\mu\nu\rho\sigma}\nabla_z$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$						$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z							0

Table 20: Trial 2-2.

	∇	$\tilde{\nabla}$	∇_ρ	$\tilde{\nabla}_\rho$	$\nabla_{\rho\sigma}$	$\nabla_{\underline{\rho}}$	∇_z
∇	0	∇_z	$-i(\nabla_{\underline{\rho}}+F_\mu)$	0	0	$-iF_{\underline{\rho}}$	iG
$\tilde{\nabla}$		0	0	$-i(\nabla_{\underline{\nu}}-F_\mu)$	0	$-i\tilde{F}_{\underline{\rho}}$	$i\tilde{G}$
∇_μ			0	$\delta_{\mu\rho}(c'\nabla_z-iF')$	$i\delta_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}}-F_\nu)$	$-iF_{\mu\underline{\rho}}$	iG_μ
$\tilde{\nabla}_\mu$				0	$-i\epsilon_{\mu\nu\rho\sigma}(\nabla_{\underline{\nu}}+F_\nu)$	$-i\tilde{F}_{\mu\underline{\rho}}$	$i\tilde{G}_\mu$
$\nabla_{\mu\nu}$					$-\epsilon_{\mu\nu\rho\sigma}\nabla_z$	$-iF_{\mu\nu\underline{\rho}}$	$iG_{\mu\nu}$
$\nabla_{\underline{\mu}}$						$-iF_{\underline{\mu\rho}}$	$iG_{\underline{\mu}}$
∇_z							0

Table 21: Trial 2-3.

reason as the Trial 2-1 ansatz.

The Trial 2-3 ansatz is shown in Table 21. And the ansatz has a different feature from the Trial 2-1 and 2-2 ansatz. In fact, the ansatz is similar to the (V_0, U_5) model ansatz. The difference between the two ansatz is the coefficient of ∇_z in $\{\nabla_\mu, \tilde{\nabla}_\nu\}$. However, the difference can be removed by the following redefinitions of ∇_A in the Trial 2-3 ansatz:

$$\nabla^{\text{new}} = c'^{\frac{1}{4}}\nabla\,, \quad \tilde{\nabla}^{\text{new}} = c'^{\frac{1}{4}}\tilde{\nabla}\,, \quad \nabla^{\text{new}}_{\mu\nu} = c'^{\frac{1}{4}}\nabla_{\mu\nu}\,,$$
$$\nabla^{\text{new}}_{\underline{\mu}} = c'^{-\frac{1}{4}}\nabla_{\mu}\,, \quad \tilde{\nabla}^{\text{new}}_{\mu} = c'^{-\frac{1}{4}}\tilde{\nabla}_\mu\,,$$
$$\nabla^{\text{new}}_{z} = c'^{\frac{1}{2}}\nabla_z\,, \quad F'^{\text{new}}_{z} = c'^{\frac{1}{2}}F'\,. \tag{3.39}$$

By the facts, it is shown that the V_0 model and U_5 model i.e. the (V_0, U_5) model where $-$ sign is chosen and $+$ sign is chosen in the right hand side of $\{\nabla_\mu, \tilde{\nabla}_\nu\}$, respectively, are equivalent as well as the equivalence between the Trial 2-3 and (V_0, U_5) model ansatz.

We eventually conclude that there is no off-shell invariant $D = N = 4$ SYM theory with central charge without any constraints as far as we consider one central charge, which is in contrast to our expectation from the results in two dimensions. But it should be noted that all variations of the model with one central charge are investigated.

We have formulated every model corresponding to each central charge U_0, U_5, V_0, and V_5. And, as can be seen in the next subsection, the models need a constraint. Although the existence of such a constraint is first understood in the model only with V_0 (the central charge treated in Ref [5] is regarded as V_0), we have shown that the models with one of the four central charges are theoretically equivalent. In another words, although the central charges except for V_0 cannot be naively regarded as five-dimensional momentum in the point of view of dimensional reduction at the algebraic level, the models with one of four central charges are theoretically equivalent.

3.4 Supermultiplets and Actions

We derive the supermultiplets and the actions for each model with supercurvature ansatz found in section 3.3. The $(V_0\,, U_5)$ and $(U_0\,, V_5)$ model are considered in section 3.4.1 and 3.4.2, respectively. Note that $N = 4$ supersymmetric theory in four dimension has generally sixteen bosonic and sixteen fermionic degrees of freedom at the off-shell level.

3.4.1 The (V_0 , U_5) model

We now consider the following superalgebra:

$$\begin{aligned}
\{s, s_\mu\} = \{\tilde{s}, \tilde{s}_\mu\} = P_\mu\,, \quad \{s_\mu, s_{\rho\sigma}\} = -\,\delta_{\mu\nu\rho\sigma} P_\nu\,, \quad \{\tilde{s}_\mu, s_{\rho\sigma}\} = \epsilon_{\mu\nu\rho\sigma} P_\nu\,,\\
\{s, \tilde{s}_\mu\} = \{\tilde{s}, s_\mu\} = \{s, s_{\mu\nu}\} = \{\tilde{s}, s_{\mu\nu}\} = 0\,,\\
2s^2 = 2\tilde{s}^2 = 0\,, \quad \{s_\mu, s_\nu\} = \{\tilde{s}_\mu, \tilde{s}_\nu\} = 0\,, \quad \{s, \tilde{s}\} = Z\,,\\
\{s_\mu, \tilde{s}_\nu\} = \mp \delta_{\mu\nu} Z\,, \quad \{s_{\mu\nu}, s_{\rho\sigma}\} = -\epsilon_{\mu\nu\rho\sigma} Z\,,
\end{aligned} \tag{3.40}$$

where $+$ represents the case $Z = U_5$ and $-$ represents the case $Z = V_0$ of (3.18) in the double sign.

