The edge of the wedge theorem for the sheaf of holomorphic functions of exponential type and Laplace hyperfunctions

(指数型正則関数の層に対する楔の刃の定理とラプラス超関数)

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Abstract

We establish an edge of the wedge theorem for the sheaf of holomorphic functions with exponential growth at infinity. As an application, we construct the sheaf of Laplace hyperfunctions, and we also study several properties of this sheaf.

1 Introduction

In [7], H. Komatsu introduces Laplace hyperfunctions in one variable and their Laplace transforms which play a part in solving both linear ordinary differential equations and partial differential equations. Roughly speaking, a Laplace hyperfunction is presented as a difference of boundary values of holomorphic functions with exponential growth at infinity from a complex domain to a real domain. By the theory of Laplace hyperfunctions, we can treat Laplace transforms for functions without any growth conditions in a framework of hyperfunctions.

To localize the notion of Laplace hyperfunctions is desired in order to further develop the theory of Laplace hyperfunctions. For that purpose, N. Honda and the author [1] established a vanishing theorem of cohomology groups on a pseudoconvex open subset for holomorphic functions with exponential growth at infinity. As its benefits, the sheaf of Laplace hyperfunctions

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in one variable was constructed. The aim of this article is to establish the edge of the wedge theorem for the sheaf of holomorphic functions with exponential growth at infinity. The edge of the wedge theorem plays an important role in constructing the sheaf of Laplace hyperfunctions in several variables.

The plan of the paper is as follows.

In Section 2, we review the definition of Laplace hyperfunctions with compact support and several fundamental theorems established by H. Komatsu.

In Section 3, we state the vanishing theorem on a pseudoconvex open subset for holomorphic functions of exponential type. We first define the sheaf $\mathcal{O}_{\hat{X}}^{\exp}$ of holomorphic functions of exponential type on $\hat{X}$. We also introduce the regularity condition at infinity for an open subset which are needed for the vanishing theorem. In subsection 3.4, we state the vanishing theorem. To prove the vanishing theorem, we rely on the theory of L^2 estimates for the $\overline{\partial}$ operator in Hörmander [3]. The fundamental ideas and techniques were already established in the papers T. Kawai [6] and S. Saburi [17] which treated several vanishing theorems for holomorphic functions with infra-exponential growth. We refer the reader to [1] for the details. We also give the example of the vanishing theorem does not holds without regularity condition at infinity.

In Section 4, we define the sheaf of Laplace hyperfunctions in one variable with holomorphic parameters. We also show that locally integrable functions of exponential type are regarded as Laplace hyperfunctions in subsection 4.2.

We need some preparations to establish the edge of the wedge theorem for the sheaf of holomorphic functions of exponential type. In subsection 5.1, we first show the Martineau type theorem for holomorphic functions of exponential type which is a key to prove the edge of the wedge theorem. We show the edge of the wedge theorem in subsection 5.2.

In Section 6, using the result in Section 5, we construct the sheaf of Laplace hyperfunctions on $\overline{\mathbb{R}^n}$ and show that real analytic functions of exponential type can be regarded as Laplace hyperfunctions.

In Section 7, we prove the softness of the sheaf $\mathcal{B}_{\overline{\mathbb{R}^n}}^{\exp}$.

At the end of the introduction, the author would like to express the deepest appreciation to Professor Naofumi Honda for his polite teaching and generous support. The author also grateful to Professor Hikosaburo Komatsu for the valuable lectures and advises.

2 Laplace hyperfunctions of one variable

At first, we briefly recall the definition of Laplace hyperfunctions with support in $[a, \infty]$ $(a \in \mathbb{R} \sqcup \{+\infty\})$ and several fundamental theorems established by H.Komatsu ([7] - [13]).

We denote by $\mathbb{D}^2$ the radial compactification $\mathbb{C} \sqcup S^1\infty$ of $\mathbb{C}$. The topology of $\mathbb{D}^2$ is defined in the following way. A fundamental system of neighborhoods of $\xi\infty \in S^1\infty$ consists of all the sets given by

$$\left\{ z \in \mathbb{C} \ \ , \frac{z}{|z|} \in \Gamma, \ |z| > r \right\} \sqcup \{w\infty; w \in \Gamma\} \tag{1}$$

for a neighborhood Γ of ξ in S^1 and $r > 0$.

