Mathematical Analysis on Continuous Measurements in Quantum Mechanics
（量子連続測定の数学的研究）

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Contents

Introduction

Mathematical investigations on continuous quantum measurements are presented.

In Chapter 1, we treat vector states only. This chapter include the following aspect: (i) quantum Zeno effect (QZE) by frequent measurements made by an arbitrary partition of a time interval $[0, t]$ $(t > 0)$; (ii) non-occurrence of QZE for vector states which are not in the domain of the Hamiltonian of the quantum system under consideration; (iii) asymptotic behavior of the survival probability characterizing QZE in the number N of divisions of $[0, t]$; (iv) QZE along a curve in the Hilbert space of state vectors. Chapter 1 is a joint work with Professor Asao Arai [1].

In Chapter 2, mixed states are mainly treated. By carrying out appropriate continuous quantum measurements with a family of projection operators, a unitary channel can be approximated in an arbitrary precision in the trace norm sense. In particular, the quantum Zeno effect is described as an application. In the case of an infinite dimension, although the von Neumann entropy is not necessarily continuous, the difference of the entropies between the states, as mentioned above, can be arbitrarily made small under some conditions. Chapter 2 is based on [2].

Appendix includes fundamental facts which is related to Chapter 2.

Acknowledgments

I would like to thank Professor Asao Arai for valuable comments.

Bibliography

[1] A. Arai and T. Fuda, Some Mathematical Aspects of Quantum Zeno Effect, *Lett. Math. Phys.* **100** (2012), 245–260.

[2] T. Fuda, Convergence Conditions of Mixed States and their von Neumann Entropy in Continuous Quantum Measurements, arXiv:1312.2028.

Chapter 1

Some Mathematical Aspects of Quantum Zeno Effect

1.1 Introduction

The quantum phenomenon in which, by a series of measurements, transitions to states different from the initial state (the state at time zero) are hindered or inhibited, is called the *quantum Zeno effect* (QZE) [5]. It has been reported that the QZE can be experimentally realized (e.g., [1, 2, 3]).

In this paper, we are interested in investigating general mathematical aspects associated with the QZE. To explain new features in the present work, we first review the QZE briefly. So let us consider a quantum system S whose Hamiltonian is given by a self-adjoint operator H on a complex Hilbert space $\mathcal{H}$ (so that $\mathcal{H}$ is a Hilbert space of state vectors of S). We denote the inner product and the norm of $\mathcal{H}$ by $\langle\,\cdot\,,\,\cdot\,\rangle$ (anti-linear in the first variable and linear in the second) and $\|\cdot\|$, respectively. The domain of H is denoted as $D(H)$. In what follows, we deal with vector states only. Hence we call a non-zero vector in $\mathcal{H}$ a state simply. As is well-known, by an axiom of quantum mechanics, for an initial state $\Psi \in \mathcal{H}$ with $\|\Psi\| = 1$, the state at time $t \in \mathbb{R}$ is given by the vector $e^{-itH}\Psi$, provided that no measurement is made in the time interval $[0, t]$, where i is the imaginary unit and we use the physical unit system such that $\hbar := h/2\pi$ (h is the Planck constant) is equal to 1. Hence the probability of finding the initial state Ψ by a measurement at time t is given by $|\langle\Psi, e^{-itH}\Psi\rangle|^2$. This quantity is called the survival probability of the initial state Ψ at time t.

Now, consider a time interval $[0, t]$ with $t > 0$ arbitrarily fixed and suppose that N measurements, spaced equally in time by t/N, are made, where N is an arbitrary natural number. If the state at time jt/N ($j = 1, \cdots, N$) is

$\Phi \in \mathcal{H}$, then the state at time $(j+1)t/N$ is $e^{-i\{(j+1)t/N-jt/N\}H}\Phi = e^{-itH/N}\Phi$, provided that no measurement is made in the time interval $[jt/N, (j+1)t/N]$. Hence the probability that, for all $j = 1, \cdots, N$, the measurement at time jt/N finds the initial state Ψ is given by

$$P_N(\Psi, t) := \left|\left\langle \Psi, e^{-itH/N}\Psi \right\rangle\right|^{2N}. \tag{1.1}$$

We call this quantity the *multi-time survival probability* of the initial state Ψ in the time interval $[0, t]$.

We say that *the QZE occurs with respect to the pair* $(\Psi, [0, t])$ *consisting of the initial state* Ψ *and the time interval* $[0, t]$ *in the sense of equally spaced measurement in time* if $\lim_{N\to\infty} P_N(\Psi, t) = 1$. The occurrence of the QZE of this type physically means that, for all sufficiently large N, the successive measurements for the quantum system S at the times $t/N, 2t/N, 3t/N, \cdots, Nt/N$ tend to maintain the initial state Ψ with probability ≈ 1, in other words, they tend to hinder transitions to states different from the initial state Ψ with probability ≈ 1.

Heuristically the occurrence of the QZE can be shown as follows (see, e.g., [4]). Using the formal expansion

$$e^{-itH} = I - itH - \frac{t^2}{2}H^2 + O(t^3) \quad (t \to 0),$$

one infers that, for all $\Psi \in \cap_{n=1}^{\infty} D(H^n)$,

$$\left|\left\langle \Psi, e^{-isH}\Psi \right\rangle\right|^2 = 1 - (\Delta H)_\Psi^2 s^2 + O(s^4) \quad (s \to 0), \tag{1.2}$$

where

$$(\Delta H)_\Psi := \|(H - \langle \Psi, H\Psi \rangle)\Psi\| = \sqrt{\|H\Psi\|^2 - \langle \Psi, H\Psi \rangle^2}$$

is the uncertainty of H in the state Ψ (formula (1.2) can be easily made mathematically rigorous if Ψ is an analytic vector of H). Hence, for all sufficiently large N,

$$P_N(\Psi, t) \approx \left[1 - (\Delta H)_\Psi^2 \left(\frac{t}{N}\right)^2\right]^N \approx e^{-(\Delta H)_\Psi^2 t^2/N} \approx 1.$$

In this way the occurrence of the QZE is inferred.

In the present paper, we begin with re-examining the QZE of the type described above in a mathematically rigorous and non-perturbative way in Section 2. We focus our attention on two aspects. One of them is to consider the situation where the N measurements are made in a way *not necessarily spaced equally in time*. This generalization is mathematically natural.

It may be physically meaningful too, because any measurement of the time inevitably has an error. We prove that, under such a situation too, a phenomenon regarded as a generalization of the QZE occurs, provided that the initial state is in $D(H)$ (Theorem 1.2.1). The other aspect is the possibility of non-occurrence of QZE for initial states not in $D(H)$, as suggested by the heuristic derivation of the QZE given above or the proof of Theorem 1.2.1 below. Indeed, there is an example in which the QZE does not occur for an initial state not in $D(H)$ (Example 1.2.4).

In the case of the QZE which occurs through frequent measurements spaced equally in time, it may be interesting to investigate the asymptotic expansion of the multi-time survival probability $P_N(\Psi, t)$ (see (1.1)) in the powers of $1/N$ as $N \to \infty$. This is done in Section 3. For all $\Psi \in D(H)$ and $t > 0$, we derive the asymptotic expansion of $P_N(\Psi, t)$ up to order $1/N$.

In Section 4, we consider measurements of states along a curve $\Psi(\cdot) : [0, t] \to \mathcal{H}$, a strongly continuous mapping from $[0, t]$ to $\mathcal{H}$. This is a generalization of the situation considered in Section 2, because the constant mapping : $[0, t] \ni \lambda \mapsto \Psi(\lambda) := \Psi$ can be regarded as a special case of the curve. We prove that, for every partition $\{t_0, t_1, \cdots, t_N\}$ of $[0, t]$ with $0 = t_0 < t_1 < \cdots < t_N = t$, the probability of finding the state $\Psi(t_k)$ at the time t_k $(k = 1, \cdots, N)$ in the successive measurements at the times $t_1, \cdots, t_N$ tends to 1 as $N \to \infty$ (Theorem 1.4.2). Physically this means that very frequent measurements made successively along a curve prescribed in advance change the initial state $\Psi(0)$ to the final state $\Psi(t)$ with probability ≈ 1.

In the last section, as an application of Theorem 1.4.2, we show that, for every pair (Ψ, Φ) of states in $\mathcal{H}$ with $\|\Psi\| = \|\Phi\| = 1$, there exists a curve in $\mathcal{H}$ connecting Ψ and Φ such that, through very frequent measurements at successive times given by a partition of the curve, Ψ can be transformed to Φ with probability ≈ 1. This is a refined version (in a sense) of von Neumann's discussion on a possible transformation, induced by frequent measurements, between arbitrary two states [6, Chapter 5], although the present case is restricted to vector states.

1.2 QZE for an Arbitrary Partition of Time Interval

Let $\Delta : t_0, t_1, \cdots, t_N$ $(t_j \in [0, t], j = 0, \cdots, N)$ be an arbitrary partition of the interval $[0, t]$:

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = t.$$

We set

$$\Delta_k := t_k - t_{k-1}, \quad (k = 1, \cdots, N), \qquad |\Delta| := \max_{1\le k\le N} \Delta_k.$$

Let H be a self-adjoint operator on $\mathcal{H}$. Then, for each unit vector $\Psi \in \mathcal{H}$, we define a number

$$P_\Delta(\Psi, t) := \prod_{k=1}^{N} \left|\left\langle \Psi, e^{-i\Delta_k H}\Psi\right\rangle\right|^2.$$

In the context of quantum mechanics where H represents the Hamiltonian of a quantum system, $P_\Delta(\Psi, t)$ is interpreted as the probability that, in the successive measurements at time $t_1, \cdots, t_N$ (measurements not necessarily spaced equally in time), the initial state Ψ is found.

Theorem 1.2.1 *For all $\Psi \in D(H)$ with $\|\Psi\| = 1$,*

$$\lim_{|\Delta|\to 0} P_\Delta(\Psi, t) = 1. \tag{1.3}$$

To prove this theorem, we need two lemmas.

