博士学位論文

Automorphisms of a non-type I C^*-algebra
（I型でない C*-環の自己同型）

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AUTOMORPHISMS OF A NON-TYPE I C^*-ALGEBRA
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Abstract. Glimm's theorem says that a UHF algebra is almost embedded in a separable C^*-algebra not of type I. Applying his methods we obtain a covariant version of his result; a UHF algebra with a product type automorphism is covariantly embedded in such a C^*-algebra equipped with an automorphism with full Connes spectrum.

1. Introduction

To examine the types of von Neumann algebras, either of type I, II or III, is one of the fundamental and traditional ways to classify them, ever since Murray and von Neumann build the research field of operator algebra. They defined the types of factors with a real valued function, which they called "dimension function" in [10]. In [8], Kaplansky simplified and extended this notion to all von Neumann algebras, which derives naturally the following definition of a C^*-algebra of type I: A C^*-algebra A is *of type I* if each non-zero quotient of A contains a non-zero positive element x such that xAx is commutative. A C^*-algebra is of type I if and only if its enveloping von Neumann algebra is of type I. An elementary example of a C^*-algebra (and also a von Neumann algebra) of type I is the full matrix algebra M_n, which is the set of all $n \times n$ matrices. Unfortunately, many algebras not of type I appear in mathematical physics, so we also have to treat other classes of operator algebras.

A *UHF (uniformly hyperfinite) algebra*, which is sometimes called a "Glimm algebra" because it first appeared in his thesis ([5]), is a C^*-algebra which is the inductive limit of a sequence of full matrix algebras $(M_{i_n})_n$ such that i_n divides i_{n+1} for each n with the embeddings

$$M_{i_n} \ni x \mapsto \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x \end{pmatrix} \in M_{i_{n+1}}.$$

A UHF algebra is not of type I — actually the von Neumann algebra generated by the image of a UHF algebra by the GNS representation of its unique tracial state is the hyperfinite factor of type II_1 (which is unique up to isomorphism). Although a UHF algebra is the simplest inductive limit of a sequence of C^*-algebras, it is a very important C^*-algebra, which is also simple in the mathematical sense.

Surprisingly, the σ-weak closure of a UHF algebra can be a von Neumann algebra of various type; concretely, of type II_1, II_∞ and III_λ, $0 \le \lambda \le 1$. Such a von Neumann algebra is called an *AFD (approximately finite dimensional)* von Neumann algebra. Here is an example of construction of an AFD factor of type III_λ, $0 < \lambda < 1$. Let $\phi^{(2)}$ be the $\mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{it} \end{pmatrix}$-KMS-state on M_2; i.e.

$$\phi^{(2)}(x) := \frac{\mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} x\right)}{\mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\right)}$$

for $x \in M_2$, where Tr denotes the usual trace on M_2, and set $\phi := \otimes_{n=1}^\infty \phi_n$, where $\phi_n := \phi^{(2)}$ for each n. Then it follows that the σ-weak closure of $\pi_\phi(\otimes_{n=1}^\infty M_2)$, denoted by $\pi_\phi(\otimes_{n=1}^\infty M_2)''$, where π_ϕ is the GNS representation of ϕ, is an AFD factor of type III_λ (see [13], Section 4 and [14], XVIII.1.1).

There are several equivalence conditions for C^*-algebras being of type I. In [7], Kaplansky studied CCR algebras, which are C^*-algebras such that there images by any irreducible representations consist of compact operators. A CCR algebra is of course of type I, for a C^*-algebra of compact operators has a minimal projection. But a C^*-algebra of type I is not necessarily a CCR algebra. Kaplansky defined a larger class of C^*-algebras, called GCR algebras, and in [6] Glimm proved that these two classes are equivalent.

Glimm also found other several conditions equivalent to being of type I in [6]. A part of his proof implies a celebrated theorem known as *Glimm's theorem*: For a separable C^*-algebra A which is not of type I and a UHF algebra D, there is a C^*-subalgebra B of A and a closed projection q in the enveloping von Neumann algebra of A such that $q \in B'$, $qAq = Bq$ and $Bq \simeq D$, where B' is the commutant of B. Roughly speaking, this theorem says that *any UHF algebra is almost embedded in such a C^*-algebra*. In fact, he proved this theorem only for $D = \otimes_{n=1}^\infty M_2$, known as the Fermion algebra, and Pedersen arranged his proof and generalized to the case of an arbitrary UHF algebra in [11].

According to Glimm's theorem, we are able to embed UHF algebras. How about group actions? It is still an open problem whether or not general actions of UHF algebras can be embedded. Bratteli, Kishimoto and Robinson first succeeded in embedding actions of compact groups of a special type in [2]. They embedded an action of a compact group on a UHF algebra $\otimes_{n=1}^\infty M_{k_n}$ of the form $\gamma_t = \otimes_{n=1}^\infty \mathrm{Ad}\, u_{nt}$, where $t \mapsto u_{nt}$ is a unitary representation on M_{k_n}. They call an action of this form "a product type action." Since any irreducible representation of a compact group is finite dimensional, a product type action seems standard. One decade and a half later, product type actions of $\mathbb{R}$ were embedded by Kishimoto in [9]. While $\mathbb{R}$ itself is easy to understand, non-compactness of $\mathbb{R}$ makes this embedding problem much more delicate, and the action (called "flow") need to be perturbed. In this paper, we treat the $\mathbb{Z}$-action case, i.e. the automorphism case. Since $\mathbb{Z}$ is not compact, a perturbation is also needed in this case. So the result is as follows:

Theorem 1.1. *Let A be a separable prime C^*-algebra and α an automorphism of A. Then the following are equivalent:*

(1) *The Connes spectrum $\Gamma(\alpha)$ of α is equal to $\mathbb{T}$.*

(2) *For any UHF algebra $D = \otimes_{n=1}^{\infty} M_{k_n}$, any automorphism γ of D of the form $\gamma = \otimes_{n=1}^{\infty} \mathrm{Ad}\, e^{ih_n}$, where M_{k_n} is the $k_n \times k_n$ matrix algebra with $k_n \geq 2$ and $h_n \in M_{k_n}$ a self-adjoint matrix for each n, and any $\epsilon > 0$, there is a C^*-subalgebra B of A, a unitary v in A (in $A + \mathbb{C}1$ if A is not unital) and a closed projection q of the enveloping von Neumann algebra of A which is in the commutant of B such that*

$$\|v - 1\| < \epsilon, \quad \alpha^{(v)}(B) = B,$$
$$(\alpha^{(v)})^{**}(q) = q, \quad qAq = Bq,$$
$$(Bq, (\alpha^{(v)})^{**}|Bq) \simeq (D, \gamma)$$

and for $x \in A$, $x = 0$ if and only if $xc(q) = 0$, where $\alpha^{(v)} := \alpha \circ \mathrm{Ad}\, v$ is a perturbation of α and $c(q)$ is the central cover of q and $(\alpha^{(v)})^{**}|Bq$ is the restriction of $(\alpha^{(v)})^{**}$ to Bq.

In the statement above, the σ-weakly extended automorphism of $\alpha^{(v)}$ to the enveloping von Neumann algebra of A is denoted by $(\alpha^{(v)})^{**}$, but we will later omit the stars; the same applies to representations, etc.

The Connes spectrum appears in the condition . The definition of the Connes spectrum is much complicated.

Definition 1.2. Let G be a locally compact abelian group, A a C^*-algebra, and α an action of G on A.

(1) For a subset Λ of $\Gamma = \hat{G}$, $M^\alpha(\Lambda)$ denotes a subset of A such that $x \in M^\alpha(\Lambda)$ if and only if $\int_G f(t)\alpha_t(x)dt = 0$ for any $f \in L^1(G)$ with $\mathrm{supp}\, \hat{f}$ is compact and $\mathrm{supp}\, \hat{f} \subset \Gamma \backslash \Lambda$, where $\hat{f}$ is the Fourier transform of f and $\mathrm{supp}\, \hat{f}$ the support of $\hat{f}$.

(2) We define $\mathrm{sp}\,(\alpha)$, which is called *the Arveson spectrum*, as the smallest closed subset Λ of Γ such that $M^\alpha(\Lambda) = A$.

(3) *The Connes spectrum* $\Gamma(\alpha)$ is defined by $\Gamma(\alpha) := \bigcap \mathrm{sp}\,(\alpha|B)$, where B runs over the set of G-invariant hereditary non-zero C^*-subalgebra of A.

We remark that $\sigma \in \mathrm{sp}\,(\alpha)$ is equivalent to the next condition; for $\epsilon > 0$ and a compact subset K of G, there is an $x \in A$ such that $\|\alpha_t(x) - \langle t, \sigma \rangle x\| < \epsilon$ for any $t \in K$.

It seems natural that the condition (1) is necessary when the condition (2) is true. If A was simple and $\Gamma(\alpha) \neq \mathbb{T}$, α^n would be inner in the multiplier algebra of A for some n (8.9.9 in [12]), so very few γ's would satisfy Theorem 1.1.

In the hypothesis of the theorem above, if A has a faithful irreducible representation and $\Gamma(\alpha) = \mathbb{T}$, then A is automatically not of type I. This can be proved as follows. Suppose that x is a positive element of A such that xAx is commutative. The norm closure of xAx is a hereditary sub-C^*-algebra of A, whose image of an irreducible representation is an algebra of one-dimensional operators. This contradicts $\Gamma(\alpha) = \mathbb{T}$ (by the same argument as in the proof of Lemma 2.5).

We state a straightforward corollary and end the introduction.

Corollary 1.3. *Let A be a separable prime C^*-algebra and α an automorphism of A with the Connes spectrum $\Gamma(\alpha) = \mathbb{T}$. Then, for any AFD factor M, there are an α-covariant representation π of A and a projection Q of $\pi(A)''$ with $c(Q) = 1$ such that $Q\pi(A)''Q \simeq M$.*