Now we introduce the notion of Laplace hyperfunctions with support in $[a, \infty]$. For an open subset $U \subset \mathbb{D}^2$, the set $\mathcal{O}^{\exp}_{\mathbb{D}^2}(U)$ of holomorphic functions of exponential type on U consists of a holomorphic function $F(z)$ on $U \cap \mathbb{C}$ which satisfies, for any compact set K in U,

$$|F(z)| \leq C_K e^{H_K|z|} \qquad (z \in K \cap \mathbb{C}) \tag{2}$$

with some positive constants C_K and H_K. We denote by $\mathcal{O}^{\exp}_{\mathbb{D}^2}$ the associated sheaf on $\mathbb{D}^2$ of the presheaf $\{\mathcal{O}^{\exp}_{\mathbb{D}^2}(U)\}_U$. It is easily seen that the restriction of $\mathcal{O}^{\exp}_{\mathbb{D}^2}$ to $\mathbb{C}$ is nothing but the sheaf $\mathcal{O}_{\mathbb{C}}$ of holomorphic functions on $\mathbb{C}$.

Definition 2.1 ([7]). *The space $\mathcal{B}^{\exp}_{[a,\infty]}$ of Laplace hyperfunctions with support in $[a, \infty]$ is the quotient space*

$$\frac{\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2 \setminus [a, \infty])}{\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2)}. \tag{3}$$

Every element of $\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2 \setminus [a, \infty])$ that is extendable to a holomorphic function of exponential type on $\mathbb{D}^2$ is identified with 0. Each equivalence class $[F(z)]$ represented by $F \in \mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2 \setminus [a, \infty])$ is considered to be a Laplace hyperfunction $f(x)$. The class $f(x) = [F] \in \mathcal{B}^{\exp}_{[a,\infty]}$ of an $F(z) \in \mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2 \setminus [a, \infty])$ can be considered as a boundary value of $F(z)$, and we sometimes denote it by

$$f(x) = F(x + i0) - F(x - i0). \tag{4}$$

It is an immediate consequence of this definition that the restriction of $\mathcal{B}^{\exp}_{[a,\infty]}$ to $\mathbb{R}$ is isomorphic to the set of hyperfunctions on $\mathbb{R}$ with support in $[a, \infty)$.

Theorem 2.2 ([11]). *We have the natural isomorphism*

$$\mathcal{B}^{\exp}_{[a,\infty]} \cong \frac{\mathcal{O}^{\exp}_{\mathbb{D}^2}(V \setminus [a, \infty])}{\mathcal{O}^{\exp}_{\mathbb{D}^2}(V)} \tag{5}$$

for any open neighborhood V of $[a, \infty]$ in $\mathbb{D}^2$.

Remember that the space $\mathcal{B}_{[a,\,\infty)}$ of ordinary hyperfunctions with support in $[a,\,\infty)$ is defined by

$$\mathcal{B}_{[a,\,\infty)} := \frac{\mathcal{O}_{\mathbb{C}}(\mathbb{C}\setminus[a,\,\infty))}{\mathcal{O}_{\mathbb{C}}(\mathbb{C})}. \tag{6}$$

Hence the restrictions $\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2\setminus[a,\,\infty]) \to \mathcal{O}_{\mathbb{C}}(\mathbb{C}\setminus[a,\,\infty))$ and $\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2) \to \mathcal{O}_{\mathbb{C}}(\mathbb{C})$ induce the canonical morphism $\rho:\mathcal{B}^{\exp}_{[a,\,\infty]} \to \mathcal{B}_{[a,\,\infty)}$, for which we have the followings.

Theorem 2.3 ([7]). *The morphism $\rho:\mathcal{B}^{\exp}_{[a,\,\infty]} \to \mathcal{B}_{[a,\,\infty)}$ is surjective.*

Since every ordinary hyperfunction with support in $[a,\,\infty)$ can be extended to a Laplace hyperfunction by the above theorem, we have

$$\mathcal{B}_{[a,\,\infty)} \cong \frac{\mathcal{B}^{\exp}_{[a,\,\infty]}}{\mathcal{B}^{\exp}_{\{\infty\}}}. \tag{7}$$

Here $\mathcal{B}^{\exp}_{\{\infty\}}$ denotes the set of Laplace hyperfunctions with support in $\{\infty\}$.

