Studies on the Generalized Spin-Boson Models

（一般化されたスピン-ボソンモデルに関する研究）

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Abstract

We consider an abstract model which describes an interaction of non-relativistic particles with a Bose field. We show that the essential self-adjointness of the generalized spin-boson Hamiltonian with a quadratic boson interaction for all coupling constant and the Hamiltonian is self-adjoint if it is bounded from below under some conditions. We consider a unitary transformation of the Hamiltonian. Using the unitary transformation, we give a sufficient condition for the semiboundedness of the Hamiltonian.

We are also interested in the ground states of the Hamiltonian. From the Birman-Schwinger principle, one can show that the Hamiltonian has no ground states for enough small coupling constants under a suitable condition. Finally, we apply our theorems to the Pauli-Fierz type Hamiltonian.

Contents

CHAPTER 1

Introduction

In this paper, we consider an abstract model which describes an interaction of non-relativistic particles with a Bose field. Arai and Hirkawa [**AH**] introduced an abstract non-relativistic quantum field model which is a generalization of the spin-boson model and it is called the generalized spin-boson model. Miyao and Sasaki [**MS**] added ϕ^2 term to the generalized spin-boson Hamiltonian. They showed that the Hamiltonian is self-adjoint for small coupling constants by applying the Kato-Rellich theorem. However these restrictions on the coupling constants could be removed because some other non-relativistic quantum field Hamiltonians are self-adjoint for arbitrary coupling constants. For the Nelson models, it is clear since the interaction term is infinitely small with respect to the free Hamiltonian [**Nel**]. For the Pauli-Fiertz models, Arai [**Ara1**] showed that Pauli-Fierz Hamiltonian in the dipole approximation is slf-adjoint for arbitrary values of coupling constants by means of the Nelson commutator theorem. Hiroshima [**Hir**] proved that the full Pauli-Fierz Hamiltonian is self-adjoint for all coupling constants by using the functional integration. Hasler and Herbst [**HH2**] give another proof of the self-adjointness of the Pauli-Fierz Hamiltonian by operator theoretical methods. It is known that the standard spin-boson Hamiltonian is self-adjoint for any coupling constants.

Arai and Kawano [**AK**] proved the self-adjointness of the Hamiltonian of the generalized spin-boson model for some coupling constants by using an unitary transformation and strong commutativity of some operators. However we hope that the strong commutativity should be removed. Therefore we study the self-adjointness of the Hamiltonian of the generalized spin-boson model with a quadratic boson interaction in a more general framework. In particular, we are interested in when the Hamiltonian is (essentially) self-adjoint without assuming the commutativity of some operators.

We are also interested in the ground states of the Hamiltonian. Arai and Hirokawa [**AH**] proved that, if the particle Hamiltonian has a compact resolvent, then the total Hamiltonian has a ground state. However, the condition of compact resolvent is strong, because a self-adjoint operator which is bounded from below and has a compact resolvent has a ground state and its ground energy is in the discrete spectrum. It is known that a total Hamiltonian may have a ground state even if the non-perturbative operator has no ground states. A typical example is a quantum mechanical system whose Hamiltonian is given by the Schrödinger operator $H(\lambda) := -\Delta + \lambda V$ on $L^2(\mathbb{R}^d)$. Indeed, it is known that $H(0)$ has no ground states but $H(\lambda)$ has a ground state for some class of potential V. We call it the enhanced binding if such a phenomenon occur.

Hiroshima and Spohn [**HS**] studied enhanced binding in the case of the Pauli-Fierz model in the dipole approximation. They showed that the enhanced binding occurs for large coupling constants under suitable assumptions. On the other hand, Hainzl, Vougalter and Vugalter [**HVV**] proved that the enhanced binding occurs for small coupling constants in the Pauli-Fierz model without the dipole approximation. We are interested in whether the enhanced binding occurs or not in the GSB model. Arai and Kawano [**AK**] proved that, in the GSB model, it occurs under suitable conditions if the coupling constants are in some interval. But it is not known that whether the enhanced binding occurs or not for small coupling constants.

We consider a GSB model which is slightly more general than the GSB model proposed by Arai and Hirokawa [**AH**]. We prove that, under some assumptions, enhanced binding does not occur for sufficiently small coupling constants in the GSB model. This is our main theorem in this paper. In the case where no infrared regular condition is assumed, we know that the GSB Hamiltonian has no ground states which satisfy some natural condition [**AHH**]. However, we prove that the GSB Hamiltonian has no ground states even if we assume an infrared regular condition.

Recently, Hiroshima, Spohn and Suzuki [**HSS**] proved that the enhanced binding does not occur for sufficiently small coupling constants in the Nelson model. Their proof is based on an estimate of the number of non-positive eigenvalues by Birman-Schwinger bound. In this paper, we prove the absence of ground states by using similar methods to the GSB model.

The outline of the present paper is as follows. In Section 2, we set up notation and terminology. In the third section, we define the GSB model. We first prove the essential self-adjointness of a Hamiltonian for all coupling constants under some natural conditions. By using this result, roughly speaking, we also show that semi-boundedness of the Hamiltonian implies the self-adjointness of it. These results improve the existing ones. We show that the semi-boundedness and some commutativities imply the (essential) self-adjointness of the Hamiltonian. It is also shown that unitary transformation of the Hamiltonian. This is based on the case that some operators are strong commuting. In addition we give a condition for semi-boundedness without strong commutativity. This condition for coupling constants is weaker than the condition which is obtained by using the Kato-Rellich theorem.

In Section 4, We consider the ground states of the Hamailtonian. We prove generalization of Birman-Schwinger bound and obtain a estimate of the number of non-positive eigenvalues. In the last, we apply the result to a Pauli-Fierz type model without A^2-term in the dipole approximation.

CHAPTER 2

Preliminaries

1. Operator Theory

Let $\mathcal{X}$ be a complex Hilbert space. We denote the inner product and the norm of the Hilbert space $\mathcal{X}$ by $\langle \cdot\,,\,\cdot \rangle_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$ respectively. For simplicity of notation, we may omit the subscript $\mathcal{X}$ in $\langle \cdot\,,\,\cdot \rangle_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$ if there is no confusion. In this paper, the inner product is antilinear in the first variable.

For a linear operator T on a Hilbert space $\mathcal{X}$, we denote its domain by $\mathrm{D}(T)$. In this article, "an operator" means "a linear operator" and "a subspace" means "a linear subspace". We use standard conventions for the sum and the composition of two operators:

$$\mathrm{D}(T+S) := \mathrm{D}(T) \cap \mathrm{D}(S),$$
$$\mathrm{D}(TS) := \{\Psi \in \mathrm{D}(S) \mid S\Psi \in \mathrm{D}(T)\}.$$

Definition 2.1. Let T be a operator on a Hilbert space $\mathcal{X}$.

(i) T is called a densely (resp. everywhere) defined if the domain $\mathrm{D}(T)$ is a dense set in $\mathcal{X}$ (resp. $\mathrm{D}(T) = \mathcal{H}$).

(ii) T is called a bounded operator if the following operator norm $\|T\|$ is finite, i.e,

$$\|T\| := \sup_{\psi \in \mathrm{D}(T)} \frac{\|T\psi\|}{\|\psi\|} < \infty$$

(iii) T is called a compact operator if, for all bounded set $X \subset \mathcal{X}$, $\overline{TX}$ is a compact set.

(iv) T is said to be closed if the graph

$$G(T) := \{(x, Tx) \in \mathcal{X} \times \mathcal{X} \mid x \in \mathrm{D}(T)\}$$

is closed.

(v) An operator S is said to be a extension of T if $\mathrm{D}(T) \subseteq \mathrm{D}(S)$ and $S\psi = T\psi$ for all $\psi \in \mathrm{D}(T)$ and we denote $T \subset S$.

(vi) T is called a closable operator if there exists a closed operator S which is a extension of T.

(vii) The closure $\overline{T}$ of a closable operator T is the smallest closed extension of T.

(viii) A subset $D \subset \mathrm{D}(T)$ is said be a core for closed operator T if $\overline{T|_D} = T$, here $T|_D$ is the restriction of T on the subspace D.

(ix) The adjoint T^* of densely defined operator T is defined as follows:

$$\mathrm{D}(T^*) := \left\{ \Psi \in \mathcal{X} \,\middle|\, \begin{array}{c} \text{for all } \Phi \in \mathrm{D}(T), \text{there exists a vector } \xi_\Psi \\ \text{such that } \langle \Psi, T\Phi \rangle = \langle \xi_\Psi, \Phi \rangle \end{array} \right\}$$

and $T^*\Psi := \xi_\Psi$ for $\Psi \in \mathrm{D}(T^*)$.

(x) T is called a symmetric operator if $T \subset T^*$.

(xi) A symmetric operator T is said to be bounded from below if there exists a constant $c \in \mathbb{R}$ such that for all $\psi \in \mathrm{D}(T)$

$$\langle \psi, T\psi \rangle \geq c\|\psi\|^2.$$

(xii) T is said to be a self-adjoint operator if $T = T^*$.

(xiii) T is said to be an essentially self-adjoint if the closure $\overline{T}$ is self-adjoint.

We use the following well-known result.

Theorem 2.2. Let S be a self-adjoint operator on a Hilbert space $\mathcal{X}$. Then there exists a unique spectral measure E_S on the Borel σ-algebra $\mathfrak{B}(\mathbb{R})$ on $\mathbb{R}$ such that

$$S = \int_{\mathbb{R}} \lambda dE_S(\lambda).$$

Definition 2.3. Let T be a closed operator on a Hilbert space $\mathcal{X}$.

(i) A complex number λ belongs to the resolvent set $\rho(T)$ of T if the operator $T - \lambda$ has a bounded inverse $(T - \lambda)^{-1}$ which is everywhere defined on $\mathcal{X}$.

(ii) The spectrum $\sigma(T)$ of T is the set $\mathbb{C} \setminus \rho(T)$.

(iii) A complex number λ belongs to the essential spectrum $\sigma_{\mathrm{ess}}(T)$ of T if the range $\mathrm{Ran}\,(T - \lambda)$ is not closed or $\dim \mathrm{Ker}\,(T - \lambda) = \infty$ or $\mathrm{codim}\,\mathrm{Ran}\,(T - \lambda) = \infty$, where

$$\mathrm{Ker}\,(T) := \{\psi \in \mathrm{D}(T) \,|\, T\psi = 0\},$$

$$\mathrm{Ran}\,(T) := T\mathcal{X} = \{\psi \in \mathcal{X} \,|\, \text{there exist a } \phi \in \mathrm{D}(T) \text{ such that } \psi = T\phi\}.$$

(iv) The point spectrum $\sigma_{\mathrm{p}}(T)$ of T is the set of all eigenvalues of T, i.e.,

$$\sigma_{\mathrm{p}}(T) := \{\lambda \in \mathbb{C} \,|\, \text{there exists a } \psi \in \mathrm{D}(T) \setminus \{0\} \text{ such that } T\psi = \lambda\psi\}$$

(v) The discrete spectrum $\sigma_{\mathrm{dis}}(T)$ of T is $\sigma(T) \setminus \sigma_{\mathrm{ess}}(T)$.

Definition 2.4. We say that a self-adjoint operator T has a ground state if $\inf \sigma(T) \in \sigma_{\mathrm{p}}(T)$ and a vector $\psi \in \ker(T - \inf \sigma(T)) \setminus \{0\}$ is called a ground state of T.

Let us recall the definition of relatively boundedness and relatively compactness.

Definition 2.5. Let T, S be closed operators on a Hilbert space $\mathcal{X}$.

(i) T is said to be relatively bounded with respect to an operator S or simply S-bounded if $\mathrm{D}(S) \subset \mathrm{D}(T)$ and there exist nonnegative constants $a, b \in \mathbb{R}$ such that

$$\|T\Psi\| \leq a\|S\Psi\| + b\|\Psi\|, \qquad \text{for all } \Psi \in \mathrm{D}(S). \tag{2.1}$$

The greatest lower bound a_S of all possible constants a in (2.1) will be called the relative bound of T with respect to S or simply

S-bound. In particular, if the relative bound of T with respect to S is equal to 0, T is called infinitesimally small with respect to S.

(ii) T is said to be relatively compact with respect to S or simply S-compact if $\mathrm{D}(S) \subseteq \mathrm{D}(T)$ and T is a compact map of the Hilbert space $(\mathrm{D}(S), \|\cdot\|_S)$ into the Hilbert space $\mathcal{X}$, where $\|\cdot\|_S$ is the graph norm of S, i.e.,

$$\|\Psi\|_S := \|\Psi\| + \|S\Psi\|, \quad \text{for } \Psi \in \mathrm{D}(S).$$

The next proposition is well-known fact. (see [**Sch**])

Proposition 2.6. Let T and S be closed operators on a Hilbert space $\mathcal{X}$.

(1) Suppose that $\rho(T) \neq \emptyset$. Then T is S-compact if and only if $\mathrm{D}(S) \subseteq \mathrm{D}(T)$ and $T(S-\lambda)^{-1}$ is a compact operator on $\mathcal{X}$ for some (and then for all) $\lambda \in \rho(T)$.

(2) If S is densely defined and T is S-compact, then T is S-bounded with S-bound zero.

The next result is the celebrated Kato-Rellich Theorem. It requires a relative bound strictly less than one.

Theorem 2.7. Let S be a self-adjoint operator on a Hilbert space $\mathcal{X}$. Suppose that T is a relatively S-bounded symmetric operator on $\mathcal{X}$ with S-bound $a < 1$. Then:

(1) The operator $S+T$ on $\mathrm{D}(S+T) = \mathrm{D}(S)$ is self-adjoint.

(2) If S is essentially self-adjoint on $D \subseteq \mathrm{D}(S)$, so is $S+T$ on D.

The following theorem shows that a stability of essential spectrum.