Proof. Note that an AFD factor always has a σ-weakly dense UHF subalgebra D ([4]). We use Theorem 1.1 for $\gamma =$identity and obtain B, v and q. We take a faithful state on M and restrict it on D. This state gives one on $qAq = Bq$ through the isomorphism $(Bq, \alpha^{(v)}|Bq) \simeq (D, \gamma)$, which is denoted by ψ_0. Because of a choice of γ, ψ_0 is $\alpha^{(v)}|Bq$-invariant. We define a state ψ on A by $\psi(x) := \psi_0(qxq)$ for $x \in A$. Let $(\pi_\psi, \mathcal{H}_\psi, \xi_\psi)$, $(\pi_{\psi_0}, \mathcal{H}_{\psi_0}, \xi_{\psi_0})$ be the GNS-triples of ψ and ψ_0, respectively. Set $Q := \pi_\psi(q)$. Then it follows that $Q\pi_\psi(A)Q = \pi_\psi(qAq)$, which implies $Q\pi_\psi(A)''Q = \pi_\psi(qAq)''$. We would like to assume that $\mathcal{H}_{\psi_0} \subset \mathcal{H}_\psi$ and π_ψ is an extension of π_{ψ_0}, but they are not evident by the definition of the GNS representation. So we will check that there is a natural isomorphism between $\overline{\pi_{\psi_0}(qAq)''\xi_{\psi_0}}^{\|\cdot\|}$ and $\overline{\pi_\psi(qAq)''\xi_\psi}^{\|\cdot\|}$. Once it is proved, we have

$$\mathcal{H}_{\psi_0} = \overline{\pi_{\psi_0}(qAq)''\xi_{\psi_0}}^{\|\cdot\|} \simeq \overline{\pi_\psi(qAq)''\xi_\psi}^{\|\cdot\|} \subset \overline{\pi_\psi(A)''\xi_\psi}^{\|\cdot\|} = \mathcal{H}_\psi,$$

and it works out. For any $x, y \in A$, there are $z, w \in B$ such that $qxq = zq$ and $qyq = wq$. So we have

$$\begin{aligned}
\langle \pi_\psi(qxq)\xi_\psi, \pi_\psi(qyq)\xi_\psi \rangle &= \langle \pi_\psi(zq)\xi_\psi, \pi_\psi(wq)\xi_\psi \rangle \\
&= \langle \pi_\psi(z)\xi_\psi, \pi_\psi(w)\xi_\psi \rangle \\
&= \psi(w^*z) = \psi_0(qw^*zq) \\
&= \langle \pi_{\psi_0}(zq)\xi_{\psi_0}, \pi_{\psi_0}(wq)\xi_{\psi_0} \rangle \\
&= \langle \pi_{\psi_0}(qxq)\xi_{\psi_0}, \pi_{\psi_0}(qyq)\xi_{\psi_0} \rangle,
\end{aligned}$$

since $\psi(q) = 1$ implies $\pi_\psi(q)\xi_\psi = \xi_\psi$, where $\langle \cdot, \cdot \rangle$ denotes the inner product. By the construction of ψ_0, it follows that $\pi_{\psi_0}(qAq)'' \simeq M$, whence $Q\pi_\psi(A)''Q \simeq M$. Finally, since

$$\begin{aligned}
c(Q)\mathcal{H}_\psi &= \overline{\pi_\psi(A)Q\pi_\psi(A)\xi_\psi}^{\|\cdot\|} \\
&\supset \overline{\pi_\psi(A)Q\xi_\psi}^{\|\cdot\|} = \overline{\pi_\psi(A)\xi_\psi}^{\|\cdot\|} = \mathcal{H}_\psi,
\end{aligned}$$

we have $c(Q) = 1$. □

Notations. For a Hilbert space $\mathcal{H}$, $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathcal{H}$, $\mathbf{B}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$, and $\mathbf{K}(\mathcal{H})$ the set of compact operators on $\mathcal{H}$. For a C^*-algebra A, A_{sa} denotes the set of self-adjoint elements in A, A_+ the set of positive elements in A, A_1 the unit ball of A, and $\mathcal{U}(A)$ the set of unitary elements in A (in $A + \mathbb{C}1$ if A is not unital). We denote by A^{**} the enveloping von

Neumann algebra of A. When A is in some von Neumann algebra, A' denotes the commutant of A and A'' the double commutant of A, which is equal to the σ-weak closure of A. For a unitary U in $\mathbf{B}(\mathcal{H})$, E_U denotes the spectral measure (on $\mathbb{T}$) of U. For a state ϕ of a C^*-algebra A, π_ϕ denotes the GNS representation of ϕ, and $\operatorname{supp}\phi \in A^{**}$ the support projection of ϕ. For a function f, $\operatorname{supp} f$ denotes the support of f.

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2. Proof of the main theorem

We can prove that (2) implies (1) easily, so we prove it first. (We put stars for σ-weakly continuous extensions of automorphisms only in this proof.) Let $D := \otimes_{n=1}^{\infty} M_{k_n}$, where $k_n := 2$ for each n. Set $u_n := \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i\theta} \end{pmatrix}$ for each n, where θ is an arbitrary irrational number independent on n, and define an automorphism of D by $\gamma := \otimes \mathrm{Ad}\, u_n$. We get an isomorphism $(qAq, (\alpha^{(v)})^{**}|qAq) \simeq (D, \gamma)$, where q and v are obtained by the condition (2). Let τ be the tracial state on qAq. Then it follows that $\pi_\tau(qAq)''$ is the hyperfinite II_1-factor. Since $\sum_{n=1}^{\infty} |1 - |(1 + e^{2m\pi i\theta})/2|| = \infty$, the σ-weakly continuous extension of $(\alpha^{(v)})^m$ to $\pi_\tau(qAq)''$ is outer for any $m \in \mathbb{Z}\backslash\{0\}$ ([3], 1.3.7). Define a state ψ on A by $\psi(x) := \tau(qxq)$ for $x \in A$. We remark that we can regard π_ψ as an extension of π_τ by the same reason as in the proof of Corollary 1.3. Since $\psi \circ (\alpha^{(v)})^{**} = \psi$, we can extend $(\alpha^{(v)})^{**}$ to $\pi_\psi(A)''$. Since $(\alpha^{(v)})^{**}(\pi_\psi(q)) = \pi_\psi(q)$, it follows that $((\alpha^{(v)})^{**})^m$ on $\pi_\psi(A)''$ is also outer for each m, whence $\Gamma(\alpha^{**}) = \mathbb{T}$ since $\mathbb{Z}$ is discrete. Therefore we have $\Gamma(\alpha) = \mathbb{T}$ (see [12], 8.8.9).

We will show that (1) implies (2) from now.

Theorem 2.1 (Kadison's transitivity)**.** *Let A be a C^*-algebra and π an irreducible representation of A on a Hilbert space $\mathcal{H}$. For $T \in \mathbf{B}(\mathcal{H})$, a finite dimensional subspace $\mathcal{K} \subset \mathcal{H}$ and $\epsilon > 0$, There is an $x \in A$ such that*

$$\pi(x)|\mathcal{K} = T|\mathcal{K} \qquad \text{and} \qquad \|x\| \leq \|T\| + \epsilon.$$

If $T = T^*$, then x can also be chosen so that $x = x^*$.

Kadison's transitivity says that any bounded linear map coincides on a finite dimensional subspace with an element in the σ-weakly dense C^*-algebra of the von Neumann algebra of bounded linear maps. Before we begin the proof, we present Kadison's transitivity in the following form.

Lemma 2.2. *For any $\epsilon > 0$ and any natural number m, there is a $\delta > 0$ such that the following holds:*

Let A be a C^*-algebra, π an irreducible representation on a Hilbert space $\mathcal{H}$, and V a unitary in $\mathbf{B}(\mathcal{H})$. Let $(\xi_1, \cdots, \xi_m)$ be a finite family of mutually orthogonal unit vectors in $\mathcal{H}$. If $\|V\xi_j - \xi_j\| < \delta$ for $j = 1, \cdots, m$, there is a v in $\mathcal{U}(A)$ such that $\|v - 1\| < \epsilon$ and $\pi(v)\xi_j = V\xi_j$ for $j = 1, \cdots, m$.

To prove this, we prepare the following lemma.

Lemma 2.3. *Let m, n be natural numbers and $\epsilon > 0$. Let $(\xi_1, \cdots, \xi_m)$ and $(\eta_1, \cdots, \eta_n)$ be two families of unit vectors such that ξ_j's are mutually orthogonal and $|\langle \eta_i, \xi_j \rangle| < \epsilon$, $|\langle \eta_i, \eta_k \rangle| < \epsilon$ for any $j = 1, \cdots m$ and $i, k = 1, \cdots, n$, $i \neq k$. Then there is a family $(\eta'_1, \cdots, \eta'_n)$ of unit vectors in the finite dimensional subspace spanned by $\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n$ such that $(\xi_1, \cdots, \xi_m, \eta'_1, \cdots, \eta'_n)$ is an orthogonal family of unit vectors and $\|\eta_i - \eta'_i\| < r_{mn}\epsilon$ for $i = 1, \cdots, n$, where r_{mn} is a positive real number dependent on m and n.*

Proof. We recall the process of the Gram-Schmidt orthogonalization. Define $\eta''_1 := \eta_1 - \sum_{j=1}^{m} \langle \eta_1, \xi_j \rangle \xi_j$. Then we have $\langle \eta''_1, \xi_j \rangle = 0$ for $j = 1, \cdots, m$ and

$$\|\eta''_1 - \eta_1\| \leq \sum_{j=1}^{m} |\langle \eta_1, \xi_j \rangle| < m\epsilon.$$

And define $\eta'_1 := \eta''_1 / \|\eta''_1\|$. Since $1 - m\epsilon < \|\eta''_1\| \leq 1$, we have

$$\begin{aligned} \|\eta'_1 - \eta_1\| &\leq \|\eta'_1 - \eta''_1\| + \|\eta''_1 - \eta_1\| \\ &\leq (1 - \|\eta''_1\|) \|\eta'_1\| + \|\eta''_1 - \eta_1\| \\ &< m\epsilon + m\epsilon = 2m\epsilon. \end{aligned}$$

When $\eta'_1, \cdots, \eta'_{i-1}$ have already defined for $2 \leq i \leq n$, set $\eta''_i := \eta_i - \sum_{j=1}^{m} \langle \eta_i, \xi_j \rangle \xi_j - \sum_{\ell=1}^{i-1} \langle \eta_i, \eta'_\ell \rangle \eta'_\ell$ and $\eta'_i := \eta''_i / \|\eta''_i\|$. As above, it follows that $\langle \eta'_i, \xi_j \rangle = 0$ for $j = 1, \cdots, m$, $\langle \eta'_i, \eta'_\ell \rangle = 0$ for $\ell = 1, \cdots, i-1$, and $\|\eta'_i - \eta_i\| < r_i\epsilon$ for all i, when we set $r_1 := 2m$ and $r_i := 2(m + i - 1 + r_1 + r_2 + \cdots + r_{i-1})$ for $2 \leq i \leq n$. Since the sequence $(r_i)_i$ is obviously increasing, we have $r_i = 2(m + i - 1 + r_1 + \cdots + r_{i-1}) \leq 2nr_{i-1}$, whence $r_i \leq 2m(2n)^{n-1}$ for $1 \leq i \leq n$. ∎