We now consider the supercurvature ansatz in Table 12. The following relations can be derived by the Jacobi identities:

$$\begin{aligned}
\nabla_{(\mu} F_{\nu)} = \delta_{\mu\nu} \tilde{\nabla} W\,, \quad \tilde{\nabla}_{(\mu} F_{\nu)} = -\delta_{\mu\nu} \nabla W\,,\\
\nabla F_\mu = \pm\frac{1}{2}\tilde{\nabla}_\mu W\,, \quad \tilde{\nabla} F_\mu = \mp\frac{1}{2}\nabla_\mu W\,, \quad \nabla_{\mu\nu} W = -\epsilon_{\mu\nu\rho\sigma} \nabla_\rho F_\sigma\,,\\
\nabla_{[\mu} F_{\nu]} = \epsilon_{\mu\nu\rho\sigma} \tilde{\nabla}_\rho F_\sigma\,, \quad \tilde{\nabla}_{[\mu} F_{\nu]} = \epsilon_{\mu\nu\rho\sigma} \nabla_\rho F_\sigma\,, \quad \nabla_{\mu\nu} F_\rho = -\delta_{\mu\nu\rho\sigma} \nabla F_\sigma + \epsilon_{\mu\nu\rho\sigma} \tilde{\nabla} F_\sigma\,,\\
F_{\underline{\mu}} = -i\nabla F_\mu\,, \quad \tilde{F}_{\underline{\mu}} = i\tilde{\nabla} F_\mu\,, \quad F_{\mu\underline{\nu}} = -i\nabla_\nu F_\mu\,, \quad \tilde{F}_{\mu\underline{\nu}} = i\tilde{\nabla}_\nu F_\mu\,,\\
F_{\mu\nu\underline{\rho}} = -i(\delta_{\mu\nu\rho\sigma} \nabla F_\sigma + \epsilon_{\mu\nu\rho\sigma} \tilde{\nabla} F_\sigma)\,, \quad F_{\underline{\mu\nu}} = \nabla_{[\mu} \nabla F_{\nu]} + i[F_\mu, F_\nu]\,,\\
G = \tilde{G} = G_{\mu\nu} = 0\,, \quad G_\mu = 2\tilde{\nabla} F_\mu\,, \quad \tilde{G}_\mu = -2\nabla F_\mu\,, \quad G_{\underline{\mu}} = \frac{i}{2}(\nabla G_\mu + \tilde{\nabla} \tilde{G}_\mu)\,,\\
Z F_\mu = \frac{1}{2}(\nabla G_\mu - \tilde{\nabla} \tilde{G}_\mu)\,, \quad Z W = 2i\nabla_{\underline{\mu}} F_\mu + 2\nabla \tilde{\nabla} W\,.
\end{aligned} \tag{3.41}$$

The component fields are defined as

$$\begin{aligned}
F_\mu| = \phi_\mu\,, \quad W| = A\,, \quad \nabla F_\mu| = \lambda_\mu\,, \quad \tilde{\nabla} F_\mu| = \tilde{\lambda}_\mu\,,\\
\nabla_\mu F_\nu| = \delta_{\mu\nu}\rho + \rho_{\mu\nu}\,, \quad \tilde{\nabla}_\mu F_\nu| = \delta_{\mu\nu}\tilde{\rho} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\rho_{\rho\sigma}\,, \quad \nabla\tilde{\nabla} W| = H\,, \quad G_{\underline{\mu}}| = g_\mu\,, \quad \nabla_z F_\mu| = H_\mu\,,
\end{aligned} \tag{3.42}$$

where ϕ_μ, A, H, g_μ, H_μ, are bosonic, and ρ, $\tilde{\rho}$, λ_μ, $\tilde{\lambda}_\mu$, and $\rho_{\mu\nu}$ are fermionic fields. Table 22 shows the supertransformations of each component field. The off-shell closure of supertransformations up to gauge transformations is shown with the following constraint on the component fields:

$$iD_\mu g_\mu + [\phi_\mu, H_\mu] \mp 2\{\rho, \tilde{\rho}\} + 2\{\lambda_\mu, \tilde{\lambda}_\mu\} \mp \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\{\rho_{\mu\nu}, \rho_{\rho\sigma}\} \pm \frac{i}{2} D^+_\mu D^-_\mu A \pm \frac{1}{2}[A, H] = 0\,. \tag{3.43}$$

Note that the following formulae are useful to confirm the off-shell closure:

$$\delta_{\mu\nu\alpha\gamma}\epsilon_{\rho\sigma\beta\gamma} + \delta_{\rho\sigma\alpha\gamma}\epsilon_{\mu\nu\beta\gamma} = \delta_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\,, \tag{3.44}$$

$$\delta_{\mu\nu\alpha\gamma}\delta_{\rho\sigma\beta\gamma} + \epsilon_{\rho\sigma\alpha\gamma}\epsilon_{\mu\nu\beta\gamma} = \delta_{\alpha\beta}\delta_{\mu\nu\rho\sigma}\,. \tag{3.45}$$

Because of the constraint, the degrees of freedom of g_μ can be regarded as three. The bosonic degrees of freedom at the off-shell level is thus sixteen (ϕ_μ, A_μ, A, H, g_μ, H_μ). Note that the gauge field A_μ has three bosonic degrees of freedom at the off-shell level.