Theorem 2.8. Let S be a closed operator on a Hilbert space $\mathcal{X}$. Suppose that T is a S-compact operator on $\mathcal{X}$. Then

$$\sigma_{\mathrm{ess}}(S) = \sigma_{\mathrm{ess}}(S+T)$$

2. Some Facts on an Abstract Boson Fock Space

In this paper, we consider the quantum system interacting with Bose fields. To describe the Bose fields, one uses the Boson Fock space over a Hilbert space $\mathcal{X}$:

$$\begin{aligned}\mathscr{F}_{\rm b}(\mathcal{X}) :&= \bigoplus_{n=0}^{\infty} \otimes_{\rm s}^{n} \mathcal{X} \\ &= \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \;\middle|\; \begin{array}{c} \text{for all } n \in \mathbb{N},\ \psi^{(n)} \in \otimes_{\rm s}^{n} \mathcal{X} \\ \text{and } \ \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty \end{array} \right\},\end{aligned}$$

where $\bigotimes_{\rm s}^{n} \mathcal{X}$ is the n-fold symmetric tensor product of $\mathcal{X}$, i.e., for the symmetrization operator $\mathcal{S}_n := (1/n!) \sum_{\sigma \in \mathfrak{S}_n} U_\sigma$ on $\bigotimes^n \mathcal{X}$, where $\mathfrak{S}_n$ is the symmetric group of degree n and $U_\sigma(\psi_1, \ldots, \psi_n) := (\psi_{\sigma(1)}, \ldots, \psi_{\sigma(n)})$, $\bigotimes_{\rm s}^{n} \mathcal{X} := \mathcal{S}_n(\otimes^n \mathcal{X})$ with $\bigotimes_{\rm s}^{0} \mathcal{X} = \mathbb{C}$. Let us define the finite particle subspace

$$\begin{aligned}\mathscr{F}_{\rm b,0}(\mathcal{X}) :&= \left\{ \psi \in \mathscr{F}_{\rm b}(\mathcal{X}) \;\middle|\; \begin{array}{c} \text{there exists a number } n_0 \in \mathbb{N} \\ \text{such that } \psi^{(n)} = 0 \text{ for all } n \geq n_0 \end{array} \right\} \\ &= \coprod_{n=0}^{\infty} \otimes_{\rm s}^{n} \mathcal{X}.\end{aligned}$$

($\coprod$ is the algebraic coproduction) $\mathscr{F}_{\rm b,0}(\mathcal{X})$ is dense in $\mathscr{F}_{\rm b}(\mathcal{X})$ and a fundamental subspace in the Fock space. Similarly, for a linear subspace $\mathcal{D} \subset \mathcal{X}$, we define a linear subspace $\mathscr{F}_{\rm b,\,fin}(\mathcal{D}) \subset \mathscr{F}_{\rm b,0}(\mathcal{X})$;

$$\mathscr{F}_{\rm b,\,fin}(\mathcal{D}) := \coprod_{n=0}^{\infty} \widehat{\otimes}_{\rm s}^{n} \mathcal{D}.$$

(Here, $\widehat{\otimes}$ is the algebraic tensor product and $\widehat{\otimes}_{\rm s}\mathcal{X} := \mathcal{S}_n(\widehat{\otimes}^n \mathcal{X})$.)

Basic objects on $\mathscr{F}_{\rm b}(\mathcal{X})$ are the creation and annihilation operators. For any $f \in \mathcal{X}$, the creation operator $a^*(f)$ is defined to be a densely defined closed linear operator on $\mathscr{F}_b(\mathcal{X})$ such that all for

$\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in D\big(a(f)^*\big) \supset \mathscr{F}_{\mathrm{b},0}(\mathcal{X})$,

$$\big(a(f)^*\psi\big)^{(0)} = 0,$$
$$\big(a(f)^*\psi\big)^{(n)} = \sqrt{n} S_n\Big(f \otimes \psi^{(n-1)}\Big), \qquad \text{for } n \geq 1.$$

This creation operator takes the n-particle subspace $\otimes_s^n \mathcal{X}$ into the $(n+1)$-particle subspace $\otimes_s^{n+1}\mathcal{X}$. The adjoint of the creation operator is called the annihilation operator and denoted by $a(g)$ $(g \in \mathcal{X})$. The creation operator $a^*(f)$ is the adjoint of annihilation operator $a(f)$. For all $f, g \in \mathcal{X}$, these operators obey the following relations:

$$[a(f), a^*(g)] = \langle f\,,\, g\rangle_{\mathcal{X}}, \quad [a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0 \tag{2.2}$$

on $\mathscr{F}_{\mathrm{b},0}(\mathcal{X})$, where $[X, Y] := XY - YX$.

The Segal field operator is defined as

$$\phi(f) := \frac{a(f) + a^*(f)}{\sqrt{2}}, \qquad f \in \mathcal{X}.$$

This operator $\phi(f)$ is known to be essentially self-adjoint on $\mathscr{F}_{\mathrm{b},0}(\mathcal{X})$ [**RS2**, Theorem X.41 (a)]. We denote its closure by the same symbol $\phi(f)$. From equalities (2.2), we have the following identity on $\mathscr{F}_{\mathrm{b},0}(\mathcal{X})$,

$$[\phi(f), \phi(g)] = i \operatorname{Im}\langle f\,,\, g\rangle, \qquad f,\ g \in \mathcal{X}. \tag{2.3}$$

Moreover, we see that

$$e^{i\phi(f)}e^{i\phi(g)} = e^{i \operatorname{Im}(f\,,\,g)}e^{i\phi(g)}e^{i\phi(f)}, \quad f, g \in \mathcal{X}, \tag{2.4}$$

which is called the Weyl relations of $\{\phi(f) \,|\, f \in \mathcal{X}\}$. For the proof of this equality, we refer to [**RS2**, Theorem X.41 (c)]

Definition 2.9. Let S, T be self-adjoint operators on a Hilbert space $\mathcal{X}$. We say that S and T strongly commute if their spectral measure E_S and E_T commute, that is, for all $I, J \in \mathfrak{B}(\mathbb{R})$,

$$E_S(I)E_T(J) = E_T(J)E_S(I)$$

THEOREM 2.10. Let S, T be self-adjoint operators on a Hilbert space $\mathcal{X}$. The followings are equivalent:

(1) S and T strongly commute.
(2) $(S-\lambda)^{-1}(T-\mu)^{-1} = (T-\mu)^{-1}(S-\lambda)^{-1}$ for all $\lambda \in \rho(S)$, $\mu \in \rho(T)$.
(3) $e^{isS}e^{itT} = e^{itT}e^{isS}$ for all $s, t \in \mathbb{R}$.

REMARK 2.11. From the equation (2.4), we see that $\phi(f)$ and $\phi(g)$ are strongly commutative if $\langle f, g\rangle \in \mathbb{R}$.

The second quantization of a densely defined closable operator S is denoted by $\mathrm{d\Gamma(S)}$ and defined by

$$\mathrm{d\Gamma(S)} := \bigoplus_{\mathrm{n}=0}^{\infty} \mathrm{S}^{(\mathrm{n})},$$

where $S^{(n)}$ is defined as follows:

$$S^{(0)} := 0,$$

$$S^{(n)} := \overline{\sum_{j=1}^{n} I \otimes \cdots \otimes I \otimes \underset{(j\,\mathrm{th})}{S} \otimes I \otimes \cdots \otimes I \Big|_{\hat{\otimes}_s^n D(T)}}, \quad \text{if } \; n \geq 1.$$

(I denotes the identity operator.) The domain of the second quantization operator $\mathrm{d}\Gamma(S)$ is

$$\Big\{\psi \in \mathcal{F}_{\mathrm{b}}(\mathcal{X}) \mid \psi^{(n)} \in \mathrm{D}(S^{(n)}),\ \sum_{n=0}^{\infty} \|S^{(n)}\psi^{(n)}\| < \infty\Big\}.$$

It is easy to see that, if S is self-adjoint or nonnegative, then so is $\mathrm{d\Gamma(S)}$. The next lemma describes well known properties of $\phi(f)$ and $\mathrm{d}\Gamma(S)$ (see, e.g., [**Ara2, HH1**]).

LEMMA 2.12. Let S be a densely defined, injective, nonnegative self-adjoint operator on a Hilbert space $\mathcal{X}$.

(i) If $f \in \mathrm{D}(S^{-1/2})$, then $\mathrm{D}(\mathrm{d}\Gamma(S)^{1/2}) \subseteq \mathrm{D}(a(f)) \cap \mathrm{D}(a^*(f))$ and for all $\psi \in \mathrm{D}(\mathrm{d}\Gamma(S))$

$$\|a(f)\psi\| \leq \|S^{-1/2}f\|\|\mathrm{d}\Gamma(S)^{1/2}\psi\|,$$
$$\|a(f)^*\psi\| \leq \|S^{-1/2}f\|\|\mathrm{d}\Gamma(S)^{1/2}\psi\| + \|f\|\|\psi\|.$$

(ii) If $f \in \mathrm{D}(S^{-1/2})$, then $\mathrm{D}(\mathrm{d}\Gamma(S)^{1/2}) \subseteq \mathrm{D}(\phi(f))$ and

$$\|\phi(f)\big(\mathrm{d}\Gamma(S)+1\big)^{-1/2}\| \leq \sqrt{2}(\|f\| + \|S^{-1/2}f\|\big). \tag{2.5}$$

(iii) If $f, g \in \mathrm{D}(S^{-1/2})$, then $\mathrm{D}\big(\mathrm{d}\Gamma(S)\big) \subseteq \mathrm{D}\big(\phi(f)\phi(g)\big)$ and

$$\|\phi(f)\phi(g)\big(\mathrm{d}\Gamma(S)+1\big)^{-1}\| \leq 4(\|f\|+\|S^{-1/2}f\|\big)(\|g\|+\|S^{-1/2}g\|\big). \tag{2.6}$$

(iv) If $f \in \mathrm{D}(S)$, then

$$\big[\mathrm{d}\Gamma(S)\,,\,\phi(f)\big] = -i\phi(iSf) \qquad \text{on } \mathcal{F}_{\mathrm{b},0}(\mathcal{X}) \cap \mathrm{D}\big(\mathrm{d}\Gamma(S)\big). \tag{2.7}$$

CHAPTER 3

Self-adjointness of the GSB Hamiltonians

1. Definition of a Hamiltonian

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. We take a Hilbert space

$$\mathscr{F} := \mathcal{H} \otimes \mathscr{F}_{\mathrm{b}}(\mathcal{K}).$$

Let A_0 be a self-adjoint operator on $\mathcal{H}$ which is a free Hamiltonian of a quantum system, A_1 an A_0-bounded symmetric operator on $\mathcal{H}$, W an injective, self-adjoint and nonnegative operator on $\mathcal{K}$ which is a one-particle Hamiltonian of the Bose field, B_j $(j = 1, \ldots, n)$ be self-adjoint operators on $\mathcal{H}$ such that $\mathrm{D}(A_0) \cap \bigcap_{j=1}^{n} D(B_j)$ is dense in $\mathcal{H}$, $f_j \in \mathcal{K}$ $(j = 1, \ldots, m)$, $g_j \in \mathcal{K}$ $(j = 1, \ldots, n)$ and $\lambda, \mu \in \mathbb{R}$. We consider the following operator as the total Hamiltonian of the coupled system:

$$H(\lambda,\mu) := A_0 \otimes I + A_1 \otimes I + I \otimes \mathrm{d}\Gamma(W) + \lambda \sum_{j=1}^{n} B_j \otimes \phi(g_j) + \mu \sum_{j=1}^{m} I \otimes \phi(f_j)^2. \tag{3.1}$$

This Hamiltonian $H(\lambda,\mu)$ was studied by Miyao and Sasaki [**MS**]. In the case of $\mu = 0$, it was introduced in [**AH**] and called the generalized spin-boson (abbreviated as GSB) Hamiltonian.

Example 3.1. (Spin-Boson Model) Let $\mathcal{H} = \mathbb{C}^2$, $\mathcal{K} = L^2(\mathbb{R}^3)$, $J = N = 1$, σ_i $(i = 1, 2, 3)$ be Pauli matrices, $A = \mu\sigma_3/2$ ($\mu > 0$ is constant), W be the multiplication operator $\omega(k) = |k|$, $B_1 = \sigma_1$. In this case, the GSB Hamiltonian becomes

$$H_{SB}(\lambda) := \frac{\mu}{2}\sigma_3 \otimes I + I \otimes \mathrm{d}\Gamma(\omega) + \lambda\sigma_1 \otimes \phi(g).$$

This Hamiltonian is called the spin-boson Hamiltonian. It is the model of a two-level atom interacting with neutral scalar fields. There are so many mathematical and physical researches on this type of Hamiltonians that we know many properties of this Hamiltonian. For a deeper discussion of spin boson models, we refer the reader to [**AH, Spo**] .

In this section, we need the following conditions.

(H_1) A_0 is a nonnegative self-adjoint operator on $\mathcal{H}$ and $B_1, \ldots, B_n$ are $A_0^{1/2}$-bounded symmetric operators.

(H_2) W is a nonnegative, injective and self-adjoint operator.

(H_3) There exists a core D for A_0 such that $\mathrm{D} \subset \bigcap_j \big(\mathrm{D}(A_0 B_j) \cap \mathrm{D}(B_j A_0)\big)$ and $[A_0, B_j]|_{\mathrm{D}}$ is $A_0^{1/2}$-bounded for each j.

(H_4) f_j and $g_j \in \mathrm{D}(W^{-1/2}) \cap \mathrm{D}(W)$ for all j

Conditions (H_1), (H_2), and (H_4) are standard condition in the GSB model.

2. (Essential) Self-Adjointness

In this section, we study the self-adjointness of the operator $H(\lambda, \mu)$. For simplicity, we set

$$H_0(\lambda, \mu) := A_0 \otimes I + I \otimes \mathrm{d}\Gamma(W) + \lambda \sum_{j=1}^{n} B_j \otimes \phi(g_j) + \mu \sum_{j=1}^{m} I \otimes \phi(f_j)^2,$$

$$H_{00} := H_0(0, 0) + 1 = A_0 \otimes I + I \otimes \mathrm{d}\Gamma(W) + 1.$$

In what follows, we may write an operator T (resp. S) for $T \otimes I$ (resp. $I \otimes S$). Let us first prove that the essential self-ajointness of $H_0(\lambda, \mu)$. First, we study the self-adjointness of the operator $H_0(\lambda, \mu)$ instead of the full Hamiltonian $H_0(\lambda, \mu)$.

Proposition 3.2. Suppose that (H_1)-(H_4) hold. Then $H_0(\lambda, \mu)$ is essentially self-adjoint on any core for H_{00}.