We give the definition of Laplace transforms and inverse Laplace transforms for Laplace hyperfunctions.

Definition 2.4 ([7]). *The Laplace transform $\widehat{f}(\lambda)$ of a Laplace hyperfunction $f(x)=[F]\in\mathcal{B}^{\exp}_{[a,\,\infty]}$ is defined by the integral*

$$\widehat{f}(\lambda) := \int_C e^{-\lambda z}F(z)dz, \tag{8}$$

where the path C of the integration is composed of a ray from $e^{i\alpha}\infty(-\pi/2 < \alpha < 0)$ to a point $c<a$ and a ray from c to $e^{i\beta}\infty(0<\beta<\pi/2)$.

It follows from Pólya's theorem ([16]) that the Laplace transform with origin at $c\in\mathbb{C}$

$$\widehat{m}_c(\lambda) = \int_c^{\infty} e^{-\lambda z}m(z)dz \tag{9}$$

of an $m(z)\in\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2)$ is a holomorphic function outside a convex compact set. Hence the Laplace transform $\widehat{f}(\lambda)$ does not depend on a choice of F.

Theorem 2.5 ([7]). *The Laplace transformation $\mathcal{L}$ is an isomorphism of linear spaces*

$$\mathcal{L}\,:\,\mathcal{B}^{\exp}_{[a,\,\infty]} \longrightarrow \mathcal{L}\mathcal{B}^{\exp}_{[a,\,\infty]}, \tag{10}$$

where $\mathcal{LB}^{\exp}_{[a,\infty]}$ *is the space of all holomorphic functions* $\widehat{f}(\lambda)$ *of exponential type defined on a neighborhood* Ω *of the semi-circle* $\{e^{i\theta}; |\theta| < \pi/2\}$ *in* $\mathbb{D}^2$ *which satisfies*

$$\overline{\lim_{\rho\to\infty}} \frac{\log|\widehat{f}(\rho e^{i\theta})|}{\rho} \le -a\cos\theta, \quad |\theta| < \pi/2. \tag{11}$$

For $\widehat{f}(\lambda) \in \mathcal{LB}^{\exp}_{[a,\infty]}$, *the inverse image* $\mathcal{L}^{-1}\widehat{f}$ *is given by*

$$\left[\frac{1}{2\pi\sqrt{-1}}\int_{\Lambda}^{\infty} e^{\lambda z}\widehat{f}(\lambda)d\lambda\right] \in \mathcal{B}^{\exp}_{[a,\infty]}, \tag{12}$$

where Λ *is a fixed point in* $\Omega \cap \mathbb{C}$ *and the path of the integration is taken in* $\Omega \cap \mathbb{C}$.

3 The vanishing theorem for holomorphic functions of exponential type

The purpose of the section is to review the vanishing theorem for cohomology groups on a pseudoconvex open subset with coefficients in the sheaf of holomorphic functions with exponential growth at infinity. All the proofs for the theorems in this section are given in [1].

3.1 Sheaf of holomorphic functions of exponential type on $\hat{X}$

Let $n \in \mathbb{N}$. We first introduce *the radial compactification* $\mathbb{D}^{2n}$ of $\mathbb{C}^n$.

Definition 3.1. *We denote by* $\mathbb{D}^{2n}$ *the radial compactification* $\mathbb{C}^n \sqcup S^{2n-1}\infty$ *of* $\mathbb{C}^n$, *where* S^{2n-1} *is the real* $(2n-1)$*-dimensional unit sphere. Let* D *be a closed unit ball in* $\mathbb{C}^n$ *which is considered as a real* $2n$*-dimensional topological manifold with the boundary* S^{2n-1}, *and let* $\rho : D \to \mathbb{D}^{2n}$ *be the bijection defined by*

$$\rho(z) = \begin{cases} \dfrac{z}{1-|z|} \in \mathbb{C}^n, & \text{if } z \in D^\circ \\ \\ z \in S^{2n-1}\infty, & \text{if } z \in \partial D \end{cases}.$$

Then $\mathbb{D}^{2n}$ *is equipped with the topology so that* ρ *becomes a topological isomorphism.*

Note that any closed subset in $\mathbb{D}^{2n}$ is compact.