	s	$\tilde{s}$	s_μ	$\tilde{s}_\mu$
ϕ_ρ	λ_ρ	$\tilde{\lambda}_\rho$	$\delta_{\mu\rho}\rho + \rho_{\mu\rho}$	$\delta_{\mu\rho}\tilde{\rho} + \frac{1}{2}\epsilon_{\mu\rho\alpha\beta}\rho_{\alpha\beta}$
A_ρ	$-i\lambda_\rho$	$i\tilde{\lambda}_\rho$	$-i\delta_{\mu\rho}\rho + i\rho_{\mu\rho}$	$i\delta_{\mu\rho}\tilde{\rho} - \frac{i}{2}\epsilon_{\mu\rho\alpha\beta}\rho_{\alpha\beta}$
A'	$-\tilde{\rho}$	ρ	$\mp\tilde{\lambda}_\mu$	$\pm\lambda_{[}\mu]$
λ_ρ	0	g^+_ρ	$\frac{i}{2}[D^+_\mu, D^-_\rho] - \delta_{\mu\rho}H'$	$\frac{i}{4}\epsilon_{\mu\rho\alpha\beta}[D^+_\alpha, D^+_\beta]$
$\tilde{\lambda}_\rho$	$-g^-_\rho$	0	$-\frac{i}{4}\epsilon_{\mu\rho\alpha\beta}[D^-_\alpha, D^-_\beta]$	$-\frac{i}{2}[D^-_\mu, D^+_\rho] + \delta_{\mu\rho}(\frac{i}{2}[D^-_\sigma, D^+_\sigma] + H')$
ρ	H'	0	0	$\mp g^+_\mu - iD^-_\mu A'$
$\tilde{\rho}$	0	$-\frac{i}{2}[D^-_\mu, D^+_\mu] - H'$	$\pm g^-_\mu + iD^+_\mu A'$	0
$\rho_{\rho\sigma}$	$-\frac{i}{2}[D^+_\rho, D^+_\sigma]$	$\frac{i}{4}\epsilon_{\rho\sigma\alpha\beta}[D^-_\alpha, D^-_\beta]$	$\epsilon_{\mu\nu\rho\sigma}(\mp g^+_\nu - iD^-_\nu A')$	$\delta_{\mu\nu\rho\sigma}(\pm g^-_\nu + iD^+_\nu A')$
H'	0	$-iD^-_\mu\tilde{\lambda}_\mu$	$-iD^+_\mu\rho$	$-\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}D^+_\nu\rho_{\rho\sigma}$
g^+_ρ	$\mp(\frac{i}{2}\epsilon_{\mu\nu\alpha\beta}D^+_\sigma\rho_{\alpha\beta}$ $-iD^-_\rho\tilde{\rho} + i[A, \lambda_\rho])$	0	$-i\delta_{\mu\rho}D^-_\nu\tilde{\lambda}_\nu + iD^-_\rho\tilde{\lambda}_\mu$	$-iD^-_\mu\lambda_\rho - i\epsilon_{\mu\rho\alpha\beta}D^+_\alpha\tilde{\lambda}_\beta$
g^-_ρ	0	$\pm(iD^-_\sigma\rho_{\rho\sigma}$ $-iD^+_\rho\rho + i[A, \tilde{\lambda}_\rho])$	$i\epsilon_{\mu\rho\alpha\beta}D^-_\alpha\lambda_\beta + iD^+_\mu\tilde{\lambda}_\rho$	$i\delta_{\mu\rho}D^+_\nu\lambda_\nu - iD^+_\rho\lambda_\mu$

	$s_{\mu\nu}$	Z
ϕ_ρ	$-\delta_{\mu\nu\rho\sigma}\lambda_\sigma + \epsilon_{\mu\nu\rho\sigma}\tilde{\lambda}_\sigma$	H_ρ
A_ρ	$-i\delta_{\mu\nu\rho\sigma}\lambda_\sigma - i\epsilon_{\mu\nu\rho\sigma}\tilde{\lambda}_\sigma$	g_ρ
A'	$-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\rho_{\rho\sigma}$	$\frac{i}{2}[D^-_\mu, D^+_\mu] + H$
λ_ρ	$\epsilon_{\mu\nu\rho\sigma}g^-_\sigma$	$\mp(-iD^-_\mu\tilde{\rho} + \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}D^+\rho_{\rho\sigma} + i[A, \lambda_\mu])$
$\tilde{\lambda}_\rho$	$\delta_{\mu\nu\rho\sigma}g^+_\sigma$	$\mp(-iD^+_\mu\rho + iD^-_\nu\rho_{\mu\nu} + i[A, \tilde{\lambda}_\mu])$
ρ	$\frac{i}{2}[D^-_\mu, D^-_\nu]$	$-iD^-_\mu\tilde{\lambda}_\mu$
$\tilde{\rho}$	$\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}[D^+_\rho, D^+_\sigma]$	$-iD^+_\mu\lambda_\mu$
$\rho_{\rho\sigma}$	$\frac{i}{2}\delta_{\mu\nu\alpha\gamma}\delta_{\rho\sigma\beta\gamma}[D^-_\alpha, D^+_\beta] + \delta_{\mu\nu\rho\sigma}H'$	$-i\epsilon_{\mu\nu\rho\sigma}D^-_\rho\lambda_\sigma - iD^+_{[\mu}\tilde{\lambda}_{\nu]}$
H'	$iD^-_{[\mu}\lambda_{\nu]}$	$iD^-_\mu g^-_\mu + 2i\{\lambda_\mu, \tilde{\lambda}_\mu\}$
g^+_ρ	$\mp\epsilon_{\mu\nu\rho\sigma}(-iD^+_\sigma\rho + iD^-_\alpha\rho_{\sigma\alpha} + i[A, \lambda_\sigma])$	$\mp i(\frac{i}{2}D^-_\nu D^+_\nu D^-_\mu + \epsilon_{\mu\nu\rho\sigma}\{\tilde{\lambda}_\nu, \rho_{\rho\sigma}\} + 2\{\rho, \lambda\} + D^-_\mu H' + 2[A', g^+_\mu])$
g^-_ρ	$\mp\delta_{\mu\nu\rho\sigma}(-iD^-_\sigma\tilde{\rho} + \frac{i}{2}\epsilon_{\sigma\gamma\alpha\beta}D^+_\gamma\rho_{\alpha\beta} + i[A, \lambda_\sigma])$	$\mp i(\frac{i}{2}D^-_\nu D^+_\mu D^+_\nu + 2\{\lambda_\nu, \rho_{\mu\nu}\} + 2\{\tilde{\rho}, \tilde{\lambda}_\mu\} + D^+_\mu H' + 2[A', g^-_\mu])$

Table 22: Supertransformations of the (V_0, U_5) model, where $g^\pm_\mu \equiv \frac{1}{2}(ig_\mu \pm H_\mu)$, $A' \equiv \frac{1}{2}A$ and $H' \equiv \frac{1}{2}H$.

