Mathematical Studies on Dirac Operators with a Variable Mass with Application to the Chiral Quark Soliton Model

A Dissertation presented by

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February, 2014

Abstract

The Hamiltonian of the Chiral Quark Soliton model (CQS) in nuclear physics is described by the following Dirac type operator

$$H_{\text{CQS}} := -i\sum_{j=1}^{3}\alpha_j D_j \otimes 1_2 + m\beta \otimes 1_2 e^{i\sum_{j=1}^{3} F\gamma_5\otimes\sigma_j n_j},$$

on the Hilbert space $L^2(\mathbb{R}^3;\mathbb{C}^4)\otimes\mathbb{C}^2$.

Let us compare it to the usual free Dirac operator with mass m

$$H_{\text{UD}} := -i\sum_{j=1}^{3}\alpha_j D_j + m\beta,$$

on the Hilbert space $L^2(\mathbb{R}^3;\mathbb{C}^4)$.

Here i is the imaginary unit, $\alpha_1,\alpha_2,\alpha_3$ and $\beta=\alpha_4$ are 4×4 Dirac matrices and 1_n denotes the $n\times n$ unit matrix, D_j $(j=1,2,3)$ is the generalized partial differential operator in the space variable x_j ($\mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$), $m>0$ denotes the mass of a quark,

$$\gamma_5 := -i\alpha_1\alpha_2\alpha_3,$$

$F:\mathbb{R}^3\to\mathbb{R}$ is called a *profile function*, Borel measurable, finite for almost everywhere (a.e.) $\mathbf{x}\in\mathbb{R}^{\mathbf{3}}$, σ_1,σ_2 and σ_3 are the Pauli matrices and $n_j:\mathbb{R}^3\to\mathbb{R}$ is a Borel measurable function such that

$$\sum_{j=1}^{3} n_j(x)^2 = 1$$

for a.e. $\mathbf{x}\in\mathbb{R}^{\mathbf{3}}$.

The main difference in the above operators are the mass term. The mass term of H_{CQS} is spatially variable in general. Hence, the CQS model may be regarded as a model of Dirac particle with a variable mass.

This thesis is mainly based on the paper [3]. The main purpose of this work is to build a model, which can be an abstract d-dimensional extension of the CQS model and under suitable conditions to investigate its Hamiltonian's self-adjoint property, supersymmetric aspects and spectral properties. We will name the Hamiltonian of this extended model by *"d-dimensional Dirac operator with a variable mass"*. The Hamiltonian of a *d-dimensional chiral quark soliton model* is defined as follows:

$$H := -i\sum_{j=1}^{d}\alpha_j D_j + \alpha_{d+1}e^{i\Phi}M.$$

Here $d\geq 2$ is natural number,

$$N_d := \begin{cases} 2^{d/2} & \text{for } d \text{ even} \\ 2^{(d+1)/2} & \text{for } d \text{ odd,} \end{cases}.$$

$\alpha_j,\quad j=1,\dots,d+1$ are $N_d\times N_d$ Hermitian matrices satisfying

$$\{\alpha_j,\alpha_k\} = 2\delta_{jk}1_{N_d},\quad j,k=1,\cdots,d+1.$$

Let $\mathcal{K}$ be a finite dimensional Hilbert space. We denote by $\mathcal{F}_{\text{s.a.}}$ the set of self-adjoint operators Φ on $L^2(\mathbb{R}^d; \mathbb{C}^{N_d} \otimes \mathcal{K})$ such that the mapping $:\mathbb{R}^d \ni \mathbf{x} \to (\Phi(\mathbf{x})+i)^{-1}$ is measurable. $\Phi(\cdot), \quad M(\cdot) \in \mathcal{F}_{\text{s.a.}}$ Φ and M be the direct integrals of $\Phi(\cdot), \quad M(\cdot)$ respectively over $\mathbb{R}^d$.

The *"d-dimensional Dirac operator with a variable mass"* acts on

$$\mathcal{H} := L^2(\mathbb{R}^d; \mathbb{C}^{N_d}) \otimes \mathcal{K}.$$

In this work we will give a simple condition for H to be self-adjoint and discuss supersymmetric aspects and the spectrum of H. Also we give a condition for H to be a supercharge of a supersymmetric quantum mechanical model. In that case, $\ker H$, the kernel of H, describes the supersymmetric states. Hence it is interesting and important to analyze $\ker H$. We will prove that, under some condition, $\ker H$ is trivial: $\ker H = \{0\}$. In the case where H is a supercharge, this means that there is no supersymmetric state, namely, the supersymmetry is spontaneously broken. We are concerned with a unitary equivalence of H to a gauge theoretic Dirac operator. This may be physically interesting. Using this structure, we find another condition for the kernel of H to be trivial. We identify the essential spectrum of H under a suitable condition. In the last, we will discuss the number of eigenvalues of H in the interval $(-m, m)$ with $m > 0$ being a constant.

Acknowledgement

I am heartily thankful to Prof. Asao Arai for suggesting very interesting studies of Dirac type operators and his continuous support in all stages of my research. His guidance helped me in all the time of research and writing of this dissertation.

I would like to thank every one who has helped me. Also I and my family would like to express warmly thanks to Japanese government and all Japanese people for giving us a chance to study and live in a beautiful country Japan.

Contents

Chapter 1

Introduction

1.1 Dirac operator

In particle physics, the Dirac equation is a relativistic wave equation formulated by British physicist Paul Dirac in 1928 (e.g., see [11], [18]). It describes fields corresponding to elementary spin $\frac{1}{2}$ particles as a vector of four complex numbers, in contrast to the Schrdinger equation which describes a field of only one complex value. The Dirac equation is consistent with both the principles of quantum mechanics and the theory of special relativity, and was the first theory to account fully for relativity in the context of quantum mechanics. The equation also implied the existence of a new form of matter, antimatter, hitherto unsuspected and unobserved, and actually predicted its experimental discovery. It also provided a theoretical justification for the introduction of several-component wave functions in Pauli's phenomenological theory of spin. Moreover, in the limit of zero mass, the Dirac equation reduces to the Weyl equation.

The equation to describe a relativistic wave equation of a free electron must be (e.g., see [11]) Lorentz invariant, first order in time derivative and energy E must be calculated by the following formula

$$E = c\sqrt{m^2c^2 + |\mathbf{p}|^2} \tag{1.1}$$

Here c is the light speed, m is the mass of the particle and $\mathbf{p} = (p_1, p_2, p_3)$ is the momentum.

From non-relativistic theory, for the energy E and momentum $\mathbf{p}$ we have the following substutions

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \mathbf{p} \to -i\hbar\nabla, \tag{1.2}$$

here $t \in \mathbb{R}$, $\quad \mathbf{x} = (x_1, x_2, x_3)$ is coordinate, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ and $\hbar$ is the Planck constant divided by 2π. So Dirac reconsidered the energy-momentum relation (1.1) and before translating it to quantum mechanics with the substitution (1.2), he linearized it and wrote:

$$E = c\sum_{j=1}^{3}\alpha_j p_j + mc^2\beta \equiv \boldsymbol{\alpha}\cdot\mathbf{p} + mc^2\beta. \tag{1.3}$$

Here $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β have to be determined from (1.1). Indeed, (1.1) can be satisfied $\alpha_j \quad (j = 1, 2, 3)$ and β anti commuting $n \times n$ matrices (Dirac matrices). Comparing E^2 in (1.3) with (1.1) we will find the following relations:

$$\begin{aligned} &\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}\mathbf{1_n} \quad (j, k = 1, 2, 3) \\ &\alpha_j\beta + \beta\alpha_j = \mathbf{0_n} \quad (j = 1, 2, 3) \\ &\beta^2 = \mathbf{1_n}. \end{aligned} \tag{1.4}$$

Here we denote δ_{jk} is the Kronecker delta, $\mathbf{1_n}$ and $\mathbf{0_n}$ are $n \times n$ unit and zero matrices . The $\alpha_j \quad (j = 1, 2, 3)$ and β should be Hermitian. For representation of $\alpha_j \quad (j = 1, 2, 3)$ and β matrices we will use the following matrices, which are named the Paul matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.5}$$

The following representation was introduced by Dirac and named the standard representation:

$$\beta := \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}, \quad \alpha_j := \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix}, \quad j = 1, 2, 3. \tag{1.6}$$

If one "translates" the equation (1.3) to quantum mechanics, one obtains the Dirac equation

$$i\hbar \frac{\partial}{\partial t} \psi(t, \mathbf{x}) = H_{FD}(m) \psi(t, \mathbf{x}). \tag{1.7}$$

The operator

$$H_{FD}(m) := -i\hbar c \left(\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3} \right) + mc^2 \beta = -i\hbar c \boldsymbol{\alpha} \cdot \nabla + mc^2 \beta \tag{1.8}$$

is named the free Dirac operator and it acts on $\mathbb{C}^4$-valued wavefunctions

$$\psi(t, \mathbf{x}) := \begin{pmatrix} \psi_1(t, \mathbf{x}) \\ \vdots \\ \psi_4(t, \mathbf{x}) \end{pmatrix}. \tag{1.9}$$

The square of the free Dirac operator is

$$H_{FD}(m)^2 = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) + m^2 = -\Delta + m^2 \tag{1.10}$$

the Laplacian operator with mass in $\mathbb{R}^3$.

1.2 d-dimensional free Dirac operator

In this section we will define a *"d-dimensional"* free Dirac operator on $\mathbb{R}^d$. From now we will use the physical unit system where the light speed c and $\hbar$ are equal to 1.

Let σ_j be the Paul matrices and

$$\gamma_1^0 := I_2 \quad \gamma_1^j := \sigma_j \quad j = 1, 2, 3. \tag{1.11}$$

Then by the following recursive formula

$$\gamma_n^0 := I_2 \otimes \gamma_{n-1}^0$$

$$\gamma_n^j := \sigma_1 \otimes \gamma_{n-1}^j, \quad j = 1, ...2n-1$$

$$\gamma_n^{2n} := \sigma_2 \otimes \gamma_{n-1}^0, \quad \gamma_n^{2n+1} := (-i)^n \gamma_n^1 \gamma_n^2 \cdots \gamma_n^{2n}$$

we can build $2^n \times 2^n$ anticommuting Hermitian $2n+1$ distinctive matrices $\{\gamma_n^j\}, \quad j = 1, \ldots, 2n+1$ (e.g., [1]). For notational simplicity for fixed number n,we denote γ_n^k by $\alpha_k, \quad k = 1, \ldots, 2n+1$ and name it 2^n *dimensional Dirac matrices.*

Let $d \geq 2$ be a natural number,

$$N_d := \begin{cases} 2^{d/2} & \text{for } d \text{ even} \\ 2^{(d+1)/2} & \text{for } d \text{ odd} \end{cases}. \tag{1.12}$$

and $\{\alpha_j\}_{j=1}^{d+1}$ be $N_d \times N_d$ Dirac matrices. We denote by D_j the generalized partial differential operator in the variable x_j ($\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$), acting in $L^2(\mathbb{R}^d)$. The d-dimensional generalized Laplacian

$$\Delta := \sum_{j=1}^{d} D_j^2 \tag{1.13}$$

on $L^2(\mathbb{R}^d)$ is a non-positive self-adjoint operator.