Proof of Lemma 2.2. We would like to use Kadison's transitivity for self-adjoint operators, but it makes this problem difficult that the initial space and the range space of an operator are not the same in general. So we have to find a unitary W which is equal to V on the subspace spanned by $\xi_1, \cdots, \xi_m$ and whose initial space is a finite dimensional subspace containing $\xi_1, \cdots, \xi_m$ and coincides with its range space. We may assume that $\epsilon < 1/2$. Let $\mathcal{F}$ be the finite-dimensional subspace of $\mathcal{H}$ spanned by $\xi_1, \cdots, \xi_m$ and $V\xi_1, \cdots, V\xi_m$. Let $\eta_1, \cdots, \eta_n$ be unit vectors such that $(\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_n)$ is an orthonormal basis of $\mathcal{F}$. Since

$$|\langle \eta_i, V\xi_j \rangle| = |\langle \eta_i, V\xi_j \rangle - \langle \eta_i, \xi_j \rangle| \leq \|V\xi_j - \xi_j\| < \delta$$

for $i = 1, \cdots, n$ and $j = 1, \cdots, m$, it follows from Lemma 2.3 that there is a family $(\eta'_1, \cdots, \eta'_n)$ of unit vectors in $\mathcal{F}$ such that $(V\xi_1, \cdots, V\xi_m, \eta'_1, \cdots, \eta'_n)$ is an orthonormal basis of $\mathcal{F}$ and $\|\eta_i - \eta'_i\| < r_{mn}\epsilon$ for $i = 1, \cdots, n$, where r_{mn} is a positive real number dependent on m and n. Let W be a unitary on $\mathcal{F}$ determined by $W\xi_j := V\xi_j$ for $j = 1, \cdots, m$ and $W\eta_i := \eta'_i$ for $i = 1, \cdots, n$. When we set $\delta := \epsilon/(2\sqrt{n}r_{mn})$, it follows that $\|W - 1\| < \epsilon/2$. Define $T := -i \log W = i \sum_{n=1}^{\infty} (W - 1)^n / n$ on $\mathcal{F}$. Then we have $\|T\| < -\log(1 - \epsilon/2)$. We extend T to a self-adjoint operator on $\mathcal{H}$ by setting $T = 0$ on the orthogonal complement of $\mathcal{F}$. We also denote this extended operator by T and define $W = e^{iT}$ on $\mathcal{H}$. Let P be the projection onto $\mathcal{F}$. By Kadison's transitivity for a self-adjoint operator, there is an $a \in A_{sa}$ such that $TP = \pi(a)P$ and $\|a\| < -\log(1-\epsilon/2)$. By the construction of T, we have $TP = PTP = \pi(a)P = P\pi(a)P$. Hence it follows that

$$\begin{aligned} WP = PWP = P\sum_{n=0}^{\infty}\frac{(iT)^n}{n!}P &= \sum_{n=0}^{\infty}\frac{(iPTP)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(iP\pi(a)P)^n}{n!} = P\sum_{n=0}^{\infty}\frac{(i\pi(a))^n}{n!}P \\ &= Pe^{i\pi(a)}P = e^{i\pi(a)}P \end{aligned}$$

and

$$\left\|e^{ia}-1\right\| \leq e^{\|a\|}-1 < e^{-\log(1-\epsilon/2)}-1 < \epsilon.$$

Now $v := e^{ia}$ is a desired unitary. □

From now on, Let π be a faithful α-covariant irreducible representation of A on a Hilbert space $\mathcal{H}$ and U the implementing unitary of α. The existence of such a π is proved in [1]. Note that every pair of a C^*-algebra and its automorphism does not have a faithful covariant irreducible representation in the case where $\Gamma(\alpha) \neq \mathbb{T}$. Here is an example. Let A_θ be the irrational rotation algebra for an irrational number θ, i.e. A_θ is the universal C^*-algebra generated by two unitaries u and v which satisfy the relation $uv = e^{2\pi i\theta}vu$, and α the automorphism of A_θ defined by

$$\alpha(u) := -u, \quad \alpha(v) := e^{\pi i\theta}v.$$

Suppose that A_θ has a faithful irreducible representation σ which satisfy $\mathrm{Ad}U \circ \sigma = \sigma \circ \alpha$, where U is the implementing unitary. Since $\mathrm{Ad}U^2 \circ \sigma(u) = \sigma \circ \alpha^2(u) = \sigma(u) = (\mathrm{Ad}\sigma(u) \circ \sigma)(u)$ and $\mathrm{Ad}U^2 \circ \sigma(v) = \sigma \circ \alpha^2(v) = \sigma(e^{2\pi i\theta}v) = \sigma(uvu^*) = (\mathrm{Ad}\sigma(u) \circ \sigma)(v)$, it follows that $\mathrm{Ad}U^2 = \mathrm{Ad}\sigma(u)$. Thus $U^2\sigma(u)^*$ is in the commutant of $\sigma(A_\theta)$, which is equal to $\mathbb{C}$ since σ is irreducible. We take a $\lambda \in \mathbb{C}$ so that $U^2 = \lambda\sigma(u)$. Then we have

$$\lambda\sigma(u) = U^2 = UU^2U^* = U\lambda\sigma(u)U^* = \lambda(\sigma\circ\alpha)(u) = -\lambda\sigma(u),$$

which is absurd.

We recall crossed products and dual actions.

Definition 2.4.

(1) Let A be a C^*-algebra, G an action of G on A. Let $C_{c,\alpha}(G, A)$ denote the set of continuous maps from G to A with compact supports. When we define involution and convolution on $C_{c,\alpha}(G, A)$ by

$$x^*(t) := \Delta(t)^{-1}\alpha_t(x(t^{-1})^*)$$

$$(x * y)(t) := \int_G x(s)\alpha_s(y(s^{-1}t))ds$$

for $x, y \in C_{c,\alpha}(G, A)$ and $t \in G$, $C_{c,\alpha}(G, A)$ becomes a $*$-algebra, where $\Delta : G \to \mathbb{R}$ is the modular function of G. With the norm $\|x\|_1 := \int \|x(t)\| \, dt$ for $x \in C_{c,\alpha}(G, A)$, $C_{c,\alpha}(G, A)$ is a normed algebra with an isometric involution. We denote its completion by $L^1_\alpha(G, A)$. The universal representation $(\pi_u, \mathcal{H}_u)$ of $L^1_\alpha(G, A)$ is the direct sum of all non-degenerate representations of $L^1_\alpha(G, A)$. We define $A \rtimes_\alpha G$ as the norm closure of $\pi_u(L^1_\alpha(G, A))$ in $\mathbf{B}(\mathcal{H}_u)$. We call $A \rtimes_\alpha G$ the crossed product of the triple (A, G, α).

(2) Let A be a C^*-algebra, G a locally compact abelian group, and α an action of G on A. For $x \in C_{c,\alpha}(G, A)$, $\sigma \in \hat{G}$ and $t \in G$, we define

$$(\hat{\alpha}_\sigma(x))(t) := \langle t, \sigma \rangle x(t).$$

Then $\hat{\alpha}_\sigma$ extends to an automorphism of $A \rtimes_\alpha G$ for each $\sigma \in \hat{G}$ and $\hat{\alpha}$ becomes an action of $\hat{G}$ on $A \rtimes_\alpha G$. We call $\hat{\alpha}$ the dual action of α.

Note that an $\hat{\alpha}$-invariant ideal of $A \rtimes_\alpha \mathbb{Z}$ induces a non-trivial α-invariant ideal of A by $y \mapsto I(y) := \int_{\mathbb{T}} \hat{\alpha}_t(y) dt$ for y in the $\hat{\alpha}$-invariant ideal, where this integral converges in the norm topology since $\mathbb{T}$ is compact (see the proof of [12], 7.9.6).

Lemma 2.5. *$(\pi \rtimes U)(A \rtimes_\alpha \mathbb{Z})$ has no non-zero compact operators, where $\pi \rtimes U \colon A \rtimes_\alpha \mathbb{Z} \to \mathbf{B}(\mathcal{H})$ is a homomorphism defined by $\pi \rtimes U(y) := \sum_{n \in \mathbb{Z}} \pi(y(n)) U^n$ for $y \in C_0(\mathbb{Z}, A)$.*

Proof. At first, we show that $\pi(A)$ has no non-zero compact operators. We may identify A with $\pi(A)$ and assume that A is an irreducible subalgebra of $\mathbf{B}(\mathcal{H})$. Suppose that A has a non-zero compact operator. Since A is irreducible, A contains $\mathbf{K}(\mathcal{H})$. It is obvious that $\alpha(\mathbf{K}(\mathcal{H})) = \mathbf{K}(\mathcal{H})$. But, since $\alpha|_{\mathbf{K}(\mathcal{H})}$ is inner in $\mathbf{B}(\mathcal{H})$ (see the proof of [12], 8.7.4), the Connes spectrum of $\alpha|_{\mathbf{K}(\mathcal{H})}$ is equal to $\{0\}$ (see [12], 8.9.10; note that the multiplier algebra of $\mathbf{K}(\mathcal{H})$ is $\mathbf{B}(\mathcal{H})$). This contradicts $\Gamma(\alpha) = \mathbb{T}$.

Next we show that $(\pi \rtimes U)(A \rtimes_\alpha \mathbb{Z})$ has no non-zero compact operators. If $(\pi \rtimes U)(A \rtimes_\alpha \mathbb{Z})$ has a non-zero compact operator $K := (\pi \rtimes U)(K') \geq 0$, then $\pi(I(K'))$ is a non-zero compact operator in $\pi(A)$, which contradicts the last paragraph. □

For an element u in $\mathcal{U}(A)$, we define

$$U^{(u)} := U\pi(u).$$

Then it follows that $\mathrm{Ad}\, U^{(u)} \circ \pi = \pi \circ \alpha^{(u)}$.

Note that since $\Gamma(\alpha) = \mathbb{T}$, it follows that $\mathrm{sp}(U) = \mathbb{T}$, where $\mathrm{sp}(U)$ is the spectrum of U, and $\pi \rtimes U$ is faithful.

Lemma 2.6. *For any $\epsilon > 0$, there are a u in $\mathcal{U}(A)$ and a unit vector ξ_0 in $\mathcal{H}$ such that $\|u - 1\| < \epsilon$ and $U^{(u)} \xi_0 = \xi_0$.*

Proof. Using the functional calculus, there is an H in $\mathbf{B}(\mathcal{H})_{sa}$ such that $U = e^{iH}$. Let $\delta > 0$. Applying Weyl's theorem, there is a compact operator K in $\mathbf{B}(\mathcal{H})_{sa}$

such that $\|K\| < \delta$ and $H - K$ is diagonal. Since

$$\frac{d}{ds}(e^{-isH}e^{is(H-K)}) = -e^{-isH}iKe^{is(H-K)},$$

we have

$$V := e^{-iH}e^{i(H-K)} = -\int_0^1 e^{-isH}iKe^{is(H-K)}ds + 1.$$

Then it follows that $\|V - 1\| \leq \|K\| < \delta$. Since UV is diagonal and $\mathrm{sp}(U) = \mathbb{T}$, there are a $\lambda \in \mathbb{T}$ and a unit vector $\xi_0 \in \mathcal{H}$ such that $\|\lambda - 1\| < \delta$ and $UV\xi_0 = \lambda\xi_0$. Thus we have

$$\|U^*\xi_0 - \xi_0\| \leq \|U^*\xi_0 - V\xi_0\| + \|V\xi_0 - \xi_0\| \leq |\lambda - 1| + \|V - 1\| < 2\delta.$$

Now we can find a desired unitary u by Lemma 2.2. □

According to this lemma, we may assume that there is a unit vector ξ_0 in $\mathcal{H}$ such that $U\xi_0 = \xi_0$ because U will be perturbed again later. Let ω be the pure state defined by $\omega(x) := \langle \pi(x)\xi_0, \xi_0 \rangle$ for x in A.

We define

$$T := \{e \in A | 0 \leq e \leq 1, \pi(e)\xi_0 = \xi_0, \text{ and } \exists a \in A : ea = a, \pi(a)\xi_0 = \xi_0\}.$$

Note that we can always take an a from T in this definition. In fact, for $e \in T$ and $a \in A$ such that $ea = a$ and $\pi(a)\xi_0 = \xi_0$, it follows that $ef(a) = f(a)$ and $\pi(f(a))\xi_0 = \xi_0$, where $f(t) := 2t$ $(0 \leq t \leq 1/2)$, $:= 1$ $(1/2 \leq t \leq 1)$. It is obvious that $f(a) \in T$.