For an Abelian gauge group, the constraint (3.43) becomes simply as

$$\partial_\mu g_\mu \pm \frac{1}{2}\partial_\mu\partial_\mu A = 0\,, \tag{3.46}$$

which can be solved as

$$g_\mu \pm \frac{1}{2}\partial_\mu A = \partial_\nu B_{\mu\nu}\,, \quad B_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}B_{\rho\sigma}\,. \tag{3.47}$$

It is understood that the degrees of freedom of $B_{\mu\nu}$ is, in fact, three. For a non-Abelian gauge group, the constraint cannot be solved locally [6].

Finally, one can find the action as

$$\begin{aligned} S = \int d^4x \mathrm{Tr}\Big(&\frac{1}{2}D_\mu\phi_\nu D_\mu\phi_\nu + \frac{1}{4}F_{\mu\nu}^2 \pm \frac{1}{2}(g_\mu^2 + H_\mu^2) + H(iD_\mu\phi_\mu + \frac{1}{2}H) - \frac{1}{2}(D_\mu\phi_\mu)^2 \\ &-2i\rho D_\mu^+\lambda_\mu - 2i\tilde{\rho}D_\mu^-\tilde{\lambda}_\mu - 2i\rho_{\mu\nu}(D_\mu^-\lambda_\nu + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}D_\rho^+\tilde{\lambda}_\sigma) - 2iA\{\lambda_\mu, \tilde{\lambda}_\mu\} - \frac{1}{4}[\phi_\mu, \phi_\nu]^2\Big)\,. \end{aligned} \tag{3.48}$$

It is worth mentioning that this action cannot be derived by the superspace.

3.4.2 The (U_0 , V_5) model

We now consider the following superalgebra:

$$\begin{gathered} \{s, s_\mu\} = \{\tilde{s}, \tilde{s}_\mu\} = P_\mu\,, \quad \{s_\mu, s_{\rho\sigma}\} = -\,\delta_{\mu\nu\rho\sigma}P_\nu\,, \quad \{\tilde{s}_\mu, s_{\rho\sigma}\} = \epsilon_{\mu\nu\rho\sigma}P_\nu\,, \\ \{s, \tilde{s}_\mu\} = \{\tilde{s}, s_\mu\} = \{s, s_{\mu\nu}\} = \{\tilde{s}, s_{\mu\nu}\} = 0\,, \\ 2s^2 = 2\tilde{s}^2 = Z\,, \quad \{s_\mu, s_\nu\} = \{\tilde{s}_\mu, \tilde{s}_\nu\} = \pm\delta_{\mu\nu}Z\,, \quad \{s, \tilde{s}\} = 0\,, \\ \{s_\mu, \tilde{s}_\nu\} = 0\,, \quad \{s_{\mu\nu}, s_{\rho\sigma}\} = \delta_{\mu\nu\rho\sigma}Z\,, \end{gathered} \tag{3.49}$$

where + represents the case $Z = U_0$ and − represents the case $Z = V_5$ of (3.18) in the double sign.

We now consider the supercurvature ansatz in Table 13. The following relations can be derived by the Jacobi identities:

$$\begin{gathered} \nabla_{(\mu}F_{\nu)} = \pm\delta_{\mu\nu}\tilde{\nabla}W\,, \quad \tilde{\nabla}_{(\mu}F_{\nu)} = \mp\delta_{\mu\nu}\nabla W\,, \\ \nabla F_\mu = \frac{1}{2}\tilde{\nabla}_\mu W\,, \quad \tilde{\nabla}F_\mu = -\frac{1}{2}\nabla_\mu W\,, \quad \nabla_{\mu\nu}W = \mp\tilde{\nabla}_{[\mu}F_{\nu]}\,, \\ \nabla_{[\mu}F_{\nu]} = \epsilon_{\mu\nu\rho\sigma}\tilde{\nabla}_\rho F_\sigma\,, \quad \tilde{\nabla}_{[\mu}F_{\nu]} = \epsilon_{\mu\nu\rho\sigma}\nabla_\rho F_\sigma\,, \quad \nabla_{\mu\nu}F_\rho = -\delta_{\mu\nu\rho\sigma}\nabla F_\sigma + \epsilon_{\mu\nu\rho\sigma}\tilde{\nabla}F_\sigma\,, \\ F_{\underline{\mu}} = i\tilde{\nabla}F_\mu\,, \quad \tilde{F}_{\underline{\mu}} = -i\nabla F_\mu\,, \quad F_{\mu\underline{\nu}} = -i\tilde{\nabla}_\nu F_\mu\,, \quad \tilde{F}_{\mu\underline{\nu}} = i\nabla_\nu F_\mu\,, \\ F_{\mu\nu\underline{\rho}} = i(\delta_{\mu\nu\rho\sigma}\tilde{\nabla}F_\sigma + \epsilon_{\mu\nu\rho\sigma}\nabla F_\sigma)\,, \quad F_{\underline{\mu\nu}} = \epsilon_{\mu\nu\rho\sigma}\nabla_\rho\nabla F_\sigma + i[F_\mu, F_\nu]\,, \\ G = \nabla W\,, \quad \tilde{G} = \tilde{\nabla}W\,, \quad G_{\mu\nu} = \nabla_{\mu\nu}W\,, \quad G_\mu = \tilde{G}_\mu = 0\,, \quad G_{\underline{\mu}} = i\nabla_\mu G\,, \\ ZF_\mu = \nabla_\mu\tilde{G}\,, \end{gathered} \tag{3.50}$$