The d dimensional free Dirac operator with mass m on $L^2(\mathbb{R}^d; \mathbb{C}^{N_d})$ is defined by

$$H_m := -i \sum_{j=1}^{d} \alpha_j D_j + m\alpha_{d+1}. \tag{1.14}$$

The operator, square of H_m, is positive and

$$H_m^2 = -\Delta + m^2. \tag{1.15}$$

Theorem 1.1 *The d dimensional free Dirac operator H_m is essentially self-adjoint on the dense domain $\mathcal{C}_l^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{C}^{N_d})$ and self-adjoint on the Sobelov space*

$$D(H_m) = \mathbf{H}^1(\mathbb{R}^d; \mathbb{C}^{N_d}) = \bigcap_{j=1}^{d} D(D_j).$$

Its spectrum is purely absolutely continuous and

$$\sigma(H_m) = (-\infty; -m] \cup [m; \infty). \tag{1.16}$$

Proof.
See [Theorem 1.1][18]. □

1.3 Dirac type operator in Chiral Quark Soliton Model

Since in 1964 Murray Gell-Mann and George Zweig had predicted quark, physics have studied many quark models. The QCS model is a relativistic quark model (e.g., [7]). The Hamiltonian of CQS model is an abstract Dirac type operator with matrix-valued mass term (e.g., [16]).

$$H_{\text{CQS}} := -i \sum_{j=1}^{3} \alpha_j D_j \otimes 1_2 + m\beta \otimes 1_2 e^{i \sum_{j=1}^{3} F\gamma_5 \otimes \sigma_j n_j},$$

on the Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathbb{C}^2$.

Here $\alpha_j, \quad j = 1, 2, 3$ and β are 4×4 the Dirac matrices, $\gamma_5 := -i\alpha_1\alpha_2\alpha_3$, $\sigma_j, \quad j = 1, 2, 3$ are the Paul matrices, $F : \mathbb{R}^3 \to \mathbb{R}$ measurable, a.e. (almost everymhere) finite, $n_j : \mathbb{R}^3 \to \mathbb{R} \quad j = 1, 2, 3,$ measureable with $\sum_{j=1}^{3} |n_j(\mathbf{x})|^2 = 1 \quad$ a.e $\mathbf{x} \in \mathbb{R}^3$. The function F is called *profile function.*

$H_{\rm CQS}$ is self-adjoint on $D(H_{\rm CQS}) = D(H_0)$ by Kato-Rellich theorem.

Let $n_j(\mathbf{x})$, $j = 1, 2, 3$ is continuously differentiable on $\mathbb{R}^3$, $(n_1(\mathbf{x}), n_2(\mathbf{x})) \neq (0, 0)$,

$$\xi(\mathbf{x}) := \frac{\sigma_1 n_2(\mathbf{x}) - \sigma_2 n_1(\mathbf{x})}{\sqrt{n_1(\mathbf{x})^2 + n_2(\mathbf{x})^2}} \tag{1.17}$$

and $\Gamma(\mathbf{x}) := i\gamma_5\beta \otimes \xi(\mathbf{x})$ acting on $\mathbb{C}^4 \otimes \mathbb{C}^2$. Then $(\Gamma\phi)(\mathbf{x}) := \Gamma(\mathbf{x})\phi(\mathbf{x})$ is self-adjoint, $\Gamma^2 = 1$ on $\mathcal{H}$ and unitary.

Theorem 1.2 *([4, Prop 2.1]) If ξ is constant matrix then $\forall\psi \in D(H_{\rm CQS}), \Gamma\psi \in D(H_{\rm CQS})$ and*

$$\{\Gamma, H_{\rm CQS}\}\psi = 0, \quad \psi \in D(H_{\rm CQS}). \tag{1.18}$$

From (1.2) implies $H_{\rm CQS} = -\Gamma H_{\rm CQS}\Gamma^*$ and in suitable condition $H_{\rm CQS}$ can be a generator of supersymmetry with grading operator Γ.

Theorem 1.3 *Suppose that*

$$\lim_{|\mathbf{x}|\to\infty} F(\mathbf{x}) = 0$$

then

$$\sigma(H_{CQS}) = (-\infty; -m] \cup [m; \infty).$$

1.4 d-dimensional version of Chiral Quark Soliton Model

Let $d \geq 2$ be a natural number and H_0 be free massless Dirac operator on $L^2(\mathbb{R}^d; \mathbb{C}^{N_d})$,

$$H_0 := -i\sum_{j=1}^{d} \alpha_j D_j. \tag{1.19}$$

then

$$H_0^2 = -\Delta. \tag{1.20}$$

Let $\mathcal{K}$ be a separable complex Hilbert space and

$$\mathcal{H} := L^2(\mathbb{R}^d; \mathbb{C}^{N_d}) \otimes \mathcal{K} \cong L^2(\mathbb{R}^d; \mathbb{C}^{N_d} \otimes \mathcal{K}) \cong \int_{\mathbb{R}^d}^{\oplus} \mathbb{C}^{N_d} \otimes \mathcal{K} dx, \tag{1.21}$$

where each $\cong$ means the natural Hilbert space isomorphism and $\int_{\mathbb{R}^d}^{\oplus} \mathbb{C}^{N_d} \otimes \mathcal{K} dx$ denotes the constant fibre direct integral with fiber $\mathbb{C}^{N_d} \otimes \mathcal{K}$ (e.g., [15, §XIII.16]). Each linear operator A on $L^2(\mathbb{R}^d)$ is extended as the direct sum $\oplus^{N_d} A$ on $L^2(\mathbb{R}^d; \mathbb{C}^{N_d}) = \oplus^{N_d} L^2(\mathbb{R}^d)$. For notational simplicity, we denote it by A again.

Every densely defined closable linear operator T on $L^2(\mathbb{R}^d; \mathbb{C}^{N_d})$ (resp. $\mathcal{K}$) has a tensor product extension $T \otimes I$ (resp. $I \otimes T$) to $\mathcal{H}$ (I denotes identity). But we write it T simply if there is no danger of confusion.

We denote by $\mathcal{F}_{\rm s.a.}$ the set of mappings $\Phi(\cdot)$ from $\mathbb{R}^d$ to the set of self-adjoint operators on $\mathbb{C}^{N_d} \otimes \mathcal{K}$ such that the mapping: $\mathbb{R}^d \ni x \mapsto (\Phi(x) + i)^{-1}$ is measurable. By a general theorem (e.g., [15, Theorem XIII.85-(i)]), for each $\Phi(\cdot) \in \mathcal{F}_{\rm s.a.}$, the direct integral

$$\Phi := \int_{\mathbb{R}^d}^{\oplus} \Phi(x) dx \tag{1.22}$$

is self-adjoint.

To introduce a mass operator, let $M(\cdot) \in \mathcal{F}_{\text{s.a.}}$ such that, for a.e.$x \in \mathbb{R}^d$ $M(x)$ is a *bounded* operator on $\mathbb{C}^{N_d} \otimes \mathcal{K}$, and set

$$M := \int_{\mathbb{R}^d}^{\oplus} M(x)dx. \tag{1.23}$$

We use this self-adjoint operator as an extended mass (variable in the space $\mathbb{R}^d$) of the quantum particle of our model (a Dirac particle). Note that M is *not necessarily bounded.* The Hamiltonian H of our model, a d-dimensional version of the CQS model, is defined as follows:

$$H := H_0 + V \tag{1.24}$$

with

$$V := \alpha_{d+1} e^{i\Phi} M. \tag{1.25}$$

Definition 1.4 Let A and B be self-adjoint operators on $\mathcal{X}$.

(i) A and B are said to *strongly commute* if their spectral measures commute.

(ii) A and B are said to *strongly anticommute* [12, 19] if, for all $t \in \mathbb{R}$, $e^{itB}A \subset Ae^{-itB}$. (it is shown from e.g. [12] that this definition is in fact symmetric in A and B).

The next lemma summarizes some basic facts on strongly commuting (resp. anticommuting) self-adjoint operators:

Lemma 1.5 *Let A and B be self-adjoint operators on $\mathcal{X}$.*

(i) *A and B strongly commute if and only if, for all $t, s \in \mathbb{R}$, $e^{itB}e^{isA} = e^{isA}e^{itB}$.*

(ii) *A and B strongly commute if and only if, for all $t \in \mathbb{R}$, $e^{itB}A = Ae^{itB}$.*

(iii) *Let A be bounded. Then A and B strongly commute if and only if $AB \subset BA$.*

(iv) *Let A be bounded. Then A and B strongly anticommute if and only if $AB \subset -BA$.*

Proof.

(i) well known (e.g., [13, Theorem VIII.13]).

(ii) $\forall \phi \in D(A)$ by [13, Theorem VIII.7(c)]

$$\lim_{s \to 0} \frac{e^{itB}e^{isA} - e^{itB}}{is}\phi = e^{itB}A\phi. \tag{1.26}$$

From (i)

$$e^{itB}A\phi = \lim_{s \to 0} \frac{e^{itA} - I}{is} e^{itB}\phi. \tag{1.27}$$

Then by [13, Theorem VIII.7(d)] $e^{itB}\phi \in D(A)$ and (ii) holds.

(iii) Using (ii) and same to proof of (ii).

(iv) Same to proof of (iii).

□

In the work, we assume the followings:

(A.1) $\alpha_{d+1}\Phi \subset -\Phi\alpha_{d+1}$

(A.2) $\alpha_{d+1}M \subset M\alpha_{d+1}$.

(A.3) Φ and M strongly commute.

(A.4) The operator M is $(-\Delta)^{1/2}$-bounded:

$$\|M\psi\|^2 \le a^2\|(-\Delta)^{1/2}\psi\|^2 + b^2\|\psi\|^2, \quad \forall\psi \in D((-\Delta)^{1/2})$$

with constants $0 \le a < 1$ and $b \ge 0$.

Remark 1.6 In the abstract CQS model [10], the strong commutativity of M and H_0 as well as the boundedness and the strict positivity of M is assumed. But, in our model, they are not assumed.

Lemma 1.7

(i) *Condition (A.1) holds if and only if α_{d+1} and Φ strongly anticommute.*

(ii) *Condition (A.1) is equivalent to the operator equality $\alpha_{d+1}\Phi = -\Phi\alpha_{d+1}$.*

(iii) *Condition (A.2) holds if and only if α_{d+1} and M strongly commute.*

(iv) *Condition (A.2) is equivalent to the operator equality $\alpha_{d+1}M = M\alpha_{d+1}$.*

Proof.

(i) This follows from Lemma 1.5-(iv).