Lemma 1.2.2

$$\lim_{|\Delta|\to 0} \sum_{k=1}^{N} \Delta_k^2 = 0. \tag{1.4}$$

Proof. By direct computations, we have

$$\sum_{k=1}^{N} \Delta_k^2 = t^2 - 2S_\Delta$$

with

$$S_\Delta := t_1(t_2 - t_1) + t_2(t_3 - t_2) + \cdots + t_{N-1}(t_N - t_{N-1}).$$

Note that

$$\lim_{|\Delta|\to 0} S_\Delta = \int_0^t x dx = \frac{t^2}{2}.$$

Hence (1.4) follows. ∎

Lemma 1.2.3 *For each $s \in \mathbb{R}$ and all $\Psi \in D(H)$ with $\|\Psi\| = 1$,*

$$\left|\left\langle \Psi, e^{-isH}\Psi\right\rangle\right|^2 \ge 1 - s^2\|H\Psi\|^2. \tag{1.5}$$

Proof. Putting

$$\alpha := \left\langle \Psi, (e^{-isH} - 1)\Psi \right\rangle,$$

we have

$$\left\langle \Psi, e^{-isH}\Psi \right\rangle = 1 + \alpha.$$

Hence

$$\begin{aligned} \left|\left\langle \Psi, e^{-isH}\Psi \right\rangle\right|^2 &\geq 1 + \alpha + \alpha^* \\ &= 1 + \left\langle \Psi, (e^{-isH} + e^{isH} - 2)\Psi \right\rangle \\ &= 1 - 2\beta \end{aligned}$$

with

$$\beta := \langle \Psi, (1 - \cos(sH))\Psi \rangle = \int_{\mathbb{R}} (1 - \cos(s\lambda)) d\|E_H(\lambda)\Psi\|^2, \tag{1.6}$$

where $E_H(\cdot)$ is the spectral measure of H. One has

$$0 \leq 1 - \cos x \leq \frac{x^2}{2}, \quad \forall x \in \mathbb{R}. \tag{1.7}$$

Hence

$$0 \leq \beta \leq \frac{s^2}{2} \int_{\mathbb{R}} \lambda^2 d\|E_H(\lambda)\Psi\|^2 = \frac{s^2}{2}\|H\Psi\|^2. \tag{1.8}$$

Thus (1.5) follows. ∎

Proof of Theorem 1.2.1

By the Schwarz inequality and the unitarity of e^{-isH} ($\forall s \in \mathbb{R}$), we have $\left|\left\langle \Psi, e^{-i\Delta_k H}\Psi \right\rangle\right| \leq 1$. Hence $P_\Delta(\Psi, t) \leq 1$, which implies that

$$\limsup_{|\Delta|\to 0} P_\Delta(\Psi, t) \leq 1. \tag{1.9}$$

By Lemma 1.2.3, we have

$$\left|\left\langle \Psi, e^{-i\Delta_k H}\Psi \right\rangle\right|^2 \geq 1 - \Delta_k^2\|H\Psi\|^2.$$

For each $a > 1$, we have

$$1 - x \geq e^{-ax}, \quad \forall x \in \left[0, \frac{\log a}{a}\right]. \tag{1.10}$$

Hence, taking $|\Delta|$ such that $|\Delta|^2\|H\Psi\|^2 \leq \log a/a$, we have

$$P_\Delta(\Psi,t) \geq e^{-a\sum_{k=1}^{N}\Delta_k^2\|H\Psi\|^2}.$$

By this estimate and Lemma 1.2.2, we obtain

$$\liminf_{|\Delta|\to 0} P_\Delta(\Psi,t) \geq 1,$$

which, combined with (1.9), gives (1.3). ∎

We remark that the condition $\Psi \in D(H)$ in Theorem 1.2.1 is optimal. A counter example is given as follows.

Example 1.2.4 We consider the case where $\mathcal{H} = L^2(\mathbb{R})$ and H is the Hamiltonian H_0 of a free quantum particle with mass $m > 0$ moving in the one-dimensional space $\mathbb{R}$, i.e., $H_0 := p^2/2m$, $p := -iD_x$ with D_x being the generalized differential operator on $L^2(\mathbb{R})$ (in the variable $x \in \mathbb{R}$). Let $c > 0$ be a constant and $\psi_0 \in L^2(\mathbb{R})$ be such that its L^2-Fourier transform $\hat{\psi}_0$ takes the form

$$\hat{\psi}_0(k) = \sqrt{\frac{2c}{\pi}}\sqrt{\frac{|k|}{k^4+c^2}}, \quad k \in \mathbb{R}.$$

It is easy to see that $\|\psi_0\| = 1$ and $\psi_0 \notin D(H_0)$. Moreover, we have for all $s \in \mathbb{R}$

$$\left\langle \psi_0, e^{-isH_0}\psi_0 \right\rangle = \frac{4c}{\pi}\int_0^\infty \frac{ke^{-isk^2/2m}}{k^4+c^2}dk = e^{-|s|c/2m}.$$

Hence, for all $t > 0$

$$\prod_{k=1}^{N}\left|\left\langle \psi_0, e^{-i\Delta_k H_0}\psi_0 \right\rangle\right|^2 = e^{-tc/m}.$$

Therefore

$$\lim_{|\Delta|\to 0}\prod_{k=1}^{N}\left|\left\langle \psi_0, e^{-i\Delta_k H_0}\psi_0 \right\rangle\right|^2 = e^{-tc/m} < 1.$$

Thus, in this case, Theorem 1.2.1 does not hold, physically meaning that the quantum Zeno effect does not occur. We also note that, for every $\varepsilon > 0$,

$$\lim_{|\Delta|\to 0}\prod_{k=1}^{N}\left|\left\langle \psi_0, e^{-i\Delta_k H_0}\psi_0 \right\rangle\right|^2 < \varepsilon$$

if $c > -(m/t)\log\varepsilon$.

1.3 Asymptotics of the Multi-time Survival Probability $P_N(\Psi, t)$ as $N \to \infty$

In the case of the QZE caused by frequent measurements spaced equally in time, the asymptotic behavior (in $1/N$) of the multi-time survival probability $P_N(\Psi, t)$ as $N \to \infty$ is interesting. It may be natural to infer that the asymptotic expansion of $P_N(\Psi, t)$ in $1/N$ has the following form:

$$P_N(\Psi, t) = 1 + c_1(\Psi, t)\frac{1}{N} + c_2(\Psi, t)\frac{1}{N^2} + \cdots \quad (N \to \infty) \tag{1.11}$$

with $c_n(\Psi, t) \in \mathbb{R}$ ($n = 1, 2, \cdots$) being constants independent of N, expecting that each $c_n(\Psi, t)$ may have a physical meaning. In this section, we are concerned with this aspect and prove the following result:

Theorem 1.3.1 *Let $t > 0$. Then, for all $\Psi \in D(H)$ with $\|\Psi\| = 1$,*

$$P_N(\Psi, t) = 1 - t^2(\Delta H)^2_\Psi \frac{1}{N} + o\left(\frac{1}{N}\right), \quad (N \to \infty). \tag{1.12}$$

Remark 1.3.2 The asymptotic formula (1.12) is only up to the first order $1/N$. But we conjecture that it is possible to find higher order asymptotics in $1/N$. We consider this aspect in a separate paper.

To prove Theorem 1.3.1, we need two lemmas:

Lemma 1.3.3 *For all $s \in \mathbb{R}$ and $\Psi \in D(H)$,*

$$\lim_{N\to\infty} N^2 \left\langle \Psi, \left(1 - \cos\frac{sH}{N}\right)\Psi \right\rangle = \frac{1}{2}s^2\|H\Psi\|^2, \tag{1.13}$$

$$\lim_{N\to\infty} N \left\langle \Psi, \sin\frac{sH}{N}\Psi \right\rangle = s\left\langle \Psi, H\Psi \right\rangle. \tag{1.14}$$

Proof. We have

$$N^2 \left\langle \Psi, \left(1 - \cos\frac{sH}{N}\right)\Psi \right\rangle = \int_{\mathbb{R}} N^2 \left(1 - \cos\frac{s\lambda}{N}\right) d\left\langle \Psi, E_H(\lambda)\Psi \right\rangle.$$

It is easy to see that

$$\lim_{N\to\infty} N^2 \left(1 - \cos\frac{s\lambda}{N}\right) = \frac{1}{2}s^2\lambda^2.$$

By (1.7), we have

$$0 \leq N^2 \left(1 - \cos \frac{s\lambda}{N}\right) \leq \frac{1}{2} s^2 \lambda^2.$$

By functional calculus, we have

$$\int_{\mathbb{R}} \frac{1}{2} s^2 \lambda^2 d \langle \Psi, E_H(\lambda)\Psi \rangle = \frac{1}{2} s^2 \|H\Psi\|^2 < \infty.$$

Hence, by the Lebesgue dominated convergence theorem, we obtain (1.13).

We next prove (1.14). We have

$$N \left\langle \Psi, \sin \frac{sH}{N} \Psi \right\rangle = \int_{\mathbb{R}} N \sin \frac{s\lambda}{N} d \langle \Psi, E_H(\lambda)\Psi \rangle .$$

By the elementary inequality $|\sin x| \leq |x|, \forall x \in \mathbb{R}$, we obtain

$$\left| N \sin \frac{s\lambda}{N} \right| \leq |s\lambda|.$$

By the Schwarz inequality, we have

$$\int_{\mathbb{R}} |\lambda| d \langle \Psi, E_H(\lambda)\Psi \rangle \leq \|\Psi\| \left(\int_{\mathbb{R}} \lambda^2 d \langle \Psi, E_H(\lambda)\Psi \rangle \right)^{\frac{1}{2}} = \|\Psi\| \cdot \|H\Psi\| < \infty.$$

Moreover,

$$\lim_{n \to \infty} N \sin \frac{s\lambda}{N} = s\lambda.$$

Thus, by the Lebesgue dominated convergence theorem, we obtain (1.14). ∎

Lemma 1.3.4 *For all $n \in \mathbb{N}$,*

$$(1-x)^n \geq 1 - nx, \qquad 0 \leq \forall x \leq 1, \tag{1.15}$$

$$(1-x)^n \leq 1 - nx + (nx)^2 e^{nx}, \quad \forall x \geq 0. \tag{1.16}$$

Proof. Inequality (1.15) is elementary. As for (1.16), we proceed as follows: For all $x \geq 0$ and $n \geq 2$,

$$\begin{aligned} (1-x)^n &= 1 - nx + \sum_{k=2}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} (-1)^k x^k \\ &\leq 1 - nx + \sum_{k=2}^{n} \frac{n^k}{k!} x^k \leq 1 - nx + (nx)^2 \sum_{k=0}^{n-2} \frac{(nx)^k}{k!} \\ &\leq 1 - nx + (nx)^2 e^{nx}. \end{aligned}$$

Thus (1.16) holds. ∎

Proof of Theorem 1.3.1

Let

$$a_N := \left\langle \Psi, \left(1 - \cos \frac{tH}{N}\right) \Psi \right\rangle, \quad b_N := \left\langle \Psi, \sin \frac{tH}{N} \Psi \right\rangle.$$

Then $a_N, b_N \in \mathbb{R}$ and $\left\langle \Psi, e^{-i\frac{t}{N}H}\Psi \right\rangle = 1 - a_N - ib_N$. Hence

$$\left| \left\langle \Psi, e^{-i\frac{t}{N}H}\Psi \right\rangle \right|^2 = (1-a_N)^2 + b_N^2 = 1 - \frac{q_N}{N^2},$$

where

$$q_N := 2N^2 a_N - (Nb_N)^2 - \frac{(N^2 a_N)^2}{N^2}.$$

Therefore we have

$$P_N(\Psi, t) = \left(1 - \frac{q_N}{N^2}\right)^N.$$

By Lemma 1.3.3, we have

$$\lim_{N\to\infty} N^2 a_N = \frac{1}{2} t^2 \|H\Psi\|^2, \quad \lim_{N\to\infty} N b_N = t \langle \Psi, H\Psi \rangle.$$