The component fields are then defined as

$$\begin{gathered} F_\mu| = \phi_\mu\,, \quad W| = A\,, \quad \nabla F_\mu| = \lambda_\mu\,, \quad \tilde{\nabla}F_\mu| = \tilde{\lambda}_\mu\,, \quad \nabla_\mu F_\nu| = \delta_{\mu\nu}\rho + \rho_{\mu\nu}\,, \\ \tilde{\nabla}_\mu F_\nu| = \delta_{\mu\nu}\tilde{\rho} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\rho_{\rho\sigma}\,, \quad \nabla\tilde{\nabla}W| = H\,, \quad G_{\underline{\mu}}| = g_\mu\,, \quad ZF_\mu| = H_\mu\,, \quad ZW| = K\,, \end{gathered} \tag{3.51}$$

	s	$\tilde{s}$
ϕ_ρ	λ_ρ	$\tilde{\lambda}_\rho$
A_ρ	$i\tilde{\lambda}_\rho$	$-i\lambda_\rho$
A'	$\mp\tilde{\rho}$	$\pm\rho$
λ_ρ	H'_ρ	g'_ρ
$\tilde{\lambda}_\rho$	$-g'_\rho$	H'_ρ
ρ	$-\frac{i}{2}D_\mu\phi_\mu$	$\pm K'$
$\tilde{\rho}$	$\mp K'$	$-\frac{i}{2}D_\mu\phi_\mu$
$\rho_{\rho\sigma}$	$-\frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}F^-_{\alpha\beta} - \frac{i}{2}D_{[\rho}\phi_{\sigma]}$	$F^-_{\rho\sigma} - \frac{i}{2}\epsilon_{\rho\sigma\alpha\beta}D_\alpha\phi_\beta$
K'	$\pm\frac{i}{2}(D_\mu\tilde{\lambda}_\mu + [\phi_\mu, \lambda_\mu])$	$\mp\frac{i}{2}(D_\mu\lambda_\mu - [\phi_\mu, \tilde{\lambda}_\mu])$
g'_ρ	$\pm\frac{i}{2}(D_\rho\tilde{\rho} - \frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}D_\sigma\rho_{\alpha\beta}$ $-[\phi_\rho, \rho] - [\phi_\sigma, \rho_{\rho\sigma}] \pm 2[A', \tilde{\lambda}_\rho])$	$\mp\frac{i}{2}(D_\rho\rho - D_\sigma\rho_{\rho\sigma} + [\phi_\rho, \tilde{\rho}]$ $+\frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}[\phi_\sigma, \rho_{\alpha\beta}] \pm 2[A', \lambda_\rho])$
H'_ρ	$\mp\frac{i}{2}(D_\rho\rho - D_\sigma\rho_{\rho\sigma} + [\phi_\rho, \tilde{\rho}]$ $+\frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}[\phi_\sigma, \rho_{\alpha\beta}] \pm 2[A', \lambda_\rho])$	$\mp\frac{i}{2}(D_\rho\tilde{\rho} - \frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}D_\sigma\rho_{\alpha\beta}$ $-[\phi_\rho, \rho] - [\phi_\sigma, \rho_{\rho\sigma}] \pm 2[A', \tilde{\lambda}_\rho])$

	s_μ	$\tilde{s}_\mu$
ϕ_ρ	$\delta_{\mu\rho}\rho + \rho_{\mu\rho}$	$\delta_{\mu\rho}\tilde{\rho} + \frac{1}{2}\epsilon_{\mu\rho\alpha\beta}\rho_{\alpha\beta}$
A_ρ	$-i\delta_{\mu\rho}\tilde{\rho} + \frac{i}{2}\epsilon_{\mu\rho\alpha\beta}\rho_{\alpha\beta}$	$i\delta_{\mu\rho}\rho - i\rho_{\mu\rho}$
A'	$-\tilde{\lambda}_\mu$	$\lambda_{[\mu]}$
λ_ρ	$\frac{1}{2}\epsilon_{\mu\rho\alpha\beta}F^-_{\alpha\beta} + \frac{i}{2}\delta_{\mu\rho}D_\nu\phi_\nu - \frac{i}{2}D_{(\mu}\phi_{\rho)}$	$F^+_{\mu\rho} \pm \delta_{\mu\rho}K' + \frac{i}{2}\epsilon_{\mu\rho\alpha\beta}D_\alpha\phi_\beta$
$\tilde{\lambda}_\rho$	$-F^+_{\mu\rho} \pm \delta_{\mu\rho}K' + \frac{i}{2}\epsilon_{\mu\rho\alpha\beta}D_\alpha\phi_\beta$	$-\frac{1}{2}\epsilon_{\mu\rho\alpha\beta}F^-_{\alpha\beta} + \delta_{\mu\rho}\frac{i}{2}D_\nu\phi_\nu - \frac{i}{2}D_{(\mu}\phi_{\rho)}$
ρ	$-\frac{i}{2}D_\mu\phi_\mu$	$\mp\frac{i}{2}g_\mu$
$\tilde{\rho}$	$\pm g_\mu$	$\frac{1}{2}H_\mu$
$\rho_{\rho\sigma}$	$\pm\frac{1}{2}\delta_{\mu\nu\rho\sigma}H_\nu \mp \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}g_\nu$	$\pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}H_\nu \pm \frac{i}{2}\delta_{\mu\nu\rho\sigma}g_\nu$
K'	$\pm\frac{i}{2}(D_\mu\tilde{\rho} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}D_\nu\rho_{\rho\sigma} - [\phi_\mu, \rho] - [\phi_\nu, \rho_{\mu\nu}])$	$\mp\frac{i}{2}(D_\mu\rho - D_\nu\rho_{\mu\nu} + [\phi_\mu, \tilde{\rho}] + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}[\phi_\nu, \rho_{\rho\sigma}])$
g'_ρ	$-\frac{i}{2}(\delta_{\mu\rho}(D_\nu\tilde{\lambda}_\nu + [\phi_\nu, \lambda_\nu]) - D_{(\mu}\tilde{\lambda}_{\rho)}$ $+[\phi_{[\mu}, \lambda_{\rho]}] - \epsilon_{\mu\rho\alpha\beta}(D_\alpha\lambda_\beta + [\phi_\alpha, \tilde{\lambda}_\beta]))$	$-\frac{i}{2}(-\delta_{\mu\rho}(D_\nu\lambda_\nu - [\phi_\nu, \tilde{\lambda}_\nu]) + D_{(\mu}\lambda_{\rho)}$ $+[\phi_{[\mu}, \tilde{\lambda}_{\rho]}] + \epsilon_{\mu\rho\alpha\beta}(D_\alpha\tilde{\lambda}_\beta - [\phi_\alpha, \lambda_\beta]))$
H'_ρ	$-\frac{i}{2}(\delta_{\mu\rho}(D_\nu\lambda_\nu - [\phi_\nu, \tilde{\lambda}_\nu]) + D_{[\mu}\lambda_{\rho]}$ $+[\phi_{(\mu}, \tilde{\lambda}_{\rho)}] + \epsilon_{\mu\rho\alpha\beta}(D_\alpha\tilde{\lambda}_\beta - [\phi_\alpha, \lambda_\beta]))$	$-\frac{i}{2}(\delta_{\mu\rho}(D_\nu\tilde{\lambda}_\nu + [\phi_\nu, \lambda_\nu]) + D_{[\mu}\tilde{\lambda}_{\rho]}$ $-[\phi_{(\mu}, \lambda_{\rho)}] + \epsilon_{\mu\rho\alpha\beta}(D_\alpha\lambda_\beta + [\phi_\alpha, \tilde{\lambda}_\beta]))$