(ii) Assume (A.1). Let $\psi \in D(\Phi\alpha_{d+1})$. Then $\eta := \alpha_{d+1}\psi \in D(\Phi)$. Hence, by (A.1), $\alpha_{d+1}\eta \in D(\Phi)$. But, since $\alpha_{d+1}^2 = I$, we have $\alpha_{d+1}\eta = \psi$. Hence $\psi \in D(\Phi)$. Therefore $D(\Phi\alpha_{d+1}) \subset D(\alpha_{d+1}\Phi)$. Thus $D(\Phi\alpha_{d+1}) = D(\alpha_{d+1}\Phi)$. Hence the desired operator equality holds.

(iii) This follows from Lemma 1.5-(iii).

(iv) Simliar to the proof of the part (ii).

□

1.5 Organization of the dissertation

This dissertation is based mainly on the joint work [3] of Arai Asao and the candidate. In Chapter 2 is concerned self-adjointness and supersymmetric aspects of H. In 2.1 is given suitable conditions for H is self-adjoint. In 2.2 is discussed supersymmetric aspect. In [3], was considered only odd dimensions of $\mathbb{R}^d$. In this dissertation is discussed ever d is even. For supersymmetric operators, the kernel, i.e. supersymmetric states are very interesting and important. In 2.3 we will show in our case the symmetry is spontaneously broken. In 2.4 we concern about unitary equivalence of H to a gauge theoretic Dirac operator. This may be physically interesting.

Chapter 3 is concerned spectrum of H. In 3.1 is considered essential spectrum. In 3.2 we discuss the number of eigenvalues in the interval $(-m, m)$.

Finally some remarks of typographical nature. Chapter, section and subsection are numbered in arabic numerals. Equations are numbered sequentially within a chapter number. Also mathematica (*i.e.,* lemmas, theorems,...) are numbered sequentially within a chapter number.We use □ notation to signify the and of a proof.

Chapter 2

Hamiltonian of d-dimensional CQS Model

2.1 Self-adjointness

We define

$$H_M := H_0 + \alpha_{d+1} M. \tag{2.1}$$

If M is a constant operator with $m > 0$, then H_m represents the free Dirac operator with a constant mass m. H_m is self-adjoint with $D(H_m) = D(H_0) = \cap_{j=1}^{d} D(D_j)$ (Theorem(1.1)) and bijective with $\|H_m^{-1}\| = 1/m$.

Lemma 2.1 *Assume (A.4). Let $m > 0$ be a constant. Then MH_m^{-1} is bounded with*

$$\|MH_m^{-1}\| \leq \max\left\{a, \frac{b}{m}\right\}. \tag{2.2}$$

Proof.
It is well known or easy to see that, for all $\psi \in D(H_m) = D(H_0)$,

$$\|H_m\psi\|^2 = \|(-\Delta)^{1/2}\psi\|^2 + m^2\|\psi\|^2$$

Hence, by (A.4), we have

$$\|M\psi\|^2 \leq a^2\|H_m\psi\| + (b^2 - a^2m^2)\|\psi\|.$$

This implies the following:
(i) if $am \leq b$, then $\|MH_m^{-1}\|^2 \leq a^2 + (b^2 - a^2m^2)/m^2 = b^2/m^2$
(ii) if $am \geq b$, then $\|MH_m^{-1}\| \leq a$. Thus (2.2) follows. □

Lemma 2.2 *Assume (A.1)–(A.4). Then:*

(i) *V is self-adjoint with $D(V) = D(M)$.*

(ii) *H is self-adjoint with $D(H) = D(H_0)$ and the subspace*

$$\mathcal{D}_0 := C_0^\infty(\mathbb{R}^d; \mathbb{C}^{N_d})\hat{\otimes}\mathcal{K} \tag{2.3}$$

($\hat{\otimes}$ means algebraic tensor product) is a core of H.

Proof.

(i) Since $D(V) = D(M)$ and $D(M)$ is dense, V is densely defined. Since $\alpha_{d+1}e^{i\Phi}$ is bounded, it follows that $V^* = Me^{-i\Phi}\alpha_{d+1}$. By (A.1) and Lemma 1.5-(iv), we have

$$e^{-i\Phi}\alpha_{d+1} = \alpha_{d+1}e^{i\Phi}. \tag{2.4}$$

By (A.2), (A.3), Lemma 1.5-(ii) and Lemma 1.7-(iv), we have

$$V^* = M\alpha_{d+1}e^{i\Phi} = \alpha_{d+1}Me^{i\Phi} = \alpha_{d+1}e^{i\Phi}M = V. \tag{2.5}$$

Hence V is self-adjoint.

(ii) By (A.4), we have for all $\psi \in D((-\Delta)^{1/2})$

$$\|V\psi\| = \|M\psi\| \le a\|(-\Delta)^{1/2}\psi\| + b\|\psi\|.$$

Note that

$$\|(-\Delta)^{1/2}\psi\| = \|H_0\psi\|.$$

Hence $\|V\psi\| \le a\|H_0\psi\| + b\|\psi\|$. Here $0 \le a < 1$. Thus, by the Kato-Rellich theorem (e.g., [14, Theorem X.12]), H is self-adjoint with $D(H) = D(H_0)$ and every core of H_0 is a core of H. It is well known that the subspace $C_0^\infty(\mathbb{R}^d;\mathbb{C}^{N_d})$ is a core of H_0 as a linear operator on $L^2(\mathbb{R}^d;\mathbb{C}^{N_d})$. Hence the subspace $\mathcal{D}_0$ defined by (2.3) is a core of H_0 as a linear operator on $\mathcal{H}$. Thus it is a core of H too.

□

Remark 2.3 One of the other sufficient conditions for H to be essentially self-adjoint is as follows: *Assume (A.1)–(A.3) and* $\mathrm{ess.sup}_{|x|<R}\|M(x)\| < \infty$ *for all* $R > 0$*. Then* H *is essentially self-adjoint on* $\mathcal{D}_0$. The proof is similar to that of [18, Theorem 4.3].

2.2 Supersymmetric aspects

As is well known, the standard free Dirac operator $-i\sum_{j=1}^{3}\alpha_j D_j + m\beta$ on $L^2(\mathbb{R}^3;\mathbb{C}^4)$ with constant mass $m \ge 0$ and its suitably perturbed ones have supersymmetry, i.e., they are respectively a supercharge with the grading operator $i\beta\gamma_5$ [18, §5.5]. From this point of view, it would be interesting to investigate if the Hamiltonian H of the present model has supersymmetry. Indeed, it was shown that the Hamiltonian of the CQS model as well as that of the GCQS model has supersymmetry [2, 4]. In this section we see that a supersymmetric structure similar to that of the CQS (GCQS) model exists in our model.

Let

$$T_d := \begin{cases} 2^{(d+1)/2} & d\text{−odd} \\ 2^{d/2+1} & d\text{−even} \end{cases} \tag{2.6}$$

In this section, we will use $T_d \times T_d$ Dirac matrix representation. Then there are $d+2$ distinct Dirac matrices for odd number d, $d+3$ distinct Dirac matrices for even number d.

Let

$$\gamma_5^{(d)} := \begin{cases} i^{d(d-1)/2}\alpha_1\cdots\alpha_d & \text{d−}odd \\ i^{d(d+1)/2}\alpha_1\cdots\alpha_d\alpha_{d+2} & \text{d−}even \end{cases} \tag{2.7}$$

Then γ_5^d is self-adjoint with

$$(\gamma_5^{(d)})^2 = 1_{T_d}. \tag{2.8}$$

Then we have

$$\alpha_j \gamma_5^{(d)} = \gamma_5^{(d)} \alpha_j \ (j = 1, \cdots, d), \quad \{\alpha_{d+1}, \gamma_5^{(d)}\} = 0. \tag{2.9}$$

Let $\xi : \mathbb{R}^d \to \mathfrak{B}(\mathcal{K})$ be Borel measurable such that, for a.e. $x \in \mathbb{R}^d$, $\xi(x)$ is self-adjoint with

$$\xi(x)^2 = I. \tag{2.10}$$

Then

$$\|\xi(x)\| = 1, \quad \text{a.e.} x \in \mathbb{R}^d. \tag{2.11}$$

Let $\Gamma(\cdot) : \mathbb{R}^d \to \mathfrak{B}(\mathbb{C}^{T_d} \otimes \mathcal{K})$ be such that

$$\Gamma(x) := i\gamma_5^{(d)} \alpha_{d+1} \otimes \xi(x), \quad \text{a.e.} x \in \mathbb{R}^d. \tag{2.12}$$

Then

$$\Gamma := \int_{\mathbb{R}^d}^{\oplus} \Gamma(x) dx \tag{2.13}$$

is self-adjoint with

$$\Gamma^2 = I. \tag{2.14}$$

Hence Γ is a grading operator on $\mathcal{H}$. The following proposition shows that, under some additional condition for $\xi(x)$, H has supersymmetry with respect to Γ:

Proposition 2.4 *Assume (A.1)–(A.4). Suppose that ξ is strongly differentiable on $\mathbb{R}^d$ with*

$$C_j := \sup_{x \in \mathbb{R}^d} \|D_j \xi(x)\| < \infty, \quad j = 1, \cdots, d. \tag{2.15}$$

Then

$$\Gamma H \subset -H\Gamma \tag{2.16}$$

if and only if

$$\sum_{j=1}^{d} \gamma_5^{(d)} \alpha_{d+1} \alpha_j D_j \xi(x) = i \left(\gamma_5^{(d)} \otimes \xi(x) e^{i\Phi(x)} M(x) - M(x) e^{-i\Phi(x)} \gamma_5^{(d)} \otimes \xi(x) \right), \quad \text{a.e.} x \in \mathbb{R}^d. \tag{2.17}$$

In that case, the spectrum $\sigma(H)$ and the point spectrum $\sigma_{\rm p}(H)$ of H are respectively symmetric with respect to the origin $0 \in \mathbb{R}$.