Lemma 2.7. *There is a decreasing sequence $(e_N)_N$ in T such that $e_N e_{N+1} = e_{N+1}$ for any $N = 1, 2, \cdots$ and $e_N \searrow \mathrm{supp}\omega$, i.e. $\mathrm{supp}\omega = \inf_N e_N$.*

Proof. Since $p := \mathrm{supp}\omega$ is a closed projection (see [12], 3.13.6), there is a decreasing sequence $(y_n)_n$ in the unit ball of A_+ such that $y_n \searrow p$. Put $y := \sum_{n=1}^{\infty} 2^{-n}y_n$, which is in the unit ball of A_+. Then, for any state ψ on A, it follows that $\psi(y) = 1$ if and only if $\psi(p) = 1$. This implies, by setting $\psi := \langle \pi(\cdot)\eta, \eta \rangle$, that for $\eta \in H$, it follows that $\pi(y)\eta = \eta$ if and only if $\eta \in \mathbb{C}\xi_0$. Thus the spectral projection of y (in A^{**}) corresponding to the eigenvalue 1 is p. We define a sequence of continuous functions on $[0, 1]$ by

$$f_N(t) := \begin{cases} 0 & (0 \leq t \leq 1 - \frac{1}{2^N}) \\ 2^{N+1}t - 2^N & (1 - \frac{1}{2^N} \leq t \leq 1 - \frac{1}{2^{N+1}}) \\ 1 & (1 - \frac{1}{2^{N+1}} \leq t \leq 1) \end{cases}.$$

and set $e_N := f_N(y)$. Then $(e_N)_N$ is a decreasing sequence whose infimum is the spectral projection of y corresponding to the eigenvalue 1, which is p. Since $\pi(p)\xi_0 = \xi_0$, $y \geq p$ and $f_N(1) = 1$, we have $\pi(e_N)\xi_0 = \xi_0$, whence $e_N \in T$. □

Since $\omega(\alpha(p)) = \omega(p) = 1$, it follows that $\alpha(p) \geq p$. Taking α^{-1} instead of α, we have that $\alpha(p) = p$.

Note that for an arbitrary positive element x in T such that $x \geq p$, this decreasing sequence can be taken so that $x \geq e_N$ for each N. We will check it. Since a state of a hereditary subalgebra extends uniquely to a state of the whole algebra (see [12], 3.1.6), the restriction of ω to the hereditary subalgebra $B := \{y \in A | xy = yx = y\}$ of A is also pure. Thus we can take the sequence $(y_n)_n$ from B in the argument above. Then we have $e_N \leq x$.

Lemma 2.8. *If $f \in \ell^1(\mathbb{Z})$ satisfies that $f \geq 0$ and $\|f\|_{\ell^1(\mathbb{Z})} = 1$, it follows that*

$$\lim_{M\to\infty} \|\alpha_f(e_N)e_M - e_M\| = 0,$$
$$\lim_{M\to\infty} \|e_N\alpha_f(e_M) - \alpha_f(e_M)\| = 0$$

for each N, where we define $\alpha_f(x) := \sum_{n=-\infty}^{\infty} f(n)\alpha^n(x)$ for $f \in \ell^1(\mathbb{Z})$ and $x \in A$.

Proof. Suppose that the first equality is not valid. Then there is a $\delta > 0$ such that there are infinitely many M's which satisfy

$$\left\|(\alpha_f(e_N) - 1)e_M^2(\alpha_f(e_N) - 1)\right\| > \delta.$$

Since $(e_M^2)_M$ is decreasing (because $(e_M)_M$ is decreasing and $e_M e_{M+1} = e_{M+1}$ for any M), this inequality holds for every M. We can take a state ϕ_M on A such that

$$\phi_M((\alpha_f(e_N) - 1)e_M^2(\alpha_f(e_N) - 1)) > \delta$$

for every M. Since $(e_M^2)_M$ is decreasing because

$$\begin{aligned} e_M^2 &= e_M^{1/2} e_M e_M^{1/2} \geq e_M^{1/2} e_{M+1} e_M^{1/2} \\ &= e_{M+1}^{1/2} e_M e_{M+1}^{1/2} \geq e_{M+1}^{1/2} e_{M+1} e_{M+1}^{1/2} = e_{M+1}^2, \end{aligned}$$

we have

$$\phi_{M'}((\alpha_f(e_N) - 1)e_M^2(\alpha_f(e_N) - 1)) > \delta$$

for any $M' > M$. Taking a cluster point, we can find a state ϕ on A such that

$$\phi((\alpha_f(e_N) - 1)e_M^2(\alpha_f(e_N) - 1)) \geq \delta$$

for any M, whence

$$\phi((\alpha_f(e_N) - 1)p(\alpha_f(e_N) - 1)) \geq \delta,$$

where $p := \text{supp}\omega$. On the other hand, since $\alpha(p) = p$ and $e_N p = p$, we have

$$(\alpha_f(e_N) - 1)p = \sum_{n=-\infty}^{\infty} f(n)(\alpha^n(e_N p) - p) = 0,$$

which is a contradiction. The second equality follows similarly. □

Lemma 2.9. *It follows that*

$$\|\pi(e_N)E_U(q - \epsilon, q + \epsilon)\| = 1$$

for any q in $\mathbb{T}$, $\epsilon > 0$ and $N = 1, 2, \cdots$.

Proof. Let λ denote the canonical embedding of $C^*(\mathbb{Z})$ into the multiplier algebra $M(A \rtimes_\alpha \mathbb{Z})$. For any $g \in C^*(\mathbb{Z})$, since $(\|e_N\lambda(g)e_N\|)_N$ is a decreasing sequence,

$$\rho(g) := \lim_{N\to\infty} \|e_N\lambda(g)e_N\|$$

exists. We will show that ρ is a C^*-norm on $C^*(\mathbb{Z})$, whence $\rho(g) = \|g\|$ for $g \in C^*(\mathbb{Z})$ because a C^*-norm on a C^*-algebra is unique.

For any $g \in C^*(\mathbb{Z})$ and any $f \in \ell^1(\mathbb{Z})$ such that $f \geq 0$ and $\|f\|_{\ell^1(\mathbb{Z})} = 1$, since, for any N,

$$\begin{aligned}\lim_M \|e_M\lambda(g)e_M\| &= \lim_M \|e_M\alpha_f(e_N)\lambda(g)\alpha_f(e_N)e_M\| \\ &\leq \|\alpha_f(e_N)\lambda(g)\alpha_f(e_N)\|\end{aligned}$$

by Lemma 2.8, it follows that $\rho(g) \leq \lim_N \|\alpha_f(e_N)\lambda(g)\alpha_f(e_N)\|$. We can prove $\rho(g) \geq \lim_N \|\alpha_f(e_N)\lambda(g)\alpha_f(e_N)\|$ similarly, so we have

$$\rho(g) = \lim_{N\to\infty} \|\alpha_f(e_N)\lambda(g)\alpha_f(e_N)\|$$

for g in $C^*(\mathbb{Z})$ and $f \in \ell^1(\mathbb{Z})$ such that $f \geq 0$ and $\sum_{n=-\infty}^{\infty} f(n) = 1$. For any g, h in $C^*(\mathbb{Z})$ and $\epsilon > 0$, there is an f in $\ell^1(\mathbb{Z})$ such that $f \geq 0$, $\sum_{n=-\infty}^{\infty} f(n) = 1$, $\|[\lambda(g), \alpha_f(e_N)]\| < \epsilon$ and $\|[\lambda(h), \alpha_f(e_N)]\| < \epsilon$, where $[x, y] := xy - yx$. We will check it. For $g, h \in \ell^1(\mathbb{Z})$, we take a natural number L such that

$$\max\{\sum_{n=-\infty}^{-L-1} |g(n)| + \sum_{n=L+1}^{\infty} |g(n)|, \sum_{n=-\infty}^{-L-1} |h(n)| + \sum_{n=L+1}^{\infty} |h(n)|\} < \epsilon/4.$$

Set $R := \max\{|g(-L)|, |g(-L+1)|, \cdots, |g(L)|, |h(-L)|, \cdots, |h(L)|\}$ and choose a natural number K such that $K > \max\{1/\epsilon, R\}$. We define

$$f(n) := \begin{cases} \frac{1}{4L(2L+1)K^2} & (1 \leq n \leq 4L(2L+1)K^2) \\ 0 & (\text{otherwise}) \end{cases}.$$

Then we have

$$\begin{aligned}|g(n)| \sum_{m=-\infty}^{\infty} |f(m-n) - f(m)| &= 2|n||g(n)| \frac{1}{4L(2L+1)K^2} \\ &\leq 2LR \frac{1}{4L(2L+1)K^2} \\ &< \frac{\epsilon}{2(2L+1)} \qquad (-L \leq n \leq L)\end{aligned}$$

and $\sum_{m=-\infty}^{\infty} |f(m-n) - f(m)| \leq 2$ for any $n \in \mathbb{Z}$, whence

$$\begin{aligned}
\|[\lambda(g), \alpha_f(e_N)]\| &\leq \sum_{n=-\infty}^{\infty} |g(n)| \, \|\alpha^n(\alpha_f(e_N)) - \alpha_f(e_N)\| \\
&\leq \sum_{n=-\infty}^{\infty} |g(n)| \sum_{m=-\infty}^{\infty} |f(m-n) - f(m)| \\
&< (2L+1) \cdot \frac{\epsilon}{2(2L+1)} + 2 \cdot \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

Similarly it follows that $\|[\lambda(h), \alpha_f(e_N)]\| < \epsilon$. Thus, for g, h in $C^*(\mathbb{Z})$, we have

$$\begin{aligned}
\rho(gh) &= \lim_{N\to\infty} \left\|\alpha_f(e_N)^2 \lambda(g)\lambda(h)\alpha_f(e_N)^2\right\| \\
&\leq \lim_{N\to\infty} \left\|\alpha_f(e_N)\lambda(g)\alpha_f(e_N)^2\lambda(h)\alpha_f(e_N)\right\| + \epsilon(\|g\| + \|h\|) \\
&\leq \rho(g)\rho(h) + \epsilon(\|g\| + \|h\|),
\end{aligned}$$

whence $\rho(gh) \leq \rho(g)\rho(h)$. It also follows that $\rho(g^*g) = \rho(g)^2$ for g in $C^*(\mathbb{Z})$ since

$$\begin{aligned}
&\left\|\alpha_f(e_N)^2 \lambda(g^*)\lambda(g)\alpha_f(e_N)^2\right\| \\
&\quad \leq \left\|\alpha_f(e_N)\lambda(g^*)\alpha_f(e_N)^2\lambda(g)\alpha_f(e_N)\right\| + 2\epsilon \|g\| \\
&\quad = \|\alpha_f(e_N)\lambda(g)\alpha_f(e_N)\|^2 + 2\epsilon \|g\| \\
&\quad \leq \left\|\alpha_f(e_N)^2 \lambda(g^*)\lambda(g)\alpha_f(e_N)^2\right\| + 4\epsilon \|g\|
\end{aligned}$$

for any $\epsilon > 0$ and f in $\ell^1(\mathbb{Z})$ such that $f \geq 0$, $\sum_{n=-\infty}^{\infty} f(n) = 1$ and $\|[\lambda(g), \alpha_f(e_N)]\| < \epsilon$. So we can conclude that ρ is a C^*-semi-norm.