	$s_{\mu\nu}$	Z
ϕ_ρ	$-\delta_{\mu\nu\rho\sigma}\lambda_\sigma + \epsilon_{\mu\nu\rho\sigma}\tilde{\lambda}_\sigma$	H_ρ
A_ρ	$i\delta_{\mu\nu\rho\sigma}\tilde{\lambda}_\sigma + i\epsilon_{\mu\nu\rho\sigma}\lambda_\sigma$	g_ρ
A'	$\mp\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\rho_{\rho\sigma}$	$2K'$
λ_ρ	$\delta_{\mu\nu\rho\sigma}H'_\sigma + \epsilon_{\mu\nu\rho\sigma}g'_\sigma$	$\mp i(D_\rho\rho - D_\sigma\rho_{\rho\sigma} + [\phi_\rho, \tilde{\rho}] + \frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}[\phi_\sigma, \rho_{\alpha\beta}])$
$\tilde{\lambda}_\rho$	$\delta_{\mu\nu\rho\sigma}g'_\sigma - \epsilon_{\mu\nu\rho\sigma}H'_\sigma$	$\mp i(D_\rho\tilde{\rho} - \frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}D_\sigma\rho_{\alpha\beta} - [\phi_\rho, \rho] - [\phi_\sigma, \rho_{\rho\sigma}])$
ρ	$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^-_{\rho\sigma} - \frac{i}{2}D_{[\mu}\phi_{\nu]}$	$-i(D_\rho\lambda_\rho - [\phi_\rho, \tilde{\lambda}_\rho] - i[A, \rho])$
$\tilde{\rho}$	$F^-_{\mu\nu} + \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}D_\rho\phi_\sigma$	$-i(D_\rho\tilde{\lambda}_\rho + [\phi_\rho, \lambda_\rho] - [A, \tilde{\rho}])$
$\rho_{\rho\sigma}$	$\epsilon_{\mu\nu\alpha\gamma}\delta_{\rho\sigma\beta\gamma}F^+_{\alpha\beta} + \frac{i}{2}\delta_{\mu\nu\alpha\gamma}\delta_{\rho\sigma\beta\gamma}D_{(\alpha}\phi_{\beta)}$ $-\frac{i}{2}\delta_{\mu\nu\rho\sigma}D_\alpha\phi_\alpha \mp \epsilon_{\mu\nu\rho\sigma}K'$	$-i(D_{[\mu}\lambda_{\nu]} + [\phi_{[\mu}, \tilde{\lambda}_{\nu]}] + \epsilon_{\mu\nu\rho\sigma}(D_\rho\tilde{\lambda}_\sigma$ $-[\phi_\rho, \lambda_\sigma]) - [A, \rho_{\mu\nu}])$
K'	$\pm\frac{i}{2}(D_{[\mu}\tilde{\lambda}_{\nu]} - [\phi_{[\mu}, \lambda_{\nu]}] + \epsilon_{\mu\nu\rho\sigma}(D_\rho\lambda_\sigma + [\phi_{rho}, \tilde{\lambda}_\sigma]))$	$\pm i(\{\lambda, \lambda\} + \{\tilde{\lambda}, \tilde{\lambda}\} - D_\rho g'_\rho + [\phi_\rho, H'_\rho] \pm [A, K'])$
g'_ρ	$\pm\frac{i}{2}\delta_{\mu\nu\rho\sigma}(D_\sigma\tilde{\rho} - \frac{1}{2}\epsilon_{\sigma\gamma\alpha\beta}D_\gamma\rho_{\alpha\beta} - [\phi_\sigma, \rho] - [\phi_\alpha, \rho_{\sigma\alpha}] \pm [A, \tilde{\lambda}_\sigma])$ $\pm\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}(D_\sigma\rho - D_\alpha\rho_{\sigma\alpha} + [\phi_\sigma, \tilde{\rho}] + \frac{1}{2}\epsilon_{\sigma\gamma\alpha\beta}[\phi_\gamma, \rho_{\alpha\beta}] \pm [A, \lambda_\sigma])$	$\mp i(-\frac{1}{2}D_\sigma F_{\rho\sigma} \pm D_\rho K' - \frac{i}{2}[\phi_\sigma, D_\rho\phi_\sigma] + \{\rho, \lambda_\rho\}$ $+\{\tilde{\rho}, \tilde{\lambda}_\rho\} + \{\lambda_\sigma, \rho_{\rho\sigma}\} + \frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}\{\tilde{\lambda}_\sigma, \rho_{\alpha\beta}\})$
H'_ρ	$\pm\frac{i}{2}\delta_{\mu\nu\rho\sigma}(D_\sigma\rho - D_\alpha\rho_{\sigma\alpha} + [\phi_\sigma, \tilde{\rho}] + \frac{1}{2}\epsilon_{\sigma\gamma\alpha\beta}[\phi_\gamma, \rho_{\alpha\beta}] \pm [A, \lambda_\sigma])$ $\mp\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}(D_\sigma\tilde{\rho} - \frac{1}{2}\epsilon_{\sigma\gamma\alpha\beta}D_\gamma\rho_{\alpha\beta} - [\phi_\sigma, \rho] - [\phi_\alpha, \rho_{\sigma\alpha}] \pm [A, \tilde{\lambda}_\sigma])$	$\mp i(-\frac{i}{2}D_\sigma D_\sigma\phi_\rho + \frac{i}{2}[\phi_\sigma, [\phi_\rho, \phi_\sigma]] \mp [\phi_\rho, K'] + \{\rho, \tilde{\lambda}_\rho\}$ $-\{\tilde{\rho}, \lambda_\rho\} - \{\tilde{\lambda}_\sigma, \rho_{\rho\sigma}\} + \frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}\{\lambda_\sigma, \rho_{\alpha\beta}\})$