Let m be a non-negative integer and $X := \mathbb{C}^{n+m}$. We denote by $\hat{X}$ the partial radial compactification $\mathbb{D}^{2n} \times \mathbb{C}^m$ of $\mathbb{C}^{n+m}$, and we also denote by X_∞ the closed subset $\hat{X} \setminus X$ in $\hat{X}$. Let $p_1 : \hat{X} = \mathbb{D}^{2n} \times \mathbb{C}^m \to \mathbb{D}^{2n}$ (resp. $p_2 : \hat{X} = \mathbb{D}^{2n} \times \mathbb{C}^m \to \mathbb{C}^m$) be the canonical projection to the first (resp. second) space. A family of fundamental neighborhoods of a point $(z_0, w_0) \in X \subset \hat{X}$ consists of

$$B_\epsilon(z_0, w_0) := \{(z, w) \in X;\, |z - z_0| < \epsilon,\, |w - w_0| < \epsilon\} \tag{13}$$

for $\epsilon > 0$, and that of $(z_0, w_0) \in X_\infty$ consists of a product of an open cone and an open ball given by

$$G_r(\Gamma, w_0) := \left(\left\{z \in \mathbb{C}^n;\, |z| > r,\, \frac{z}{|z|} \in \Gamma\right\} \cup \Gamma\infty\right) \times \left\{w \in \mathbb{C}^m;\, |w - w_0| < \frac{1}{r}\right\}, \tag{14}$$

where $r > 0$ and Γ runs through open neighborhoods of z_0 in $S^{2n-1}\infty$.

Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. We define the sheaf of holomorphic functions of exponential type on $\hat{X}$.

Definition 3.2. *Let Ω be an open subset in $\hat{X}$. The set $\mathcal{O}^{\exp}_{\hat{X}}(\Omega)$ of holomorphic functions of exponential type on Ω consists of a holomorphic function $f(z, w)$ on $\Omega \cap X$ which satisfies, for any compact set K in Ω,*

$$|f(z, w)| \le C_K e^{H_K|z|} \qquad ((z, w) \in K \cap X) \tag{15}$$

with some positive constants C_K and H_K. We denote by $\mathcal{O}^{\exp}_{\hat{X}}$ the associated sheaf on $\hat{X}$ of the presheaf $\{\mathcal{O}^{\exp}_{\hat{X}}(\Omega)\}_\Omega$.

For any open subset $\Omega \subset \hat{X}$, we can take an exhausting family $\{\Omega_k\}_k$ of Ω satisfying the conditions below.

1. Ω_k is an open subset of Ω, and the union of Ω_k is equal to Ω.
2. $\overline{\Omega_k}$ is a compact set and $\overline{\Omega_k} \subset \Omega_{k+1}$ $(k = 1, 2, \dots)$.
3. Each Ω_k is a finite union of open subsets given by either (13) or (14).

Then $f \in \mathcal{O}^{\exp}_{\hat{X}}(\Omega)$ if and only if the estimate (15) holds with $K = \overline{\Omega_k}$ $(k = 1, 2, \dots)$. In particular, if $\Omega \subset X$, then each $\overline{\Omega_k}$ is bounded in X and the estimate (15) is always satisfied. Hence we see that the restriction of $\mathcal{O}^{\exp}_{\hat{X}}$ to X is nothing but the sheaf $\mathcal{O}_X$ of holomorphic functions on X.

3.2 Soft resolution for the sheaf $\mathcal{O}_{\hat{X}}^{\exp}$

We review a soft resolution for the sheaf $\mathcal{O}_{\hat{X}}^{\exp}$.

Definition 3.3. *For an open subset $\Omega \subset \hat{X}$, we denote by $\mathcal{G}(\Omega)$ the set of real valued continuous functions $\varphi(z,w)$ on $\Omega \cap X$ that satisfy, for any compact set K in Ω,*

$$\varphi(z,w) \leq \alpha_K + \beta_K |z| \qquad ((z,w) \in K \cap X) \tag{16}$$

with some positive constants α_K and β_K.