PROOF. Let D' be a core for W and $D_0 := D \widehat{\otimes} \mathscr{F}_{\mathrm{b,\,fin}}(D')$. Then D_0 is a core for H_{00}. To prove this proposition we use Nelson's commutator theorem [**RS2**, Theorem X.37]. We verify that $H_0(\lambda,\mu)$ and H_{00} satisfy the condition of the commutator theorem. In the following inequalities, C denotes a constant which may change from one inequality to the next. By using (i) and (ii) in Lemma 2.12, we have the following inequalities for all $\Psi \in D_0$,

$$\begin{aligned}
\|H_0(\lambda,\mu)\Psi\| \le\ & \big\|\big(A_0\otimes I + I\otimes \mathrm{d}\Gamma(W)\big)\Psi\big\| + |\lambda|\sum_{j=1}^{n}\big\|B_j\otimes\phi(g_j)\Psi\big\| \\
& + |\mu|\sum_{j=1}^{m}\big\|I\otimes\phi(f_j)^2\Psi\big\| \\
\le\ & \|\big(A_0\otimes I + I\otimes \mathrm{d}\Gamma(W)+1\big)\Psi\| \\
& + C\sum_{j=1}^{n}\|(A_0+1)^{1/2}\otimes\big(\mathrm{d}\Gamma(W)+1\big)^{1/2}\Psi\| \\
& + C\sum_{j=1}^{m}\|I\otimes(\mathrm{d}\Gamma(W)+1)\Psi\| \\
\le\ & \|H_{00}\Psi\| + C\big\langle (A_0+1)\otimes I\ \Psi\,,\ I\otimes\big(\mathrm{d}\Gamma(W)+1\big)\Psi\big\rangle^{1/2} \\
& + C\|H_{00}\Psi\| \\
\le\ & C\|H_{00}\Psi\| \\
& + C\|(A_0+1)\otimes I\ \Psi\|^{1/2}\|I\otimes(\mathrm{d}\Gamma(W)+1)\Psi\|^{1/2} \\
\le\ & C\|H_{00}\Psi\| \\
& + C\Big(\|(A_0+1)\otimes I\ \Psi\| + \|I\otimes(\mathrm{d}\Gamma(W)+1)\Psi\|\Big) \\
\le\ & C\|H_{00}\Psi\|
\end{aligned}$$

Similarly,

$$\begin{aligned}
& |\langle H_0(\lambda,\mu)\Psi\,,\ H_{00}\Psi\rangle - \langle H_{00}\Psi\,,\ H_0(\lambda,\mu)\Psi\rangle| \\
& \le |\lambda|\sum_{j=1}^{n}\Big|\big\langle B_j\otimes\phi(g_j)\Psi\,,\ H_{00}\Psi\big\rangle - \big\langle H_{00}\Psi\,,\ B_j\otimes\phi(g_j)\Psi\big\rangle\Big|
\end{aligned}$$

$$
\begin{aligned}
&+ |\mu| \sum_{j=1}^{m} \Big| \big\langle \phi(f_j)^2 \Psi \,,\, \mathrm{d}\Gamma(W)\Psi \big\rangle - \big\langle \mathrm{d}\Gamma(W)\Psi \,,\, \phi(f_j)^2 \Psi \big\rangle \Big| \\
&\le |\lambda| \sum_{j=1}^{n} \Big| \big\langle I \otimes \phi(g_j)\Psi \,,\, [B_j \,,\, A_0]\Psi \big\rangle + \big\langle B_j \Psi \,,\, I \otimes [\phi(g_j) \,,\, \mathrm{d}\Gamma(W)]\Psi \big\rangle \Big| \\
&+ |\mu| \sum_{j=1}^{m} \Big| \Big\langle \big[\phi(f_j), \mathrm{d}\Gamma(W)\big]\Psi, \phi(f_j)\Psi \Big\rangle - \Big\langle \phi(f_j)\Psi, \big[\mathrm{d}\Gamma(W), \phi(f_j)\big]\Psi \Big\rangle \Big| \\
&\le |\lambda| \sum_{j=1}^{n} \Big(\|I \otimes \phi(g_j)\Psi\| \, \|[B_j \,,\, A_0]\Psi\| + \|B_j \Psi\| \, \|I \otimes \phi(iWg_j)\Psi\| \Big) \\
&+ |\mu| \sum_{j=1}^{m} \Big| \langle \phi(iWf_j)\Psi \,,\, \phi(f_j)\Psi \rangle + \langle \phi(f_j)\Psi \,,\, \phi(iWf_j)\Psi \rangle \Big| \\
&\le |\lambda| C \sum_{j=1}^{n} \|I \otimes \big(\mathrm{d}\Gamma(W) + 1\big)^{1/2}\Psi\| \Big(\|[B_j \,,\, A_0]\Psi\| + \|B_j \Psi\| \Big) \\
&+ |\mu| C \sum_{j=1}^{m} \|I \otimes \big(\mathrm{d}\Gamma(W) + 1\big)^{1/2}\Psi\|^2 \\
&\le 2n|\lambda| C \|I \otimes \big(\mathrm{d}\Gamma(W) + 1\big)^{1/2}\Psi\| \, \|(A_0 + 1)^{1/2} \otimes I \, \Psi\| \\
&+ m|\mu| C \|I \otimes \big(\mathrm{d}\Gamma(W) + 1\big)^{1/2}\Psi\|^2 \\
&\le (2n|\lambda| + m|\mu|) C \|H_{00}^{1/2}\Psi\|^2.
\end{aligned}
$$

By Nelson's commutator theorem, the operator $H_0(\lambda, \mu)$ is essentially self-adjoint on the subspace D and any core for H_{00}. ■

The next corollary follows immediately from the last proposition,

Corollary 3.3. Suppose that (H_1)-(H_4) hold. If A_1 is an infinitesimall small with respect ot A_0, then $H(\lambda, \mu)$ is essentially self-adjoint on any core for H_{00}.

We next show the self-adjointness of $H_0(\lambda, \mu)$. From the previous proposition, we see that $H_0(\lambda, \mu)$ is essentially self-adjoint on the domain $\mathrm{D}(H_{00})$ for any coupling constant under some condition. We infer that $H_0(\lambda, \mu)$ is self-adjoint for any coupling constant under suitable condition even if A_0 and B_j are not commutative. Here we do not show that the self-adjontness of the Hamiltonian for all coupling constant. However, in the next theorem, we prove that the self-adjointness of $H_0(\lambda, \mu)$ follows from the semi-boundedness of $H_0(\lambda, 0)$ under natural conditions.

Theorem 3.4. Suppose that (H_1)-(H_4) hold. Assume, in addition, the following conditions hold:

(i) the core $D \subset \bigcap_j \mathrm{D}(A_0 B_j) \cap \mathrm{D}(A_0^2)$ and $[A_0^{1/2}, B_j]|_{\mathrm{D}}$ is bounded;

(ii) $g_j \in \mathrm{D}(W^{-1/2}) \cap \mathrm{D}(W^2)$ for all j.

If $H_0(\lambda, 0)$ is bounded from below for some λ, then for all λ' with $|\lambda'| < |\lambda|$ and $\mu \geq 0$, $H_0(\lambda', \mu)$ is a self-adjoint operator and $\mathrm{D}\big(H_0(\lambda', \mu)\big) = \mathrm{D}(H_{00})$.

Proof. Without loss of generality, we can assume that $0 < \lambda' < \lambda$. It follows from Proposition 3.2 that $H_0(\lambda', \mu)$ is essentially self-adjoint on $\mathrm{D}(H_{00})$ and $\mathrm{D}(H_{00}) \subseteq \mathrm{D}\big(\overline{H_0(\lambda', \mu)}\big)$.

Hence we only have to verify that $\mathrm{D}\big(\overline{H(\lambda', \mu)}\big) \subseteq \mathrm{D}(H_{00})$. Since $0 < \lambda' < \lambda$, there exists a positive number $\eta < 1$ such that $\lambda = \lambda'/(1-\eta)$. Let D' be a core for W^2, $D_0 := D \,\widehat{\otimes}\, \mathscr{F}_{\mathrm{b,\,fin}}(D')$, $\mu' := \mu/(1-\eta)$, c the infimum of the spectrum of $H_0(\lambda, 0)$, that is, $c := \inf \sigma\big(H_0(\lambda, 0)\big)$, and $\gamma := \sup_j \|\overline{[A_0^{1/2},\, B_j]}\|$. For all $\Psi \in D_0$,

$$
\begin{aligned}
&\|\big(H_0(\lambda', \mu) + \eta\big)\Psi\|^2 - \frac{\eta^2}{2}\|H_{00}\Psi\|^2 \\
&\geq \frac{\eta^2}{2}\|H_{00}\Psi\|^2 + (1-\eta)^2\|H_0(\lambda, \mu')\Psi\|^2 \\
&\quad + \eta(1-\eta)\Big(\langle H_{00}\Psi\,,\, H_0(\lambda, \mu')\Psi\rangle + \langle H_0(\lambda, \mu')\Psi\,,\, H_{00}\Psi\rangle\Big) \\
&\geq \frac{\eta^2}{2}\|H_{00}\Psi\|^2 \\
&\quad + \eta(1-\eta)\Big(2\big\langle A_0^{1/2}\Psi, H_0(\lambda, 0)A_0^{1/2}\Psi\big\rangle + 2\,\mathrm{Re}\,\big\langle A_0^{1/2}\Psi, [A_0^{1/2}, H_0(\lambda, 0)]\Psi\big\rangle \\
&\qquad + 2\big\langle\big(\mathrm{d}\Gamma(W)+1\big)^{1/2}\Psi, H_0(\lambda, 0)\big(\mathrm{d}\Gamma(W)+1\big)^{1/2}\Psi\big\rangle \\
&\qquad + 2\,\mathrm{Re}\,\big\langle\big(\mathrm{d}\Gamma(W)+1\big)^{1/2}\Psi, \big[\big(\mathrm{d}\Gamma(W)+1\big)^{1/2}, H_0(\lambda, 0)\big]\Psi\big\rangle\Big)
\end{aligned}
$$

$$+\,\eta(1-\eta)\mu' \sum_{j=1}^{m} 2\,\mathrm{Re}\,\big\langle \big(\mathrm{d}\Gamma(W)+1\big)\Psi\,,\ \phi(f_j)^2\Psi\big\rangle.$$

By the semi-boundedness of $H_0(\lambda, 0)$, we have

$$\langle A_0^{1/2}\Psi\,,\ H_0(\lambda,0)A_0^{1/2}\Psi\rangle \geq c\|A_0^{1/2}\Psi\|^2,$$

$$\langle(\mathrm{d}\Gamma(W)+1)^{1/2}\Psi, H_0(\lambda,0)(\mathrm{d}\Gamma(W)+1)^{1/2}\Psi\rangle \geq c\|(\mathrm{d}\Gamma(W)+1)^{1/2}\Psi\|^2.$$

Hence we get the following inequality.

$$\begin{aligned}
&\|\big(H_0(\lambda',\mu)+\eta\big)\Psi\|^2 - \frac{\eta^2}{2}\|H_{00}\Psi\|^2 \\
&\geq \frac{\eta^2}{2}\|H_{00}\Psi\|^2 \\
&\quad + \eta(1-\eta)\Big(2c\|A_0^{1/2}\Psi\|^2 + 2c\|\big(\mathrm{d}\Gamma(W)+1\big)^{1/2}\Psi\|^2\Big) \\
&\quad - 2\eta(1-\eta)\|A_0^{1/2}\Psi\|\,\big\|[H(\lambda,0)\,,\ A_0^{1/2}]\Psi\big\| \\
&\quad + \eta(1-\eta)\Big\langle \Psi\,,\ \Big[\big(\mathrm{d}\Gamma(W)+1\big)^{1/2},\ \big[\big(\mathrm{d}\Gamma(W)+1\big)^{1/2},\ H_0(\lambda,0)\big]\Big]\Psi\Big\rangle \\
&\quad + 2\eta(1-\eta)\mu' \sum_{j=1}^{m} \big\langle \phi(f_j)\Psi\,,\ \big(\mathrm{d}\Gamma(W)+1\big)\phi(f_j)\Psi\big\rangle \\
&\quad + 2\eta(1-\eta)\mu' \sum_{j=1}^{m} \mathrm{Re}\,\big\langle [\phi(f_j)\,,\ \mathrm{d}\Gamma(W)]\Psi\,,\ \phi(f_j)\Psi\big\rangle \\
&\geq \frac{\eta^2}{2}\|H_{00}\Psi\|^2 \\
&\quad + 2c\eta(1-\eta)\Big(\|A_0^{1/2}\Psi\|^2 + \big\|\big(\mathrm{d}\Gamma(W)+1\big)^{1/2}\Psi\big\|^2\Big) \\
&\quad - 2\eta(1-\eta)|\lambda| \sum_{j=1}^{n} \|A_0^{1/2}\Psi\|\,\|[A_0^{1/2},\ B_j]\otimes\phi(g_j)\Psi\| && (3.2) \\
&\quad - \eta(1-\eta)|\lambda| \sum_{j=1}^{n} \big\|B_j\Psi\big\| \\
&\qquad\qquad \times \Big\|\Big[\big(\mathrm{d}\Gamma(W)+1\big)^{1/2}\!,\ \big[\big(\mathrm{d}\Gamma(W)+1\big)^{1/2}\!,\ \phi(g_j)\big]\Big]\Psi\Big\| && (3.3) \\
&\quad - \eta(1-\eta)|\mu'| \sum_{j=1}^{m} \|\Psi\|\,\big\|\big[\,[\mathrm{d}\Gamma(W)\,,\ \phi(f_j)]\,,\ \phi(f_j)\big]\Psi\big\|. && (3.4)
\end{aligned}$$

In the following arguments, we show that each term of (3.2)–(3.4) is greater than or equal to $-C(\|A_0^{1/2}\Psi\|^2 + \|I \otimes \big(d\Gamma(W)+1\big)^{1/2}\Psi\|^2)$ for some positive constant C.