Table 23: Supertransformations of the (U_0, V_5) model, where $g'_\mu \equiv \frac{i}{2}(g_\mu - D_\mu A)$, $H' \equiv \frac{1}{2}(H_\mu - i[A, \phi_\mu])$, $A' \equiv \frac{1}{2}A$, $K' \equiv \frac{1}{4}K$ and $F^\pm_{\mu\nu} \equiv \frac{1}{2}(F_{\mu\nu} \pm i[\phi_\mu, \phi_\nu])$.

where ϕ_μ, A, H, g_μ, H_μ, and K are bosonic, and ρ, $\tilde{\rho}$, λ_μ, $\tilde{\lambda}_\mu$, and $\rho_{\mu\nu}$ are fermionic fields. Table 23 shows the supertransformations of each component field. In the step of calculating the $s_{\mu\nu}$ transformations of ρ and $\rho_{\mu\nu}$, the following result is derived:

$$\begin{aligned} s_{\mu\nu}(\delta_{\rho\sigma}\rho + \rho_{\rho\sigma}) =& \nabla_{\mu\nu}\nabla_\rho F_\sigma| \\ &= \delta_{\rho\sigma}(\frac{1}{4}\epsilon_{\mu\nu\alpha\beta}(F_{\alpha\beta} - i[\phi_\alpha,\phi_\beta]) - \frac{i}{2}D_{[\mu}\phi_{\nu]}) && (3.52) \\ &-\frac{1}{2}\delta_{\rho\sigma\rho'\sigma'}\epsilon_{\mu\nu\rho'\beta}(F_{\sigma'\beta} + i[\phi_{\sigma'},\phi_\beta]) + \frac{i}{2}\delta_{\rho\sigma\rho'\sigma'}\delta_{\mu\nu\rho'\beta}D_{(\sigma'}\phi_{\beta)} && (3.53) \\ &\pm\frac{1}{2}\delta_{\mu\nu\rho\sigma}H \mp \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}K\,, && (3.54) \end{aligned}$$

which means

$$s_{\mu\nu}\rho = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}(F_{\rho\sigma} - i[\phi_\rho,\phi_\sigma]) - \frac{i}{2}D_{[\mu}\phi_{\nu]}\,, \tag{3.55}$$

$$\begin{aligned} s_{\mu\nu}\rho_{\rho\sigma} =& \frac{1}{2}\delta_{\mu\nu\alpha\gamma}\epsilon_{\rho\sigma\beta\gamma}(F_{\alpha\beta} + i[\phi_\alpha,\phi_\beta]) + \frac{i}{2}\delta_{\mu\nu\alpha\gamma}\delta_{\rho\sigma\beta\gamma}D_{(\alpha}\phi_{\beta)} \\ &\pm\frac{1}{2}\delta_{\mu\nu\rho\sigma}H \mp \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}K\,. \end{aligned} \tag{3.56}$$

On the other hand, the calculation of $s_{\mu\nu}$ transformations of $\tilde{\rho}$ also derives the $s_{\mu\nu}$ transformations of $\rho_{\mu\nu}$ as

$$\begin{aligned} s_{\mu\nu}(\delta_{\rho\sigma}\tilde{\rho} + \frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}\rho_{\alpha\beta}) =& \nabla_{\mu\nu}\tilde{\nabla}_\rho F_\sigma| \\ &= \delta_{\rho\sigma}(\frac{1}{2}(F_{\mu\nu} - i[\phi_\mu,\phi_\nu]) + \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}D_\alpha\phi_\beta) \\ &+\frac{1}{2}\epsilon_{\rho\sigma\alpha\beta}(\frac{1}{2}\delta_{\mu\nu\alpha'\gamma}\epsilon_{\alpha\beta\beta'\gamma}(F_{\alpha'\beta'} + i[\phi_{\alpha'},\phi_{\beta'}]) \\ &+\frac{i}{2}\delta_{\mu\nu\alpha'\gamma}\delta_{\alpha\beta\beta'\gamma}D_{(\alpha'}\phi_{\beta')} - i\delta_{\mu\nu\alpha\beta}D_\gamma\phi_\gamma \mp \frac{1}{2}\delta_{\mu\nu\alpha\beta}H \mp \frac{1}{4}\epsilon_{\mu\nu\alpha\beta}K)\,. \end{aligned} \tag{3.57}$$

which means

$$s_{\mu\nu}\tilde{\rho} = \frac{1}{2}(F_{\mu\nu} - i[\phi_\mu,\phi_\nu]) + \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}D_\rho\phi_\sigma\,, \tag{3.58}$$

$$\begin{aligned} s_{\mu\nu}\rho_{\rho\sigma} =& \frac{1}{2}\delta_{\mu\nu\alpha\gamma}\epsilon_{\rho\sigma\beta\gamma}(F_{\alpha\beta} + i[\phi_\alpha,\phi_\beta]) + \frac{i}{2}\delta_{\mu\nu\alpha\gamma}\delta_{\rho\sigma\beta\gamma}D_{(\alpha}\phi_{\beta)} \\ &-i\delta_{\mu\nu\rho\sigma}D_\alpha\phi_\alpha \mp \frac{1}{2}\delta_{\mu\nu\rho\sigma}H \mp \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}K\,. \end{aligned} \tag{3.59}$$

From consistency of (3.56) and (3.59), one can results

$$H = \mp iD_\mu\phi_\mu\,, \tag{3.60}$$

which implies

$$\nabla\tilde{\nabla}W = \mp i\nabla_{\underline{\mu}}F_\mu\,. \tag{3.61}$$

This relation can be regarded as a more sophisticated relation derived from Jacobi identities than (3.50).