Hence

$$\lim_{N\to\infty} q_N = t^2 \|H\Psi\|^2 - t^2 |\langle \Psi, H\Psi\rangle|^2 = t^2 (\Delta H)_\Psi^2.$$

Moreover, it follows from Lemma 1.3.4 that, if $q_N \le N$, then

$$1 - \frac{q_N}{N} \le \left(1 - \frac{q_N}{N^2}\right)^N \le 1 - \frac{q_N}{N} + \left(\frac{q_N}{N}\right)^2 e^{\frac{q_N}{N}},$$

which implies that

$$q_N - \frac{q_N^2}{N} e^{\frac{q_N}{N}} \le N\left(1 - P_N(\Psi, t)\right) \le q_N.$$

Hence

$$\lim_{N\to\infty} N\left(1 - P_N(\Psi, t)\right) = \lim_{N\to\infty} q_N = t^2 (\Delta H)_\Psi^2. \tag{1.17}$$

Putting

$$c_N := t^2 (\Delta H)_\Psi^2 \frac{1}{N} - \left(1 - P_N(\Psi, t)\right),$$

we have

$$P_N(\Psi, t) = 1 - t^2 (\Delta H)_\Psi^2 \frac{1}{N} + c_N.$$

By (1.17), we have $\lim_{N\to\infty} N c_N = 0$, which means that $c_N = o(1/N)\ (N \to \infty)$. Thus (1.12) holds. ∎

1.4 General Mathematical Structure Behind QZE

In this section, as a generalization of the QZE considered in Section 1.2, we consider the physical situation where measurements for states are made along a curve in the Hilbert space $\mathcal{H}$.

Let $\Psi(\cdot) : [0,t] \to \mathcal{H}$ (a mapping from $[0,t]$ to $\mathcal{H}$) such that $\|\Psi(\lambda)\| = 1$, $\forall \lambda \in [0,t]$ and consider

$$P_\Delta(\Psi(\cdot),t) := \prod_{k=1}^{N} \left| \left\langle \Psi(t_k), e^{-i\Delta_k H}\Psi(t_{k-1}) \right\rangle \right|^2 . \tag{1.18}$$

This quantity is physically interpreted as the probability that, in the successive measurement at time $t_1, \cdots, t_N$, the state $\Psi(t_k)$ is found at time t_k $(k = 1, \cdots, N)$.

Remark 1.4.1 For a unit vector $\Psi \in \mathcal{H}$, one can consider a constant mapping $\Psi_{\text{const}}(\cdot) : [0,t] \to \mathcal{H}$ defined by $\Psi_{\text{const}}(\lambda) := \Psi$, $\forall \lambda \in [0,t]$. In this case, we have $P_\Delta(\Psi_{\text{const}}(\cdot),t) = P_\Delta(\Psi,t)$, i.e., the case considered in Section 1.2. Thus $P_\Delta(\Psi(\cdot),t)$ is a generalization of $P_\Delta(\Psi,t)$.

Theorem 1.4.2 *Let $\Psi(\cdot) : [0,t] \to \mathcal{H}$ such that, for all $\lambda \in [0,t]$, $\Psi(\lambda) \in D(H)$ and $\|\Psi(\lambda)\| = 1$. Assume the following:*

$$\xi := \sup_{0 \le \lambda \le t} \|H\Psi(\lambda)\| < \infty, \tag{1.19}$$

$$\eta := \sup_{\substack{\lambda,\nu \in [0,t] \\ \lambda \ne \nu}} \frac{\|\Psi(\lambda) - \Psi(\nu)\|}{|\lambda - \nu|} < \infty, \tag{1.20}$$

$$\lim_{|\Delta| \to 0} \sum_{k=1}^{N} \mathrm{Re}\, \langle \Psi(t_k) - \Psi(t_{k-1}), \Psi(t_{k-1}) \rangle = 0, \tag{1.21}$$

where, for a complex number z, $\mathrm{Re}\, z$ *denotes its real part. Then*

$$\lim_{|\Delta| \to 0} P_\Delta(\Psi(\cdot),t) = 1 \tag{1.22}$$

Remark 1.4.3 Condition (1.20) implies that $\|\Psi(\lambda) - \Psi(\nu)\| \le \eta|\lambda - \mu|, \forall \lambda, \mu \in [0,t]$ (Lipschitz continuity). In particular, $\Psi(\cdot)$ is strongly continuous, so that the mapping $\Psi(\cdot) : [0,t] \to \mathcal{H}$ is a curve in $\mathcal{H}$.

Proof. By the Schwarz inequality and the unitarity of e^{-isH} ($\forall s \in \mathbb{R}$), we have

$$|\left\langle \Psi(t_k), e^{-i\Delta_k H}\Psi(t_{k-1})\right\rangle|^2 \leq 1. \tag{1.23}$$

Hence

$$P_\Delta(\Psi(\cdot), t) \leq 1. \tag{1.24}$$

For $k = 1, \cdots, N$, we set

$$\begin{aligned}
a_k &:= \left\langle \Psi(t_{k-1}), (e^{-i\Delta_k H} - 1)\Psi(t_{k-1})\right\rangle, \\
b_k &:= \left\langle \Psi(t_k) - \Psi(t_{k-1}), (e^{-i\Delta_k H} - 1)\Psi(t_{k-1})\right\rangle, \\
c_k &:= \langle \Psi(t_k) - \Psi(t_{k-1}), \Psi(t_{k-1})\rangle .
\end{aligned}$$

Then we have

$$\begin{aligned}
|\left\langle \Psi(t_k), e^{-i\Delta_k H}\Psi(t_{k-1})\right\rangle|^2 &= |1 + a_k + b_k + c_k|^2 \\
&\geq 1 + 2\mathrm{Re}\, a_k + 2\mathrm{Re}\, b_k + 2\mathrm{Re}\, c_k.
\end{aligned} \tag{1.25}$$

By (1.6), (1.8) and (1.19), we have

$$1 + 2\mathrm{Re}\, a_k \geq 1 - \Delta_k^2 \|H\Psi(t_{k-1})\|^2 \geq 1 - \Delta_k^2 \xi^2 \tag{1.26}$$

By the Schwarz inequality, we have

$$|\mathrm{Re}\, b_k| \leq |b_k| \leq \|\Psi(t_k) - \Psi(t_{k-1})\| \cdot \|(e^{-i\Delta_k H} - 1)\Psi(t_{k-1})\|.$$

Assumption (1.20) implies that

$$\|\Psi(t_k) - \Psi(t_{k-1})\| \leq \Delta_k \eta.$$

On the other hand, we have

$$\|(e^{-i\Delta_k H} - 1)\Psi(t_{k-1})\|^2 = \int_{\mathbb{R}} |e^{-i\Delta_k \lambda} - 1|^2 d\|E_H(\lambda)\Psi(t_{k-1})\|^2.$$

Using the elementary inequality

$$|e^{-ix} - 1|^2 \leq x^2, \quad \forall x \in \mathbb{R},$$

we obtain

$$\begin{aligned}
\int_{\mathbb{R}} |e^{-i\Delta_k \lambda} - 1|^2 d\|E_H(\lambda)\Psi(t_{k-1})\|^2 &\leq \int_{\mathbb{R}} \Delta_k^2 \lambda^2 d\|E_H(\lambda)\Psi(t_{k-1})\|^2 \\
&= \Delta_k^2 \|H\Psi(t_{k-1})\|^2.
\end{aligned}$$

Hence

$$\|(e^{-i\Delta_k H} - 1)\Psi(t_{k-1})\| \le \Delta_k \|H\Psi(t_{k-1})\| \le \Delta_k \xi$$

Therefore

$$|\mathrm{Re}\, b_k| \le \xi\eta\Delta_k^2.$$

Thus we obtain

$$2\mathrm{Re}\, b_k \ge -2\xi\eta\Delta_k^2 \tag{1.27}$$

By estimates (1.26) and (1.27), we have

$$1 + 2\mathrm{Re}\, a_k + 2\mathrm{Re}\, b_k + 2\mathrm{Re}\, c_k \ge 1 - \left\{(\xi^2 + 2\xi\eta)\Delta_k^2 - 2\mathrm{Re}\, c_k\right\}.$$

Note that, by (1.23) and (1.25), $1 \ge 1 + 2\mathrm{Re}\, a_k + 2\mathrm{Re}\, b_k + 2\mathrm{Re}\, c_k$. Hence $(\xi^2 + 2\xi\eta)\Delta_k^2 - 2\mathrm{Re}\, c_k \ge 0$. We also have

$$|\mathrm{Re}\, c_k| \le |c_k| \le \|\Psi(t_k) - \Psi(t_{k-1})\| \le \eta\Delta_k$$

Hence

$$0 \le (\xi^2 + 2\xi\eta)\Delta_k^2 - 2\mathrm{Re}\, c_k \le (\xi^2 + 2\xi\eta)\Delta_k^2 + 2\eta\Delta_k \le (\xi^2 + 2\xi\eta)|\Delta|^2 + 2\eta|\Delta|. \tag{1.28}$$

Let $a > 1$ be a constant and take $|\Delta|$ such that

$$(\xi^2 + 2\xi\eta)|\Delta|^2 + 2\eta|\Delta| \le \frac{\log a}{a}.$$

Then, by (1.28), we have for $k = 1, \cdots, N$

$$0 \le (\xi^2 + 2\xi\eta)\Delta_k^2 - 2\mathrm{Re}\, c_k \le \frac{\log a}{a}$$

Hence, by (1.10), we obtain

$$1 - \left\{(\xi^2 + 2\xi\eta)\Delta_k^2 - 2\mathrm{Re}\, c_k\right\} \ge \exp[-a\left\{(\xi^2 + 2\xi\eta)\Delta_k^2 - 2\mathrm{Re}\, c_k\right\}].$$

Therefore

$$\begin{aligned} P_\Delta(\Psi(\cdot), t) &\ge \prod_{k=1}^{N} \exp[-a\left\{(\xi^2 + 2\xi\eta)\Delta_k^2 - 2\mathrm{Re}\, c_k\right\}] \\ &= \exp\left[-a\left\{(\xi^2 + 2\xi\eta)\sum_{k=1}^{N}\Delta_k^2 - 2\sum_{k=1}^{N}\mathrm{Re}\, c_k\right\}\right] \end{aligned}$$

By Lemma 1.2.2 and (1.21),

$$\lim_{|\Delta|\to 0}\left\{(\xi^2 + 2\xi\eta)\sum_{k=1}^{N}\Delta_k^2 - 2\sum_{k=1}^{N}\mathrm{Re}\, c_k\right\} = 0$$

Thus $\liminf_{|\Delta|\to 0} P_\Delta(\Psi(\cdot), t) \ge 1$, which, combined with (1.24), yields (1.22). ∎

Corollary 1.4.4 *Let $\Psi(\cdot) : [0,t] \to \mathcal{H}$ be a strongly differentiable mapping from $[0,t]$ to $\mathcal{H}$ such that the following conditions hold:*

(i) *For all $\lambda \in [0,t]$, $\Psi(\lambda) \in D(H)$ and $\|\Psi(\lambda)\| = 1$.*

(ii) *(1.19) holds and*

$$\delta := \sup_{0 \le \lambda \le t} \|\Psi'(\lambda)\| < \infty, \tag{1.29}$$

where $\Psi'(\cdot)$ denotes the strong derivative of $\Psi(\cdot)$.