Clearly $\mathcal{G}(\Omega)$ is a directed set with respect to the partial order $f \leq g \iff f(z,w) \leq g(z,w)$ for $(z,w) \in \Omega \cap X$.

Definition 3.4. *Let Ω be an open subset in $\hat{X}$. We denote by $L^2_{\mathcal{G}}(\Omega)$ the set of locally square integrable functions f on $\Omega \cap X$ satisfying*

$$\int_{\Omega \cap X} |f(z,w)|^2 e^{-\varphi(z,w)} d\lambda < +\infty \tag{17}$$

for some $\varphi \in \mathcal{G}(\Omega)$.

We denote by $L^{2,(p,q)}_{\mathcal{G}}(\Omega)$ the set of (p,q)-forms on $\Omega \cap X$ with coefficients in $L^2_{\mathcal{G}}(\Omega)$. Moreover we set

$$\tilde{L}^{2,(p,q)}_{\mathcal{G}}(\Omega) := \left\{ f \in L^{2,(p,q)}_{\mathcal{G}}(\Omega);\, \overline{\partial} f \in L^{2,(p,q+1)}_{\mathcal{G}}(\Omega) \right\}.$$

The presheaf $\{L^2_{\mathcal{G}}(\Omega)\}_{\Omega}$ $\left(\text{resp. } \left\{L^{2,(p,q)}_{\mathcal{G}}(\Omega)\right\}_{\Omega} \text{ and } \left\{\tilde{L}^{2,(p,q)}_{\mathcal{G}}(\Omega)\right\}_{\Omega}\right)$ forms a sheaf on $\hat{X}$. We denote it by $\mathcal{L}^2_{\mathcal{G}}$ (resp. $\mathcal{L}^{2,(p,q)}_{\mathcal{G}}$ and $\tilde{\mathcal{L}}^{2,(p,q)}_{\mathcal{G}}$). Note that these sheaves are soft.

Proposition 3.5 ([1]). *We have the following soft resolution for the sheaf $\mathcal{O}_{\hat{X}}^{\exp}$ on $\hat{X}$.*

$$0 \to \mathcal{O}_{\hat{X}}^{\exp} \to \tilde{\mathcal{L}}^{2,(0,0)}_{\mathcal{G}} \xrightarrow{\overline{\partial}_0} \tilde{\mathcal{L}}^{2,(0,1)}_{\mathcal{G}} \xrightarrow{\overline{\partial}_1} \dots \xrightarrow{\overline{\partial}} \tilde{\mathcal{L}}^{2,(0,n+m)}_{\mathcal{G}} \to 0. \tag{18}$$

Hence we see that flabby dim $\mathcal{O}_{\hat{X}}^{\exp} \leq \dim \hat{X} + 1$.

3.3 Regularity condition at ∞ for an open subset in $\hat{X}$

We introduce the regularity condition at ∞ for an open subset in $\hat{X}$ which are needed for the vanishing theorem for the sheaf $\mathcal{O}^{\mathrm{exp}}_{\hat{X}}$.

Definition 3.6 ([1]). *For a subset A in $\hat{X}$, we define the set $\mathrm{clos}^1_\infty(A) \subset X_\infty$ as follows. A point $(z, w) \in X_\infty$ belongs to $\mathrm{clos}^1_\infty(A)$ if there exist points $\{(z_k, w_k)\}_{k\in\mathbb{N}}$ in $A \cap X$ that satisfy the following two conditions.*

1. $(z_k, w_k) \to (z, w)$ *in* $\hat{X}$.
2. $\frac{|z_{k+1}|}{|z_k|} \to 1$ *as* $k \to \infty$.

We set $N^1_\infty(A) := X_\infty \setminus \mathrm{clos}^1_\infty(X \setminus A)$. An open subset U in $\hat{X}$ is said to be regular at ∞ if $N^1_\infty(U) = U \cap X_\infty$ is satisfied.