We will check that ρ is non-degenerate. Note that $g \in \ell^1(\mathbb{Z})_+$ means that there is a $g_0 \in \ell^1(\mathbb{Z})$ such that $g = g_0^* * g_0$, which implies that $\hat{g} = \hat{g_0}^* \hat{g}_0$, where $\hat{g}$ is the Fourier transform of g. So it is easier to calculate $\hat{g} \in C(\mathbb{T})$ than $g \in \ell^1(\mathbb{Z})$. At first, since

$$\begin{aligned}
\hat{g}(t) &= \langle \sum_n g(n) e^{int} \xi_0, \xi_0 \rangle = \langle \sum_n g(n) e^{int} U^n \pi(e_N)\xi_0, \pi(e_N)\xi_0 \rangle \\
&= \langle (\pi \rtimes U)(e_N \hat{\alpha}_t(\lambda(g)) e_N)\xi_0, \xi_0 \rangle
\end{aligned}$$

for $g \in \ell^1(\mathbb{Z})$ and $\ell^1(\mathbb{Z})$ is dense in $C^*(\mathbb{Z})$, the same equality holds for any $g \in C^*(\mathbb{Z})$. Suppose that $\rho(g) = 0$ for $g \in C^*(\mathbb{Z})$. We may assume that $g \geq 0$. Since

$$\begin{aligned}
\hat{g}(t) &= \langle (\pi \rtimes U)(e_N \hat{\alpha}_t(\lambda(g)) e_N)\xi_0, \xi_0 \rangle \\
&\leq \|(\pi \rtimes U)(e_N \hat{\alpha}_t(\lambda(g)) e_N)\| = \|e_N \hat{\alpha}_t(\lambda(g)) e_N\| \\
&= \|\hat{\alpha}_t(e_N \lambda(g) e_N)\| = \|e_N \lambda(g) e_N\| \\
&\to \rho(g) = 0
\end{aligned}$$

for any $t \in \mathbb{R}/2\pi\mathbb{Z}$, it follows that $g = 0$. Thus ρ is a C^*-norm.

Let h be an element of $C^*(\mathbb{Z})$ such that $\hat{h} \geq 0$, $\left\|\hat{h}\right\| = 1$ and $\operatorname{supp}\hat{h} \subset (q - \epsilon, q + \epsilon)$. Then we have

$$\begin{aligned}
\|\pi(e_N)E_U(q-\epsilon, q+\epsilon)\|^2 &= \|\pi(e_N)E_U(q-\epsilon, q+\epsilon)\pi(e_N)\| \\
&\geq \left\|\pi(e_N)\hat{h}(U)\pi(e_N)\right\| \\
&= \|(\pi \rtimes U)(e_N\lambda(h)e_N)\| = \|e_N\lambda(h)e_N\| \\
&\to \rho(\lambda(h)) = \|h\| = 1.
\end{aligned}$$

Now we reach the assertion. □

Lemma 2.10. *For any $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\|U\xi - e^{iq}\xi\| < \delta$ for a unit vector ξ in $\mathcal{H}$ and a q in $\mathbb{R}/2\pi\mathbb{Z}$ ($\simeq \mathbb{T}$), then*

$$\|E_U(q-\epsilon, q+\epsilon)\xi\| > 1 - \epsilon.$$

Proof. For a unit vector ξ in $\mathcal{H}$ and a q in $\mathbb{R}/2\pi\mathbb{Z}$, we define a probability measure $\mu := \mu_{\xi,q}$ on $\mathbb{R}/2\pi\mathbb{Z}$ by $\mu(S) = \langle E_U(S+q)\xi, \xi\rangle$. Since

$$\begin{aligned}
\langle U\xi, e^{iq}\xi\rangle &= \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{i(p-q)}\, d\langle E_U(p)\xi, \xi\rangle \\
&= \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{ip}\, d\langle E_U(p+q)\xi, \xi\rangle \\
&= \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{ip}\, d\mu(p),
\end{aligned}$$

it follows that

$$\left\|U\xi - e^{iq}\xi\right\|^2 = 2\int_{\mathbb{R}/2\pi\mathbb{Z}} (1 - \cos p)\, d\mu(p).$$

Thus, if $\|U\xi - e^{iq}\xi\| < \delta$, then

$$1 - \int \cos p \, d\mu(p) < \delta^2/2.$$

Suppose that the assertion is false. Then there are an $\epsilon > 0$, a sequence $(\xi_m)_m$ of unit vectors in $\mathcal{H}$, and a sequence $(q_m)_m$ in $\mathbb{R}/2\pi\mathbb{Z}$ such that

$$\begin{aligned}
&\lim_{m\to\infty} \left\|U\xi_m - e^{iq_m}\xi_m\right\| = 0, \\
&\|E_U(q_m - \epsilon, q_m + \epsilon)\xi_m\| \leq 1 - \epsilon.
\end{aligned}$$

Then, by taking a weak cluster point of $(\mu_{\xi_m,q_m})_m$ (in the dual of $C(\mathbb{R}/2\pi\mathbb{Z})$), we can find a measure μ on $\mathbb{R}/2\pi\mathbb{Z}$ such that

$$\mu(\mathbb{R}/2\pi\mathbb{Z}) \leq 1, \quad \mu(-\epsilon, \epsilon) \leq (1-\epsilon)^2, \quad \int \cos p\, d\mu(p) = 1.$$

The first and third conditions imply that μ is the Dirac measure at $p = 0$, which contradicts the second condition. Thus we have reached the assertion. □

Lemma 2.11. *If $x \in A$ satisfies $xp = 0$, where p is the support projection of $\omega = \langle \pi(\cdot)\xi_0, \xi_0\rangle$, then it follows that $\|xe_N\| \to 0$ as $N \to \infty$.*

Proof. For a state ϕ on A, we define $f_N(\phi) := \phi(xe_N^2x^*)$. Since $(e_N^2)_N$ is also decreasing, $(f_N(\phi))_N$ converges to $\phi(xpx^*) = 0$. Since $f_N(\phi)$ is continuous for each N as a function on the state space of A with the weak* topology, which is compact, it follows that $(f_N)_N$ converges uniformly to 0. Thus we have $\|xe_N^2x^*\| = \sup_\phi f_N(\phi) \to 0$, whence $\|xe_N\| \to 0$ as $N \to \infty$. □

Lemma 2.12. *Let x be an element of T and β an automorphism of A and V a unitary such that $V\xi_0 = \xi_0$ and $\mathrm{Ad}V \circ \pi = \pi \circ \beta$. Then for any $\epsilon > 0$ there exists a $b \in T$ such that $xb = b$ and $\|\beta(b) - b\| < \epsilon$.*

Proof. At first, note that since $V\xi_0 = \xi_0$ implies $\omega(\beta(p)) = 1$, we have $\beta(p) = p$, where $p := \mathrm{supp}\omega$. Let $(e_N)_N$ be a decreasing sequence for ξ_0 as before. Let f be a function on $\mathbb{Z}$ such that $f \geq 0, \sum_{n\in\mathbb{Z}} f(n) = 1$, and $\sum_{n\in\mathbb{Z}} |f(n-1) - f(n)| < \epsilon$ (for example, $f(n) = 1/(2N+1)$ for $-N \leq n \leq N$, $= 0$ otherwise), and let $b_N = \beta_f(e_N)$. Then we have

$$\pi(b_N)\xi_0 = \xi_0,$$
$$\|\beta(b_N) - b_N\| < \epsilon.$$

Take an element $c \in T$ such that $cx = c$. Then it follows that $(c-1)p = 0$ since $\pi(p)$ is the one-dimensional projection onto $\mathbb{C}\xi_0$ (here 1 is in the unitization of A when A is non-unital). By Lemma 2.11, we have

$$\|ce_N - e_N\| \to 0.$$

By Lemma 2.8, it follows that

$$\|cb_N - b_N\| \to 0.$$

Let

$$g_N(t) := \begin{cases} \frac{N}{N-1}t & (0 \leq t \leq 1 - 1/N) \\ 1 & (1 - 1/N \leq t \leq 1) \end{cases}.$$

Then it follows that $\sup_{0\leq t\leq 1} |g_N(t) - t| \to 0$. Now $b := g_N(cb_Nc)$ for a sufficiently large N satisfies all of the conditions of the lemma. □

Next we have the key lemma for proving the main theorem. Before that, we have important definitions.

Definition 2.13 ([12], 6.6.1)**.** Let A be a C^*-algebra.

(1) A sequence $(x_{nj})_{n=1,2,\cdots,0\leq j<k_n}$ in A is a quasi-matrix system if $(x_{nj})_{n,j}$ satisfies for any n,

$$x_{n0} \geq 0, \qquad \|x_{nj}\| = 1 \qquad \text{for } 0 \leq j < k_n,$$
$$x_{ni}^* x_{nj} = 0 \qquad \text{for } i \neq j,$$
$$x_{ni}x_{nj} = 0 \qquad \text{for } j \neq 0,$$
$$x_{nj}^* x_{nj} x_{n0} = x_{n0} \qquad \text{for } 1 \leq j < k_n,$$
$$x_{n0}x_{n+1,j} = x_{n+1,j}x_{n0} = x_{n+1,j} \qquad \text{for } 0 \leq j < k_{n+1}.$$

(2) A sequence $(x_{nj})_{n=1,2,\cdots,0\le j<k_n}$ in A is a matrix system if $(x_{nj})_{n,j}$ is a quasi-matrix system and satisfies for any n,

$$x_j^*x_j = x_0 \qquad \text{if } j \neq 0$$
$$x_{n+1,0} + \sum_{i=1}^{k_{n+1}-1} x_{n+1,i}x_{n+1,i}^* = x_{n0}.$$

When $(x_{nj})_{n=1,2,\cdots,0\le j<k_n}$ is a matrix system in a C^*-algebra A, the C^*-subalgebra of A generated by $(x_{nj})_{n,j}$ is isomorphic to the UHF algebra $D = \otimes_{n=1}^{\infty} M_{k_n}$ by

$$x_{1i_1}x_{2i_2}\cdots x_{ni_n}x_{n\ell_n}^*x_{n-1,\ell_{n-1}}^*\cdots x_{1\ell_1}^* \mapsto E_{i_1\ell_1}\otimes\cdots\otimes E_{i_n\ell_n} \in M_{k_1}\otimes\cdots\otimes M_{k_n} \subset D$$

for $0 \le i_m, \ell_m < k_n$, where $E_{i\ell}$ denotes the (i,ℓ)-matrix unit of M_{k_n}. Here x_{nj} expresses the $(j,0)$-matrix unit

$$j> \begin{pmatrix} 0 & & \cdots & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}$$

of M_{k_n}. (To avoid mistaking indexes, we call the top of a matrix "the 0-th row" and the left end of one "the 0-th column".) Note that $x_{n0}x_{n+1,j} = x_{n+1,j}x_{n0} = x_{n+1,j}$ for $0 \le j < k_{n+1}$ implies that x_{n0} is a unit for $M_{k_{n+1}} \subset D$.

Lemma 2.14. *Let A be a separable C^*-algebra, α an automorphism on A, π a faithful α-covariant irreducible representation of A on a Hilbert space $\mathcal{H}$, U the implementing unitary for α, and ξ_0 a unit vector such that $U\xi_0 = \xi_0$. Let $(p_1, p_2, \cdots, p_m)$ be a sequence in $\mathbb{R}/2\pi\mathbb{Z}$ and $(x_0, x_1, \cdots, x_m)$ a sequence in A_1 with $x_0 \in T$ such that*

$$U\pi(x_k)\xi_0 = e^{ip_k}\pi(x_k)\xi_0,$$
$$x_j^*x_k = 0 \qquad \text{if } j \neq k,$$
$$x_jx_k = 0 \qquad \text{if } k \neq 0,$$
$$x_j^*x_jx_0 = x_0 \qquad \text{if } j \neq 0$$

for $j, k = 0, 1, \cdots, m$, where $p_0 = 0$. Let $(q_1, q_2, \cdots, q_n)$ be a sequence in $\mathbb{R}/2\pi\mathbb{Z}$ and $\epsilon > 0$.