Proof.
Since the subspace $\mathcal{D}_0$ given by (2.3) is a core of H by Lemma 2.2-(ii), (2.16) is equivalent to that, for all $\psi \in \mathcal{D}_0$, $\Gamma\psi \in D(H)$ and

$$\Gamma H \psi = -H \Gamma \psi. \tag{2.18}$$

Let $\psi \in \mathcal{D}_0$. Then

$$(\Gamma\psi)(x) = i\gamma_5^{(d)} \alpha_{d+1} \otimes \xi(x) \psi(x), \quad \text{a.e.} x \in \mathbb{R}^d.$$

It follows that the $\mathbb{C}^{T_d} \otimes \mathcal{K}$-valued function: $x \mapsto (\Gamma\psi)(x)$ is strongly differentiable on $\mathbb{R}^d$ with

$$D_j(\Gamma\psi)(x) = i\gamma_5^{(d)} \alpha_{d+1} \otimes (D_j \xi(x)) \psi(x) + i\gamma_5^{(d)} \alpha_{d+1} \otimes \xi(x) D_j \psi(x), \quad j = 1, \cdots, d.$$

Hence

$$\|D_j(\Gamma\psi)(x)\|^2 \le 2(C_j^2 \|\psi(x)\|^2 + \|D_j \psi(x)\|^2),$$

which implies that $D_j\Gamma\psi \in \mathcal{H}$ and hence $\Gamma\psi \in D(H_0) = D(H)$. Moreover, we have

$$(H_0\Gamma\psi)(x) = -\gamma_5^{(d)}\alpha_{d+1}\left(\sum_{j=1}^{d}\alpha_j D_j\xi(x)\right)\psi(x) - (\Gamma H_0\psi)(x).$$

and

$$(V\Gamma\psi)(x) = -(\Gamma V\psi)(x) + i\left(\gamma_5^{(d)} \otimes \xi(x)e^{i\Phi(x)}M(x) - M(x)e^{-i\Phi(x)}\gamma_5^{(d)} \otimes \xi(x)\right)\psi(x).$$

Hence

$$\begin{aligned}(H\Gamma\psi)(x) \quad &= \quad -(\Gamma H\psi)(x) - \sum_{j=1}^{d}\gamma_5^{(d)}\alpha_{d+1}\alpha_j \otimes (D_j\xi(x))\psi(x) \\ &\quad +i\left(\gamma_5^{(d)} \otimes \xi(x)e^{i\Phi(x)}M(x) - M(x)e^{-i\Phi(x)}\gamma_5^{(d)} \otimes \xi(x)\right)\psi(x).\end{aligned}$$

Therefore, $H\Gamma\psi = -\Gamma H\psi$ for all $\psi \in \mathcal{D}_0$ if and only if

$$\begin{aligned}&\sum_{j=1}^{d}\gamma_5^{(d)}\alpha_{d+1}\alpha_j \otimes (D_j\xi(x))\psi(x) \\ &= i\left(\gamma_5^{(d)} \otimes \xi(x)e^{i\Phi(x)}M(x) - M(x)e^{-i\Phi(x)}\gamma_5^{(d)} \otimes \xi(x)\right)\psi(x), \forall\psi \in \mathcal{D}_0. \qquad (2.19)\end{aligned}$$

By the original assumption for $M(\cdot)$, $M(x) \in \mathfrak{B}(\mathbb{C}^{T_d} \otimes \mathcal{K})$ for a.e.$x \in \mathbb{R}^d$. Thereore (2.19) is equivalent to (2.17).

By (2.14) and $\Gamma^* = \Gamma$, one easily sees that (2.16) is in fact equivalent to operator equality $\Gamma^* H\Gamma = -H$. Hence H is unitarily equivalent to $-H$. This implies the symmetry of $\sigma(H)$ and $\sigma_{\mathrm{p}}(H)$ with respect to the origin. □

Remark 2.5 Proposition 2.4 gives a generalization of [2, Theorem 1] and clarifies a condition for H to have supersymmetry.

It may be difficult in general to show the existence of self-adjoint, unitary solutions $\xi(x)$ to operator equation (2.17). Here we only note the following fact:

Lemma 2.6 *Assume (A.1)–(A.3). Suppose that*

$$\gamma_5^{(d)}\Phi(x) \subset \Phi(x)\gamma_5^{(d)}, \quad \text{a.e.}x \in \mathbb{R}^d, j = 1, \cdots, d, \qquad (2.20)$$

$\xi = \xi(x)$ is independent of $x \in \mathbb{R}^d$ and

$$(I \otimes \xi)\Phi(x) \subset -\Phi(x)(I \otimes \xi), \qquad (2.21)$$

$$(\gamma_5^{(d)} \otimes \xi)M(x) \subset M(x)(\gamma_5^{(d)} \otimes \xi), \quad \text{a.e.}x \in \mathbb{R}^d. \qquad (2.22)$$

Then ξ is a solution to (2.17).

Proof.
Since ξ is a constant operator, $D_j\xi = 0$. By Lemma 1.5-(iii), (2.20) implies the strong commutativity of $\gamma_5^{(d)}$ and Φ. Hence $\gamma_5^{(d)}e^{i\Phi(x)} = e^{i\Phi(x)}\gamma_5^{(d)}$ for a.e.$x \in \mathbb{R}^d$. By (2.21) and Lemma 1.5-(iv), $(I \otimes \xi)e^{i\Phi(x)} = e^{-i\Phi(x)}(I \otimes \xi)$. We also have (2.22) and the strong commutativity of Φ and M. Hence

$$(\gamma_5^{(d)} \otimes \xi)e^{i\Phi(x)}M(x) = M(x)(\gamma_5^{(d)} \otimes \xi)e^{i\Phi(x)} = M(x)e^{-i\Phi}(\gamma_5^{(d)} \otimes \xi). \qquad (2.23)$$

Thus (2.17) holds with the both sides being zero. □

Additionally we make a remark on the converse of Lemma 2.6. For this purpose, we need a lemma:

Lemma 2.7 *Let T_j $(j = 1, \cdots, d)$ be a densely defined closed linear operator on $\mathcal{K}$. Suppose that*

$$\sum_{j=1}^{d} \alpha_j \otimes T_j = 0 \quad on\ \cap_{j=1}^{d} D(\alpha_j \otimes T_j). \tag{2.24}$$

Then, for all $j = 1, \ldots, d$, $T_j = 0$ on $\cap_{j=1}^{d} D(T_j)$.

Proof.
Eq.(2.24) implies that, for all $u \in \cap_{j=1}^{d} D(T_j)$ and $v \in \mathcal{K}$, $\sum_{j=1}^{d} \langle v, T_j u \rangle \alpha_j = 0$. Since $\{\alpha_j\}_{j=1}^{d}$ is linearly independent, it follows that $\langle v, T_j u \rangle = 0, j = 1, \cdots, d$. Hence $T_j u = 0, j = 1, \cdots, d$. □

The following lemma gives a sufficient condition for a solution to (2.17) to be a constant operator:

Lemma 2.8 *Assume (A.1)–(A.3). Let $\xi(x)$ be strongly differentiable on $\mathbb{R}^d$ with (2.15) and be a solution to (2.17). Suppose that (2.20)–(2.22) hold. Then ξ is independent of $x \in \mathbb{R}^d$.*

Proof.
As in the proof of Lemma 2.6, (2.20)–(2.22) imply (2.23). Hence the right hand side of (2.17) vanishes, so that $\sum_{j=1}^{d} \gamma_5^{(d)} \alpha_{d+1} \alpha_j \otimes D_j \xi(x) = 0$, which implies that $\sum_{j=1}^{d} \alpha_j \otimes D_j \xi(x) = 0$. By Lemma 2.7, $D_j \xi(x) = 0, j = 1, \cdots, d$, which implies that ξ is independent of x. □

We have from Proposition 2.4 and Lemma 2.6 the following result:

Corollary 2.9 *Assume (A.1)–(A.4). Suppose that $\xi = \xi(x)$ is independent of $x \in \mathbb{R}^d$ and that (2.20)–(2.22) hold. Then H has supersymmetry with respect to Γ.*

2.3 Vanishing theorems of the kernel of H

In supersymmetric quantum mechanics with a supercharge Q, a non-zero vector in $\ker Q$ is called a *supersymmetric state*. If the kernel of Q vanishes, i.e., $\ker Q = \{0\}$, then the supersymmetry is said to be *spontaneously broken*. It turns out that, in supersymmetric quantum mechanics, it is importnat to investigate $\ker Q$. Thus we are led to consider $\ker H$ in view of Proposition 2.4. This would be interesting even if H does not have supersymmetry (note that H does not necessarily have supersymmetry).

To investigate $\ker H$, we also need an additional condition:

(A.5) (i) For each $f \in \mathbb{C}^{N_d} \otimes \mathcal{K}$, the function: $x \mapsto M(x)f$ is strongly differentiable on $\mathbb{R}^d$ and, for all $x \in \mathbb{R}^d$, $M(x)$ commutes with α_j $(j = 1, \cdots, d)$.

(ii) There exists a constant $\mu_0 > 0$ such that

$$M(x)^2 - i \sum_{j=1}^{d} \alpha_j \alpha_{d+1} D_j M(x) \geq \mu_0^2, \quad \forall x \in \mathbb{R}^d \tag{2.25}$$

as an operator inequality on $\mathbb{C}^{N_d} \otimes \mathcal{K}$ (note that, by the principle of uniform boundedness, the strong partial derivative $D_j M(x)$ is a bounded operator on $\mathbb{C}^{N_d} \otimes \mathcal{K}$ for each $x \in \mathbb{R}^d$ and hence, under (A.2) and condition (A.5)-(i), the opeartor $M(x)^2 - i \sum_{j=1}^{d} \alpha_j \alpha_{d+1} D_j M(x)$ on $\mathbb{C}^{N_d} \otimes \mathcal{K}$ is a bounded self-adjoint operator).

For a linear operator L on a Hilbert space, we denote the resolvent set of L by $\rho(L)$.

Lemma 2.10 *Assume (A.2), (A.4) and (A.5). Then H_M defined by (2.1) is self-adjoint with $D(H_M) = D(H_0)$ and*

$$\|H_0\psi\|^2 + \mu_0^2\|\psi\|^2 \le \|H_M\psi\|^2, \quad \psi \in D(H_0). \tag{2.26}$$

In particular, $0 \in \rho(H_M)$ with operator-norm bound

$$\|H_M^{-1}\| \le \frac{1}{\mu_0} \tag{2.27}$$

and $H_0H_M^{-1}$ is bounded with

$$\|H_0H_M^{-1}\| \le 1. \tag{2.28}$$

Moreover, MH_M^{-1} is bounded with

$$\|MH_M^{-1}\| \le a + \frac{b}{\mu_0}. \tag{2.29}$$

Proof.
The self-adjointness of H_M follows from that of H with $\Phi = 0$. For all $\psi \in \mathcal{D}_0$, using the anticommutativity of α_j with α_{d+1} and the commutativity of $M(x)$ with α_{d+1} and α_j ($j = 1, \cdots, d$), we have

$$\begin{aligned} \|H_M\psi\|^2 &= \|H_0\psi\|^2 + \|M\psi\|^2 + \sum_{j=1}^d \langle\psi, (-iD_jM)\alpha_j\alpha_{d+1}\psi\rangle \\ &\ge \|H_0\psi|^2 + \int_{\mathbb{R}^d} \left\langle \psi(x), \left(M(x)^2 - i\sum_{j=1}^d \alpha_j\alpha_{d+1}D_jM(x)\right)\psi(x)\right\rangle dx \\ &\ge \|H_0\psi\|^2 + \mu_0^2\|\psi\|^2. \end{aligned}$$

Hence (2.26) holds for all $\psi \in \mathcal{D}_0$. Since $\mathcal{D}_0$ is a core of H_M, this inequality extends to all $\psi \in D(H_0)$. In particular, we have

$$\mu_0\|\psi\| \le \|H_M\psi\|, \quad \psi \in D(H_0). \tag{2.30}$$

This implies that the self-adjoint operator H_M is bijective with (2.27).