Then (1.22) holds.

Proof. By Theorem 1.4.2, it is sufficient to prove that (1.20) and (1.21) hold. By the strong differentiability, we have for all $\lambda, \nu \in [0,t]$

$$\Psi(\lambda) - \Psi(\nu) = \int_\nu^\lambda \Psi'(s)ds, \tag{1.30}$$

where the integral is taken in the sense of Bochner integral. Hence

$$\|\Psi(\lambda) - \Psi(\nu)\| \le \left| \int_\nu^\lambda \|\Psi'(s)\| ds \right| \le \delta |\lambda - \nu|.$$

Thus (1.20) holds.

By (1.30), we have

$$\Psi(t_k) - \Psi(t_{k-1}) = \int_{t_{k-1}}^{t_k} \Psi'(\lambda) d\lambda, \quad k = 1, \cdots, N.$$

Let $\chi_{(t_{k-1},t_k]}$ be the characteristic function of the interval $(t_{k-1}, t_k]$. Then

$$\begin{aligned}
\sum_{k=1}^N \mathrm{Re}\, \langle \Psi(t_k) - \Psi(t_{k-1}), \Psi(t_{k-1}) \rangle &= \mathrm{Re} \sum_{k=1}^N \left\langle \int_{t_{k-1}}^{t_k} \Psi'(\lambda) d\lambda, \Psi(t_{k-1}) \right\rangle \\
&= \mathrm{Re} \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \langle \Psi'(\lambda), \Psi(t_{k-1}) \rangle \, d\lambda \\
&= \mathrm{Re} \sum_{k=1}^N \int_0^t \chi_{(t_{k-1},t_k]}(\lambda) \langle \Psi'(\lambda), \Psi(t_{k-1}) \rangle \, d\lambda \\
&= \mathrm{Re} \int_0^t \left\langle \Psi'(\lambda), \sum_{k=1}^N \chi_{(t_{k-1},t_k]}(\lambda) \Psi(t_{k-1}) \right\rangle d\lambda
\end{aligned}$$

For all $\lambda \in (0, t]$, we have

$$\left| \left\langle \Psi'(\lambda), \sum_{k=1}^{N} \chi_{(t_{k-1}, t_k]}(\lambda)\Psi(t_{k-1}) \right\rangle \right| \leq \delta \sum_{k=1}^{N} \chi_{(t_{k-1}, t_k]}(\lambda) = \delta,$$

$$\lim_{|\Delta| \to 0} \left\langle \Psi'(\lambda), \sum_{k=1}^{N} \chi_{(t_{k-1}, t_k]}(\lambda)\Psi(t_{k-1}) \right\rangle = \langle \Psi'(\lambda), \Psi(\lambda) \rangle .$$

Hence, by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
& \lim_{|\Delta| \to 0} \mathrm{Re} \int_0^t \left\langle \Psi'(\lambda), \sum_{k=1}^{N} \chi_{(t_{k-1}, t_k]}(\lambda)\Psi(t_{k-1}) \right\rangle d\lambda \\
= & \ \mathrm{Re} \int_0^t \langle \Psi'(\lambda), \Psi(\lambda) \rangle \, d\lambda = \frac{1}{2} \int_0^t \frac{d}{d\lambda} \|\Psi(\lambda)\|^2 d\lambda \\
= & \ \frac{1}{2}(\|\Psi(t)\|^2 - \|\Psi(0)\|^2) = 0.
\end{aligned}$$

Thus (1.21) holds. ∎

Example 1.4.5 Let A be a self-adjoint operator on $\mathcal{H}$ and Ψ_0 be a vector in $\mathcal{H}$ satisfying the following conditions:

$$\Psi_0 \in D(A) \cap \bigcap_{0 \leq \lambda \leq t} D(He^{-i\lambda A}), \quad \sup_{0 \leq \lambda \leq t} \|He^{-i\lambda A}\Psi_0\| < \infty, \quad \|\Psi_0\| = 1.$$

Then one can define a mapping $\Psi(\cdot) : [0, t] \to \mathcal{H}$ by

$$\Psi(\lambda) := e^{-i\lambda A}\Psi_0, \quad \lambda \in [0, t].$$

It is obvious that the mapping $\Psi(\cdot)$ satisfies condition (i) in Corollary 1.4.4. Moreover, $\Psi(\cdot)$ is strongly differentiable on $[0, t]$ and

$$\|\Psi'(\lambda)\|^2 \ = \ \|Ae^{-i\lambda A}\Psi_0\|^2 = \|A\Psi_0\|^2,$$

so that $\Psi(\cdot)$ satisfies condition (ii) in Corollary 1.4.4 too. Thus, for this $\Psi(\cdot)$, (1.22) holds.

1.5 Transition Between Arbitrary Two States by Measurements

We fix two unit vectors Ψ and Φ in $D(H)$ arbitrarily. Then one can define a strongly differentiable mapping $\Psi(\cdot) : [0, t] \to \mathcal{H}$ connecting Ψ and Φ as follows.

(1) The case where Ψ and Φ are linearly dependent

In this case, there exists a constant $\alpha \in [0, 2\pi)$ such that $\Phi = e^{i\alpha}\Psi$. Then, defining $\Psi(\cdot) : [0, t] \to \mathcal{H}$ by

$$\Psi(\lambda) := e^{i\frac{\alpha\lambda}{t}}\Psi, \quad \lambda \in [0, t], \tag{1.31}$$

we see that $\Psi(\cdot)$ is strongly differentiable on $[0, t]$ with $\Psi(0) = \Psi$ and $\Psi(t) = \Phi$.

(2) The case where Ψ and Φ are linearly independent

In this case, let

$$\Xi := \frac{\Phi - \langle\Phi, \Psi\rangle\Psi}{\|\Phi - \langle\Phi, \Psi\rangle\Psi\|}. \tag{1.32}$$

Then $\{\Psi, \Xi\}$ is an orthonormal system in $\mathcal{H}$. It follows that there exist constants $\alpha, \beta, \gamma \in [0, 2\pi)$ such that

$$\Phi = (\cos\alpha)e^{i\beta}\Psi + (\sin\alpha)e^{i\gamma}\Xi.$$

Using this fact, we define $\Psi(\cdot) : [0, t] \to \mathcal{H}$ by

$$\Psi(\lambda) := \left(\cos\frac{\alpha\lambda}{t}\right)e^{i\frac{\beta\lambda}{t}}\Psi + \left(\sin\frac{\alpha\lambda}{t}\right)e^{i\frac{\gamma\lambda}{t}}\Xi, \quad \lambda \in [0, t]. \tag{1.33}$$

It is easy to see that the mapping $\Psi(\cdot)$ is strongly differentiable on $[0, t]$ with $\Psi(0) = \Psi$ and $\Psi(t) = \Phi$.

Proposition 1.5.1 *Let $\Psi(\cdot)$ be defined by (1.31) or (1.33). Then $\Psi(\cdot)$ satisfies all the assumptions of Corollary 1.4.4 with $\Psi(0) = \Psi$ and $\Psi(t) = \Phi$.*

Proof. We see from the definition of $\Psi(\cdot)$ that, for all $\lambda \in [0, t]$, $\Psi(\lambda) \in D(H)$ and $\|\Psi(\lambda)\| = 1$, and $\Psi(\cdot)$ is strongly differentiable on $[0, t]$ with $\Psi(0) = \Psi$ and $\Psi(t) = \Phi$.

We first consider the case where Ψ and Φ are linearly dependent. In this case, we have

$$\|H\Psi(\lambda)\| = \|H\Psi\|,$$
$$\|\Psi'(\lambda)\| = \frac{\alpha}{t}.$$

Hence condition (ii) in Corollary 1.4.4 are satisfied.

Next, let Ψ and Φ be linearly independent. Then

$$\|H\Psi(\lambda)\| \leq \|H\Psi\| + \|H\Xi\|,$$
$$\|\Psi'(\lambda)\| = \left\| \left(-\frac{\alpha}{t}\sin\frac{\alpha\lambda}{t} + i\frac{\beta}{t}\cos\frac{\alpha\lambda}{t}\right) e^{i\frac{\beta\lambda}{t}}\Psi + \left(\frac{\alpha}{t}\cos\frac{\alpha\lambda}{t} + i\frac{\gamma}{t}\sin\frac{\alpha\lambda}{t}\right) e^{i\frac{\gamma\lambda}{t}}\Xi \right\|$$
$$\leq \frac{2\alpha+\beta+\gamma}{t}.$$

Hence condition (ii) in Corollary 1.4.4 are satisfied. ∎

Corollary 1.4.4 and Proposition 1.5.1 immediately lead one to the following fact:

Corollary 1.5.2 *For the mapping* $\Psi(\cdot)$ *defined by (1.31) or (1.33),* $\lim_{|\Delta|\to 0} P_\Delta(\Psi(\cdot), t) = 1$.

Corollary 1.5.2 may be interpreted as follows: For every pair (Ψ, Φ) of states in $\mathcal{H}$ with $\|\Psi\| = \|\Phi\| = 1$, there exists a curve in $\mathcal{H}$ connecting Ψ and Φ such that, through very frequent measurements at successive times given by a partition of this curve, the state Ψ can be transformed to Φ with probability ≈ 1.

Acknowledgments

T. Fuda would like to thank S. Futakuchi, T. Muroi and M. Nagasaka for valuable comments. This work was supported by the Grant-In-Aid 21540206 for scientific research from Japan Society for the Promotion of Science (JSPS).

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Chapter 2

Convergence Conditions of Mixed States and their von Neumann Entropy in Continuous Quantum Measurements

2.1 Introduction

The quantum Zeno effect (QZE) is a quantum effect which was shown by Misra and Sudarshan in [5]. This effect demonstrates that, in quantum mechanics, continuous measurements can freeze a state. Of course, this effect is peculiar to quantum mechanics. Such an effect is not observed in classical mechanics. The QZE has been extensively investigated by many researchers since its discovery.

Recently, some general mathematical aspects of quantum Zeno effect were investigated in [2]. In particular, continuous measurements of a state along a certain curve in a Hilbert space were considered. Roughly speaking, continuous measurements made along a curve prescribed in advance change the initial state to the final state with probability 1. This fact includes the QZE as a special case. However, in the paper [2], it is assumed that states under consideration are vector states.

In this paper, we show that a result similar to one in [2] holds with respect to mixed states too. By considering a mixed state, its von Neumann entropy can also be considered. In the case where the Hilbert space under consideration is infinite dimensional, the von Neumann entropy is not necessarily continuous with respect to the trace norm. Hence, by continuous measurements, even if the initial state converges to the final state in the trace norm sense, it does not always mean that the von Neumann entropy converges too. Moreover, the set of density operators with finite entropy is a first category [10]. Hence, it is meaningful to investigate convergence conditions of the von Neumann entropy in our continuous measurements.