Then there exist a sequence $(y_0, y_1, \cdots, y_n)$ in A with $y_0 \in T$ and $\|y_\ell\| = 1$ for $\ell = 0, 1, \cdots, n$, and v in $\mathcal{U}(A)$ such that $\|v - 1\| < \epsilon$,

$$x_0y_\ell = y_\ell x_0 = y_\ell,$$
$$y_j^*y_\ell = 0 \qquad \text{if } j \neq \ell,$$
$$y_jy_\ell = 0 \qquad \text{if } \ell \neq 0,$$
$$y_j^*y_jy_0 = y_0 \qquad \text{if } j \neq 0$$

for $j, \ell = 0, 1, \cdots, n$ *and*

$$U^{(v)}\pi(x_k y_\ell)\xi_0 = e^{i(p_k+q_\ell)}\pi(x_k y_\ell)\xi_0,$$
$$\left\|(\alpha^{(v)}(x_k y_\ell) - e^{i(p_k+q_\ell)}x_k y_\ell)y_0\right\| < \epsilon,$$

for $k = 0, 1, \cdots, m$ *and* $\ell = 0, 1, \cdots, n$ *with* $q_0 = 0$.

Using this lemma inductively, we get a quasi-matrix system $(x_{nj})_{n,j}$ in A. In this lemma, x_j and y_ℓ correspond to x_{nj} and $x_{n+1,\ell}$ of a quasi-matrix system, respectively. In the proof of Theorem 1.1 below, we find a projection $q \in A^{**}$ such that $(x_{nj}q)_{n,j}$ is a matrix system, which implies that the UHF algebra $D = \otimes_{n=1}^{\infty} M_{k_n}$ is almost embedded in A.

Proof of Lemma 2.14. We may assume that $(q_\ell - \epsilon, q_\ell + \epsilon)$, $\ell = 1, \cdots, n$ are identical or mutually disjoint. Let $(e_N)_N$ be a decreasing sequence in T associated with ξ_0 as before. We may suppose that $e_1 = x_0$. By Lemma 2.11, we can take a sufficiently large number N such that

$$\left\|(\alpha(x_k) - e^{ip_k}x_k)e_N\right\| < \epsilon$$

for $k = 0, 1, \cdots, m$. Let P be the spectral projection of $\pi(e_N)$ corresponding to the eigenvalue 1. Then we have $\pi(e_N)P = P$ and $\pi(e_{N+1})P = \pi(e_{N+1})$. By Lemma 2.9, it follows for $\ell = 1, \cdots, n$ that $\|\pi(e_{N+1})E_U(q_\ell - \epsilon, q_\ell + \epsilon)\pi(e_{N+1})\| = 1$. Since $\pi(e_{N+1})P = \pi(e_{N+1})$, we have $\|PE_U(q_\ell - \epsilon, q_\ell + \epsilon)P\| = 1$ for $\ell = 1, \cdots, n$. So there is a unit vector η_ℓ in $P\mathcal{H}$ for each $\ell = 1, \cdots, n$ such that $1 - \langle E_U(q_\ell - \epsilon, q_\ell + \epsilon)\eta_\ell, \eta_\ell\rangle < \epsilon^2$, which is equivalent to

$$\|E_U(q_\ell - \epsilon, q_\ell + \epsilon)\eta_\ell - \eta_\ell\| < \epsilon.$$

Since $E_U(q_\ell - \epsilon, q_\ell + \epsilon)E_U(q_k - \epsilon, q_k + \epsilon) = 0$ for $q_\ell \neq q_k$, we have $|\langle \eta_\ell, \eta_k\rangle| < 2\epsilon$. For $q_{\ell_1} = \cdots = q_{\ell_r} (= q_\ell)$, we want to take a mutually orthogonal family $(\eta_{\ell_1}, \cdots, \eta_{\ell_r})$ such that $\eta_{\ell_j} \in P\mathcal{H}$ and $1 - \langle E_U(q_\ell - \epsilon, q_\ell + \epsilon)\eta_{\ell_j}, \eta_{\ell_j}\rangle < \epsilon^2$ for $j = 1, \cdots, r$. We show that $\|(PE_U(q_\ell - \epsilon, q_\ell + \epsilon)P)|\mathcal{H}'\| = 1$ for $\ell = 1, \cdots, n$ and any subspace $\mathcal{H}'$ of $\mathcal{H}$ whose orthogonal complement is finite-dimensional. By Lemma 2.9, it follows that

$$\begin{aligned}&\left\|\pi(e_{N+1})E_U(q_\ell - \frac{\epsilon}{2}, q_\ell + \frac{\epsilon}{2})\pi(e_{N+1})\right\| \\ &\quad = \|\pi(e_{N+1})E_U(q_\ell - \epsilon, q_\ell + \epsilon)\pi(e_{N+1})\| = 1.\end{aligned}$$

Let $h : \mathbb{R}/2\pi\mathbb{Z} \to [0, 1]$ be a continuous function such that $h = 1$ on $(q_\ell - \epsilon/2, q_\ell + \epsilon/2)$ and $h = 0$ on the complement of $(q_\ell - \epsilon, q_\ell + \epsilon)$. Then it follows that

$$\begin{aligned}&\pi(e_{N+1})E_U(q_\ell - \epsilon, q_\ell + \epsilon)\pi(e_{N+1}) \\ &\quad \geq \pi(e_{N+1})h(U)\pi(e_{N+1}) \\ &\quad \geq \pi(e_{N+1})E_U(q_\ell - \epsilon/2, q_\ell + \epsilon/2)\pi(e_{N+1}),\end{aligned}$$

which implies that $\|\pi(e_{N+1})h(U)\pi(e_{N+1})\| = 1$. By Lemma 2.5, we have

$$\|Q(\pi(e_{N+1})h(U)\pi(e_{N+1}))\| = 1,$$

where $Q : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{H})/\mathbf{K}(\mathcal{H})$ is the quotient map. Hence it follows that

$$\begin{aligned}
\|PE_U(q_\ell - \epsilon, q_\ell + \epsilon)P + K\| &\geq \|Q(PE_U(q_\ell - \epsilon, q_\ell + \epsilon)P)\| \\
&\geq \|Q(\pi(e_{N+1})E_U(q_\ell - \epsilon, q_\ell + \epsilon)\pi(e_{N+1}))\| \\
&\geq \|Q(\pi(e_{N+1})h(U)\pi(e_{N+1}))\| \\
&= 1
\end{aligned}$$

for any $K \in \mathbf{K}(\mathcal{H})$. For a finite rank projection F such that $F \leq P$, let

$$K := -(FE_U(q_\ell - \epsilon, q_\ell + \epsilon)P + PE_U(q_\ell - \epsilon, q_\ell + \epsilon)F - FE_U(q_\ell - \epsilon, q_\ell + \epsilon)F).$$

Then it follows that K is a finite rank operator, and hence

$$\begin{aligned}
&\|(P - F)E_U(q_\ell - \epsilon, q_\ell + \epsilon)(P - F)\| \\
&\quad = \|PE_U(q_\ell - \epsilon, q_\ell + \epsilon)P + K\| = 1.
\end{aligned}$$

Thus we can take a desired family $(\eta_{\ell_1}, \cdots, \eta_{\ell_r})$ inductively. We use the Gram-Schmidt orthogonalization for all η_ℓ's. By Lemma 2.3, we have $\|E_U(q_\ell - \epsilon, q_\ell + \epsilon)\eta_\ell - \eta_\ell\| < r_n\epsilon$ after this process, where r_n is a positive real number dependent on n.

By Kadison's transitivity, there exists a y_ℓ in A such that $\|y_\ell\| = 1$ and

$$\pi(y_\ell)\xi_0 = \eta_\ell$$

for $\ell = 1, 2, \cdots, n$. There also exists a b in A_+ such that

$$\pi(b)\eta_\ell = (\ell + 1)\eta_\ell$$

for $\ell = 0, 1, \cdots, n$, where $\eta_0 = \xi_0$. Since $\pi(e_N)P = P$, we may replace b by $e_N b e_N$, and hence we may assume that $x_0 b = b$. Let $(f_0, f_1, \cdots, f_n)$ be a sequence of non-negative functions in $C_0(0, \infty)$ with norm 1 such that $f_\ell(\ell + 1) = 1$ and $\mathrm{supp}(f_\ell) \subset (\ell + 1/2, \ell + 3/2)$ for $\ell = 0, 1, \cdots, n$. Then, since

$$\pi(f_\ell(b)y_\ell f_0(b))\eta_0 = \eta_\ell,$$

we may replace y_ℓ by $f_\ell(b)y_\ell f_0(b)$ for $\ell = 0, 1, \cdots, n$. Then it follows that $x_0 y_\ell = y_\ell x_0 = y_\ell$, $y_j^* y_\ell = 0$ for $j \neq \ell$, and $y_j y_\ell = 0$ for $j, \ell = 1, 2, \cdots, n$ besides the original conditions $\pi(y_\ell)\xi_0 = \eta_\ell$ and $\|y_\ell\| = 1$.

Since $\langle \pi(y_1^* y_1)\xi_0, \xi_0 \rangle = \|\eta_1\| = 1$ and $\|\pi(y_1^* y_1)\xi_0\| \leq 1$ (because $\|y_1\| = \|\xi_0\| = 1$), it follows that $\pi(y_1^* y_1)\xi_0 = \xi_0$. Let f be a non-negative function in $C_0(0, \infty)$ such that $f(t) = t^{-1/2}$ around $t = 1$ and $tf(t)^2 \leq 1$ for all $t > 0$. Then we have $\pi(f(y_1^* y_1))\xi_0 = \xi_0$, and so

$$\pi(y_1 f(y_1^* y_1))\xi_0 = \eta_1.$$

Replacing y_1 by $y_1 f(y_1^* y_1)$, it follows that $y_1^* y_1 \in T$, since $tf(t)^2 \equiv 1$ around $t = 1$. Take a $z_1 \in T$ such that $y_1^* y_1 z_1 = z_1$. Replacing y_2 by $y_2 z_1 f(z_1 y_2^* y_2 z_1)$, it follows that $y_2 y_1^* y_1 = y_2$ and $y_2^* y_2 \in T$. Take a $z_2 \in T$ such that $y_2^* y_2 z_2 = z_2$. Inductively, we replace y_i by $y_i z_{i-1} f(z_{i-1} y_i^* y_i z_{i-1})$ and obtain a $z_i \in T$. Set $y_0 := z_n$. Then we have $y_0 y_\ell^* y_\ell = y_0$ and $y_0 y_\ell = 0$ for $\ell = 1, 2, \cdots, n$. Thus $(y_0, \cdots, y_n)$ satisfies the first four conditions.