Inequality (2.26) implies also that, for all $\psi \in D(H_0)$, $\|H_0\psi\| \le \|H_M\psi\|$. Hence $H_0H_M^{-1}$ is bounded with (2.28).

By (A.4) and $\|(-\Delta)^{1/2}\psi\| = \|H_0\psi\|$ for all $\psi \in D((-\Delta)^{1/2}) = D(H_0)$, we have $\|M\psi\| \le a\|H_0\psi\| + b\|\psi\|$. Hence, for all $\phi \in \mathcal{H}$,

$$\begin{aligned} \|MH_M^{-1}\phi\| &\le a\|H_0H_M^{-1}\phi\| + b\|H_M^{-1}\phi\| \le (a\|H_0H_M^{-1}\| + b\|H_M^{-1}\|)\|\phi\| \\ &\le \left(a + \frac{b}{\mu_0}\right)\|\phi\|. \end{aligned}$$

Thus (2.29) holds. □

Lemma 2.11 *Let A be a self-adjoint operator on a complex Hilbert space $\mathcal{X}$. Then*

$$\|e^{iA} - 1\| = 2\left\|\sin\frac{A}{2}\right\|. \tag{2.31}$$

Proof.
By the functional calculus, one has $e^{iA}-1 = 2ie^{iA/2}\sin(A/2)$. Hence $\|e^{iA}-1\| = 2\|e^{iA/2}\sin(A/2)\|$. Since $e^{iA/2}$ is unitary, one has $\|e^{iA/2}\sin(A/2)\| = \|\sin(A/2)\|$. Thus (2.31) holds. □

Theorem 2.12 *Assume (A.1)–(A.5) and*

$$\text{ess.sup}_{x\in\mathbb{R}^d}\left\|\sin\frac{\Phi(x)}{2}\right\| < \frac{1}{2(a+b\mu_0^{-1})}. \tag{2.32}$$

Then $\ker H = \{0\}$ *and* $0 \in \rho(H)$.

Moreover, the constant

$$\gamma(H) := \inf_{\psi\in D(H),\|\psi\|=1}\|H\psi\| \tag{2.33}$$

is strictly positive, $\gamma(H) \in \sigma(H)$ *or* $-\gamma(H) \in \sigma(H)$, *and*

$$\sigma(H) \subset (-\infty, -\gamma(H)] \cup [\gamma(H), \infty). \tag{2.34}$$

Proof.
The operator H is written as

$$H = H_M + \alpha_{d+1}(e^{i\Phi}-1)M = KH_M$$

with

$$K := I + \alpha_{d+1}(e^{i\Phi}-1)MH_M^{-1}.$$

By applying Lemma 2.11 with $A = \Phi(x)$, we have

$$\|e^{i\Phi(x)}-1\| = 2\left\|\sin\frac{\Phi(x)}{2}\right\|. \tag{2.35}$$

Therefore, for all $\psi \in \mathcal{H}$,

$$\|\alpha_{d+1}(e^{i\Phi}-1)MH_M^{-1}\psi\| \le 2\,\text{ess.sup}_{x\in\mathbb{R}^d}\|\sin(\Phi(x)/2)\|\|MH_M^{-1}\|\|\psi\|.$$

By this estimate and (2.29), we obtain

$$\|\alpha_{d+1}(e^{i\Phi}-1)MH_M^{-1}\| \le 2(\text{ess.sup}_{x\in\mathbb{R}^d}\|\sin(\Phi(x)/2)\|)(a+b\mu_0^{-1}).$$

Hence, by (2.32), we obtain $\|\alpha_{d+1}(e^{i\Phi}-1)MH_M^{-1}\| < 1$. This implies that K is bijective with bounded inverse K^{-1}. Thus H is bijective with $H^{-1} = H_M^{-1}K^{-1}$ being bounded. Hence $\ker H = \{0\}$ and $0 \in \rho(H)$.

We set $b_H := \|H^{-1}\|$. If $|\lambda| < 1/b_H$, then λ is in $\rho(H)$. Therefore

$$\sigma(H) \subset (-\infty, -b_H^{-1}] \cup [b_H^{-1}, \infty).$$

It is obvious that, for all $\psi \in D(H)$ with $\|\psi\| = 1$, $1 \le b_H\|H\psi\|$. This implies that $b_H\gamma(H) \ge 1$. On the other hand, we have from (2.33) $\|\psi\| \ge \gamma(H)\|H^{-1}\psi\|, \forall\psi \in \mathcal{H}$. Hence $b_H\gamma(H) \le 1$. Therefore $b_H^{-1} = \gamma(H)$. Thus (2.34) holds and $\gamma(H) > 0$. Since $b_H \in \sigma(H^{-1})$ or $-b_H \in \sigma(H^{-1})$, it follows that $\gamma(H) = b_H^{-1} \in \sigma(H)$ or $-\gamma(H) \in \sigma(H)$. □

Remark 2.13 *Under the same assumption as in Theorem 2.12, H is Fredholm* (the proof is easy).

We next consider a perturbation of $\Phi(\cdot)$. Let $\eta(\cdot) \in \mathcal{F}_{\text{s.a.}}$ such that, for a.e. $x \in \mathbb{R}^d$, $\eta(x)$ is bounded and strongly commutes with $\Phi(x)$ and $M(x)$. Then, for a.e.$x \in \mathbb{R}^d$,

$$\Phi_\eta(x) := \Phi(x) + \eta(x) \tag{2.36}$$

is self-adjoint on $\mathbb{C}^{N_d} \otimes \mathcal{K}$ and

$$\Phi_\eta := \int_{\mathbb{R}^d}^{\oplus} \Phi_\eta(x) dx \tag{2.37}$$

is a self-adjoint operator on $\mathcal{H}$.

The quantity

$$\kappa(H) := \sup_{\psi \in D(H), \|\psi\|=1} \frac{\|M\psi\|}{\|H\psi\|}, \tag{2.38}$$

may be infinite. But we have:

Lemma 2.14 *Under the assumption of Theorem 2.12,* $0 < \kappa(H) < \infty$.

Proof.
Since H is closed with $D(H) = D(H_0) \cap D(V) (= D(H_0))$ and $\|V\psi\| = \|M\psi\|, \psi \in D(H)$, it follows from the closed graph theorem that there exists a constant $c > 0$ such that

$$\|M\psi\| \le c(\|H\psi\| + \|\psi\|), \quad \psi \in D(H).$$

Let $\psi \in D(H)$ with $\|\psi\| = 1$. Then, by Theorem 2.12, we have $\|H\psi\| \ge \gamma(H) > 0$. Hence

$$\frac{\|M\psi\|}{\|H\psi\|} \le c + \frac{c}{\gamma(H)}.$$

Thereore $\kappa(H) \le c + c/\gamma(H) < \infty$. If $\kappa(H) = 0$, then $\|M\psi\| = 0$ for all $\psi \in D(M) = D(H_0)$. Hence $M = 0$. But this contradicts condition (A.5). □

Theorem 2.15 *Assume (A.1)–(A.5) and (2.32). Suppose that*

$$\text{ess.sup}_{x\in\mathbb{R}^d} \left\| \sin \frac{\eta(x)}{2} \right\| < \frac{1}{2\kappa(H)} \tag{2.39}$$

Let

$$H_\eta := H_0 + \alpha_{d+1} e^{i\Phi_\eta} M. \tag{2.40}$$

Then $\ker H_\eta = \{0\}$ *and* $0 \in \rho(H_\eta)$. *Moreover, the last statement on* $\gamma(H)$ *and* $\sigma(H)$ *in Theorem 2.12 holds with* H *replaced by* H_η.

Proof.
We write

$$H_\eta = H + W, \quad W := \alpha_{d+1}(e^{i\Phi_\eta} - e^{i\Phi})M.$$

By the strong commutativity of $\Phi(x)$ and $\eta(x)$, we have for a.e.$x \in \mathbb{R}^d$

$$e^{i\Phi_{\eta(x)}} - e^{i\Phi(x)} = e^{i\Phi(x)}(e^{i\eta(x)} - 1) = 2ie^{i\Phi(x)}e^{i\eta(x)/2}\sin(\eta(x)/2).$$

Hence, for all $\psi \in D(H_0)$

$$\|W\psi\| \le 2\text{ess.sup}_{x\in\mathbb{R}^d} \left\| \sin \frac{\eta(x)}{2} \right\| \|M\psi\|.$$

We have $\|M\psi\| \le \kappa(H)\|H\psi\|$. Hence

$$\|W\psi\| \le C_\eta \|H\psi\|$$

with

$$C_\eta := 2\kappa(H)\text{ess.sup}_{x\in\mathbb{R}^d}\left\|\sin\frac{\eta(x)}{2}\right\|.$$

Hence W is H-bounded. By Remark 2.13, H is Fredholm. Condition (2.39) is equivalent to $C_\eta < 1$. Hence, by a stability theorem (e.g., [9, Chapter IV,Theorem 5.22]), H_η is Fredholm and $\dim\ker H_\eta \le \dim\ker H = 0$. Therefore $\ker H_\eta = \{0\}$. It follows from this fact and the self-adjointness of H_η that $\text{Ran}(H_\eta) = \mathcal{H}$. Hence $0 \in \rho(H_\eta)$. Then the last statement of the present theorem can be proved in the same way as in the proof of the corresponding part in Theorem 2.12. □

2.4 Unitary equivalence to a gauge theoretic Dirac operator and a vanishing theorem for $\ker H$

In the papers [2, 4], it was shown that, under a suitable condition, the Hamiltonian of the CQS (GCQS) model is unitarily transformed to a Dirac operator which is simpler in a sense. In this section, we show that those structures are unified into a simple general structure.

We introduce a class of $\Phi(\cdot)$:

$$\mathcal{F} := \{\Phi(\cdot) \in \mathcal{F}_{\text{s.a.}} \ \mid \ e^{\pm i\Phi(\cdot)/2} \text{ is strongly differentiable and } \sup_{x\in\mathbb{R}^d}\|E_j(x)\| < \infty,\ j = 1,\cdots,d\}, \tag{2.41}$$

where

$$E_j(x) := D_j e^{-i\Phi(x)/2} \tag{2.42}$$

denotes the strong partial derivative of $e^{-i\Phi(x)/2}$ in x_j. For $\Phi(\cdot) \in \mathcal{F}$, one can define a bounded linear operator

$$A_j := i\int_{\mathbb{R}^d}^{\oplus} e^{i\Phi(x)/2}E_j(x)dx \tag{2.43}$$

on $\mathcal{H}$.