In Section 2, we begin with defining the "continuous quantum measurements" as a certain type of quantum channel. We use two types of quantum channels and a combination of them. By doing so, a concept of "continuous quantum measurements" are defined clearly. We consider conditions for pointwise convergence and trace norm convergence. We apply obtained results to the QZE.

In Section 3, we consider the von Neumann entropy in infinite dimension. We show that the convergence conditions of the von Neumann entropy in continuous quantum measurement which considered in Section 2. Here, Simon's convergence theorem [4] plays a central role.

2.2 Continuous measurements for mixed states

2.2.1 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space of state vectors of a quantum system $\mathcal{S}$. We denote the inner product and the norm of $\mathcal{H}$ by $\langle\,\cdot\,,\,\cdot\,\rangle$ (anti-linear in the first variable and linear in the second) and $\|\cdot\|$, respectively. Let $d(\leq\infty)$ be the dimension of $\mathcal{H}$. We denote all bounded linear operators, all compact operators, all trace-class operators, all density operators, and all unitary operators on $\mathcal{H}$ by $\mathfrak{B}(\mathcal{H}), \mathfrak{C}(\mathcal{H}), \mathfrak{T}(\mathcal{H}), \mathfrak{S}(\mathcal{H}), and\, \mathfrak{U}(\mathcal{H})$, respectively. A mixed state of $\mathcal{S}$ is represented as an element of $\mathfrak{S}(\mathcal{H})$. We denote the trace norm by $\|\cdot\|_1 := \mathrm{Tr}|\cdot|$. The Hamiltonian of the quantum system $\mathcal{S}$ is given by a self-adjoint operator H which is time independent. The domain of H is denoted as $D(H)$.

Let us consider the following two maps on $\mathfrak{S}(\mathcal{H})$:

(1) (**Unitary channel**)

Let U be a unitary operator on $\mathcal{H}$ and $\mathcal{E}_U$ be a map on $\mathfrak{S}(\mathcal{H})$ which is given by

$$\mathcal{E}_U\rho := U\rho U^*,\ \forall\rho\in\mathfrak{S}(\mathcal{H}).$$

In particular, in the case $U = e^{-itH}$ $(t\in\mathbb{R})$, we denote $\mathcal{E}_{e^{-itH}}$ by $\mathcal{E}_t$.

(2) (**Projection channel**)

Let $\mathfrak{P} := \{P_n\}_n$ be a family of projection operators on $\mathcal{H}$ with $P_m \perp P_n$ $(m \neq n), I = \sum_n P_n$, and $\mathcal{E}_{\mathfrak{P}}$ be a map on $\mathfrak{S}(\mathcal{H})$ which is given by

$$\mathcal{E}_{\mathfrak{P}}\rho := \sum_n P_n \rho P_n, \ \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

Now, consider a state $\rho \in \mathfrak{S}(\mathcal{H})$ fixed and suppose that one of the Schatten decompositions is given by

$$\rho = \sum_{n=1}^{d} \lambda_n |\Psi_n\rangle\langle\Psi_n|, \tag{2.1}$$

where, for all $\Psi, \Phi \in \mathcal{H}$, we denote the operator $\langle\Psi, \cdot\ \rangle\Phi$ by $|\Phi\rangle\langle\Psi|$. In (2.1), we allow $\lambda_n = 0$ to take Ψ_n such that $\{\Psi_n\}_{n=1}^d$ is a complete orthonormal system (CONS). We remark that it is not necessarily $\lambda_n \geq \lambda_{n+1}$ in this representation.

Let us consider a time interval $[0, \tau]$ with $\tau > 0$. For the decomposition (2.1), consider a CONS of $\mathcal{H}$ denoted by $\{\Psi_n(t)\}_{n=1}^d$ which is parametrized by $t \in [0, \tau]$ with $\Psi_n(0) = \Psi_n$ $(1 \leq \forall n \leq d)$. If $n \in \mathbb{N}$ is fixed, then $\Psi_n(\cdot)$ is a map from $[0, \tau]$ to $\mathcal{H}$.

We define

$$\mathfrak{P}(t) := \{|\Psi_n(t)\rangle\langle\Psi_n(t)|\}_{n=1}^d, \quad (t \in [0, \tau]). \tag{2.2}$$

Let $\Delta : t_0, t_1, \cdots, t_N$ $(t_j \in [0, \tau], j = 0, \cdots, N)$ be an arbitrary partition of the interval $[0, \tau]$:

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = \tau.$$

We set

$$\Delta_k := t_k - t_{k-1}, \quad (k = 1, \cdots, N), \qquad |\Delta| := \max_{1 \leq k \leq N} \Delta_k,$$

and define

$$\rho_\Delta(\tau) := \mathcal{E}_{\mathfrak{P}(t_N)} \circ \mathcal{E}_{\Delta_N} \circ \mathcal{E}_{\mathfrak{P}(t_{N-1})} \circ \mathcal{E}_{\Delta_{N-1}} \circ \cdots \circ \mathcal{E}_{\mathfrak{P}(t_1)} \circ \mathcal{E}_{\Delta_1}\rho. \tag{2.3}$$

In the context of quantum mechanics where $\rho_\Delta(\tau)$ is interpreted as the posterior state that, in the successive measurements at time $t_1, \cdots, t_N$ by using the family of projection operators $\mathfrak{P}(t_1), \cdots, \mathfrak{P}(t_N)$, respectively. We remark that $\rho_\Delta(\tau)$ is dependent on the form of decomposition (2.1).

If $\rho_\Delta(\tau)$ converges with respect to $|\Delta| \to 0$ in a certain sense, we call such a measurements of a series "continuous quantum measurements".

By direct computations, we have

$$\rho_\Delta(\tau) = \sum_k \lambda_{\Delta,k} \, |\Psi_k(\tau)\rangle\langle\Psi_k(\tau)| \tag{2.4}$$

with

$$\lambda_{\Delta,k} := \sum_{k_0,\cdots,k_{N-1}} \lambda_{k_0} \prod_{j=1}^{N} \left|\langle \Psi_{k_j}(t_j), e^{-i\Delta_j H} \Psi_{k_{j-1}}(t_{j-1})\rangle\right|^2, \quad (k_N = k). \tag{2.5}$$

2.2.2 Pointwise convergence

Let us consider a convergence condition of $\lambda_{\Delta,k}$ in the case $|\Delta| \to 0$.

Let

$$\gamma_{\Delta,k} := \prod_{j=1}^{N} \left|\langle \Psi_k(t_j), e^{-i\Delta_j H} \Psi_k(t_{j-1})\rangle\right|^2, \tag{2.6}$$

$$\epsilon_{\Delta,k} := \sum_{\substack{k_0,\cdots,k_{N-1} \\ \exists l \in \{0,\cdots,N-1\}, k_l \neq k}} \lambda_{k_0} \prod_{j=1}^{N} \left|\langle \Psi_{k_j}(t_j), e^{-i\Delta_j H} \Psi_{k_{j-1}}(t_{j-1})\rangle\right|^2, \tag{2.7}$$

so that

$$\lambda_{\Delta,k} = \lambda_k \gamma_{\Delta,k} + \epsilon_{\Delta,k}. \tag{2.8}$$

Theorem 2.2.1 *Assume that there exists $k \in \mathbb{N}$ such that the following conditions hold:*

$$\forall \lambda \in [0,\tau], \quad \Psi_k(\lambda) \in D(H), \tag{2.9}$$

$$\xi_k := \sup_{0 \le \lambda \le \tau} \|H\Psi_k(\lambda)\| < \infty, \tag{2.10}$$

$$\eta_k := \sup_{\substack{\lambda,\nu \in [0,\tau] \\ \lambda \neq \nu}} \frac{\|\Psi_k(\lambda) - \Psi_k(\nu)\|}{|\lambda - \nu|} < \infty, \tag{2.11}$$

$$\lim_{|\Delta| \to 0} \sum_{j=1}^{N} \mathrm{Re}\, \langle \Psi_k(t_j) - \Psi_k(t_{j-1}), \Psi_k(t_{j-1})\rangle = 0. \tag{2.12}$$

Then we have

$$\lim_{|\Delta| \to 0} \lambda_{\Delta,k} = \lambda_k. \tag{2.13}$$

Remark 2.2.2 *Condition (2.11) implies that* $\|\Psi_k(\lambda) - \Psi_k(\nu)\| \leq \eta_k|\lambda - \mu|, \forall\lambda, \mu \in [0, \tau]$ *(Lipschitz continuity). In particular,* $\Psi_k(\cdot)$ *is strongly continuous, so that the mapping* $\Psi_k(\cdot) : [0, t] \to \mathcal{H}$ *is a curve in* $\mathcal{H}$*.*

Proof. By using [2, THEOREM 4.2], the assumptions (2.9)–(2.12) imply that

$$\lim_{|\Delta|\to 0} \gamma_{\Delta,k} = 1. \tag{2.14}$$

On the other hand, we can estimate $\epsilon_{\Delta,k}$ as follows.

$$\begin{aligned}
\epsilon_{\Delta,k} &= \sum_{l=0}^{N-1} \sum_{\substack{k_0,\cdots,k_{N-1} \\ \forall i>l, k_i=k, k_l\neq k}} \lambda_{k_0} \prod_{j=1}^{N} \left|\langle \Psi_{k_j}(t_j), e^{-i\Delta_j H}\Psi_{k_{j-1}}(t_{j-1})\rangle\right|^2 && (2.15) \\
&= \sum_{l=0}^{N-1} \epsilon_{\Delta,k}(l) && (2.16)
\end{aligned}$$

where $\epsilon_{\Delta,k}(l)$ is given by

$$\begin{aligned}
\epsilon_{\Delta,k}(0) &= \prod_{j=2}^{N} \left|\langle \Psi_k(t_j), e^{-i\Delta_j H}\Psi_k(t_{j-1})\rangle\right|^2 \\
&\times \sum_{\substack{k_0 \\ k_0\neq k}} \left|\langle \Psi_{k_1}(t_1), e^{-i\Delta_1 H}\Psi_{k_0}(t_0)\rangle\right|^2 \lambda_{k_0}, && (2.17) \\
\epsilon_{\Delta,k}(l) &= \prod_{j=l+2}^{N} \left|\langle \Psi_k(t_j), e^{-i\Delta_j H}\Psi_k(t_{j-1})\rangle\right|^2 \\
&\times \sum_{\substack{k_l \\ k_l\neq k}} \left|\langle \Psi_k(t_{l+1}), e^{-i\Delta_{l+1} H}\Psi_{k_l}(t_l)\rangle\right|^2 \\
&\times \sum_{k_{l-1}} \left|\langle \Psi_{k_l}(t_l), e^{-i\Delta_l H}\Psi_{k_{l-1}}(t_{l-1})\rangle\right|^2 \\
&\vdots \\
&\times \sum_{k_0} \left|\langle \Psi_{k_1}(t_1), e^{-i\Delta_1 H}\Psi_{k_0}(t_0)\rangle\right|^2 \lambda_{k_0}, && (2.18) \\
&(1 \leq l \leq N-2)
\end{aligned}$$

$$\begin{aligned}
\epsilon_{\Delta,k}(N-1) &= \sum_{\substack{k_{N-1}\\ k_{N-1}\neq k}} \left|\langle \Psi_k(t_{l+1}), e^{-i\Delta_{l+1}H}\Psi_{k_l}(t_l)\rangle\right|^2 \\
&\times \sum_{k_{N-2}} \left|\langle \Psi_{k_{N-1}}(t_{N-1}), e^{-i\Delta_{N-1}H}\Psi_{k_{l-2}}(t_{l-2})\rangle\right|^2 \\
&\vdots \\
&\times \sum_{k_0} \left|\langle \Psi_{k_1}(t_1), e^{-i\Delta_1 H}\Psi_{k_0}(t_0)\rangle\right|^2 \lambda_{k_0}, \qquad (2.19)
\end{aligned}$$

respectively.