Since

$$\begin{aligned}\left\|U\eta_\ell - e^{iq_\ell}\eta_\ell\right\| &\le 2\left\|E_U(q_\ell-\epsilon, q_\ell+\epsilon)\eta_\ell - \eta_\ell\right\| \\ &\qquad + \left\|UE_U(q_\ell-\epsilon, q_\ell+\epsilon)\eta_\ell - e^{iq_\ell}E_U(q_\ell-\epsilon, q_\ell+\epsilon)\eta_\ell\right\| \\ &< 2r_n\epsilon + \left\|\int_{q_\ell-\epsilon}^{q_\ell+\epsilon}(e^{it}-e^{iq_\ell})dE_U(t)\eta_\ell\right\| \\ &\le (2r_n+2)\epsilon,\end{aligned}$$

it follows that

$$\begin{aligned}&\left\|U\pi(x_k y_\ell)\xi_0 - e^{i(p_k+q_\ell)}\pi(x_k y_\ell)\xi_0\right\| \\ &\le \left\|\pi(\alpha(x_k))U\eta_\ell - e^{iq_\ell}\pi(\alpha(x_k))\eta_\ell\right\| + \left\|\pi(\alpha(x_k))\eta_\ell - e^{ip_k}\pi(x_k)\eta_\ell\right\| \\ &\le \left\|U\eta_\ell - e^{iq_\ell}\eta_\ell\right\| + \left\|(\alpha(x_k) - e^{ip_k}x_k)e_N\right\| \\ &< (2r_n+3)\epsilon.\end{aligned}$$

Since the $(m+1)(n+1)$ unit vectors $\pi(x_k y_\ell)\xi_0$, $k=0,\cdots,m$, $\ell=0,\cdots,n$ are mutually orthogonal, and

$$U\pi(x_k y_0)\xi_0 = e^{ip_k}\pi(x_k y_0)\xi_0$$

for $k=0,\cdots,m$ and

$$\left\|U\pi(x_k y_\ell)\xi_0 - e^{i(p_k+q_\ell)}\pi(x_k y_\ell)\xi_0\right\| < (2r_n+3)\epsilon$$

for $k=0,\cdots,m$, $\ell=1,\cdots,n$, we can use Lemma 2.2 for a unitary V such that $V\pi(x_k y_l)\xi_0 := e^{i(p_k+q_l)}U^*\pi(x_k y_l)\xi_0$ and $\pi(x_k y_\ell)\xi_0$, $k=0,\cdots,m$ and $\ell=0,\cdots,n$ to obtain a $v\in\mathcal{U}(A)$ as required except for the last condition. Since $y_0\ge p$, there is another decreasing sequence $(e'_N)_N$ such that $e'_1=y_0$ and $e'_N\searrow p$. By Lemma 2.11, there is a sufficiently large number N_0 such that

$$\left\|(\alpha^{(v)}(x_k y_\ell) - e^{i(p_k+q_\ell)}x_k y_\ell)e'_{N_0}\right\| < \epsilon.$$

We replace y_0 by e'_{N_0} and end the proof. □

Proof of Theorem 1.1. Since a unitary matrix can be diagonalized by some unitary matrix, we may assume, up to conjugacy, that γ is of the form

$$\gamma = \bigotimes_{n=1}^{\infty}\mathrm{Ad}\ \mathrm{diag}(1, e^{ip_{n1}}, \cdots, e^{ip_{n,k_n-1}})$$

on $D=\otimes_{n=1}^{\infty}M_{k_n}$, where $\mathrm{diag}(\lambda_1,\cdots,\lambda_k)$ means the diagonal matrix whose (i,i) component is λ_i. We define $p_{n0}:=0$ for $n=0,1,\cdots$.

We have fixed a unit vector $\xi_0\in\mathcal{H}$ such that $U\xi_0=\xi_0$. We choose an $e\in T$. Let (μ_n) be a strictly decreasing sequence of positive numbers such that

$$nk_1k_2\cdots k_n\mu_n < 1$$

and let $\epsilon_n := \mu_n - \mu_{n+1}$.

Using Lemma 2.14 inductively, we will find suitable elements $v_m\in\mathcal{U}(A)$ for $m=0,1,\cdots$, and $x_{mj}\in A$ for $m=0,1,\cdots$ and $0\le j<k_m$. When $m=0$ (note that we can set $k_0:=1$), we define $v_0:=1$ and $x_{00}:=e$.

Suppose $v_n \in \mathcal{U}(A)$ and $x_{nj} \in A$ for $0 \le j < k_n$ are already defined for $n \le m$ so that $x_{n0} \in T$, $\|x_{nj}\| = 1$, $\|v_n - 1\| < \epsilon_n$, and

$$\begin{aligned} U^{(\overline{v_m})}\pi(w_i^{(m)})\xi_0 &= e^{ip_i^{(m)}}\pi(w_i^{(m)})\xi_0, \\ w_j^{(m)*}w_k^{(m)} &= 0 \qquad \text{if } j \ne k, \\ w_j^{(m)}w_k^{(m)} &= 0 \qquad \text{if } k \ne 0, \\ w_j^{(m)*}w_j^{(m)}w_0^{(m)} &= w_0^{(m)} \qquad \text{if } j \ne 0, \end{aligned}$$

for all $i, j, k \in X_m$ and $0 = (0, 0, \cdots, 0) \in X_m$, where

$$\begin{aligned} \overline{v_m} &:= v_1 v_2 \cdots v_m, \\ X_m &:= \{i = (i_1, i_2, \cdots, i_m) | 0 \le i_n < k_n\}, \\ w_i^{(m)} &:= x_{1i_1}x_{2i_2}\cdots x_{mi_m} \qquad \text{for } i = (i_1, i_2, \cdots, i_m) \in X_m, \\ p_i^{(m)} &:= p_{1i_1} + p_{2i_2} + \cdots + p_{mi_m} \qquad \text{for } i = (i_1, i_2, \cdots, i_m) \in X_m, \end{aligned}$$

and $\overline{v_0} := 1$, $X_0 := \{0\}$, $w_0^{(0)} := e$ and $p_0^{(0)} := 0$ for $m = 0$. Then, there exist $x_{m+1,\ell} \in A$ for $0 \le \ell < k_{m+1}$ and $v_{m+1} \in \mathcal{U}(A)$ such that $x_{m+1,0} \in T$, $\|x_{m+1,j}\| = 1$, $\|v_{m+1} - 1\| < \epsilon_{m+1}$, and

$$\begin{aligned} w_0^{(m)}x_{m+1,\ell} = x_{m+1,\ell}w_0^{(m)} &= x_{m+1,\ell}, \\ x_{m+1,j}^*x_{m+1,\ell} &= 0 \qquad \text{if } j \ne \ell, \\ x_{m+1,j}x_{m+1,\ell} &= 0 \qquad \text{if } \ell \ne 0, \\ x_{m+1,j}^*x_{m+1,j}x_{m+1,0} &= x_{m+1,0} \qquad \text{if } j \ne 0 \end{aligned}$$

for all $j, \ell = 0, 1, \cdots, k_{m+1} - 1$ and

$$\begin{aligned} U^{(\overline{v_{m+1}})}\pi(w_k^{(m)}x_{m+1,\ell})\xi_0 &= e^{i(p_k^{(m)}+p_{m+1,\ell})}\pi(w_k^{(m)}x_{m+1,\ell})\xi_0, \\ \left\|(\alpha^{(\overline{v_{m+1}})}(w_k^{(m)}x_{m+1,\ell}) - e^{i(p_k^{(m)}+p_{m+1,\ell})}w_k^{(m)}x_{m+1,\ell})x_{m+1,0}\right\| &< \epsilon_{m+1} \end{aligned}$$

for $k \in X_m$ and $\ell = 0, 1, \cdots, k_{m+1} - 1$. Since $w_k^{(m)}x_{m+1,\ell} = w_{(k,\ell)}^{(m+1)}$, where $(k, \ell) \in X_{m+1}$, it follows that

$$\begin{aligned} w_j^{(m+1)*}w_k^{(m+1)} &= 0 \qquad \text{if } j \ne k, \\ w_j^{(m+1)}w_k^{(m+1)} &= 0 \qquad \text{if } k \ne 0, \\ w_j^{(m+1)*}w_j^{(m+1)}w_0^{(m+1)} &= w_0^{(m+1)} \qquad \text{if } j \ne 0, \\ U^{(\overline{v_{m+1}})}\pi(w_i^{(m+1)})\xi_0 &= e^{ip_i^{(m+1)}}\pi(w_i^{(m+1)})\xi_0, \\ \left\|(\alpha^{(\overline{v_{m+1}})}(w_k^{(m+1)}) - e^{ip_k^{(m+1)}}w_k^{(m+1)})w_0^{(m+1)}\right\| &< \epsilon_{m+1}, \end{aligned}$$

for $i, j, k \in X_{m+1}$, where we used $w_0^{(m)}x_{m+1,\ell} = x_{m+1,\ell}w_0^{(m)} = x_{m+1,\ell}$. Since $w_0^{(n-1)}x_{n0} = x_{n0}w_0^{(n-1)} = x_{n0}$ for any n implies $w_0^{(n)} = x_{n0}$, we have $(x_{nj})_{n=1,2,\cdots,0\le j<k_n}$ is a quasi-matrix system.

We define

$$\overline{v} := \lim_m \overline{v_m} = v_1 v_2 \cdots .$$

Then it follows that $\|\overline{v} - 1\| < \mu_1$, and

$$\begin{aligned}
&\left\|(\alpha^{(\overline{v})}(w_i^{(m)}) - e^{ip_i^{(m)}} w_i^{(m)}) w_0^{(m)}\right\| \\
&\quad < \left\|(\alpha^{(\overline{v_m})}(w_i^{(m)}) - e^{ip_i^{(m)}} w_i^{(m)}) w_0^{(m)}\right\| + 2\mu_{m+1} \\
&\quad < 2\mu_m
\end{aligned}$$

for $i \in X_m$. Note that since $U^{(\overline{v_{m+1}})}\xi_0 = \xi_0$, it follows that $U^{(\overline{v})}\xi_0 = \xi_0$, which implies $\alpha^{(\overline{v})}(p) = p$.