Remark 2.16 If $\Phi(\cdot) \in \mathcal{F}$ sucht that $\Phi(x)$ and $\Phi(x')$ commute for a.e.$x, x' \in \mathbb{R}^d$, then $E_j(x) = -ie^{-i\Phi(x)/2}D_j\Phi(x)/2$ and hence

$$A_j = \frac{1}{2}\int_{\mathbb{R}^d}^{\oplus} D_j\Phi(x)dx.$$

Lemma 2.17 *For each $j = 1,\cdots,d$, A_j is a bounded self-adjoint operator on $\mathcal{H}$.*

Proof.
Since $\text{ess.sup}_{x\in\mathbb{R}^d}\|e^{i\Phi(x)/2}E_j(x)\| = \text{ess.sup}_{x\in\mathbb{R}^d}\|E_j(x)\| < \infty$, A_j is bounded. We have

$$A_j^* = \frac{(-i)}{2}\int_{\mathbb{R}^d}^{\oplus}(D_j e^{i\Phi(x)/2})e^{-i\Phi(x)/2}dx.$$

Differentiating the identity $e^{i\Phi(x)/2}e^{-i\Phi(x)/2} = I$ in x_j, we have

$$e^{i\Phi(x)/2}E_j(x) = -(D_j e^{\Phi(x)/2})e^{-i\Phi(x)/2}. \tag{2.44}$$

Hence $A_j^* = A_j$. $\square$

For $\Phi(\cdot) \in \mathcal{F}$, we define an operator:

$$H' := \sum_{j=1}^{d} \alpha_j(-iD_j - A_j) + \alpha_{d+1}M = H_M - \sum_{j=1}^{d} \alpha_j A_j. \tag{2.45}$$

Lemma 2.18 *Assume (A.4). Let $\Phi(\cdot) \in \mathcal{F}$. Suppose that*

$$\alpha_j \Phi \subset \Phi \alpha_j, \quad j = 1, \cdots, d. \tag{2.46}$$

Then H' is self-adjoint and every core of H_0 is a core of H'.

Proof.
Under condition (A.4), H_M is self-adjoint. By (2.46), we have $\alpha_j A_j = A_j \alpha_j$ $(j = 1, \cdots, d)$. Hence, by Lemma 2.17, $-\sum_{j=1}^{d} \alpha_j A_j$ is a bounded self-adjoint operator. Hence the Kato-Rellich theorem yields the desired result. $\square$

We note that, if one regards $\mathbf{A} := (A_1, \cdots, A_d)$ as a (non-commutative) gauge potential, then H' is a gauge theoretic Dirac operator with gauge potential $\mathbf{A}$.

Let

$$U := e^{i\Phi/2}, \tag{2.47}$$

which is unitary. The following theorem shows that, under a suitable condition, H is unitarily equivalent to a gauge theoretic Dirac operator H':

Theorem 2.19 *Assume (A.1)–(A.4) and (2.46). Let $\Phi(\cdot) \in \mathcal{F}$. Then*

$$UHU^{-1} = H'. \tag{2.48}$$

Proof.
We have

$$UHU^{-1} = -i \sum_{j=1}^{d} (U\alpha_j U^{-1}) UD_j U^{-1} + U\alpha_{d+1} e^{i\Phi} U^{-1} (UMU^{-1}).$$

By (2.46) and Lemma 1.5, $U\alpha_j U^{-1} = \alpha_j$. By (A.3) and Lemma 1.5, $UMU^{-1} = M$. By (A.1) and Lemma 1.5-(iv), $U\alpha_{d+1} e^{i\Phi} U^{-1} = \alpha_{d+1} e^{-i\Phi/2} e^{i\Phi} e^{i\Phi/2} = \alpha_{d+1}$. Moreover

$$UD_j U^{-1} = D_j - iA_j.$$

Hence (2.48) holds. $\square$

The following theorem gives another sufficient condition for $ker H$ to be trivial:

Theorem 2.20 *Assume (A.1)–(A.4) and (2.46). Let $\Phi(\cdot) \in \mathcal{F}$ and*

$$\sum_{j=1}^{d} \text{ess.sup}_{x \in \mathbb{R}^d} \|E_j(x)\| < \mu_0. \tag{2.49}$$

Then $\ker H = \{0\}$ *and* $0 \in \rho(H)$.

Proof.
We write $H' = H_M + X$ with $X := -\sum_{j=1}^{d} \alpha_j A_j$. Then

$$\|X\| \leq \sum_{j=1}^{d} \|\alpha_j A_j\| \leq \sum_{j=1}^{d} \text{ess.sup}_{x\in\mathbb{R}^d} \|E_j(x)\|.$$

By this estimate and (2.49), $\|XH_M^{-1}\| < 1$. Hence H' is bijective and $0 \in \rho(H')$. In particular, $\ker H' = \{0\}$. On the other hand, (2.48) implies that $\rho(H') = \rho(H)$ and $\ker H = U^{-1} \ker H'$. Thus $0 \in \rho(H)$ and $\ker H = \{0\}$. □

Chapter 3

Spectrum of d-dimensional Dirac operator with variable mass

3.1 Essential spectrum of H

We recall that In this section, we consider the essential spectrum of H. For a self-adjoint operator S on a Hilbert space, we denote by $\sigma_{\mathrm{ess}}(S)$ the essential spectrum of S.

Lemma 3.1 *Let $\dim \mathcal{K} < \infty$ and $m > 0$ be a constant. Let $V(\cdot) : \mathbb{R}^d \to \mathfrak{B}(\mathbb{C}^{N_d} \otimes \mathcal{K})$ be Borel measurable satisfying the following conditions:*

(i) *The operator $V := \int_{\mathbb{R}^d}^{\oplus} V(x)dx$ is relatively bounded with respect to H_0.*

(ii)
$$\lim_{|x|\to\infty} \|V(x)\| = 0.$$

(iii) *The operator $H_m + V$ on $\mathcal{H}$ is self-adjoint.*

Then
$$\sigma_{\mathrm{ess}}(H_m + V) = (-\infty, -m] \cup [m, \infty). \tag{3.1}$$

Proof.
For each $R > 0$, we denote by χ_R the characteristic function of the set $\{x \in \mathbb{R}^d | \, |x| < R\}$. As in the case of the 3-dimensional free Dirac operator (cf. [18, Lemma 4.6]), one can show that $|H_m|^{-k}\chi_R$ is compact for all $k > 0$ as an operator on $L^2(\mathbb{R}^d; \mathbb{C}^{N_d})$. Since $\dim \mathcal{K} < \infty$, it follows that $|H_m|^{-k}\chi_R$ is compact as an operator on $\mathcal{H}$. Hence, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(H_m - z)^{-1}\chi_R = ((H_m - z)^{-1}|H_m|)|H_m|^{-1}\chi_R$ is compact. Since $H_m + V$ is self-adjoint with $D(H_m + V) = D(H_m) = D(H_0)$ and hence closed, there exists a constant $c > 0$ such that $\|V\psi\| \le c(\|(H_m + V)\psi\| + |\psi\|), \forall \psi \in D(H_m)$. Hence $V(H_m + V - z)^{-1}$ is bounded. Therefore we have

$$(H_m + V - z)^{-1} - (H_m - z)^{-1} = -(H_m - z)^{-1}V(H_m + V - z)^{-1} = -W_R - X_R,$$

where $W_R := (H_m - z)^{-1}\chi_R[V(H_m + V - z)^{-1}], X_R = (H_m - z)^{-1}(1 - \chi_R)V(H_m + V - z)^{-1}$. By the fact mentioned above, W_R is compact. We have

$$\|X_R\| \le \frac{1}{|\mathrm{Im}\, z|^2}\|(1 - \chi_R)V\|.$$

By condition (ii), for every $\varepsilon > 0$, there exists a constant $R > 0$ such that, for all a.e.$x \in \mathbb{R}^d$ with $|x| \ge R$, $\|V(x)\| < \varepsilon$, i.e., $\mathrm{ess.sup}_{|x|\ge R}\|V(x)\| \le \varepsilon$, which implies that $\|(1 - \chi_R)V\| \le \varepsilon$.

Hence $\lim_{R\to\infty}\|X_R\| = 0$. Therefore $(H_m + V - z)^{-1} - (H_m - z)^{-1}$ is compact. Hence, by Weyl's essential spectrum theorem (e.g., [15, Theorem XIII.14]), $\sigma_{\rm ess}(H_m + V) = \sigma_{\rm ess}(H_m)$. On the other hand, as in the case of the 3-dimensional free Dirac operator [18, Theorem 1.1], one can show that $\sigma(H_m) = (-\infty, -m] \cup [m, \infty)$. Thus (3.1) holds. □

Theorem 3.2 *Let* $\dim \mathcal{K} < \infty$. *Assume (A.1)–(A.4). Suppose that there exists a constant* $m \in \mathbb{R}$ *satisfying*

$$\lim_{|x|\to\infty} \|M(x) - m\| = 0 \tag{3.2}$$

and

$$\lim_{|x|\to\infty} \left\|\sin\frac{\Phi(x)}{2}\right\| = 0. \tag{3.3}$$

Then

$$\sigma_{\rm ess}(H) = (-\infty, -m] \cup [m, \infty). \tag{3.4}$$

Proof.
We write

$$H = H_m + V_1 + V_2$$

with

$$V_1 := \alpha_{d+1}(M - m), \quad V_2 := \alpha_{d+1}(e^{i\Phi} - 1)M.$$

It is obvious that V_1 and V_2 are relatively bounded with respect to H_0 and

$$\lim_{|x|\to\infty} \|V_1(x)\| = \lim_{|x|\to\infty} \|M(x) - m\| = 0.$$

As for V_2, we have

$$\|V_2(x)\| \le \|M(x)\|\|e^{i\Phi(x)} - 1\| \le 2\|M(x)\| \left\|\sin\frac{\Phi(x)}{2}\right\|.$$

Hence, by (3.2) and (3.3), we have $\lim_{|x|\to\infty}\|V_2(X)\| = 0$. Therefore $\lim_{|x|\to\infty}\|V_1(x)+V_2(x)\| = 0$. Thus we can apply Lemma 3.1 to obtain (3.4). □

If $\Phi(\cdot)$ is in the class $\mathcal{F}$ introduced in Section 2.4, then we can obtain a sufficient condition for (3.4) hold:

Theorem 3.3 *Let* $\dim \mathcal{K} < \infty$. *Assume (A.1)–(A.4), (2.46) and (3.2). Let* $\Phi(\cdot) \in \mathcal{F}$. *Suppose that*

$$\lim_{|x|\to\infty} \|E_j(x)\| = 0. \tag{3.5}$$

Then (3.4) holds.