First, we estimate the term (3.2). Since $[A_0^{1/2},\, B_j]$ is bounded,

$$
\begin{aligned}
&\|A_0^{1/2}\Psi\|\,\|[A_0^{1/2},\, B_j]\otimes\phi(g_j)\Psi\| \\
&\qquad\qquad \le \gamma\|A_0^{1/2}\Psi\|\,\|I\otimes\phi(g_j)\Psi\| \\
&\qquad\qquad \le C\|A_0^{1/2}\Psi\|\,\big\|\big(d\Gamma(W)+1\big)^{1/2}\Psi\big\| \\
&\qquad\qquad \le C\Big(\|A_0^{1/2}\Psi\|^2 + \big\|\big(d\Gamma(W)+1\big)^{1/2}\Psi\big\|^2\Big).
\end{aligned}
$$

Next, we consider the term (3.3). It is known that for a nonnegative self-adjoint operator T on a Hilbert space,

$$
T^{1/2}\psi = \Big(\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(T+\lambda)^{-1}d\lambda\Big)T\psi \tag{3.5}
$$

for any $\psi \in \mathrm{D}(T)$ (see [**RS1**, Chapter VIII Problem 50 (c)]). Since $\Psi \in D_0$ and $g_j \in \mathrm{D}(W^2)$, it is easy to see that $(d\Gamma(W)+1)^{1/2}\Psi$, $\phi(g_j)\Psi$, $\phi(g_j)(d\Gamma(W)+1)^{1/2}\Psi$, and $[(d\Gamma(W)+1)^{1/2},\ \phi(g_j)]\Psi$ are in $\mathrm{D}(d\Gamma(W))$. Using the formula (3.5), we can calculate as follow.

$$
\begin{aligned}
&\Big[\big(d\Gamma(W)+1\big)^{1/2},\ \big[\big(d\Gamma(W)+1\big)^{1/2},\ \phi(g_j)\big]\Big]\Psi \\
&= \frac{1}{\pi^2}\int_0^\infty dt\int_0^\infty ds\ \frac{1}{\sqrt{ts}}\Big[\big(d\Gamma(W)+1+s\big)^{-1}\big(d\Gamma(W)+1\big)\,, \\
&\qquad\qquad\qquad \big[\big(d\Gamma(W)+1+t\big)^{-1}\big(d\Gamma(W)+1\big)\,,\ \phi(g_j)\big]\Big]\Psi \\
&= \frac{1}{\pi^2}\int_0^\infty dt\int_0^\infty ds\ \sqrt{ts}\big(d\Gamma(W)+1+s\big)^{-1}\big(d\Gamma(W)+1+t\big)^{-1} \\
&\qquad\qquad \times\phi(W^2g_j)\big(d\Gamma(W)+1+t\big)^{-1}\big(d\Gamma(W)+1+s\big)^{-1}\Psi.
\end{aligned}
$$

Thus we get a bound of the term (3.3) from the following computation:

$$
\begin{aligned}
&\Big\|\Big[\big(d\Gamma(W)+1\big)^{1/2},\ \big[\big(d\Gamma(W)+1\big)^{1/2},\ \phi(g_j)\big]\Big]\Psi\Big\| \\
&\le \frac{1}{\pi^2}\int_0^\infty dt\int_0^\infty ds\frac{\sqrt{ts}}{(1+t)(1+s)}
\end{aligned}
$$

$$\times \left\| \phi(W^2 g_j) \big(\mathrm{d}\Gamma(W) + 1 + t \big)^{-1} \big(\mathrm{d}\Gamma(W) + 1 + s \big)^{-1} \Psi \right\|$$

$$\leq C \left(\frac{1}{\pi} \int_0^\infty ds \frac{\sqrt{s}}{(1+s)^2} \right)^2 \left\| \big(\mathrm{d}\Gamma(W) + 1 \big)^{1/2} \Psi \right\|.$$

Using Lemma 2.12 (iii) and the identity (2.3), we have an inequality about the term (3.4),

$$\left| \Big\langle \Psi \, , \, \Big[\big[\mathrm{d}\Gamma(W) \, , \, \phi(f_j) \big] \, , \, \phi(f_j) \Big] \Psi \Big\rangle \right| \leq |\langle W f_j \, , \, f_j \rangle| \|\Psi\|^2.$$

Hence we see that

$$\begin{aligned}
&\| \big(H_0(\lambda', \mu) + \eta \big) \Psi \|^2 - \frac{\eta^2}{2} \| H_{00} \Psi \|^2 \\
&\geq \frac{\eta^2}{2} \| H_{00} \Psi \|^2 - C \Big(\| A_0^{1/2} \Psi \|^2 + \| \big(\mathrm{d}\Gamma(W) + 1 \big)^{1/2} \Psi \|^2 + \| \Psi \|^2 \Big) \\
&= \frac{\eta^2}{2} \| H_{00} \Psi \|^2 - C \langle \Psi \, , \, H_{00} \Psi \rangle - C \| \Psi \|^2 \\
&\geq -C \| \Psi \|^2.
\end{aligned}$$

Since D_0 is a core for $H(\lambda', \mu)$, the above inequality implies

$$\mathrm{D}\big(\overline{H(\lambda', \mu)} \big) \subseteq \mathrm{D}(H_{00}).$$

Thus $H_0(\lambda', \mu)$ is self-adjoint and $\mathrm{D}\big(H(\lambda', \mu) \big) = \mathrm{D}(H_{00})$. ■

3. Unitary Transformation

In this part, we consider the unitary transformation of the following operator on a Hilbert space $\mathcal{H} \otimes \mathscr{F}_{\mathrm{b}}(\mathcal{K})$:

$$H(\lambda) := A \otimes I + I \otimes \mathrm{d}\Gamma(W) + \lambda \sum_{j=1}^n B_j \otimes \phi(g_j).$$

First, we consider the weak differentiability of a Heisenberg operator. Let $\mathcal{X}$ be a Hilbert sapce, H a self-adjoint operator and S a symmetric operator on $\mathcal{X}$. We recall that the Heisenberg operator of S with respect to H is defined

$$S(t) =: e^{itH} S e^{-itH}, \quad t \in \mathbb{R}.$$

Proposition 3.5. Let H, S and $\mathcal{X}$ be as above. Suppose that there exist a operator T on $\mathcal{X}$ such that the followings hold:

(1) $\mathrm{D}(T) \subseteq \mathrm{D}(S)$ and there exist constants $a, b \geq$ such that

$$\|S\psi\| \leq a\|T\psi\| + b\|\psi\|, \quad \psi \in \mathrm{D}(T).$$

(2) $e^{itH}\mathrm{D}(T) \subseteq \mathrm{D}(T)$ for all $t \in \mathbb{R}$ and $\lim_{t\to 0} T(e^{itH} - 1)\psi = 0$ for all $\psi \in \mathrm{D}(T) \cap \mathrm{D}(H)$.

Then, for all $\psi, \phi \in \mathrm{D}(T) \cap \mathrm{D}(H)$, the function $f(t) = \langle \psi, S(t)\phi \rangle$ on $\mathbb{R}$ is differentiable and

$$\begin{aligned} f'(t) &= \frac{d}{dt}\langle \psi, S(t)\phi \rangle \\ &= i\Big\{ \big\langle He^{-itH}\psi, Se^{-itH}\phi \big\rangle - \big\langle Se^{-itH}\psi, He^{-itH}\phi \big\rangle \Big\}. \end{aligned} \tag{3.6}$$

Proof. From the assumption (2), we see that $e^{itH}\mathrm{D}(T) \cap \mathrm{D}(H) \subseteq \mathrm{D}(T) \cap \mathrm{D}(H)$ for all $t \in \mathbb{R}$. We put $F_s := (e^{-isH} - 1)/s$ and $G_s := e^{-isH} - 1$ for $s \in \mathbb{R} \setminus \{0\}$. Then, for $s \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} \frac{f(t+s) - f(t)}{s} &= \frac{1}{s}\Big(\big\langle e^{-i(t+s)H}\psi, Se^{-i(t+s)H}\phi \big\rangle - \big\langle e^{-itH}\psi, Se^{-itH}\phi \big\rangle \Big) \\ &= \big\langle F_s e^{-itH}\psi, Se^{-i(t+s)H}\phi \big\rangle \\ &\quad + \frac{1}{s}\Big(\big\langle e^{-itH}\psi, Se^{-i(t+s)H}\phi \big\rangle - \big\langle e^{-itH}\psi, Se^{-itH}\phi \big\rangle \Big) \\ &= \big\langle F_s e^{-itH}\psi, SG_s e^{-itH}\phi \big\rangle + \big\langle F_s e^{-itH}\psi, Se^{-itH}\phi \big\rangle \\ &\quad + \big\langle Se^{-itH}\psi, F_s e^{-itH}\phi \big\rangle. \end{aligned}$$

By the assumption (1), we see that

$$\begin{aligned} |\big\langle F_s e^{-itH}\psi, SG_s e^{-itH}\phi \big\rangle| &\leq \|F_s e^{-itH}\psi\| \|SG_s e^{-itH}\phi\| \\ &\leq \|F_s\psi\| \big(a\|TG_s e^{-itH}\phi\| + b\|G_s e^{-itH}\phi\| \big). \end{aligned} \tag{3.7}$$

Since $F_s\psi \to -iH\psi$ and $TG_s\psi \to 0$ as $s \to 0$ in norm topology for all $\psi \in \mathrm{D}(T) \cap \mathrm{D}(H)$, from the assumption (2) and the inequality (3.7), we obtain

$$\lim_{s\to 0}\langle F_s e^{-itH}\psi, SG_s e^{-itH}\phi\rangle = 0$$

and equation (3.6). ■

REMARK 3.6. In the Proposition 3.5, if the operator T is self-adjoint and strongly commute with H, then the condition (2) hold.

In this section, we assume that following conditions:

(A_1) A is a symmetric operator on a Hilbert space $\mathcal{H}$.

(A_2) W is a nonnegative injective self-adjoint operator on a Hilbert space $\mathcal{K}$.

(A_3) $B_1, \dots, B_n$ are strongly commuting self-adjoint operators on $\mathcal{H}$.

(A_4) $g_j \in \mathrm{D}(W^{-3/2}) \cap \mathrm{D}(W)$ and $\langle W^{-1}g_j, W^{-1}g_l\rangle \in \mathbb{R}$ for all $1 \le j, l \le n$.

(A_5) there exists a dense subspace $D \subseteq \bigcap_{t\ge 1, j}(\mathrm{D}(B_jA) \cap \mathrm{D}(AB_j) \cap \mathrm{D}(AB_j(A+t)^{-1}))$ such that $[B_j, A]|_{\mathrm{D}}$ is bounded for each j.

(A_6) there exists a nonnegative self-adjoint operator V on $\mathcal{H}$ such that the followings hold:

(1) the set D is a core for V.

(2) A is V-bounded.

(3) B_j is $V^{1/2}$-bounded for each j.

(4) $e^{itB_j\otimes\phi(iW^{-1}g_j)}\big(\mathrm{D}(V\otimes I)\big) \subseteq \mathrm{D}(V\otimes I)$ for all $t \in \mathbb{R}$.

For simplicity of notation, we set

$$\begin{aligned}
F &:= V\otimes I + I\otimes \mathrm{d}\Gamma(W),\\
T &:= \overline{\sum_{j=1}^{n} B_j \otimes \phi(iW^{-1}g_j)},\\
U(\lambda) &:= e^{i\lambda T},\\
R(g,B) &:= \sum_{j,k=1}^{n}\langle g_j, W^{-1}g_k\rangle B_jB_k,\\
\delta A(\lambda) &:= U(\lambda)(A\otimes I)U(-\lambda) - A\otimes I.
\end{aligned}$$

REMARK 3.7. Under the conditions (A_3) and (A_4), we see that $\{B_j \otimes \phi(iW^{-1}g_j)\}_{j=1}^n$ is a family of strongly commuting self-adjoint operators. Therefore the operator T is self-adjoint and $U(\lambda)$ is an unitary operator.

Next lemma is well known fact.

LEMMA 3.8. Let $\mathcal{X}$ be a Hilbert space, W a nonnegative, injective self-adjoint operator on $\mathcal{X}$, and $g \in \mathrm{D}(W)$. Then

$$e^{i\phi(ig)}\mathrm{D}\big(\mathrm{d}\Gamma(W)\big) = \mathrm{D}\big(\mathrm{d}\Gamma(W)\big)$$

and

$$e^{i\phi(ig)}\mathrm{d}\Gamma(W)e^{-i\phi(ig)} = \mathrm{d}\Gamma(W) + \phi(Wg) + \frac{1}{2}\langle g, Wg\rangle.$$

From above lemmas, we have the following theorem.

THEOREM 3.9. Suppose that condition (A_1)-(A_6) hold. Then, for all $t \in \mathbb{R}$,

$$U(\lambda)\mathrm{D}(F) = \mathrm{D}(F)$$

and, for all $\Psi \in \mathrm{D}(F)$,

$$\begin{aligned} &U(\lambda)H(\lambda)U(-\lambda)\Psi \\ &\quad = \big(A \otimes I - \frac{\lambda^2}{2}R(g,B) \otimes I + I \otimes \mathrm{d}\Gamma(W) + \delta A(\lambda)\big)\Psi. \end{aligned} \tag{3.8}$$

PROOF. By the strong commutativity of the family $\{B_j\}$, there exists a n-dimensional spectral measure $E_{\mathbf{B}}$ such that, for all Borel sets $I_j \subseteq \mathbb{R}$ ($j = 1, \ldots, n$), $E(I_1 \times \cdots \times I_n) = E_{B_1}(I_1) \cdots E_{B_n}(I_n)$ and $B_j = \int_{\mathbb{R}^n} \xi_j dE_B(\xi)$ ($\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$). Let $\Psi = u \otimes \psi$ and $\Phi = v \otimes \phi \in \mathrm{D}(V) \mathbin{\widehat{\otimes}} \mathrm{D}(\mathrm{d}\Gamma(W))$. Then

$$\langle I \otimes \mathrm{d}\Gamma(W)\Psi, U(\lambda)\Phi\rangle = \int_{\mathbb{R}^n} \big\langle \mathrm{d}\Gamma(W)\psi, e^{-i\phi(iG_\xi(g))}\phi\big\rangle d\langle u, E(\xi)v\rangle,$$

where $G_\xi(g) = \lambda \sum_{j=1}^n \xi_j W^{-1} g_j$. By Lemma 3.8, we see that

$$\begin{aligned}&\left\langle \mathrm{d}\Gamma(W)\psi, e^{-i\phi(iG_\xi(g))}\phi \right\rangle \\ &\quad = \left\langle \psi, e^{-i\phi(iG_\xi(g))}\Big(\mathrm{d}\Gamma(W) + \phi(WG_\xi(g)) + \frac{1}{2}\big\langle G_\xi(g), WG_\xi(g)\big\rangle\Big)\phi \right\rangle.\end{aligned}$$

Hence

$$\begin{aligned}&\langle I \otimes \mathrm{d}\Gamma(W)\Psi, U(\lambda)\Phi\rangle \\ &\quad = \left\langle \Psi, U(\lambda)\Big(I \otimes \mathrm{d}\Gamma(W) + \lambda \sum_{j=1}^n B_j \otimes \phi(g_j) + \frac{\lambda^2}{2} R(g, B) \otimes I\Big)\Phi \right\rangle.\end{aligned} \tag{3.9}$$