The off-shell closure of supertransformations up to gauge transformations is shown with the following constraint on the component fields:

$$
\begin{aligned}
& iD_\mu g_\mu - [\phi_\mu, H_\mu] - \frac{i}{2} D_\mu D_\mu A - \frac{i}{2}[\phi_\mu, [\phi_\mu, A]] \mp \frac{1}{4}[A, K] \\
& \mp\{\rho, \rho\} \mp \{\tilde{\rho}, \tilde{\rho}\} \mp \frac{1}{2}\{\rho_{\mu\nu}, \rho_{\mu\nu}\} - \{\lambda_\mu, \lambda_\mu\} - \{\tilde{\lambda}_\mu, \tilde{\lambda}_\mu\} = 0\,. \qquad (3.62)
\end{aligned}
$$

Because of the constraint, the degrees of freedom of g_μ can be regarded as three. The bosonic degrees of freedom at the off-shell level is thus sixteen (ϕ_μ, A_μ, A, H, g_μ, H_μ).

For an Abelian gauge group, the constraint (3.62) becomes

$$
\partial_\mu g_\mu - \frac{1}{2}\partial_\mu \partial_\mu A = 0\,, \qquad (3.63)
$$

which can be solved similarlity to the $(V_0\,,\,U_5)$ model as

$$
g_\mu - \frac{1}{2}\partial_\mu A = \partial_\nu B_{\mu\nu}\,, \quad B_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} B_{\rho\sigma}\,. \qquad (3.64)
$$

For a non-Abelian gauge group, the constraint cannot be solved locally [6].

Finally, one can find an action for the non-Abelian gauge group case as

$$
\begin{aligned}
S = \int d^4x \mathrm{Tr}\Big(& \frac{1}{2} D_\mu \phi_\nu D_\mu \phi_\nu - \frac{1}{4} F_{\mu\nu}^2 \pm \frac{1}{2}(g_\mu - D_\mu A)^2 \mp \frac{1}{2}(H_\mu - i[A, \phi_\mu])^2 - \frac{1}{8}K^2 \\
& -2i\rho(D_\mu \lambda_\mu - [\phi_\mu, \tilde{\lambda}_\mu]) - 2i\tilde{\rho}(D_\mu \tilde{\lambda}_\mu + [\phi_\mu, \lambda_\mu]) - 2i\rho_{\mu\nu}(D_\mu \lambda_\nu + [\phi_\mu, \tilde{\lambda}_\nu]) \\
& -i\epsilon_{\mu\nu\alpha\beta}\rho_{\mu\nu}(D_\alpha \tilde{\lambda}_\beta - [\phi_\alpha, \lambda_\beta]) \pm iA(\{\lambda_\mu, \lambda_\mu\} + \{\tilde{\lambda}_\mu, \tilde{\lambda}_\mu\}) + \frac{1}{4}[\phi_\mu, \phi_\nu]^2 \Big)\,. \qquad (3.65)
\end{aligned}
$$

The action cannot be derived by the superspace. In this section, we found two types of $D = N = 4$ SYM formulation with a central charge with a constraint.

4 Conclusion and discussions

We have constructed off-shell invariant $N = 2$ twisted SYM theories with a gauged central charge in two dimensions. Depending on the supercurvature ansatz, we have formulated A and B models. In the A model, the superalgebra is closed at off-shell level with an extra constraint (2.45) similarly to $D = N = 4$ USp(4) SYM theory with central charge [5].

On the other hand, we have found two types of B model whose superalgebra is closed at the off-shell level without any constraints by including dependent-central-charges. This feature gives a new aspect to SYM theory with gauged central charge and vector-tensor multiplets. In principle, the Dirac-Kähler twisting procedure is not necessarily needed for the investigations in this thesis, and one can expect that the results with twisted SUSY are also valid with ordinary SUSY. It is noted, however, that the success is done with a help of twisting procedure. With the twisted form of superalgebra, the technique of dependent-central-charges is naturally found. In the ordinary representation of superalgebra, this technique is not characteristic.

Consideration of dimensional reduction from four-dimensional $N = 1$ theory may be helpful to realize the reason why the models without constraints are allowed. But it is not trivial whether the B models can be obtained by dimensional reduction of some mother theory as discussed in Section 2.4, and we need more detailed investigations to answer the question.

We have then applied the technique of dependent-central-charges to off-shell invariant $D = N = 4$ USp(4) SYM theory with central charge. The ansatz including dependent-central-charges in the most general form is considered. However, we have eventually concluded that the four-dimensional theory does not contain such a model without constraints as far as central charge, which are dependent, are included in the algebra. However the possibility that including more than one central charge, which is independent, alters the situation remains.

On the other hand, we have also found two kinds of appropriate ansatz including one of the central charges U_0, U_5, V_0, and V_5. We have also shown that the two kinds of ansatz can be interchanged to each other by some redefinitions of gauge-supercovariant derivative in construction, which means that the two kinds of ansatz are equivalent. We have obtained the supermultiplets and actions of the two models i.e. the (V_0, U_5) model and the (U_0, V_5) model, which means that we have formulated each model with one central charge U_0, U_5, V_0, or V_5. Although only V_0 can be regarded as five-dimensional momentum from the view point of dimensional reduction by Legendre transformation as in Ref.[5], any of the models similarly needs a constraint to be off-shell invariant. In other words, the fact that a constraint is necessary is independent from which one central charge is included among the four possible central charges.

Acknowledgements

I would like to thank my supervisor prof. N. Kawamoto for his encouragement thoughout my student life in Hokkaido university. I am grateful to J. Kato, and A. Miyake for useful comments and discussions. I would like also to thank Y. Kondo for useful discussions. I am also grateful to J. Saito for insightful comments and discussions. I would like also to thank R. Nakayama for useful comments and discussions. I would like also to thank all the members of Theoretical Particle Physics Group at Hokkaido university.

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