Proof.
By (2.48), we have $\sigma_{\rm ess}(H) = \sigma_{\rm ess}(H')$. Hence we need only to prove

$$\sigma_{\rm ess}(H') = (-\infty, -m] \cup [m, \infty). \tag{3.6}$$

We write

$$H' = H_m + \alpha_{d+1}(M - m) - \sum_{j=1}^{d} \alpha_j A_j.$$

We have $\lim_{|x|\to\infty}\|\alpha_{d+1}(M(x)-m)\| = 0$. Moreover, $\|\alpha_j A_j(x)\| \le \|E_j(x)\|$. Hence $\lim_{|x|\to\infty}\|-\sum_{j=1}^{d}\alpha_j A_j(x)\| = 0$. Thus we can apply Lemma 3.1 to obtain (3.6). □

3.2 Bounds on the number of discrete eigenvalues

In this section, in view of Theorem 3.2, we consider the number of eigenvalues of H in the interval $(-m,m)$ and establish upper bounds on it. This aspect has been considered in the CQS model [4] as well as the GCQS model [2]. In this paper, we take another method, which is an extension of the method used in [5] where the number of eigenvalues of the three-dimensional Dirac operator $H_m + W$ with a *scalar* potential $W : \mathbb{R}^3 \to \mathbb{R}$ in $(-m,m)$ is considered. This extension is not difficult. But, for the sake of completeness, we present some details of it. One easily notes that the problem under consideration can be studied in a more general frame work as in Lemma 3.1. Hence we first discuss the general case.

3.2.1 A general case

Let V be as in Lemma 3.1 and

$$H(V) := H_m + V. \tag{3.7}$$

Then, by (3.1), an eigenvalue of $H(V)$ in $(-m,m)$ (if it exists) is an isolated eigenvalue of $H(V)$ with finite multiplicity. For each $\lambda \in (0,m^2)$, we denote by $N(\lambda,V)$ the number of eigenvalues in the interval $(-\sqrt{m^2-\lambda}, \sqrt{m^2-\lambda})$.

We first note an elementary fact:

Theorem 3.4 *Suppose that the assumption of Lemma 3.1 holds and that $\|V(x)\| \le \lambda/4m$ for a.e.$x \in \mathbb{R}^d$. Then $N(\lambda,V)=0$.*

Proof.
Suppose that $N(\lambda,V) \ge 1$. Then, it follows from the definition of $N := N(\lambda,V)$ that there exists an N-dimensional subspace E of $\mathcal{H}$ such that

$$\|H(V)\psi\| \le \sqrt{m^2-\lambda}\|\psi\|, \forall\psi \in E. \tag{3.8}$$

Hence

$$\|H_m\psi\| \le \|H(V)\psi\| + \|V\psi\| \le \left(\sqrt{m^2-\lambda} + \frac{\lambda}{4m}\right)\|\psi\| \le \sqrt{m^2-\frac{\lambda}{2}}\|\psi\|.$$

Hence $\|H_m\psi\|^2 \le \left(m^2-\frac{\lambda}{2}\right)\|\psi\|^2$, which is equivalent to $\|(-\Delta+\frac{\lambda}{2})^{1/2}\psi\|^2 \le 0$. This implies that $\psi = 0$. But this is a contradiction. □

In view of Theorem 3.4, we define, for each $\lambda > 0$, $V_\lambda : \mathbb{R}^d \to \mathfrak{B}(\mathbb{C}^{N_d} \otimes \mathcal{K})$ by

$$V_\lambda(x) := \begin{cases} V(x) & \text{if } \|V(x)\| > \dfrac{\lambda}{4m} \\ 0 & \text{otherwise} \end{cases}.$$

For each $\lambda > 0$, the operator

$$R_\lambda := \left(-\Delta + \frac{\lambda}{2}\right)^{-1/2}$$

is a bounded self-adjoint operator. Since V is H_0-bounded, where H_0 is defined by (1.19), and $(H_0+i)R_\lambda$ is bounded, it follows that VR_λ and $V_\lambda R_\lambda$ are bounded operators on $\mathcal{H}$. Also H_0R_λ is bounded with $\|H_0R_\lambda\| \le 1$. Hence the following operators $T_{\lambda j}$ $(j = 1,2,3,5)$ are in $\mathfrak{B}(\mathcal{H})$:

$$T_{\lambda 1} := (H_0R_\lambda)^* V_\lambda R_\lambda, \tag{3.9}$$
$$T_{\lambda 2} := (V_\lambda R_\lambda)^* H_0R_\lambda, \tag{3.10}$$
$$T_{\lambda 3} := m\alpha_{d+1}R_\lambda V_\lambda R_\lambda, \tag{3.11}$$
$$T_{\lambda 4} := mR_\lambda V_\lambda R_\lambda \alpha_{d+1}, \tag{3.12}$$
$$T_{\lambda 3} := (V_\lambda R_\lambda)^* V_\lambda R_\lambda. \tag{3.13}$$

We set

$$v(x) := \|V(x)\|, \quad v_\lambda(x) := \|V_\lambda(x)\|, \quad \text{a.e} x \in \mathbb{R}^d. \tag{3.14}$$

For a compact operator A on a Hilbert space, we denote the nonincreasing sequence of the singular values of A (repeated with multiplicity) by $\mu_n(A)$ $(n \in \mathbb{N})$. For $f \in L^p(\mathbb{R}^d)$, we set $\|f\|_{L^p} := (\int_{\mathbb{R}^d} |f(x)|^p dx)^{1/p}$.

Lemma 3.5 *Let $d \geq 3$ and suppose that the assumption of Lemma 3.1 holds and $v \in L^d(\mathbb{R}^d) \cap L^{d/2}(\mathbb{R}^d)$. Then, for all $j = 1, 2, 3, 4, 5$, $T_{\lambda j}$ is compact. Moreover, there exists a constant $C > 0$ independent of V and $\lambda > 0$ such that, for all $n \in \mathbb{N}$,*

$$\mu_n(T_{\lambda j}) \leq C\|v_\lambda\|_{L^d} n^{-1/d} \quad (j = 1, 2), \tag{3.15}$$

$$\mu_n(T_{\lambda j}) \leq C\|v_\lambda^{1/2}\|_{L^d}^2 n^{-2/d} \quad (j = 3, 4), \tag{3.16}$$

$$\mu_n(T_{\lambda 5}) \leq C\|v_\lambda\|_{L^d}^2 n^{-2/d}. \tag{3.17}$$

Proof.
By the weak Hausdorff–Young inequality (e.g., [14, p.32]) and the condition $d \geq 3$, one can easily see that the Fourier transform g_λ of the function:$\mathbb{R}^d \ni k \mapsto (k^2 + \lambda/2)^{-1/2}$ is in $L_{\mathrm{w}}^{p'}(\mathbb{R}^d)$ (the weak $L^{p'}$ space on $\mathbb{R}^d$) with $1/p' = 1 - 1/d$ and $\|g_\lambda\|_{p',\mathrm{w}} \leq c_d$, where $\|\cdot\|_{p',\mathrm{w}}$ denotes the "pseudo" norm of $L_{\mathrm{w}}^{p'}(\mathbb{R}^d)$ and c_d is a constant independent of $\lambda > 0$. By Cwikel's theorem [6, §3] and the condition $v \in L^d(\mathbb{R}^d)$, which implies that $v_\lambda \in L^d(\mathbb{R}^d)$, $v_\lambda R_\lambda$ is compact as an operator on $L^2(\mathbb{R}^d)$ and

$$\mu_n(v_\lambda R_\lambda) \leq K_1\|v_\lambda\|_{L^d} n^{-1/d}, \quad n \in \mathbb{N},$$

where $K_1 > 0$ is a constant independent of V, $\lambda > 0$ and $n \in \mathbb{N}$. Since $\dim \mathcal{K} < \infty$, it follows that $v_\lambda R_\lambda$ is compact also as an operator on $\mathcal{H}$. Let

$$B_\lambda(x) := \begin{cases} \frac{V(x)}{v(x)} & \text{if } v(x) > \lambda/4m \\ 0 & \text{otherewise} \end{cases}.$$

Then B_λ is bounded with $\|B_\lambda(x)\| \leq 1$. We have $V_\lambda R_\lambda = B_\lambda v_\lambda R_\lambda$. Hence $V_\lambda R_\lambda$ is compact. This shows that all $T_{\lambda j}$ $(j = 1, 2, 3, 4, 5)$ are compact.

In general, for all compact operators A and bounded operators B on a Hilbert space

$$\mu_n(BA) \leq \|B\|\mu_n(A).$$

(e.g., see [17, Theorem 1.6].) Hence

$$\mu_n(V_\lambda R_\lambda) \leq \|B_\lambda\|\mu_n(v_\lambda R_\lambda) \leq K_1\|v_\lambda\|_{L^d} n^{-1/d}.$$

Therefore

$$\mu_n(T_{\lambda 1}) \leq \|H_0 R_\lambda\| K_1\|v\|_{L^d} n^{-1/d} \leq \|K_1\|v_\lambda\|_{L^d} n^{-1/d}.$$

Similarly one can show that $T_{\lambda 2}$ is compact and

$$\mu_n(T_{\lambda 2}) \leq \|K_1\|v_\lambda\|_{L^d} n^{-1/d},$$

where we have use the fact that $\mu_n(A) = \mu_n(A^*)$ for all compact operators on a Hilbert space [17, (1.3)].

As for $T_{\lambda 3}$, we write

$$T_{\lambda 3} = m\alpha_{d+1} R_\lambda v_\lambda^{1/2} B_\lambda v_\lambda^{1/2} R_\lambda.$$

By the condition $v \in L^{d/2}(\mathbb{R}^d)$, $v_\lambda^{1/2} \in L^d(\mathbb{R}^d)$. Hence, Cwikel's theorem again, $v_\lambda^{1/2} R_\lambda$ is compact and

$$\mu_n(v_\lambda^{1/2} R_\lambda) \leq K_1'\|v_\lambda^{1/2}\|_{L^d} n^{-1/d},$$

where $K_1' > 0$ is a constant independent of V and $\lambda > 0$. We have

$$\mu_n(T_{\lambda 3}) \le m\mu_n(R_\lambda v_\lambda^{1/2} B_\lambda v_\lambda^{1/2} R_\lambda).$$

In general, for all compact operators A and bounded operators D on a Hilbert space,

$$\mu_{2n+1}(A^* DA) \le \|D\|\mu_{n+1}(A)^2, \quad \mu_{2n}(A^* DA) \le \|D\|\mu_n(A)^2,$$

where we have used the fact that, for all compact operators A and B on a Hilbert space,

$$\mu_{n+k+1}(AB) \le \mu_{n+1}(A)\mu_{k+1}(B), \quad n, k \ge 0.$$

Hence

$$\mu_{2n+1}(T_{\lambda 3}) \le m\mu_{n+1}(v_\lambda R_\lambda)^2 \le m(K_1')^2 \|v_\lambda^{1/2}\|_{L^d}^2 (n+1)^{-2/d},$$
$$\mu_{2n}(T_{\lambda 3}) \le m\mu_n(v_\lambda R_\lambda)^2 \le m(K_1')^2 \|v_\lambda^{1/2}\|_{L^d}^2 n^{-2/d}.$$

which imply that

$$\mu_n(T_{\lambda 3}) \le K_2' \|v_\lambda^{1/2}\|_{L^d}^2 n^{-2/d}, \quad n \in \mathbb{N}$$

where $K_2' > 0$ is a constant independent of V, λ and n. Similarly we have

$$\mu_n(T_{\lambda 4}) \le K_2' \|v_\lambda^{1/2}\|_{L^d}^2 n^{-2/d}, \quad \mu_n(T_{\lambda 5}) \le K_3' \|v_\lambda\|_{L^d}^2 n^{-2/d}, \quad n \in \mathbb{N},$$

where $K_3' > 0$ is a constant independent of V, λ and n. Thus the desired results follw. □