It is easy to see that above equation (3.9) extends to all vectors $\Psi, \Phi \in \mathrm{D}(V) \widehat{\otimes} \mathrm{D}(\mathrm{d}\Gamma(W))$. From Lemma 2.12, for all vector $\psi \in \mathrm{D}(\mathrm{d}\Gamma(W))$ and $f \in \mathrm{D}(W^{-1/2})$, we have

$$\begin{aligned}\|a(f)\psi\| &\le \|W^{-1/2} f\| \|\mathrm{d}\Gamma(W)^{1/2}\psi\|, \\ \|a(f)^*\psi\| &\le \|W^{-1/2} f\| \|\mathrm{d}\Gamma(W)^{1/2}\psi\| + \|f\|\|\psi\|.\end{aligned}$$

Since B_j is $V^{1/2}$-bounded, it follow from the similar argument in Proposition 3.2 that

$$\begin{aligned}&\|B_j \otimes \phi(g_j)\Psi\| \\ &\le \sqrt{2}\|W^{1/2} g_j\| \|B_j \otimes \mathrm{d}\Gamma(W)^{1/2}\Psi\| + \frac{1}{\sqrt{2}}\|g_j\|\|\Psi\| \\ &\le \sqrt{2}\|W^{1/2} g_j\| \big(a_j \|V^{1/2} \otimes \mathrm{d}\Gamma(W)^{1/2}\Psi\| + b_j \|I \otimes \mathrm{d}\Gamma(W)^{1/2}\Psi\|\big) \\ &\quad + \frac{1}{\sqrt{2}}\|g_j\|\|\Psi\| \\ &\le \sqrt{2} a_j \|W^{1/2} g_j\| \|V \otimes I\Psi\|^{1/2} \|I \otimes \mathrm{d}\Gamma(W)\Psi\|^{1/2} \\ &\quad + \sqrt{2} b_j \|W^{1/2} g_j\| \|I \otimes \mathrm{d}\Gamma(W)^{1/2}\Psi\| + \frac{1}{\sqrt{2}}\|g_j\|\|\Psi\| \\ &\le \frac{a_j}{\sqrt{2}}\|W^{1/2} g_j\| \|V \otimes I\Psi\| + \frac{1}{\sqrt{2}}(a_j + 2\varepsilon b_j)\|W^{1/2} g_j\| \|I \otimes \mathrm{d}\Gamma(W)\Psi\|\end{aligned}$$

$$+\Big(d_j(\varepsilon)\sqrt{2}b_j\|W^{1/2}g_j\| + \frac{1}{\sqrt{2}}\|g_j\|\Big)\|\Psi\|$$
$$\le C\|\big(V\otimes I + I\otimes \mathrm{d}\Gamma(W)\big)\Psi\|$$

By assumptions, we see that $\sum_{j,k=1}^{n} B_j B_k$ is V-bounded. From this, we get

$$\Big\|\Big(I\otimes\mathrm{d}\Gamma(W) + \lambda\sum_{j=1}^{n} B_j\otimes\phi(g_j) + \frac{\lambda^2}{2}R(g,B)\otimes I\Big)\Psi\Big\| \le C\|(F+1)\Psi\|$$

From the fact that $\mathrm{D}(V)\,\widehat{\otimes}\,\mathrm{D}(\mathrm{d}\Gamma(W))$ is a core for $V\otimes I + I\otimes\mathrm{d}\Gamma(W)$, (3.9) extends to all $\Psi\in\mathrm{D}(F)$. Thus, for all $\Psi\in\mathrm{D}(F)$, $U(\lambda)\Psi$ is in $\mathrm{D}(I\otimes\mathrm{d}\Gamma(W))$ and

$$I\otimes\mathrm{d}\Gamma(W)\Psi = U(\lambda)\Big(I\otimes\mathrm{d}\Gamma(W) + \lambda\sum_{j=1}^{n} B_j\otimes\phi(g_j) + \frac{\lambda^2}{2}R(g,B)\otimes I\Big)\Psi.$$

Thus, $U(\lambda)\mathrm{D}(F)\subseteq\mathrm{D}(F)$. Since $U(\lambda)$ is an unitary operator, it follows that

$$\mathrm{D}(F)\subseteq U(-\lambda)\mathrm{D}(F) = U(-\lambda)\mathrm{D}(F)$$

Therefore we obtain $U(\lambda)\mathrm{D}(F) = \mathrm{D}(F)$ and it is easy to see that the equation (3.8). ■

Proposition 3.10. We assume (A_1)-(A_6). Suppose that the commutator $[V, B_j]$ is bounded on the core D for each j. Then, $\mathrm{D}(V\otimes I + I\otimes\mathrm{d}\Gamma(W))\subseteq\mathrm{D}(\delta A(\lambda))$ and

$$\|\delta A(\lambda)\Psi\| \le |t|\Big(c_3(g)\|I\otimes\mathrm{d}\Gamma(W)^{1/2}\Psi\| + \frac{1}{2}c_1(g)\|\Psi\|\Big) \qquad (3.10)$$

for all $\Psi\in\mathrm{D}(V\otimes I + I\otimes\mathrm{d}\Gamma(W))$. Here the constant $c_k(g)$ $(k=1,2,3)$ is defined as follow

$$c_k(g) := \sqrt{2}\sum_{j=1}^{n}\|[B_j, A]\|\|W^{-k/2}g_j\|.$$

Before proving above proposition, we give some lemmas.

Lemma 3.11. Let A, B be densely defined operators on $\mathcal{H}$ and W a nonnegative injective self-adjoint operator on $\mathcal{K}$. Suppose that $g \in \mathrm{D}(W^{-3/2})$ and there exist a dense subset $D \subseteq \mathrm{D}(AB) \cap \mathrm{D}(BA)$ such that $[B, A]|_{\mathrm{D}}$ is bounded. Then

$$\mathrm{D}(I \otimes \mathrm{d}\Gamma(W)^{1/2}) \subseteq \mathrm{D}(\overline{[B, A]} \otimes \phi(iW^{-1}g_j))$$

and

$$\begin{aligned} &\left\|\overline{[B, A]} \otimes \phi(iW^{-1}g)\Psi\right\| \\ &\quad \leq \left\|\overline{[B, A]}\right\|\Big(\sqrt{2}\|W^{-3/2}g\|\|I \otimes \mathrm{d}\Gamma(W)^{1/2}\Psi\| + \frac{1}{\sqrt{2}}\|W^{-1}g\|\|\Psi\|\Big) \end{aligned} \tag{3.11}$$

for all $\Psi \in \mathrm{D}(I \otimes \mathrm{d}\Gamma(W)^{1/2})$.

Proof. By Lemma 2.12, we have

$$\|\phi(f)\psi\| \leq \sqrt{2}\|W^{-1/2}f\|\|\mathrm{d}\Gamma(W)^{1/2}\psi\| + \frac{1}{2}\|f\|\|\psi\|, \tag{3.12}$$

for $f \in \mathrm{D}(W^{-1/2})$ and $\psi \in \mathrm{D}(\mathrm{d}\Gamma(W))$. Hence,

$$\begin{aligned} &\left\|\overline{[B, A]} \otimes \phi(iW^{-1}g)\Psi\right\| \leq \left\|\overline{[B, A]}\right\|\|\phi(iW^{-1}g)\psi\| \\ &\quad \leq \left\|\overline{[B, A]}\right\|\Big(\sqrt{2}\|W^{-3/2}g\|\|I \otimes \mathrm{d}\Gamma(W)^{1/2}\Psi\| + \frac{1}{\sqrt{2}}\|W^{-1}g\|\|\Psi\|\Big) \end{aligned}$$

for all $\Psi \in D \widehat{\otimes} \mathrm{D}(\mathrm{d}\Gamma(W)^{1/2})$. Since $D \widehat{\otimes} \mathrm{D}(\mathrm{d}\Gamma(W)^{1/2})$ is a core for $I \otimes \mathrm{d}\Gamma(W)^{1/2}$, the inequality (3.11) holds for all $\Psi \in \mathrm{D}(I \otimes \mathrm{d}\Gamma(W)^{1/2})$. It follows from the inequality (3.11) that $\mathrm{D}(I \otimes \mathrm{d}\Gamma(W)^{1/2}) \subseteq \mathrm{D}(\overline{[B, A]} \otimes \phi(iW^{-1}g_j))$. ■

Lemma 3.12. Let A be nonnegative self-adjoint operator on $\mathcal{H}$ and B an $A^{1/2}$-bounded operator. Suppose that $[A, B]$ is bounded operator and there exists a dense set $D \subseteq \mathrm{D}(AB) \cap \mathrm{D}(BA) \cap \bigcap_{t \geq 1} \mathrm{D}(AB(A+t)^{-1})$ which is a core for A. Then B^2 is an A-bounded operator.

Proof. Since B is an $A^{1/2}$-bounded operator, there exist positive constants c and d such that

$$\|B\Psi\| \leq c\|A^{1/2}\Psi\| + d\|\Psi\|, \qquad \text{for all } \Psi \in \mathrm{D}(A^{1/2}).$$

Hence, for all $\Psi \in D \subseteq \mathrm{D}(AB) \cap \mathrm{D}(BA)$ and $\varepsilon > 0$,

$$\begin{aligned}
\|B^2\Psi\| &\leq c\|A^{1/2}B\Psi\| + d\|B\Psi\| \\
&\leq c\big\|A(A+1)^{-1/2}\big\|\|(A+1)^{1/2}B\Psi\| + d\|B\Psi\| \\
&\leq c\Big(\big\|[(A+1)^{1/2},\, B]\Psi\big\| + \|B(A+1)^{1/2}\Psi\|\Big) \\
&\quad + d\big(\|A^{1/2}\Psi\| + d\|\Psi\|\big).
\end{aligned}$$

Since $A^{1/2}$ is an A-compact operator, for all positive constant ε, there is a positive constant $c(\varepsilon)$ depending on ε which satisfies

$$\|(A+1)^{1/2}\Psi\| \leq \varepsilon\|A\Psi\| + c(\varepsilon)\|\Psi\|, \qquad \text{for all } \Psi \in \mathrm{D}(A).$$

From the formula (3.5), we see that

$$\begin{aligned}
\|[(A+1)^{1/2}, B]\Psi\| &\leq \left\|\frac{1}{\pi}\int_0^\infty t^{-1/2}\big[(A+1+t)^{-1}(A+1), B\big]\Psi dt\right\| \\
&\leq \frac{1}{\pi}\int_0^\infty \sqrt{t}\Big\|\big[(A+1+t)^{-1}, B\big]\Psi\Big\|dt \\
&\leq \frac{1}{\pi}\int_0^\infty \sqrt{t}\Big\|\big(A+1+t)^{-1}[A, B](A+1+t)^{-1}\Psi\Big\|dt \\
&\leq \frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{(1+t)^2}dt\|[A, B]\|\|\Psi\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|B^2\Psi\| &\leq (c^2 + 2\varepsilon cd)\|A\Psi\| \\
&\qquad + \left(d^2 + 2cdc(\varepsilon) + c^2 + c\frac{1}{\pi}\int_0^\infty \frac{\sqrt{t}}{(1+t)^2}dt\|[A\,,\, B]\|\right)\|\Psi\|.
\end{aligned}$$

Since D is a core for A, B^2 is an A-bounded operator. From this, it follows that $\mathrm{D}(A) \subseteq \mathrm{D}(B^2)$. ■

Proof of Proposition 3.10. Let $T := \overline{\sum_{j=1}^{n} B_j \otimes \phi(iW^{-1}g_j)}$. For all $\Psi, \Phi \in \mathrm{D}(F)$, by Proposition 3.5, the function $\mathbb{R} \ni t \mapsto \langle \Psi, U(\lambda)(A \otimes I)U(-\lambda)\Phi \rangle$ is differentiable and

$$\frac{d}{dt}\big\langle \Psi, U(\lambda)(A\otimes I)U(-\lambda)\Phi\big\rangle$$
$$= -i\Big\{\Big\langle TU(-\lambda)\Psi, (A\otimes I)U(-\lambda)\Phi\Big\rangle - \Big\langle (A\otimes I)U(-\lambda)\Psi, TU(-\lambda)\Phi\Big\rangle\Big\} \tag{3.13}$$

It is easy to see that for all $\Psi, \Phi \in D \mathbin{\widehat{\otimes}} \mathrm{D}(\mathrm{d}\Gamma(W))$

$$\Big\langle T\Psi, (A \otimes I)\Phi\Big\rangle - \Big\langle (A \otimes I)\Psi, T\Phi\Big\rangle = \Big\langle \Psi, \sum_{j=1}^{n} \overline{[B_j, A]} \otimes \phi(iW^{-1}g_j)\Phi\Big\rangle \tag{3.14}$$

By the assumption and Lemma 3.11, $A \otimes I$, $\sum_{j=1}^{n} B_j \otimes \phi(iW^{-1}g_j)$, and $\sum_{j=1}^{n} \overline{[B_j, A]} \otimes \phi(iW^{-1}g_j)$ are F-bounded . Since $D \mathbin{\widehat{\otimes}} \mathrm{D}(\mathrm{d}\Gamma(W))$ is a core for F, we see that the equation (3.14) holds for all $\Psi, \Phi \in \mathrm{D}(F)$. Therefore, from the equation (3.13) and Lemma 3.8, we obtain

$$\frac{d}{dt}\big\langle \Psi, U(\lambda)(A \otimes I)U(-\lambda)\Phi\big\rangle$$
$$= -\Big\langle U(-\lambda)\Psi, \sum_{j=1}^{n} \overline{[B_j, A]} \otimes \phi(iW^{-1}g_j)U(-\lambda)\Phi\Big\rangle.$$

Hence,

$$\begin{aligned}
\big|\big\langle \Psi, \delta A(\lambda)\Phi\big\rangle\big| &\leq \left| \int_0^t \Big\langle U(-s)\Psi, \sum_{j=1}^{n} \overline{[B_j, A]} \otimes \phi(iW^{-1}g_j)U(-s)\Phi\Big\rangle ds\right| \\
&\leq \left| \int_0^t \big\|U(-s)\Psi\big\| \Big\| \sum_{j=1}^{n} \overline{[B_j, A]} \otimes \phi(iW^{-1}g_j)U(-s)\Phi\Big\| ds\right| \\
&\leq \sum_{j=1}^{n} \left| \int_0^t \big\|\Psi\big\| \|\overline{[B_j, A]}\| \big\|I \otimes \phi(iW^{-1}g_j)U(-s)\Phi\big\| ds\right| \\
&\leq \sum_{j=1}^{n} \left| \int_0^t \big\|\Psi\big\| \|\overline{[B_j, A]}\| \big\|I \otimes \phi(iW^{-1}g_j)\Phi\big\| ds\right|
\end{aligned}$$

$$\le |t| \sum_{j=1}^{n} \|\Psi\| \|\overline{[B_j, A]}\| \|I \otimes \phi(iW^{-1}g_j)\Phi\|,$$

where we used the strong commutativity of $I \otimes \phi(iW^{-1}g_j)$ and $U(-\lambda)$. From above inequality and (3.12), the conclusion follows. ■

4. Semiboundedness of $H_0(\lambda)$

Our next aim is to prove the semi-boundedness of $H_0(\lambda, 0)$. Next proposition gives us a sufficient condition. To simplify notation, we write

$$H_0(\lambda) := H_0(\lambda, 0) = A_0 + \mathrm{d}\Gamma(W) + \lambda \sum_{j=1}^{n} B_j \otimes \phi(g_j).$$

Proposition 3.13. Let A_0 be a nonnegative self-adjoint operator on the Hilbert space $\mathcal{H}$, $\{B_j\}_{j=1}^n$ is a family of self-adjoint operators on $\mathcal{H}$, and W a nonnegative injective self-adjoint operator on the Hilbert space $\mathcal{K}$. Suppose that A_0 has a decomposition $A_0 \subset \sum_{j=1}^n A_{0,j}$ satisfying following conditions:

(i) the operator $A_{0,j}$ is a nonnegative self-adjoint operator with $\mathrm{D}(A_0) \subset \mathrm{D}(A_{0,j})$ for all j;
(ii) each operator B_j is $A_{0,j}^{1/2}$-bounded;
(iii) there exists a dense subset

$$D_j \subset \mathrm{D}(A_{0,j}B_j) \cap \mathrm{D}(B_j A_{0,j}) \cap \bigcap_{t \ge 1} \mathrm{D}(A_{0,j}B_j(A_{0,j}+t)^{-1})$$

which is a core for $A_{0,j}$ and the commutator $[A_{0,j}, B_j]$ on D_j is bounded for each j.