By using the separability of A, we will impose another condition on the choice of $w_0^{(m)} = x_{m0}$ for each m. (We only have to replace them for sufficiently large m's.) Fix a dense sequence $(a_n)_n$ of A_{sa}. Let $(e_N)_N$ and $(f_N)_N$ be as in Lemma 2.7 and choose $a \in T$ such that $w_0^{(m)} a = a$. Set $y' := \sum 2^{-N} a e_N a$ and $z_N := f_N(y')$. Then we have $w_0^{(m)} z_N = z_N$ for all N. Let b be an element in A. Since $z_N \searrow p$, $(z_N(b - \omega(b))z_N)$ converges σ-weakly to $p(b - \omega(b))p$, which is equal to 0 since $\pi(p)$ is the 1-dimensional projection supporting ω. So the norm closure of the convex hull of $\{z_N(b - \omega(b))z_N\}$ contains 0. Thus for each $\delta > 0$ there are positive numbers $(t_i)_i$ with $\sum t_i = 1$ such that $\|\sum t_i z_{N_i}(b - \omega(b))z_{N_i}\| < \delta$. Hence whenever $N \geq N_i$ for all i, it follows that

$$\|z_N(b - \omega(b))z_N\| \leq \|z_N\| \left\|\sum t_i z_{N_i}(b - \omega(b))z_{N_i}\right\| \|z_N\| < \delta.$$

We take such an N and set $\tilde{w_0}^{(m)} := z_N$. Applying Lemma 2.12, we may assume that

$$\left\|\alpha^{(\overline{v})}(\tilde{w_0}^{(m)}) - \tilde{w_0}^{(m)}\right\| < \mu_m.$$

Set $\tilde{a_m} := \sum_{i,j \in X_m} \omega(w_j^{(m)*} a_m w_i^{(m)}) w_j^{(m)} (\tilde{w}_0^{(m)})^2 w_i^{(m)*} \in B$. Then, by setting $b = w_i^{(m)*} a_m w_j^{(m)}$ and $\delta = \mu_m/(k_1 k_2 \cdots k_m)$ in the argument above, we have

$$\begin{aligned}
&\left\|(\sum_i w_i^{(m)} (\tilde{w}_0^{(m)})^2 w_i^{(m)*})(a_m - \tilde{a_m})(\sum_j w_j^{(m)} (\tilde{w}_0^{(m)})^2 w_j^{(m)*})\right\| \\
&= \left\|\sum_{i,j} w_i^{(m)} (\tilde{w}_0^{(m)})^2 (w_i^{(m)*} a_m w_j^{(m)} - \omega(w_i^{(m)*} a_m w_j^{(m)}))(\tilde{w}_0^{(m)})^2 w_j^{(m)*}\right\| \\
&< 1/m.
\end{aligned}$$

From now on we just write $w_0^{(m)}$ instead of $\tilde{w_0}^{(m)}$.

Let p_m be the spectral projection of $w_0^{(m)} = x_{m0}$ corresponding to the eigenvalue 1. We define

$$q_m := \sum_{i \in X_m} w_i^{(m)} p_m w_i^{(m)*}$$

and for $1 \le n \le m$,

$$q_{mn} := \sum_{i \in X_{n,m}} w_i^{(m)} p_m w_i^{(m)*}$$

where $X_{n,m} := \{i \in X_m | i_1 = 0, i_2 = 0, \cdots, i_n = 0\}$. Then it follows that q_m, q_{mn} with $1 \le n \le m$ are projections in A^{**} satisfying

$$\begin{aligned}
&x_{ni} q_m = x_{ni} q_{mn} = q_m x_{ni}, \\
&x_{ni}^* x_{ni} q_m = x_{n0} q_m = q_{mn}, \\
&x_{n+1,0} q_{m'} + \sum_{i=1}^{k_{n+1}-1} x_{n+1,i} x_{n+1,i}^* q_{m'} = x_{n0} q_{m'}
\end{aligned}$$

for each $1 \le n \le m$, $0 \le i < k_n$ and $m' > n+1$. We will prove these claims. Note that if $i \in X_{n,m}$, then $w_i^{(m)} = x_{10}x_{20} \cdots x_{n0} x_{n+1,i_{n+1}} \cdots x_{mi_m} = x_{n+1,i_{n+1}} \cdots x_{mi_m}$. It can be easily shown that q_m, q_{mn} with $1 \le n \le m$ are projections by $x_{n'i} x_{n''j} = 0$ if $n' \ge n''$ and $j \ne 0$, and $x_{n'i}^* x_{n''j} = 0$ unless $n' = n''$ and $i = j$. The first and second line of the three can also be shown immediately by $x_{n'i} x_{n''j} = 0$ if $n' \ge n''$ and $j \ne 0$, and $x_{n'i}^* x_{n'i} x_{n'0} = x_{n'0}$ for each n' and $i \ne 0$. We will check the last equality. Since $x_{n0} q_{n+1} = q_{n+1} x_{n0} = x_{n0} q_{n+1,n} = q_{n+1,n} x_{n0} = q_{n+1,n}$, it follows that $p_n q_{n+1} = q_{n+1,n}$ for each n. Hence we have

$$\begin{aligned}
x_{n0} q_{n+2} &= x_{n0} q_{n+1} q_{n+2} = q_{n+1,n} q_{n+2} \\
&= \sum_{j=0}^{k_{n+1}-1} x_{n+1,j} p_{n+1} x_{n+1,j}^* q_{n+2} = \sum_{j=0}^{k_{n+1}-1} x_{n+1,j} p_{n+1} q_{n+2} x_{n+1,j}^* \\
&= \sum_{j=0}^{k_{n+1}-1} x_{n+1,j} q_{n+2,n+1} x_{n+1,j}^* = \sum_{j=0}^{k_{n+1}-1} x_{n+1,j} q_{n+2} x_{n+1,j}^* \\
&= x_{n+1,0} q_{n+2} + \sum_{i=1}^{k_{n+1}-1} x_{n+1,i} x_{n+1,i}^* q_{n+2}.
\end{aligned}$$

Multiplying $q_{m'}$ $(m' > n+1)$ by the right side, we get the desired equality.

We define

$$r_m := \sum_{i \in X_m} w_i^{(m)} w_0^{(m)^2} w_i^{(m)*} \in A.$$

Then it follows that $q_m \le r_m \le q_{m-1}$. Let $q :=$weak*-lim q_m. Since $(q_m)_m$ is a decreasing sequence, q is a closed projection in A^{**}.

For $i \in X_m$, we have that

$$\begin{aligned}
&\left\| \alpha^{(\overline{v})}(w_i^{(m)} w_0^{(m)^2} w_i^{(m)*}) - w_i^{(m)} w_0^{(m)^2} w_i^{(m)*} \right\| \\
\le &\left\| \alpha^{(\overline{v})}(w_i^{(m)}) w_0^{(m)^2} \alpha^{(\overline{v})}(w_i^{(m)*}) - w_i^{(m)} w_0^{(m)^2} w_i^{(m)*} \right\| + 2\mu_m \\
\le 2 &\left\| (\alpha^{(\overline{v})}(w_i^{(m)}) - e^{ip_i^{(m)}} w_i^{(m)}) w_0^{(m)} \right\| + 2\mu_m \\
\le 6&\mu_m,
\end{aligned}$$

which implies that

$$\left\|\alpha^{(\overline{v})}(r_m) - r_m\right\| < 6k_1k_2\cdots k_m\mu_m < 6/m.$$

Therefore we have $\alpha^{(\overline{v})}(q) = q$.

For $n \leq m$, $0 \leq i < k_n$ and $j \in X_m$, note that if $j \notin X_{n,m}$, it follows that $x_{ni}w_j^{(m)} = 0$, otherwise we can write $w_\ell^{(m)} = x_{ni}w_j^{(m)}$ for some $\ell \in X_m$ (because $w_i^{(m)} = x_{10}x_{20}\cdots x_{n0}x_{n+1,i_{n+1}}\cdots x_{mi_m} = x_{n+1,i_{n+1}}\cdots x_{mi_m}$ for $i \in X_{n,m}$ and $x_{ni}x_{n'j} = 0$ if $n \geq n', j \neq 0$). So we have

$$\begin{aligned}
&\left\|(\alpha^{(\overline{v})}(x_{ni}) - e^{ip_{ni}}x_{ni})w_j^{(m)}w_0^{(m)}\right\| \\
&\quad < \left\|(e^{-ip_j^{(m)}}\alpha^{(\overline{v})}(x_{ni}w_j^{(m)}) - e^{ip_{ni}}x_{ni}w_j^{(m)})w_0^{(m)}\right\| + 2\mu_m \\
&\quad = \left\|(\alpha^{(\overline{v})}(w_\ell^{(m)}) - e^{ip_\ell^{(m)}}w_\ell^{(m)})w_0^{(m)}\right\| + 2\mu_m \\
&\quad < 4\mu_m,
\end{aligned}$$

whenever $w_\ell^{(m)} = x_{ni}w_j^{(m)} \neq 0$. (This inequality also holds when $x_{ni}w_j^{(m)} = 0$.) Thus it follows that

$$\begin{aligned}
\left\|(\alpha^{(\overline{v})}(x_{ni}) - e^{ip_{ni}}x_{ni})q_m\right\| &\leq \sum_{j\in X_m}\left\|(\alpha^{(\overline{v})}(x_{ni}) - e^{ip_{ni}}x_{ni})w_j^{(m)}p_m\right\| \\
&< 4k_1k_2\cdots k_m\mu_m < 4/m,
\end{aligned}$$

and hence $\alpha^{(\overline{v})}(x_{ni})q = e^{ip_{ni}}x_{ni}q$.

Let B be the C^*-subalgebra of A generated by $\{\alpha^{(\overline{v})^m}(x_{ni}) | m \in \mathbb{Z}, n = 1, 2, \cdots, 0 \leq i < k_n\}$. Then it is evident that B is invariant under $\alpha^{(\overline{v})}$. Since $qx_{ni} = x_{ni}q$ and q is $\alpha^{(\overline{v})}$-invariant, we have $q\alpha^{(\overline{v})}(x_{ni}) = \alpha^{(\overline{v})}(x_{ni})q$, which implies that $q \in B'$. Since $(x_{nj})_{n=1,2,\cdots,0\leq j<k_n}$ is a quasi-matrix system and

$$\begin{aligned}
&x_{ni}^*x_{ni}q = x_{n0}q, \\
&x_{n+1,0}q + \sum_{i=1}^{k_{n+1}-1} x_{n+1,i}x_{n+1,i}^*q = x_{n0}q,
\end{aligned}$$

it follows that $(x_{nj}q)_{n=1,2,\cdots,0\leq j<k_n}$ is a matrix system and generates a UHF algebra which is isomorphic to D. Since $\alpha^{(\overline{v})}(x_{ni})q = e^{ip_{ni}}x_{ni}q$, it follows that $\{x_{nj}q\}_{n,j}$ generates Bq and $(Bq, (\alpha^{(\overline{v})})^{**}|Bq) \simeq (D, \gamma)$. And since $\|r_m(a_m - \tilde{a_m})r_m\| < 1/m$ and $r_mq = q$ (because $q = qq \geq qr_mq \geq qqq = q$ and $qr_m = r_mq$), it follows that $qAq = Bq$.

Finally we show that $xc(q) = 0$ implies $x = 0$ for $x \in A$. Since $w_i^{(n)*}x_{n0} = 0$ for $i \in X_n$ unless $i = 0$, it follows that

$$\begin{aligned}
q_nx_{n0} &= \sum_{i\in X_n} w_i^{(n)}p_nw_i^{(n)*}x_{n0} \\
&= x_{n0}p_nx_{n0}^*x_{n0} = p_n.
\end{aligned}$$

Thus we have

$$\pi(q_n)\xi_0 = \pi(q_nx_{n0})\xi_0 = \pi(p_n)\xi_0 = \xi_0,$$

which implies $\omega(q_n) = 1$ for each n. Hence it follows that $\omega(q) = 1$, which is equivalent to $q \geq p$. So it suffices to show that $xc(p) = 0$ implies $x = 0$ for $x \in A$. Since $\pi(p)\xi_0 = \xi_0$ and $c(p) \geq p$, we have $\pi(c(p))\xi_0 = \xi_0$. Thus, for any $x, y \in A$, it follows that

$$\pi(x)(\pi(y)\xi_0) = \pi(x)\pi(y)\pi(c(p))\xi_0 = \pi(xc(p))\pi(y)\xi_0 = 0.$$

Note that since π is irreducible, we have ξ_0 is a cyclic vector. Hence it follows that $\pi(x) = 0$, which implies $x = 0$. □

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