Theorem 3.6 *Let $d \ge 3$ and suppose that the assumption of Lemma 3.1 holds and $\|V(\cdot)\| \in L^d(\mathbb{R}^d) \cap L^{d/2}(\mathbb{R}^d)$. Let $\lambda \in (0, m^2)$. Then, there exists a constant $C_0 > 0$ independent of V and λ such that*

$$N(\lambda, V) \le C_0 \int_{\|V(x)\| > \lambda/4m} \left(\|V(x)\|^{d/2} + \|V(x)\|^d \right) dx. \tag{3.18}$$

Proof.
We need only to consider the case where $N := N(\lambda, V) \ge 1$. Then there exists an N-dimensional subspace E of $\mathcal{H}$ such that (3.8) holds for all $\psi \in E$. It is easy to see that $\|(V_\lambda - V)\phi\| \le (\lambda/4m)\|\phi\|, \forall \phi \in \mathcal{H}$. Let $\psi \in E$. Then, as in the proof of Theorem 3.4, we have $\|(H_m + V_\lambda)\psi\|^2 \le \left(m^2 - \frac{\lambda}{2}\right) \|\psi\|^2$, which is equivalent to the following inequality:

$$\begin{aligned} &\left\| \left(-\Delta + \frac{\lambda}{2} \right)^{1/2} \psi \right\|^2 + \langle H_0\psi, V_\lambda\psi \rangle + \langle V_\lambda\psi, H_0\psi \rangle \\ &+ m \langle \alpha_{d+1}\psi, V_\lambda\psi \rangle + m \langle V_\lambda\psi, \alpha_{d+1}\psi \rangle + \|V_\lambda\psi\|^2 \le 0. \end{aligned} \tag{3.19}$$

The subspace $F := (-\Delta + \lambda/2)^{1/2} E$ is also N-dimensional. Inequality (3.19) implies that, for all $\phi \in F$,

$$\|\phi\|^2 \le \langle \phi, T_\lambda \phi \rangle,$$

where

$$T_\lambda := -\sum_{j=1}^{5} T_{\lambda j}.$$

By Lemma 3.5, T_λ is a compact self-adjoint operator on $\mathcal{H}$. Hence, by the Hilbert–Schmidt theorem, there exists a complete orthonormal system $\{\phi_n\}_{n=1}^\infty$ of $\mathcal{H}$ and a real sequence $\{t_n\}_{n=1}^\infty$

such that $T_\lambda \phi_n = t_n \phi_n$ and $\lim_{n\to\infty} t_n = 0$. Using this fact, one sees that the number of eigenvalues t_n of T_λ with $t_n \geq 1$ is more than or equal to $\dim F = N$. Hence $\mu_N(T_\lambda) \geq 1$. Let k be the largest natural number not exceeding $(N+4)/5$. Then $5k-4 \leq N$. Hence $1 \leq \mu_N(T_\lambda) \leq \mu_{5k-4}(T_\lambda)$. On the other hand, by a general fact on singular values of the sum of two compact operators (e.g., [17, Theorem 1.7]), we have

$$\mu_{5k-4}(T_\lambda) \leq \sum_{j=1}^{5} \mu_k(T_{\lambda j}).$$

Using this fact and Lemma 3.5, we obtain

$$1 \leq 2C\|v_\lambda\|_{L^d} k^{-1/d} + 2C\|v_\lambda^{1/2}\|_{L^d}^2 k^{-2/d} + C\|v_\lambda\|_{L^d}^2 k^{-2/d}.$$

We have $k \geq N/5$. Hence

$$1 \leq C'(\|v_\lambda\|_{L^d} N^{-1/d} + \|v_\lambda^{1/2}\|_{L^d}^2 N^{-2/d} + \|v_\lambda\|_{L^d}^2 N^{-2/d}),$$

where $C' > 0$ is a constant independent of V, λ and N. This implies that $N \leq C_0(\|v_\lambda^{1/2}\|_{L^d}^d + \|v_\lambda\|_{L^d}^d)$ with a constant C_0 independent of V and λ. Thus (3.18) holds. □

As in Corollaries 1.2 and 1.3 in [5], we have from Theorem 3.6 the following results:

Corollary 3.7 *Under the same assumption as in Theorem 3.6, the number $N(V)$ of eigenvalues of $H(V)$ in $(-m, m)$ is finite and*

$$N(V) \leq C_0 \int_{\mathbb{R}^d} (\|V(x)\|^{d/2} + \|V(x)\|^d) dx. \tag{3.20}$$

Corollary 3.8 *Suppose that the assumption of Theorem 3.6 holds. Let λ_j $(j = 1, \ldots, N(V))$ be the eigenvalues of $H(V)$ in $(-m, m)$, counted with multiplicity and $\gamma > 0$ be such that*

$$f_\gamma(V) := \int_{\mathbb{R}^d} \|V(x)\|^\gamma (\|V(x)\|^{d/2} + \|V(x)\|^d) dx < \infty.$$

Then, there exists a constant $C_\gamma > 0$ such that

$$\sum_{j=1}^{N(V)} (1 - \lambda_j^2)^\gamma \leq C_\gamma f_\gamma(V). \tag{3.21}$$

3.2.2 Applications

Now we apply the results in the preceeding section to the Dirac operator H. For $\lambda \in (0, m^2)$, we denote by $N(\lambda)$ the number of eigenvalues of H in $(-\sqrt{m^2-\lambda}, \sqrt{m^2-\lambda})$.

Theorem 3.9 *Let $d \geq 3$ and $\lambda \in (0, m^2)$. Suppose that the assumption of Theorem 3.2 holds. Let*

$$F_{M,\Phi}(x) := \|M(x) - m\| + 2m \left\| \sin \frac{\Phi(x)}{2} \right\|, \quad \text{a.e.} x \in \mathbb{R}^d.$$

(i) *If $F_{M,\Phi}(x) \leq \lambda/4m$, a.e.$x \in \mathbb{R}^d$, then $N(\lambda) = 0$.*

(ii) *Suppose that* $F_{M,\Phi} \in L^{d/2}(\mathbb{R}^d) \cap L^d(\mathbb{R}^d)$*. Then there exists a positive constant* $C > 0$ *independent of* M, Φ *and* λ *such that*

$$N(\lambda) \le C \int_{F_{M,\Phi}(x)>\lambda/4m} \left(F_{M,\Phi}(x)^{d/2} + F_{M,\Phi}(x)^d\right) dx < \infty. \tag{3.22}$$

Moreover, the number N_0 *of eigenvalues of* H *in* $(-m, m)$ *obeys*

$$N_0 \le C \int_{\mathbb{R}^d} \left(F_{M,\Phi}(x)^{d/2} + F_{M,\Phi}(x)^d\right) dx < \infty. \tag{3.23}$$

Proof.
(i) We can write $H = H(V)$ with $V = \alpha_{d+1}(Me^{i\Phi} - m)$. Hence

$$\|V(x)\| = \|M(x)e^{i\Phi(x)} - m\| \le \|M(x) - m\| + m\|e^{i\Phi(x)} - 1\| = F_{M,\Phi}(x). \tag{3.24}$$

Hence, the present assumption implies that $\|V(x)\| \le \lambda/4m$ a.e.$x \in \mathbb{R}^d$. Hence, by Theorem 3.4, $N(\lambda) = 0$.

(ii) By (3.24) and the present assumption, $\|V(\cdot)\| \in L^d(\mathbb{R}^d) \cap L^{d/2}(\mathbb{R}^d)$. Thus we can apply Theorem 3.6 to obtain (3.22). Inequality (3.23) follows from (3.22) or Corollary 3.7. □

We have from Corollary 3.8 the following fact:

Corollary 3.10 *Let* $d \ge 3$*. Suppose that the assumption of Theorem 3.2 and* $F_{M,\Phi} \in L^{d/2}(\mathbb{R}^d) \cap L^d(\mathbb{R}^d)$*. Let* λ_j $(j = 1, \ldots, N_0)$ *be the eigenvalues of* H *in* $(-m, m)$*, counted with multiplicity and* $\gamma > 0$ *be such that*

$$f_\gamma(M, \Phi) := \int_{\mathbb{R}^d} F_{M,\Phi}(x)^\gamma (F_{M,\Phi}(x)^{d/2} + F_{M,\Phi}(x)^d) dx < \infty.$$

Then, there exists a constant $C_\gamma > 0$ *such that*

$$\sum_{j=1}^{N_0} (1 - \lambda_j^2)^\gamma \le C_\gamma f_\gamma(M, \Phi). \tag{3.25}$$

We can also use Theorems 2.19 and 3.3 to obtain another upper bound for $N(\lambda)$. Let

$$G_{M,\Phi}(x) := \left\| M(x) - m - \sum_{j=1}^d \alpha_{d+1}\alpha_j e^{i\Phi(x)/2} E_j(x) \right\|, \quad \text{a.e.} x \in \mathbb{R}^d. \tag{3.26}$$

Theorem 3.11 *Let* $d \ge 3$ *and* $\lambda \in (0, m^2)$*. Suppose that the assumption of Theorem 3.3 holds. Then:*

(i) *If* $G_{M,\phi}(x) \le \lambda/4m$ *for a.e.*$x \in \mathbb{R}^d$*, then* $N(\lambda) = 0$*.*

(ii) *Suppose that* $G_{M,\Phi} \in L^{d/2}(\mathbb{R}^d) \cap L^d(\mathbb{R}^d)$*. Then (3.22) and (3.23) with* $F_{M,\Phi}$ *replaced by* $G_{M,\Phi}$ *hold.*

Proof.
By Theorem 2.19, $N(\lambda)$ is equal to the number of eigenvalues of H' in $(-\sqrt{m^2 - \lambda}, \sqrt{m^2 - \lambda})$. One can write $H' = H_m + V$ with $V := \alpha_{d+1}(M - m - \sum_{j=1}^d \alpha_{d+1}\alpha_j A_j)$. We have $\|V(x)\| = G_{M,\Phi}(x)$. Thus, in the same way as in the proof of Theorem 3.6, we obtain the desired results. □

Theorem 3.11 implies the following result as in Corollary 3.10:

Corollary 3.12 *Let $d \geq 3$. Suppose that the assumption of Theorem 3.3 holds and $G_{M,\Phi} \in L^{d/2}(\mathbb{R}^d) \cap L^d(\mathbb{R}^d)$. Then (3.25) with $F_{M,\Phi}$ replaced by $G_{M,\Phi}$ holds for all $\lambda > 0$ such that $\int_{\mathbb{R}^d} G_{M,\Phi}(x)^\gamma (G_{M,\Phi}(x)^{d/2} + G_{M,\Phi}(x)^d) dx < \infty$.*

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