If $g_j \in \mathrm{D}(W^{-3/2}) \cap \mathrm{D}(W)$, $e^{itB_j \otimes \phi(iW^{-1}g_j)}\big(\mathrm{D}(A_{0,j} \otimes I)\big) \subseteq \mathrm{D}(A_{0,j} \otimes I)$ for all $t \in \mathbb{R}$ and j, $\lim_{t \to 0}(A_{0,j} \otimes I)(e^{itB_j \otimes \phi(iW^{-1}g_j)} - 1)\psi = 0$ for all $\psi \in \mathrm{D}(A_{0,j} \otimes I) \cap \mathrm{D}(I \otimes \mathrm{d}\Gamma(W))$, and there exists a $\{\lambda_j\}_{j=1}^n$ such that $0 < \lambda_j < 1$, $\sum_{j=1}^n \lambda_j = 1$, and $A_{0,j} - \lambda^2 \lambda_j^{-1} \| W^{-1/2} g_j \|^2 B_j^2/2$ is bounded from below for each j, then $H_0(\lambda)$ is bounded from below.

PROOF. By Lemma 3.12, we see that B_j^2 is an$A_{0,j}$-bouded operator. We set

$$H_{0,j}(\lambda) := A_{0,j} + \lambda_j \mathrm{d}\Gamma(W) + \lambda B_j \otimes \phi(g_j),$$
$$U_j(\lambda) := \exp(i\lambda\lambda_j^{-1} B_j \otimes \phi(iW^{-1}g_j)).$$

It follows from Theorem 3.9 that

$$U_j(-\lambda)H_{0,j}(\lambda)U_j(\lambda) = A_{0,j} - \frac{\lambda^2}{2\lambda_j}\|W^{-1/2}g_j\|^2 B_j^2 + \lambda_j \mathrm{d}\Gamma(W) + \delta A_{0,j}(\lambda).$$

Here $\delta A_{0,j}(\lambda) := U_j(-\lambda)(A_{0,j} \otimes I)U_j(\lambda) - A_{0,j} \otimes I$. By Lemma 3.10, we see that $\delta A_{0,j}(\lambda)$ is infinitesimally small with respect to $I \otimes \mathrm{d}\Gamma(W)$. Since unitary transformation preserve the spectral property, by the Kato-Rellich theorem, we see that $H_{0,j}(\lambda)$ is bounded from below for each j. Hence $H_0(\lambda) = \sum H_{0,j}(\lambda)$ is bounded from below. ■

COROLLARY 3.14. Suppose that (H_1)-(H_2) hold. Assume, in addition, the following conditions hold:

(i) there exists a dense subset

$$D \subset \mathrm{D}(A_0 B_j) \cap \mathrm{D}(B_j A_0) \cap \bigcap_{t \geq 1} \mathrm{D}(A_0 B_j (A_0 + t)^{-1})$$

which is a core for A_0 and the commutator $[A_0, B_j]$ on D is bounded for each j;

(ii) B_j is self-adjoint and $e^{itB_j \otimes \phi(iW^{-1}g_j)}\big(\mathrm{D}(A_0 \otimes I)\big) \subseteq \mathrm{D}(A_0 \otimes I)$ for all $t \in \mathbb{R}$ and j;

(iii) $\lim_{t\to 0}(A_0 \otimes I)(e^{itB_j\otimes\phi(iW^{-1}g_j)} - 1)\psi = 0$ for all $\psi \in \mathrm{D}(A_0 \otimes I) \cap \mathrm{D}(I \otimes \mathrm{d}\Gamma(W))$;

(iv) $g_j \in \mathrm{D}(W^{-3/2}) \cap \mathrm{D}(W)$ for all j.

If there exists a $\{\lambda_j\}_{j=1}^n$ such that $0 < \lambda_j < 1$, $\sum_{j=1}^n \lambda_j = 1$, and $A_0 - \lambda^2\lambda_j^{-2}\| W^{-1/2}g_j\|^2 B_j^2/2$ is bounded from below for each j, then $H_0(\lambda)$ is bounded from below.

Proof. Applying Proposition 3.13 with $A_{0,j} = \lambda_j A_0$ for each j, we obtain the desired conclusion. ■

Remark 3.15. Under the conditions of Corollary 3.14 except for existence of the $\{\lambda_j\}_{j=1}^n$, if $|\lambda| \sum_{j=1}^n c_j \|W^{-1/2} g_j\| < \sqrt{2}$, then there exists a $\{\lambda_j\}_{j=1}^n$ such that $0<\lambda_j<1$, $\sum_{j=1}^n \lambda_j = 1$, and

$$A_0 - \lambda^2 \lambda_j^{-2} \|W^{-1/2} g_j\|^2 B_j^2/2$$

is self-adjoint and bounded from below for all j. Here c_j is an $A_0^{1/2}$-bound of B_j. Indeed, there exists a $\{\lambda_j\}_{j=1}^n$ such that $0 < \lambda_j < 1$, $\sum_{j=1}^n \lambda_j = 1$, and $|\lambda| c_j \|W^{-1/2} g_j\| < \sqrt{2}\lambda_j$ for all j whenever λ satisfies

$$|\lambda| \sum_{j=1}^n c_j \|W^{-1/2} g_j\| < \sqrt{2}.$$

From the proof of Proposition 3.13, we see that relative bound of B_j^2 with respect to A_0 is less than or equal to c_j^2. Therefore, from the Kato-Rellich theorem, we have the desired $\{\lambda_j\}_{j=1}^n$. This condition is weaker than the condition (A.3) in [**AH**].

Finally, from the above results, we obtain the following corollary.

Corollary 3.16. Suppose that (H_1), (H_2) and (H_4) hold. Assume, in addition, the following conditions hold:

(i) B_j is self-adjoint and $e^{itB_j \otimes \phi(iW^{-1} g_j)} \big(\mathrm{D}(A_0 \otimes I)\big) \subseteq \mathrm{D}(A_0 \otimes I)$ for all $t \in \mathbb{R}$ and j;

(ii) $\lim_{t\to 0} (A_0 \otimes I)(e^{itB_j \otimes \phi(iW^{-1} g_j)} - 1)\psi = 0$ for all $\psi \in \mathrm{D}(A_0 \otimes I) \cap \mathrm{D}(I \otimes \mathrm{d}\Gamma(W))$;

(iii) there exists a core D for A_0 such that $D \subset \mathrm{D}(A_0^2) \cap \bigcap_j \mathrm{D}(A_0 B_j) \cap \bigcap_{t \geq 1/2} \mathrm{D}(AB_j(A_0 + t)^{-1})$ and $[A_0, B_j]|_D$ is a bounded operator for each j;

If $g_j \in \mathrm{D}(W^{-3/2}) \cap \mathrm{D}(W^2)$ for all j, $|\lambda| \sum_{j=1}^n c_j \|W^{-1/2} g_j\| < \sqrt{2}$, and $\mu \geq 0$, then $H_0(\lambda, \mu)$ is a self-adjoint operator with $\mathrm{D}\big(H_0(\lambda,\mu)\big) = \mathrm{D}\big(H(0,0)\big)$.

Proof. In the proof of Theorem 3.4, we can replace the term $A_0^{1/2}$ by $(A_0 + 1/2)^{1/2}$ and $(\mathrm{d}\Gamma(W)+1)^{1/2}$ by $(\mathrm{d}\Gamma(W)+1/2)^{1/2}$. Using the formlua (3.5), it is easy to see that $[(A_0+1/2)^{1/2}, B_j]|_D$ is bounded. Therefore we see that similar argument in the proof of Theorem 3.4 work. From this and Corollary 3.14 and Remark 3.15, we have the self-adjointness of $H_0(\lambda,\mu)$ and $\mathrm{D}\big(H_0(\lambda,\mu)\big) = \mathrm{D}(H(0,0))$. ■

Corollary 3.17. Suppose that (H_1) and (H_2). Assume, in addition, the following:

(1) A_1 is an infinitesimally small operator with respect to A_0;
(2) B_j strongly commutes with A_0 for each j;
(3) $g_j \in \mathrm{D}(W^{-3/2}) \cap \mathrm{D}(W^2)$ and $f_j \in \mathrm{D}(W^{-1/2}) \cap \mathrm{D}(W)$;
(4) $A_0 - n\lambda^2 \|W^{-1/2} g_j\|^2 B_j^2/2$ is bounded from below for each j.

Then $H(\lambda,\mu)$ is self-adjoint operator with $\mathrm{D}(H(\lambda,\mu)) = \mathrm{D}(H(0,0))$ for all $\mu \geq 0$.

Corollary 3.18. Suppose that (H_1) and (H_2). Assume, in addition, the following:

(1) A_1 is an infinitesimally small operator with respect to A_0;
(2) B_j strongly commutes with A_0 and B_k for each j,k;
(3) $g_j \in \mathrm{D}(W^{-3/2}) \cap \mathrm{D}(W^2)$ and $f_j \in \mathrm{D}(W^{-1/2}) \cap \mathrm{D}(W)$;
(4) $\langle W^{-1} g_j, W^{-1} g_k \rangle \in \mathbb{R}$ for all j,k;
(5) $A_0 - \frac{\lambda^2}{2} \sum_{j,k=1}^n \langle g_j, W^{-1} g_k \rangle B_j B_k$ is bounded from below for each j.

Then $H(\lambda,\mu)$ is self-adjoint operator with $\mathrm{D}(H(\lambda,\mu)) = \mathrm{D}(H(0,0))$ for all $\mu \geq 0$.

CHAPTER 4

Ground States of the GSB Hamiltonians

1. Absence of the Ground States

In this section, we consider ground states of the GSB Hamiltonian $H(\lambda)$ in the case where the particle Hamiltonian A has no ground states. Existence of a ground state of $H(\lambda)$ depends on whether the mass of the boson m (=$\inf \sigma(W)$) is positive or 0. In each case, under some conditions, the enhanced binding occurs and there exists a ground state for large coupling constant λ when $\mathcal{K} = L^2(\mathbb{R}^d)$ and W is a multiplication operator of continuous function [**AK**]. To consider the ground states of Hamiltonian $H(\lambda)$, we pose the following hypotheses (cf. [**AK**]):

(H-I) $g_j \in D(W^{-3/2})$ $(j = 1, \ldots, J)$ and $\langle W^{-1}g_j, W^{-1}g_l \rangle \in \mathbb{R}$ $(j, l = 1, \ldots, n)$.

(H-II) A_0 is a non-negative self-adjoint operator and A_1 is an A_0-bounded symmetric operator, that is, $D(A_0) \subset D(A_1)$ and there exist constants $a, b \geq 0$ such that

$$\|A_1 u\| \leq a\|A_0 u\| + b\|u\|, \qquad u \in D(A_0).$$

(H-III) The operator A_0 strongly commutes with each B_j $(j = 1, \ldots, n)$ and

$$D(A_0) \subset \bigcap_{j,l=1}^{J} D(B_j B_l).$$

Moreover there exist constants c_j, $d_j \geq 0$ such that

$$\|B_j u\| \leq c_j \|A_0^{1/2} u\| + d_j \|u\|, \quad (j = 1, \dots, n) \quad \text{for } u \in D(A_0^{1/2}).$$

(H-IV) The set $\{B_j\}_{j=1}^n$ is a family of strongly commuting self-adjoint operators.

(H-V) The domain of A_0 satisfies

$$D(A_0) \subset \bigcap_{j=1}^{n} \big(D(B_j A_1) \cap D(A_1 B_j) \big)$$

and $[B_j,\, A_1]|_{D(A_0)}$ $(j = 1, \dots n)$ are bounded.

(H-VI) Let

$$A_0(\lambda) := A_0 - \lambda^2 R(g, B),$$
$$A(\lambda) := A_0 + A_1 - \lambda^2 R(g, B).$$

with

$$R(g, B) := \frac{1}{2} \sum_{j,l=1}^{J} \left\langle W^{-1/2} g_j \,,\, W^{-1/2} g_l \right\rangle_{\mathcal{K}} B_j B_l$$

The set

$$\Lambda := \left\{ \lambda \in \mathbb{R} \setminus \{0\} \;\middle|\; \begin{array}{l} A_0(\lambda) \text{ and } A(\lambda) \text{ are self-adjoint} \\ \text{and bounded from below} \end{array} \right\}$$

is not empty.