By the Schwarz inequality, we have

$$\begin{aligned}
\prod_{j=l+2}^{N} \left|\langle \Psi_k(t_j), e^{-i\Delta_j H}\Psi_k(t_{j-1})\rangle\right|^2 &\leq \prod_{j=l+2}^{N} \|\Psi_k(t_j)\|^2 \cdot \|e^{-i\Delta_j H}\Psi_k(t_{j-1})\|^2 \\
&\leq 1, \qquad \forall l \in \{0, \cdots, N-2\}.
\end{aligned}$$

For all $l \geq 1$,

$$\begin{aligned}
& \sum_{k_{l-1}} \left|\langle \Psi_{k_l}(t_l), e^{-i\Delta_l H}\Psi_{k_{l-1}}(t_{l-1})\rangle\right|^2 \cdots \sum_{k_0} \left|\langle \Psi_{k_1}(t_1), e^{-i\Delta_1 H}\Psi_{k_0}(t_0)\rangle\right|^2 \lambda_{k_0} \\
\leq\ & \sum_{k_{l-1}} \left|\langle \Psi_{k_l}(t_l), e^{-i\Delta_l H}\Psi_{k_{l-1}}(t_{l-1})\rangle\right|^2 \cdots \sum_{k_0} \left|\langle e^{i\Delta_1 H}\Psi_{k_1}(t_1), \Psi_{k_0}(t_0)\rangle\right|^2 \\
\leq\ & \sum_{k_{l-1}} \left|\langle \Psi_{k_l}(t_l), e^{-i\Delta_l H}\Psi_{k_{l-1}}(t_{l-1})\rangle\right|^2 \cdots \|e^{i\Delta_1 H}\Psi_{k_1}(t_1)\|^2 \\
\leq\ & \cdots \leq 1.
\end{aligned}$$

Thus (2.16) implies that

$$\epsilon_{\Delta,k} \leq \sum_{l=0}^{N-1} \sum_{k_l, k_l\neq k} \left|\langle \Psi_k(t_{l+1}), e^{-i\Delta_{l+1}H}\Psi_{k_l}(t_l)\rangle\right|^2. \qquad (2.20)$$

In the case where $k_l \neq k$, we have $\langle \Psi_k(t_l), \Psi_{k_l}(t_l)\rangle = 0$. Hence

$$\begin{aligned}
&\sum_{\substack{k_l \\ k_l \neq k}} \left|\langle \Psi_k(t_{l+1}), e^{-i\Delta_{l+1}H}\Psi_{k_l}(t_l)\rangle\right|^2 \\
= &\sum_{\substack{k_l \\ k_l \neq k}} \left|\langle \Psi_k(t_{l+1}), (e^{-i\Delta_{l+1}H}-1)\Psi_{k_l}(t_l)\rangle + \langle \Psi_k(t_{l+1}) - \Psi_k(t_l), \Psi_{k_l}(t_l)\rangle\right|^2 \\
\leq\ & 2\sum_{\substack{k_l \\ k_l \neq k}} \Big\{\left|\langle (e^{i\Delta_{l+1}H}-1)\Psi_k(t_{l+1}), \Psi_{k_l}(t_l)\rangle\right|^2 \\
&\quad + \left|\langle \Psi_k(t_{l+1}) - \Psi_k(t_l), \Psi_{k_l}(t_l)\rangle\right|^2\Big\} \\
\leq\ & 2\left\{\|(e^{i\Delta_{l+1}H}-1)\Psi_k(t_{l+1})\|^2 + \|\Psi_k(t_{l+1}) - \Psi_k(t_l)\|^2\right\}. \qquad (2.21)
\end{aligned}$$

Let $E_H(\cdot)$ be the spectral measure of Hamiltonian H. By the spectral theorem, we have

$$\begin{aligned}
\|(e^{i\Delta_{l+1}H}-1)\Psi_k(t_{l+1})\|^2 &= \int_{\mathbb{R}} |e^{i\Delta_{l+1}x}-1|^2 d\|E_H(x)\Psi_k(t_{l+1})\|^2 \\
&\leq \int_{\mathbb{R}} \Delta_{l+1}^2 x^2 d\|E_H(x)\Psi_k(t_{l+1})\|^2 \\
&\leq \Delta_{l+1}^2 \|H\Psi_k(t_{l+1})\|^2. \qquad (2.22)
\end{aligned}$$

The assumptions (2.9)–(2.11) imply that

$$\|H\Psi_k(t_{l+1})\|^2 \leq \xi_k^2, \quad \|\Psi_k(t_{l+1}) - \Psi_k(t_l)\|^2 \leq \Delta_{l+1}^2\eta_k^2. \qquad (2.23)$$

Therefore, (2.20), (2.21), (2.22) and (2.23) implies that

$$\begin{aligned}
\epsilon_{\Delta,k} &\leq 2\sum_{l=0}^{N-1}\left\{\|(e^{i\Delta_{l+1}H}-1)\Psi_k(t_{l+1})\|^2 + \|\Psi_k(t_{l+1}) - \Psi_k(t_l)\|^2\right\} \\
&\leq 2\sum_{l=0}^{N-1}\left\{\Delta_{l+1}^2\|H\Psi_k(t_{l+1})\|^2 + \|\Psi_k(t_{l+1}) - \Psi_k(t_l)\|^2\right\} \\
&\leq 2(\xi_k^2 + \eta_k^2)\sum_{l=1}^{N}\Delta_l^2. \qquad (2.24)
\end{aligned}$$

By [2, LEMMA 2.2],

$$\lim_{|\Delta|\to 0}\sum_{l=1}^{N}\Delta_l^2 = 0.$$

Thus (2.24) implies that $\lim_{|\Delta|\to 0} \epsilon_{\Delta,k} = 0$. Hence, by (2.8) and (2.14), we obtain (2.13)

∎

Remark 2.2.3 *Assume that the conditions of Theorem 2.2.1 hold. Let $a > 1$ be a constant and take $|\Delta|$ such that*

$$(\xi_k^2 + 2\xi_k\eta_k)|\Delta|^2 + 2\eta_k|\Delta| \le \frac{\log a}{a}. \tag{2.25}$$

By the proof of [2, THEOREM 4.2],

$$\begin{aligned}&\exp\left[-a\left\{(\xi_k^2 + 2\xi_k\eta_k)\sum_{l=1}^{N}\Delta_l^2 - 2\sum_{l=1}^{N}\mathrm{Re}\,\langle\Psi_k(t_l) - \Psi_k(t_{l-1}), \Psi_k(t_{l-1})\rangle\right\}\right]\\ &\le \gamma_{\Delta,k} \le 1.\end{aligned} \tag{2.26}$$

Then, by (2.24) and (2.26), we have

$$\begin{aligned}&|\lambda_{\Delta,k} - \lambda_k| = |\lambda_k(\gamma_{\Delta,k} - 1) + \epsilon_{\Delta,k}| \le \lambda_k(1 - \gamma_{\Delta,k}) + \epsilon_{\Delta,k}\\ \le\ & 2(\xi_k^2 + \eta_k^2)\sum_{l=1}^{N}\Delta_l^2 + \lambda_k\\ -\ & \lambda_k \exp\left[-a\left\{(\xi_k^2 + 2\xi_k\eta_k)\sum_{l=1}^{N}\Delta_l^2 - 2\sum_{l=1}^{N}\mathrm{Re}\,\langle\Psi_k(t_l) - \Psi_k(t_{l-1}), \Psi_k(t_{l-1})\rangle\right\}\right].\end{aligned}$$

The following corollary can be easily proven by using [2, COROLLARY 4.4].

Corollary 2.2.4 *Assume that there exists $k \in \mathbb{N}$ such that the following conditions hold:*

$$\Psi_k(\cdot) : [0,\tau] \to \mathcal{H} \quad \textit{is a strongly differentiable mapping,} \tag{2.27}$$

$$\forall\lambda \in [0,\tau], \quad \Psi_k(\lambda) \in D(H), \tag{2.28}$$

$$\xi_k < \infty, \tag{2.29}$$

$$\sup_{0\le\lambda\le\tau} \|\Psi_k'(\lambda)\| < \infty, \tag{2.30}$$

where $\Psi_k'(\cdot)$ denotes the strong derivative of $\Psi_k(\cdot)$.

Then (2.9)–(2.12) hold. Therefore, by Theorem 2.2.1, (2.13) holds.

Example 2.2.5 *Let A be a self-adjoint operator on $\mathcal{H}$. Assume that there exists $k \in \mathbb{N}$ such that the following conditions hold:*

$$\Psi_k \in D(A) \cap \bigcap_{0 \leq \lambda \leq \tau} D(He^{-i\lambda A}), \tag{2.31}$$

$$\sup_{0 \leq \lambda \leq \tau} \|He^{-i\lambda A}\Psi_k\| < \infty, \tag{2.32}$$

$$\forall \lambda \in [o, \tau], \;\; \Psi_k(\lambda) = e^{-i\lambda A}\Psi_k. \tag{2.33}$$

In this case, by [2, EXAMPLE 4.5], (2.27)–(2.30) hold. Then by using Corollary 2.2.4, (2.9)–(2.13) hold.

2.2.3 Trace norm convergence

For the decomposition (2.1), we define

$$\rho(t) := \sum_n \lambda_n |\Psi_n(t)\rangle\langle\Psi_n(t)|, \quad \forall t \in [0, \tau]. \tag{2.34}$$

Let us consider conditions of convergence from $\rho_\Delta(\tau)$ to $\rho(\tau)$ in the trace norm sense.

Theorem 2.2.6 *Assume that the conditions (2.9)–(2.12) hold for all $k \in \mathbb{N}$ satisfying $\lambda_k > 0$.*

Then we have

$$\lim_{|\Delta| \to 0} \|\rho_\Delta(\tau) - \rho(\tau)\|_1 = 0. \tag{2.35}$$

Proof. By definition of $\rho_\Delta(\tau)$, $\rho(\tau)$, and equation (2.8), we have

$$\begin{aligned}
\|\rho_\Delta(\tau) - \rho(\tau)\|_1 &= \sum_k \langle \Psi_k(\tau), |\rho_\Delta(\tau) - \rho(\tau)|\Psi_k(\tau)\rangle \\
&= \sum_k |\lambda_{\Delta,k} - \lambda_k| \\
&= \sum_k |\lambda_k(\gamma_{\Delta,k} - 1) + \epsilon_{\Delta,k}| \\
&\leq \sum_k \lambda_k(1 - \gamma_{\Delta,k}) + \sum_k \epsilon_{\Delta,k} \\
&= \sum_k \lambda_k(1 - \gamma_{\Delta,k}) + \sum_k (\lambda_{\Delta,k} - \lambda_k\gamma_{\Delta,k}) \\
&= 2 - 2\sum_k \lambda_k\gamma_{\Delta,k}.
\end{aligned} \tag{2.36}$$

Note that

$$|\lambda_k \gamma_{\Delta,k}| \leq \lambda_k \ (\forall k \in \mathbb{N}), \quad \sum_k \lambda_k = 1.$$

The assumptions (2.9)–(2.12) imply that

$$\lim_{|\Delta| \to 0} \lambda_k \gamma_{\Delta,k} = \lambda_k \ (\forall k \in \mathbb{N}).$$

Hence, by using Lebesgue's dominated convergence theorem, we have

$$\lim_{|\Delta| \to 0} \sum_k \lambda_k \gamma_{\Delta,k} = 1.$$

Therefore, by (2.36), we obtain (2.35).

∎

Remark 2.2.7 *Assume that the conditions of Theorem 2.2.6 hold and that* $\sup_{k,\lambda_k \neq 0} \xi_k < \infty$ *and* $\sup_{k,\lambda_k \neq 0} \eta_k < \infty$ *hold. Then, for* $a > 1$*, we can take* $|\Delta|$ *such that (2.25) holds for all* k *with* $\lambda_k \neq 0$*. Then we have (2.26) for all* $k \in \mathbb{N}$ *with* $\lambda_k \neq 0$*. Hence, by (2.36), for all* $k \in \mathbb{N}$*, we obtain the following estimation:*

$$\begin{aligned} & |\lambda_{\Delta,k} - \lambda_k| \leq \|\rho_\Delta(\tau) - \rho(\tau)\|_1 \\ \leq \ & 2 - 2\sum_k \lambda_k \exp\left[-a\left\{(\xi_k^2 + 2\xi_k\eta_k)\sum_{l=1}^N \Delta_l^2 - 2\sum_{l=1}^N \mathrm{Re}\ \langle \Psi_k(t_l) - \Psi_k(t_{l-1}), \Psi_k(t_{l-1})\rangle\right\}\right]. \end{aligned}$$

The following corollary and example can be easily proven by using Corollary 2.2.4, Example 2.2.5, and Theorem 2.2.6.

Corollary 2.2.8 *Assume that the conditions (2.27)–(2.30) hold for all* $k \in \mathbb{N}$ *with* $\lambda_k > 0$*. Then we have (2.35).*

Example 2.2.9 *Let* A *be a self-adjoint operator on* $\mathcal{H}$*. Assume that the conditions (2.31)–(2.33) hold for all* $k \in \mathbb{N}$ *with* $\lambda_k > 0$*. Then we have (2.35).*

In Example 2.2.9, let us consider the case of $d < \infty$. It is easy to see that the assumptions (2.31)–(2.32) are satisfied. On the other hand, by Stone's theorem, for all $U \in \mathfrak{U}(\mathcal{H})$, there exists a self-adjoint operator A such that $U = e^{-i\tau A}$. Since $\rho(\tau) = U\rho U^*$, we have $\lim_{|\Delta| \to 0} \|\rho_\Delta(\tau) - U\rho U^*\|_1 = 0$. This fact shows that, in the case $d < \infty$, an arbitrary state in $\{U\rho U^* \mid U \in \mathfrak{U}(\mathcal{H})\}$ can be approximated (in the trace norm sense) by states obtained after an appropriate continuous measurements. In other words, in this case, we can approximate an arbitrary unitary channel by continuous quantum measurements.

2.2.4 Application to quantum Zeno effect for mixed states

Let $\Psi_k \in D(H)$ and $\Psi_k(\lambda) = \Psi_k$ $(\forall\lambda \in [0,\tau])$ holds for all $k \in \mathbb{N}$ with $\lambda_k > 0$.

This is the case where $A = 0$ in Example 2.2.9. Then (2.9)–(2.12) hold for all $k \in \mathbb{N}$ with $\lambda_k > 0$. Hence, we have (2.35).

This means that, by the series of measurement with respect to the family of the projection operators $\{|\Psi_k\rangle\langle\Psi_k|\}_k$, transitions to states different from the initial state are hindered. This can be interpreted as a quantum Zeno effect for mixed states.

2.3 Convergence condition of the von Neumann entropy

Let $\varphi : [0,\infty) \ni \lambda \mapsto -\lambda\log\lambda \in [0,\infty)$, where $\varphi(0) := 0$. Then φ is continuous, concave, and subadditive. Let $S(\rho)$ be the von Neumann entropy of $\rho \in \mathfrak{S}(\mathcal{H})$. i.e.

$$S(\rho) := \mathrm{Tr}\varphi(\rho).$$

In the case $d < \infty$, by Fannes' inequality, we have for all $\rho_1, \rho_2 \in \mathfrak{S}(\mathcal{H})$

$$\|\rho_1 - \rho_2\|_1 \le 1/e \Longrightarrow |S(\rho_1) - S(\rho_2)| \le \|\rho_1 - \rho_2\|_1 \log d + \varphi(\|\rho_1 - \rho_2\|_1).$$

Therefore the von Neumann entropy is continuous with respect to the trace norm.

On the other hand, in the case $d = \infty$, although the von Neumann entropy is lower semi-continuous with respect to the trace norm, i.e.

$$\lim_{n\to\infty} \|\rho_n - \rho\|_1 = 0 \Rightarrow S(\rho) \le \liminf_{n\to\infty} S(\rho_n),$$

it is not necessarily continuous. Moreover, it is known that the set $\{\rho \in \mathfrak{S}(\mathcal{H}) \mid S(\rho) < \infty\}$ is of the first category [10].

In what follows, we deal with the case where $d = \infty$ only.

For $\rho_\Delta(\tau)$ and ρ considered in the section 2, conditions of the convergence $S(\rho_\Delta(\tau)) \to S(\rho)$ are given by the following theorem.

Theorem 2.3.1 *Assume that the conditions (2.9)–(2.11) hold for all $k \in \mathbb{N}$, and that the condition (2.12) holds for all $k \in \mathbb{N}$ with $\lambda_k > 0$. Suppose that the following conditions hold:*

$$\xi_k \to 0, \quad \eta_k \to 0 \ (k \to \infty), \tag{2.37}$$

$$S(\rho) < \infty, \tag{2.38}$$

$$\sum_k \varphi(\xi_k^2) < \infty, \quad \sum_k \varphi(\eta_k^2) < \infty. \tag{2.39}$$

Then

$$\lim_{|\Delta|\to 0} S(\rho_\Delta(\tau)) = S(\rho(\tau)) = S(\rho). \tag{2.40}$$

Remark 2.3.2 *The function φ is monotone increasing on $[0, 1/e]$ and*

$$\xi_k^2 = \sup_{0\le\lambda\le\tau} \|H\Psi_k(\lambda)\|^2 = \sup_{0\le\lambda\le\tau} \int_{\mathbb{R}} x^2 d\|E_H(x)\Psi_k(\lambda)\|^2.$$

Hence, $\xi_k \to 0$ $(k\to\infty)$ implies that there exists $N_0 \in \mathbb{N}$ such that, for all $k > N_0$,

$$\varphi(\xi_k^2) \ge \sup_{0\le\lambda\le\tau} \varphi\left(\int_{\mathbb{R}} x^2 d\|E_H(x)\Psi_k(\lambda)\|^2\right).$$

By Jensen's inequality, we have

$$\varphi\left(\int_{\mathbb{R}} x^2 d\|E_H(x)\Psi_k(\lambda)\|^2\right) \ge \int_{\mathbb{R}} \varphi(x^2) d\|E_H(x)\Psi_k(\lambda)\|^2.$$

Hence, for all $k > N_0$,

$$\varphi(\xi_k^2) \ge \sup_{0\le\lambda\le\tau} \int_{\mathbb{R}} \varphi(x^2) d\|E_H(x)\Psi_k(\lambda)\|^2.$$

Then, we have

$$\forall k > N_0,\ \forall\lambda\in[0,\tau],\ \Psi_k(\lambda)\in D(\sqrt{\varphi(H^2)}),\ \varphi(\xi_k^2) \ge \sup_{0\le\lambda\le\tau} \|\sqrt{\varphi(H^2)}\Psi_k(\lambda)\|^2.$$

Moreover, using the estimate that

$$\begin{aligned}
\sum_k \varphi(\xi_k^2) &= \sum_{k=1}^{N_0} \varphi(\xi_k^2) + \sum_{k=N_0+1}^{\infty} \varphi(\xi_k^2) \\
&\ge \sum_{k=1}^{N_0} \varphi(\xi_k^2) + \sum_{k=N_0+1}^{\infty} \sup_{0\le\lambda\le\tau} \|\sqrt{\varphi(H^2)}\Psi_k(\lambda)\|^2 \\
&\ge \sum_{k=1}^{N_0} \varphi(\xi_k^2) + \sup_{0\le\lambda\le\tau} \sum_{k=N_0+1}^{\infty} \|\sqrt{\varphi(H^2)}\Psi_k(\lambda)\|^2,
\end{aligned}$$

we obtain

$$\begin{aligned}
& \xi_k \to 0\ (k\to\infty),\ \sum_k \varphi(\xi_k^2) < \infty \\
\Longrightarrow\quad & \exists N_0\in\mathbb{N},\ \sup_{0\le\lambda\le\tau} \sum_{k=N_0+1}^{\infty} \|\sqrt{\varphi(H^2)}\Psi_k(\lambda)\|^2 < \infty.
\end{aligned} \tag{2.41}$$

Particularly, in the case $H \in \mathfrak{B}(\mathcal{H})$, we have, for all $\Phi \in \mathcal{H}$,

$$\begin{aligned}\int_{\mathbb{R}} \varphi(x^2) d\|E_H(x)\Phi\|^2 &\leq \sup_{x\in\sigma(H)} \varphi(x^2) \int_{\mathbb{R}} d\|E_H(x)\Phi\|^2 \\ &= \sup_{x\in\sigma(H)} \varphi(x^2) \cdot \|\Phi\| < \infty.\end{aligned}$$

Hence, we obtain $\sqrt{\varphi(H^2)} \in \mathfrak{B}(\mathcal{H})$. Therefore, by (2.41), we have

$$\xi_k \to 0 \ (k \to \infty), \ \sum_k \varphi(\xi_k^2) < \infty \Longrightarrow \varphi(H^2) \in \mathfrak{T}(\mathcal{H}). \tag{2.42}$$

We remark that, in this case, if Hamiltonian H is represented as a density operator, then $\varphi(H^2) \in \mathfrak{T}(\mathcal{H})$ means $S(H^2) < \infty$.