Remark 4.1. Assume (H-I)-(H-III) and suppose that

$$a + \frac{\lambda^2}{2} \sum_{j,l=1}^{J} \left| \left(\frac{g_j}{\sqrt{\omega}} \,,\, \frac{g_l}{\sqrt{\omega}} \right) \right| c_j c_l < 1. \tag{4.1}$$

Then, $A_0(\lambda)$ and $A(\lambda)$ are self-adjoint and bounded from below. Indeed, by (H-III), we obtain

$$\|B_j B_l u\| \leq c_j c_l \|A_0 u\| + (c_j d_l + c_l d_j) \|A_0^{1/2} u\| + d_j d_l \|u\|, \quad u \in D(A_0).$$

Hence

$$\begin{aligned}\|(A_1-\lambda^2R_B)u\| \le &\left(a+\frac{\lambda^2}{2}\sum_{j,l=1}^{J}\left|\left(\frac{g_j}{\sqrt{\omega}},\frac{g_l}{\sqrt{\omega}}\right)\right|c_jc_l\right)\|A_0u\|\\ &+\lambda^2\left(\sum_{j,l=1}^{J}\left|\left(\frac{g_j}{\sqrt{\omega}},\frac{g_l}{\sqrt{\omega}}\right)\right|c_jd_l\right)\|A_0^{1/2}u\|\\ &+\left(b+\frac{\lambda^2}{2}\sum_{j,l=1}^{J}\left|\left(\frac{g_j}{\sqrt{\omega}},\frac{g_l}{\sqrt{\omega}}\right)\right|d_jd_l\right)\|u\|.\end{aligned}$$

Since $A_0^{1/2}$ is infinitesimally small with respect to A_0, the inequality (4.1) implies that $A_1-\lambda^2R_B$ has relative bound with respect to A_0 which is less than 1. Therefore the Kato-Rellich theorem implies $A(\lambda)=A_0+A_1-\lambda^2R_B$ is self-adjoint on $D\big(A(\lambda)\big)=D(A_0)$ and bounded from below. In particular, if $a<1$, then (H-VI) holds.

Let us state a fact on the self-adjointness of $H(\lambda)$.

Theorem 4.2. Assume (H-I)–(H-VI). Then, for any $\lambda\in\Lambda$, $H(\lambda)$ is self-adjoint with $D\big(H(\lambda)\big)=D(A_0\otimes I)\bigcap D(I\otimes \mathrm{d}\Gamma(W))$ and bounded from below.

Proof. From Theorem 3.9 and Proposition 3.10, we obtain the conclusion. ■

Under the some condition, it is known that the Hamiltonian has ground states. Before we state the results on ground states, we introduce assumptions and notations. We pose the following conditions.

(H-VII) The Hilbert space $\mathcal{K}=L^2(\mathbb{R}^d)$ and the operator W is the multiplication operator ω on $L^2(\mathbb{R}^d)$.

(H-VIII) The function $\omega(k)$ is continuous with

$$\lim_{|k|\to\infty}\omega(k)=\infty$$

and there exist constants $\gamma > 0$ and $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^\gamma[1 + \omega(k) + \omega(k')], \qquad k, k' \in \mathbb{R}^d.$$

(H-IX) Each g_j ($j = 1, \ldots, n$) is a continuous function.

We set

$$c_s(g) := \sqrt{2}\sum_{j=1}^{n} \|[B_j, A_1]\| \left\| \frac{g_j}{\omega^{s/2}} \right\|,$$

$$m := \inf_{k\in\mathbb{R}^d} \omega(k),$$

$$E_0\big(H(\lambda)\big) := \inf \sigma\big(H(\lambda)\big),$$

$$\Sigma_\lambda := \inf \sigma_{\mathrm{ess}}\big(A(\lambda)\big).$$

Theorem 4.3. Suppose that (H-I)-(H-IX) hold, $\lambda \in \Lambda$, and $A_0 + A_1$ has compact resolvent. Then $H(\lambda)$ has purely discrete spectrum in $[E_0(H(\lambda)), E_0(H(\lambda)) + m)$ for small coupling constants λ. In particular, $H(\lambda)$ has a ground state for small coupling constants λ.

Proof. See [**AH**]. ■

Theorem 4.4. Assume (H-I)-(H-VIII). Suppose that $\lambda \in \Lambda$ and

$$\Sigma_\lambda - E_0\big(A(\lambda)\big) > m + \frac{\lambda^2}{2}c_3(g)^2 + |\lambda|c_1(g).$$

Then $H(\lambda)$ has purely discrete spectrum in $[E_0(H(\lambda)), E_0(H(\lambda)) + m)$. In particular, $H(\lambda)$ has a ground states.

Proof. See [**AK**] ■

To study absence of ground states for small coupling constant λ, we assume the following conditions.

Assumption I. The operator A_0 has no ground states, i.e.,

$$E_0(A_0) = \inf \sigma(A_0) \notin \sigma_{\mathrm{p}}(A_0).$$

Assumption II. The operator A_1 is an A_0-compact self-adjoint non-positive operator, i.e., $A_1 \leq 0$, $D(A_0) \subset D(A_1)$ and $A_1(A_0 - z)^{-1}$ is compact operator for some $z \in \rho(A_0)$.

Assumption III. There exists a $\lambda_0 \in \Lambda$ such that $A_0(\lambda_0) - E_0(A_0)$ is a non-negative injective operator and

$$\dim \operatorname{Ker} \big(A_0(\lambda_0) + A_1 - E_0(A_0)\big) \leq \aleph_0.$$

Here $\aleph_0$ is the cardinal number of the set $\mathbb{N}$ of natural numbers.

Assumption IV. The Birman-Schwinger operator

$$K_E := |A_1|^{1/2}(A_0(\lambda_0) - E)^{-1}|A_1|^{1/2}$$

is bounded for $E < E_0(A_0)$ and there exists a compact self-adjoint operator K with $\|K\| < 1$ such that $\overline{K_E} \leq K$ for all $E < E_0(A_0)$.

Remark 4.5. It is easy to see that (H-VI) holds under Assumption II, since every relatively compact operator with respect to A_0 is infinitesimally small with respect to A_0 and Remark 4.1.

Under the above assumptions, we prove that $H(\lambda)$ has no ground states for sufficiently small $|\lambda|$. Before proving the absence of ground states, we show the following two lemmas.

Lemma 4.6. Let $\mathcal{X}$ be a Hilbert space, A a operator on $\mathcal{X}$, B a nonpositive closed operator on $\mathcal{X}$. Suppose that $B \neq 0$, $\lambda > 0$ and $E \in \rho(A)$. If $E \in \sigma_{\rm p}(A + \lambda B)$, then $\lambda^{-1} \in \sigma_{\rm p}(|B|^{1/2}(A - E)^{-1}|B|^{1/2})$.

Proof. By assumption, $\mathrm{D}(A + \lambda B) \subseteq \mathrm{D}(B) \subseteq \mathrm{D}(|B|^{1/2})$. Since $E \in \sigma_{\rm p}(A + \lambda B)$, there exists a nonzero vector $\psi \in \mathrm{D}(A + \lambda B)$ such that $(A+\lambda B)\psi = E\psi$. From this and nonpositivity of B, we have

$$|B|\psi = -B\psi = \frac{1}{\lambda}(A-E)\psi.$$

Hence, since $E \in \rho(A)$, we see that $\lambda(A-E)^{-1}|B|\psi = \psi \in \mathrm{D}(B)$ and $|B|^{1/2}\psi \in \mathrm{D}(|B|^{1/2}(A-E)^{-1}|B|^{1/2})$. Therefore we obtain

$$|B|^{1/2}(A-E)^{-1}|B|^{1/2}\big(|B|^{1/2}\psi\big) = \frac{1}{\lambda}|B|^{1/2}\psi.$$

Since $|B|^{1/2}\psi = 0$ implies $\psi \in \mathrm{Ker}\,(A-E) = \{0\}$, we see that $|B|^{1/2}\psi \neq 0$. This completes the proof. ■

Next we give a generalization of Birman-Schwinger bound. This is a key lemma to prove the absence of the ground states.

Lemma 4.7. Let $\mathcal{X}$ be a Hilbert space, A be a self-adjoint non-negative operator on $\mathcal{X}$ and B be an A-compact symmetric non-positive operator on $\mathcal{X}$. Suppose that $E_0(A) = \inf \sigma(A) \notin \sigma_{\mathrm{p}}(A)$, $K'_E := \overline{|B|^{1/2}(A-E)^{-1}|B|^{1/2}}$ is a compact operator for all $E < E_0(A)$ and $\dim \mathrm{Ker}\,(A+B-E_0(A)) \leq \aleph_0$. If there exists a compact operator $K'_0 := \lim_{E\uparrow E_0(A)} \overline{|B|^{1/2}(A-E)^{-1}|B|^{1/2}}$, then

$$\dim \mathrm{Ran}\,\Big(E_{A+B}\big(\,(-\infty, E_0(A)]\,\big)\Big) \leq \dim \mathrm{Ran}\,\big(E_{1-K'_0}(\mathbb{R}_{\leq})\big),$$

where $\mathbb{R}_{\leq} := (-\infty, 0]$.

Proof. We may assume that $E_0(A) = 0$ and $B \neq 0$. Let λ be a real number which is larger than 1. Since B is an A-compact operator, $A+B$ is a self-adjoint operator. By the stability theorem, $\sigma_{\mathrm{ess}}(A+B) = \sigma_{\mathrm{ess}}(A) \subset [0,\infty)$. Hence every $\eta \in \sigma(A)$ which is less than 0 is in the discrete spectrum of $A+B$.

Suppose that $\dim \mathrm{Ran}\, E_{A+\lambda B}(\mathbb{R}_{<}) < \dim \mathrm{Ran}\, E_{A+B}(\mathbb{R}_{\leq}) \leq \aleph_0$. Here $\mathbb{R}_{<} := (-\infty, 0)$. Set $n = \dim \mathrm{Ran}\, E_{A+\lambda B}(\mathbb{R}_{<})$ and $\{\varphi_i\}_{i=1}^{n+1}$ is an orthonormal system in $\mathrm{Ran}\, E_{A+B}(\mathbb{R}_{\leq})$. We see that for all $\phi_1, \ldots, \phi_n \in$

$\mathcal{X}$,

$$U(\phi_1, \ldots, \phi_n) := \inf_{\substack{\psi \in D(A),\ \|\psi\|=1 \\ \psi \in [\phi_1, \ldots, \phi_n]^\perp}} \langle \psi\,,\,(A+\lambda B)\psi \rangle$$

$$\leq \inf_{\substack{\psi \in [\varphi_1, \ldots, \varphi_{n+1}],\ \|\psi\|=1 \\ \psi \in [\phi_1, \ldots, \phi_n]^\perp}} \langle \psi\,,\,(A+\lambda B)\psi \rangle, \qquad (4.2)$$

wherer $[\phi_1, \ldots, \phi_n] = \{\Phi \in \mathcal{X} \mid \Phi = \sum_{i=1}^n a_i\phi_i,\ a_i \in \mathbb{C}\}$. Let P be the projection onto the subspace $[\varphi_1, \ldots, \varphi_{n+1}]$ and

$$\mu_{n+1} := \sup_{\phi_1, \ldots, \phi_n} U(\phi_1, \ldots, \phi_n).$$

Then, from the min-max principle and (4.2), the μ_{n+1} is less than or equal to the $(n+1)$st eigenvalue of $P(A+\lambda B)|_{[\varphi_1, \ldots, \varphi_{n+1}]}$. However, we see that $P(A+\lambda B)|_{[\varphi_1, \ldots, \varphi_{n+1}]}$ has no non-negative eigenvalues. Indeed, if $\xi \in [\varphi_1, \ldots, \varphi_{n+1}]$ is normalized eigenvector of $P(A+\lambda B)$, then its eigenvalue $\mu = \langle \xi\,,\,(A+B)\xi \rangle + (\lambda-1)\langle \xi\,,\,B\xi \rangle \leq 0$. But $\mu = 0$ implies $\xi \in \operatorname{Ker} A = \{0\}$, so that $\xi = 0$. This is a contradiction. Hence $\mu < 0$. We thus get $P(A+\lambda B)|_{[\varphi_1, \ldots, \varphi_{n+1}]} < 0$. This implies that $A+\lambda B$ has the $(n+1)$st eigenvalue which is less than 0. This is a contradiction to $n = \dim \operatorname{Ran} E_{A+\lambda B}(\mathbb{R}_<)$. Therefore we get an inequality

$$\dim \operatorname{Ran} E_{A+B}(\mathbb{R}_\leq) \leq \dim \operatorname{Ran} E_{A+\lambda B}(\mathbb{R}_<).$$

Let $\kappa_n(\mu)$ be the n-th eigenvalue of $A+\mu\lambda B$ for $\mu \in \mathbb{R}$, counting from $\inf \sigma(A+\mu\lambda B)$ and counting multiplicity. If $A+\mu\lambda B$ has only m eigenvalues below 0, we set $\kappa_n(\mu) = 0$ for $n \geq m+1$. Then the $\kappa_n(\mu)$ are monotone decreasing and continuous with respect to $\mu \geq 0$. Moreover, once $\kappa_n(\mu) \geq 0$, they are strictly monotone. Thus, by Lemma 4.6, we have an inequality

$$\dim \operatorname{Ran}\bigl(E_{A+\lambda B}((-\infty, E])\bigr) \leq \#\{n \in \mathbb{N} \mid \text{for some } 0 \leq \mu \leq 1,\ \kappa_n(\mu) = E\}$$

$$\leq \dim \bigcup_{0 \leq \mu \leq 1} \operatorname{Ker}\left(\mu\lambda K'_E - 1\right)$$

$$\leq \dim \operatorname{Ran}\big(E_{1-\lambda K'_E}(\mathbb{R}_<)\big)$$

for all $E<0$. From $K'_E \leq K'_0$, we obtain

$$\dim \operatorname{Ran}\big(E_{1-\lambda K'_E}(\mathbb{R}_<)\big) \leq \dim \operatorname{Ran}\big(E_{1-\lambda K'_0}(\mathbb{R}_<)\big).$$

Thus we see that

$$\begin{aligned}
\dim \operatorname{Ran}\big(E_{A+\lambda B}(\mathbb{R}_<)\big) &= \lim_{E\uparrow 0} \dim \operatorname{Ran}\big(E_{A+\lambda B}((-\infty, E])\big) \\
&\leq \lim_{E\uparrow 0} \dim \operatorname{Ran}\big(E_{1-\lambda K'_E}(\mathbb{R}_<)\big) \\
&\leq \dim \operatorname{Ran}\big(E_{1-\lambda K'_0}(\mathbb{R}_<)\big).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\dim \operatorname{Ran}\big(E_{A+B}(\mathbb{R}_\leq)\big) &\leq \lim_{\lambda\downarrow 1} \dim \operatorname{Ran}\big(E_{1-\lambda K'_0}(\mathbb{R}_<)\big) \\
&\leq \dim \operatorname{Ran}\big(E_{1-K'_0}(\mathbb{R}_\leq)\big).
\end{aligned}$$

This is the desired conclusion. ■

First, we consider the ground state of $A(\lambda)$ (see (H-VI)).