Proof. The assumption of this theorem and Theorem 2.2.6 imply that $\lim_{|\Delta|\to 0} \|\rho_\Delta(\tau) - \rho(\tau)\|_1 = 0$. Hence we have $\text{w-}\lim_{|\Delta|\to 0} \rho_\Delta(\tau) = \rho(\tau)$, where w-lim means weak limit.

By (2.8), (2.24) and $\gamma_{\Delta,k} \leq 1$, we have

$$\lambda_{\Delta,k} \leq \lambda_k + 2(\xi_k^2 + \eta_k^2) \sum_{l=1}^{N} \Delta_l^2.$$

By $\lim_{|\Delta|\to 0} \sum_{l=1}^{N} \Delta_l^2 = 0$, there exists $\delta > 0$ such that,

$$|\Delta| < \delta \Longrightarrow \sum_{l=1}^{N} \Delta_l^2 < 1/2.$$

Thus

$$\lambda_{\Delta,k} \leq \lambda_k + \xi_k^2 + \eta_k^2 \quad (|\Delta| < \delta). \tag{2.43}$$

We set

$$\sigma := \sum_k (\lambda_k + \xi_k^2 + \eta_k^2) |\Psi_k(\tau)\rangle\langle\Psi_k(\tau)|. \tag{2.44}$$

By the assumption of this theorem, $\sigma \in \mathfrak{C}(\mathcal{H})$. On the other hand, (2.43) implies that

$$\rho_\Delta(\tau) \leq \sigma \quad (|\Delta| < \delta). \tag{2.45}$$

Moreover, by the assumption of this theorem and subadditivity of φ, we have

$$\begin{aligned} S(\sigma) &= \sum_k \varphi(\lambda_k + \xi_k^2 + \eta_k^2) \qquad (2.46) \\ &\leq S(\rho) + \sum_k \varphi(\xi_k^2) + \sum_k \varphi(\eta_k^2) < \infty. \qquad (2.47) \end{aligned}$$

Hence, by Simon's dominated convergence theorem for entropy [4, THEOREM A.3], we have

$$\lim_{|\Delta|\to 0} S(\rho_\Delta(\tau)) = S(\rho(\tau)).$$

It is obvious that $S(\rho(\tau)) = S(\rho)$ holds. ∎

Remark 2.3.3 *In the proof of Theorem 2.3.1, we used that*

$$S(\rho),\ \sum_k \varphi(\xi_k^2),\ \sum_k \varphi(\eta_k^2) < \infty \Longrightarrow \sum_k \varphi(\lambda_k + \xi_k^2 + \eta_k^2) < \infty. \tag{2.48}$$

Conversely, we can show that, under condition (2.37),

$$\sum_k \varphi(\lambda_k + \xi_k^2 + \eta_k^2) < \infty \Longrightarrow S(\rho),\ \sum_k \varphi(\xi_k^2),\ \sum_k \varphi(\eta_k^2) < \infty \tag{2.49}$$

as follows. By $\lambda_k + \xi_k^2 + \eta_k^2 \to 0$ $(k \to \infty)$, we have

$$\exists N_0 \in \mathbb{N}, \forall k > N_0, \max\{\lambda_k, \xi_k^2, \eta_k^2\} \leq \lambda_k + \xi_k^2 + \eta_k^2 < 1/e.$$

Hence, by the fact that φ is a monotone increasing function on $[0, 1/e]$, we obtain

$$\max\left\{ \sum_{k=N_0+1}^{\infty} \varphi(\lambda_k),\ \sum_{k=N_0+1}^{\infty} \varphi(\xi_k^2),\ \sum_{k=N_0+1}^{\infty} \varphi(\eta_k^2) \right\} \leq \sum_{k=N_0+1}^{\infty} \varphi(\lambda_k + \xi_k^2 + \eta_k^2).$$

Therefore, we have (2.49). Thus, in Theorem 2.3.1, we can replace the condition (2.38) and (2.39) with $\sum_k \varphi(\lambda_k + \xi_k^2 + \eta_k^2) < \infty$.

Example 2.3.4 *Let A be a self-adjoint operator on $\mathcal{H}$. Assume that $A, H \in \mathfrak{C}(\mathcal{H})$, and that A and H are strongly commuting. Moreover, we assume that*

$$\forall k \in \mathbb{N},\ \forall \lambda \in [0, \tau],\ \Psi_k(\lambda) = e^{-i\lambda A}\Psi_k, \tag{2.50}$$

$$S(\rho) < \infty,\ \sum_k \varphi(\|H\Psi_k\|^2) < \infty,\ \sum_k \varphi(\|A\Psi_k\|^2) < \infty. \tag{2.51}$$

Then, the compactness, the strong commutativity of A and H, and (2.50) imply that $\xi_k = \|H\Psi_k\| \to 0,\ \eta_k = \|A\Psi_k\| \to 0$ $(k \to \infty)$. Hence, the assumption of Theorem 2.3.1 is satisfied. Hence, we have $S(\rho_\Delta(\tau)) \to S(\rho)$ $(|\Delta| \to 0)$.

In Example 2.3.4, let us consider the case of $A = 0$. The following fact can be easily seen:

$$\begin{aligned} & H \in \mathfrak{C}(\mathcal{H}),\ \Psi_k(\lambda) = \Psi_k\ (\forall k \in \mathbb{N},\ \forall \lambda \in [0,\tau]), \\ & S(\rho) < \infty,\ \sum_k \varphi(\|H\Psi_k\|^2) < \infty \\ \Longrightarrow\quad & \lim_{|\Delta|\to 0} S(\rho_\Delta(\tau)) = S(\rho). \end{aligned} \tag{2.52}$$

This is the case of QZE. We remark that, if $\{\Psi_k\}_k$ is a sequence of eigenvectors of H, we have $\sum_k \varphi(\|H\Psi_k\|^2) = \mathrm{Tr}\varphi(H^2) < \infty$. Then, in (2.52), we can replace the condition $\sum_k \varphi(\|H\Psi_k\|^2) < \infty$ with $\varphi(H^2) \in \mathfrak{T}(\mathcal{H})$.

Acknowledgments

The author would like to thank Professor Asao Arai for valuable comments.

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Chapter 3

Appendix

In this chapter, we record some properties of compact operators and the von Neumann entropy which is related to Chapter 2. We presuppose that the symbols in this Chapter are the same as Chapter 2.

3.1 Compact operators on Hilbert space

We denote all finite-rank operators on $\mathcal{H}$ by $\mathfrak{F}(\mathcal{H})$.

Theorem 3.1.1 *For all $A \in \mathfrak{B}(\mathcal{H})$, the following conditions are equivalent:*

(1) The set $\{A\Psi \mid \Psi \in \mathcal{H}, \|\Psi\| \leq 1\}$ is relatively compact.

(2) If $w\text{-}\lim_{n\to\infty} \Psi_n = \Psi$, then $\lim_{n\to\infty} \|A\Psi_n - A\Psi\| = 0$.

(3) If $\{\Psi_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is orthonormal system (ONS), then $\lim_{n\to\infty} \|A\Psi_n\| = 0$.

(4) There exists $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{F}(\mathcal{H})$ such that $\lim_{n\to\infty} \|A_n - A\| = 0$.

A bounded operator A is called compact if and only if any of Theorem 3.1.1 (1)–(4) is true.

3.2 Some properties of the von Neumann entropy

Theorem 3.2.1 (Fannes's inequality) *Suppose $d(= \dim \mathcal{H}) < \infty$ and $\rho, \sigma \in \mathfrak{S}(\mathcal{H})$. Then the following holds:*

$$\|\rho - \sigma\|_1 \leq \frac{1}{e} \Longrightarrow |S(\rho) - S(\sigma)| \leq \|\rho - \sigma\|_1 \log d + \varphi(\|\rho - \sigma\|_1). \qquad (3.1)$$

Theorem 3.2.2 *Let $\rho, \rho_n \in \mathfrak{S}(\mathcal{H})$ $(\forall n \in \mathbb{N}), U \in \mathfrak{U}(\mathcal{H}), \lambda \in (0,1)$. Then we have*

*(1) (**Positivity**) $S(\rho) \geq 0$.*

*(2) (**Unitary invariance**) $S(U\rho U^*) = S(\rho)$.*

*(3) (**Concavity**) $S(\lambda\rho_1 + (1-\lambda)\rho_2) \geq \lambda S(\rho_1) + (1-\lambda)S(\rho_2)$.*

*(4) (**Lower semi-continuity**) If $\lim_{n\to\infty} \|\rho_n - \rho\|_1 = 0$, then we have*

$$S(\rho) \leq \liminf_{n\to\infty} S(\rho_n).$$

Definition 3.2.3 *Let X be a topological space, and A a subset of X.*

*(1) A is called **nowhere dense** if $(\bar{A})^\circ = \emptyset$.*

*(2) A is called **first category** if there exists $\{A_n\}_{n=1}^\infty$ such that, for all $n \in \mathbb{N}$, A_n is nowhere dense and $A = \bigcup_{n=1}^\infty A_n$.*

*(3) A is called **second category** if A is not first category.*

Example 3.2.4 *$\mathbb{Q}$ is of first category and $\mathbb{R}\backslash\mathbb{Q}$ is of second category in $\mathbb{R}$.*

Theorem 3.2.5 (Wehrl 1976 [9]) *$\{\rho \in \mathfrak{S}(\mathcal{H}) \mid S(\rho) < \infty\}$ is of first category in $\mathfrak{S}(\mathcal{H})$ with respect to the topology which induced by trace norm.*

Theorem 3.2.6 (Dominated convergence theorem for entropy, Simon 1973 [4]) *Let $A_n, A, B \in \mathfrak{C}(\mathcal{H})$ $(\forall n \in \mathbb{N})$ and $A_n, A, B \geq 0$ $(\forall n \in \mathbb{N})$. Suppose that*

(1) $S(B) < \infty$,

(2) $w\text{-}\lim_{n\to\infty} A_n = A$,

(3) $A_n \leq B$, $(\forall n \in \mathbb{N})$.

Then we have

$$\lim_{n\to\infty} S(A_n) = S(A).$$

Theorem 3.2.7 *Let $\mathcal{E}_{\mathfrak{P}}$ be a projection channel on $\mathfrak{S}(\mathcal{H})$. Then, for all $\rho \in \mathfrak{S}(\mathcal{H})$,*

$$S(\rho) \leq S(\mathcal{E}_{\mathfrak{P}}\rho).$$

We have $S(\rho) = S(\mathcal{E}_{\mathfrak{P}}\rho)$ if and only if $\rho = \mathcal{E}_{\mathfrak{P}}\rho$.

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