Lemma 4.8. The operator $R(g,B)$ is a non-negative operator.

Proof. Let $\xi := (\xi_1, \dots \xi_J) \in \mathbb{R}^J$ and $E_B(\,\cdot\,)$ be the joint spectral measure of $\{B_j\}_{j=1}^J$. Then

$$\begin{aligned}
R(g,B) &= \frac{1}{2}\sum_{j,l}^{J} \int_{\mathbb{R}^J} \langle g_j \,,\, W^{-1} g_l\rangle \xi_j \xi_l \, dE_B(\xi) \\
&= \frac{1}{2}\int \Big\langle \sum_j^J \xi_j g_j \,,\, W^{-1} \sum_l^J \xi_l g_l \Big\rangle dE_B(\xi) \geq 0.
\end{aligned}$$

■

Remark 4.9. From Lemma 4.8 and Assumption III, it is easy to see that for all $\lambda \in \Lambda$ with $|\lambda| \leq |\lambda_0|$, $E_0(A_0) = E_0(A_0(\lambda)) \in \sigma_{\mathrm{ess}}(A_0(\lambda))$.

Lemma 4.10. Suppose that (H-I)-(H-V) and Assumptions I-IV hold. Then for all $\lambda \in \Lambda$ with $|\lambda| \le |\lambda_0|$, the operator $A(\lambda)$ has no ground states and $\inf \sigma(A(\lambda)) = E_0(A_0)$.

Proof. There is no loss of generality in assuming $E_0(A_0) = 0$. Since A_1 is a A_0-compact operator and for any $z \in \mathbb{C} \setminus \mathbb{R}$

$$A_1\big(A_0(\lambda) - z\big)^{-1} = A_1(A_0 - z)^{-1}(A_0 - z)\big(A_0(\lambda) - z\big)^{-1},$$

we see that A_1 is $A_0(\lambda)$-compact. Hence $\sigma_{\mathrm{ess}}(A(\lambda)) = \sigma_{\mathrm{ess}}(A_0(\lambda))$ for all $\lambda \in \Lambda$. From Lemma 4.8 and Assumption III, we obtain that, if $|\lambda| \le |\lambda_0|$, then $A(\lambda) \ge A(\lambda_0)$ and $\inf \sigma_{\mathrm{ess}}(A(\lambda_0)) = 0$. From this, we need only to prove that $A(\lambda_0)$ has no no non-positive eigenvalues. It is easy to see that $E_0(A) = E_0(A_0(\lambda_0)) \notin \sigma_{\mathrm{p}}(A_0(\lambda_0))$. It is known that a monotone increasing sequence of self-adjoint non-negative compact operators $\{T_n\}_{n\in\mathbb{N}}$ which is less than or equal to a self-adjoint compact operator have norm limit $\lim_{n\to\infty} T_n$ [**Kur**, problem 11,8-9]. Therefore, by Assumption IV, there exists a compact operator

$$K_0 := \lim_{E\uparrow 0} \overline{|A_1|^{1/2}(A_0(\lambda_0) - E)^{-1}|A_1|^{1/2}}.$$

According to the Lemma 4.7, we have

$$\dim \mathrm{Ran}\,\big(E_{A_0(\lambda_0)+A_1}(\mathbb{R}_{\le})\big) \le \dim \mathrm{Ran}\,\big(E_{1-K_0}(\mathbb{R}_{\le})\big).$$

But $\dim \mathrm{Ran}\,\big(E_{1-K_0}(\mathbb{R}_{\le})\big) = 0$ since $\|K_0\| < 1$. Hence we see that $A(\lambda_0)$ has no non-positive eigenvalues. ■

Next, we check the ground state of $H(0)$.

Proposition 4.11. Suppose that (H-I)-(H-V) and Assumptions I-IV hold. Then $H(0) = A + \mathrm{d}\Gamma(W)$ has no ground states.

Proof. It is easily seen that $A + \mathrm{d}\Gamma(W)$ has ground states if and only if A has ground states. Since A_1 is A_0-compact, $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A_0)$. By Lemma 4.8, we obtain $A \ge A(\lambda_0)$. But, from Lemma 4.10, we see that $A(\lambda_0) \geq E_0(A_0)$ and $A(\lambda_0)$ has no ground states. Therefore we see that $H(0)$ has no ground states. ■

We define the operator

$$T := \overline{\sum_{j=1}^{J} B_j \otimes \phi\Big(iW^{-1}g_j\Big)}.$$

By (H-I) and (H-IV), $\{B_j \otimes \phi(iW^{-1}g_j)\}_{j=1}^{J}$ is strongly commutative. Hence T is a self-adjoint operator. We set

$$U(\lambda) := e^{-i\lambda T}, \tag{4.3}$$

$$H_{00} := A_0 \otimes I + I \otimes \mathrm{d}\Gamma(W). \tag{4.4}$$

We consider the unitary transformation of the GSB Hamiltonian $H(\lambda)$ by $U(\lambda)$. Applying Theorem 3.9 and Proposition 3.10, one can show the following lemma.

Lemma 4.12. Assume (H-I)–(H-VI). Let $\lambda \in \mathbb{R}$. Then

$$U(\lambda)D(H_{00}) = D(H_{00}),$$

$$\widetilde{H}(\lambda) := U(\lambda)H(\lambda)U(\lambda)^{-1}\Psi = \Big(A(\lambda) \otimes I + I \otimes \mathrm{d}\Gamma(W) + \delta A_1(\lambda)\Big)\Psi$$

for $\Psi \in D(H_{00})$, where

$$\delta A_1(\lambda) := U(\lambda)(A_1 \otimes I)U(\lambda)^{-1} - A_1 \otimes I.$$

Now, we prove our main theorem.

Theorem 4.13. Suppose (H-I)-(H-V) and Assumptions I-IV. Then $H(\lambda)$ has no ground states for all $\lambda \in \Lambda$ with $|\lambda| \leq |\lambda_0|$.

Proof. Without loss of generality, we can assume that $\inf \sigma(A_0) = 0$. Let $\Omega = (1, 0, 0, \ldots) \in \mathscr{F}_{\mathrm{b}}(\mathcal{K})$ be the Fock vacuum and ψ be in

$D(A_0)$.

$$\begin{aligned}\langle \psi \otimes \Omega\,,\, H(\lambda)\psi \otimes \Omega\rangle &= \langle \psi \otimes \Omega\,,\, (A_0 + A_1)\psi \otimes \Omega\rangle \\ &= \langle \psi\,,\, (A_0 + A_1)\psi\rangle\end{aligned}$$

By the min-max principle, we see that $\inf \sigma\big(H(\lambda)\big) \leq \inf \sigma(A_0 + A_1)$. Since A_1 is a A_0-compact operator, $\sigma_{\text{ess}}(A_0 + A_1) = \sigma_{\text{ess}}(A_0)$. Therefore $\inf \sigma\big(H(\lambda)\big) \leq 0$. Hence if $H(\lambda)$ has ground states, the eigenvalue is non-positive. By Lemma 4.12 and positivity of the operator W, we obtain the following inequality;

$$\begin{aligned}\widetilde{H}(\lambda) &\geq A_0(\lambda) \otimes I + U(\lambda)(A_1 \otimes I)U(\lambda)^* \\ &= U(\lambda)\big((A_0(\lambda) + A_1\big) \otimes I)U(\lambda)^*.\end{aligned}$$

Thus $H(\lambda) \geq A(\lambda)$. However, Lemma 4.10 means $A(\lambda)$ has no non-positive eigenvalues and negative spectrum. Therefore $H(\lambda)$ has no non-positive eigenvalues. This prove the theorem. ■

2. The Pauli-Fierz Type Model

Now, we consider a ground state in a special case and apply Theorem 4.13. Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$, $\mathcal{K} = \oplus^N L^2(\mathbb{R}^3)$ and let $A_0 = -\Delta$, A_1 is the multiplication operator of non-positive rapidly decreasing function V on $\mathbb{R}^3$, $B_j = p_j := -iD_j$, W is the multiplication operator of a non-negative continuous real function $\omega(k)$ on $\mathbb{R}^3$ which satisfies the following:

$$\lim_{|k|\to\infty} \omega(k) = \infty$$

and there exist constants $\gamma > 0$ and $C > 0$ such taht

$$|\omega(k) - \omega(k')| \leq C|k - k'|^\gamma (1 + \omega(k) + \omega(k')), \quad k, k' \in \mathbb{R}^3,$$

and $g_j \in \mathcal{K}$ $(j = 1, 2, 3)$ satisfying $g_j/\omega^2 \in \mathcal{K}$. In this case $H(\lambda)$ becomes

$$H_{\mathrm{PF}}(\lambda) = (-\Delta + V) \otimes I + I \otimes \mathrm{d}\Gamma(W) + \lambda \sum_{j=1}^{3} p_j \otimes \phi(g_j)$$

and we easily see that (H-I)-(H-V) hold. Moreover we take $\{g_j\}_{j=1}^{3}$ as in [**AK**, Example 6.2]. Hence, for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, we can get

$$\frac{1}{2} \sum_{j,l=1}^{3} \left(\frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right) \xi_j \xi_l = G(g) \xi^2,$$

where $G(g)$ is a constant independent of ξ. The operators $R(g, B)$ and $A(\lambda)$ are of the following form:

$$R(g, B) = \frac{1}{2} \sum_{j,l=1}^{3} \left(\frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right) p_j p_l = -G(g)\Delta,$$

$$A(\lambda) = -\big(1 - \lambda^2 G(g)\big)\Delta + V.$$

Therefore it is easily seen that

$$\Lambda = \left(-\frac{1}{\sqrt{G(g)}}, 0 \right) \cup \left(0, \frac{1}{\sqrt{G(g)}} \right) \neq \emptyset$$

and (H-VI) holds.

Assumptions I and III hold clearly. In $L^2(\mathbb{R}^3)$, every multiplication operator of $V \in L^2(\mathbb{R}^3) + L^\infty_\epsilon(\mathbb{R}^3)$ is a Δ-compact operator, where

$$L^2(\mathbb{R}^3) + L^\infty_\epsilon(\mathbb{R}^3)$$
$$:= \left\{ V : \mathbb{R}^3 \to \mathbb{R} \;\middle|\; \begin{array}{l} \text{For all } \epsilon > 0, \text{ there exist} \\ V_1 \in L^2(\mathbb{R}^3) \text{ and } V_2 \in L^\infty(\mathbb{R}^3) \text{ such that} \\ V = V_1 + V_2 \text{ and } \|V_2\|_\infty < \epsilon. \end{array} \right\}.$$

Hence Assumption II holds.

We define the Rollnik norm by

$$\|V\|_{\mathcal{R}}^2 := \int_{\mathbb{R}^6} \frac{|V(x)||V(y)|}{|x - y|^2} d^3x d^3y.$$

It follows from the Hardy-Littlewood-Sobolev inequality (see [**RS2**, IX.4 Example 3]) that if $V \in L^{3/2}(\mathbb{R}^3)$, then $\|V\|_{\mathcal{R}} < \infty$. For any $E < 0$, the operator $(-\Delta - E)^{-1}$ has an integral kernel $e^{-\sqrt{-E}|x-y|}/4\pi|x-y|$. Thus, for all $E < 0$, $f \in L^2(\mathbb{R}^3)$ and $\lambda \in \Lambda$,

$$
\begin{aligned}
&\langle f\,,\, |A_1|^{1/2}(A_0(\lambda) - E)^{-1}|A_1|^{1/2} f\rangle \\
&\le \lim_{E\uparrow 0} \int \overline{f(x)}\big(|V|^{1/2}\big(-(1-\lambda^2 G(g))\Delta - E\big)^{-1}|V|^{1/2} f\big)(x) d^3x \\
&= \lim_{E'\uparrow 0} \frac{1}{1-\lambda^2 G(g)} \int \frac{\overline{f(x)}|V|^{1/2}(x) e^{-\sqrt{-E'}|x-y|}|V|^{1/2}(y) f(y)}{4\pi|x-y|} d^3y d^3x \\
&= \frac{1}{1-\lambda^2 G(g)} \int \frac{\overline{f(x)}|V|^{1/2}(x)|V|^{1/2}(y) f(y)}{4\pi|x-y|} d^3y d^3x \\
&= \frac{1}{4\pi(1-\lambda^2 G(g))}\langle f,\, K_V f\rangle.
\end{aligned}
$$

Here K_V is an integral operator:

$$K_V f(x) := \int_{\mathbb{R}^3} \frac{|V|^{1/2}(x)|V|^{1/2}(y)}{|x-y|} f(y) dy.$$

Thus we see that $|A_1|^{1/2}(A(\lambda) - E)^{-1}|A_1|^{1/2} \le (4\pi(1-\lambda^2 G(g)))^{-1} K_V$ and K_V is a self-adjoint Hilbert-Schmidt operator. Since $\|K_V\| \le \|V\|_{\mathcal{R}}$, if $\|V\|_{\mathcal{R}} < 4\pi$ and $|\lambda| < (1 - \|V\|_{\mathcal{R}}/(4\pi))^{1/2} G(g)^{-1/2}$, then we get $(4\pi(1-\lambda^2 G(g)))^{-1}\|K_V\| < 1$. Thus Assumption IV holds when $\|V\|_{\mathcal{R}} < 4\pi$. Hence we have the following theorem by Theorem 4.13.

Theorem 4.14. Assume that $V \in \mathscr{S}(\mathbb{R}^3)$ with $V \le 0$. If $\|V\|_{\mathcal{R}} < 4\pi$, then there exists a coupling constant $\lambda_0 \in \Lambda$ such that for all $|\lambda| < |\lambda_0|$, $H_{\mathrm{PF}}(\lambda)$ has no ground states.

However, from Theorem 4.4, we see that enhanced binding occurs for large coupling constants. We set

$$\lambda(g) := \frac{1}{\sqrt{G(g)}},$$

$$V_0 := \inf_{x\in\mathbb{R}^3} V(x) < 0.$$

Theorem 4.15. Consider the case $m > 0$. Suppose that

$$|V_0| > m + \frac{1}{2}\lambda(g)^2 c_3(g)^2 + \lambda(g)c_2(g).$$

Then there exists a constant δ such that, for all $|\lambda| \in (\lambda(g) - \delta, \lambda(g))$, $H_{\mathrm{PF}}(\lambda)$ has purely discrete spectrum in $[E_0(H_{\mathrm{PF}}(\lambda)), E_0(H_{\mathrm{PF}}(\lambda)) + m)$. In particular, $H_{\mathrm{PF}}(\lambda)$ has a ground state.

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