Note that, for subsets $A_1, A_2, \ldots, A_\ell$ in $\hat{X}$, we have

$$N^1_\infty(A_1 \cap \cdots \cap A_\ell) = N^1_\infty(A_1) \cap \cdots \cap N^1_\infty(A_\ell).$$

Hence a finite intersection of open subsets which are regular at ∞ is again regular at ∞. We give a sufficient condition for which an open subset becomes regular at ∞. Let A be a subset in $\hat{X}$, and we set

$$N^L_\infty(A) := \left\{(\zeta, w) \in X_\infty;\ (\zeta, w) \in \overline{(\mathbb{R}_+\zeta \times \{w\}) \cap A}\right\} \subset X_\infty, \tag{19}$$

where $\mathbb{R}_+\zeta$ is the real half line in $\mathbb{C}^n$ with direction ζ and the closure is taken in $\hat{X}$. For subsets $A_1, A_2, \ldots, A_\ell$ in $\hat{X}$, we have

$$N^L_\infty(A_1 \cup \cdots \cup A_\ell) = N^L_\infty(A_1) \cup \cdots \cup N^L_\infty(A_\ell). \tag{20}$$

Lemma 3.7 ([1]). *Let U be an open subset in $\hat{X}$. If $N^L_\infty(U) = U \cap X_\infty$ holds, then U is regular at ∞.*

A finite union of open subsets which satisfy the condition given in the above lemma is also regular at ∞ by (20). We give some examples of open subsets which are regular at ∞.

Example 3.8 ([1]). *Let U be the open set $G_r(\Gamma, 0) \cup \tilde{U}$ where $\tilde{U}$ is a bounded open subset in X and the cone $G_r(\Gamma, 0)$ was defined by (14) with $r > 0$ and Γ being an open subset in S^{2n-1}. Then U is regular at ∞ as we have $N^L_\infty(U) = U \cap X_\infty$. In particular, $\mathbb{D}^2$ and $\mathbb{D}^2 \setminus [a, +\infty]$ ($a \in [-\infty, \infty)$) are regular at ∞.*

Example 3.9 ([1]). *For the set $U := \mathbb{D}^2 \setminus \{1, 2, 3, 4, \ldots, +\infty\}$ we have $N^1_\infty(U) = S^1\infty \setminus \{+\infty\}$. Hence U is regular at ∞. However, for the set $U := \mathbb{D}^2 \setminus \{1, 2, 4, 8, 16, \ldots, +\infty\}$, U is not regular because of $N^1_\infty(U) = S^1\infty$. Note that we have $N^L_\infty(U) = S^1\infty$ for the both cases.*

3.4 Vanishing theorem for the sheaf $\mathcal{O}_{\hat{X}}^{\exp}$

Before stating the vanishing theorem, we prepare some notations. For a subset A in X, we denote by $\mathrm{dist}(p, A)$ the distance between a point p and A, i.e., $\mathrm{dist}(p,\, A) := \inf_{q \in A} |p - q|$. If A is empty, we set $\mathrm{dist}(p,\, A) = +\infty$. We also define, for $q = (z, w) \in X$,

$$\mathrm{dist}_{\mathbb{D}^{2n}}(q,\, A) := \mathrm{dist}(q,\, A \cap p_2^{-1}(p_2(q))) = \inf_{(\zeta,\, w) \in A} |z - \zeta|.$$

For an open subset $\Omega \subset \hat{X}$, we define the function by

$$\psi(p) := \min\left\{\frac{1}{2},\ \frac{\mathrm{dist}_{\mathbb{D}^{2n}}(p,\, X \setminus \Omega)}{1 + |z|}\right\} \quad \text{for } p = (z, w) \in X, \tag{21}$$

and we set
(22)

$$\Omega_\epsilon := \left\{p = (z, w) \in \Omega \cap X;\ \mathrm{dist}(p,\, X \setminus \Omega) > \epsilon,\ |w| < \frac{1}{\epsilon}\right\} \quad (\epsilon > 0).$$

Note that $\psi(p)$ is lower semicontinuous (i.e., $\{p \in X;\ \psi(p) > c\}$ is open for $c \in \mathbb{R}$) and continuous with respect to the variables z, however, it is not necessarily continuous with respect to the variables w. Furthermore, if $p_1((X \setminus \Omega) \cap p_2^{-1}(w_0))$ $(w_0 \in \mathbb{C}^m)$ is a bounded subset in $\mathbb{C}^n$, then $\psi(z,\, w_0)$ is identically equal to $\frac{1}{2}$ for a sufficiently large z. Hence the values of $\psi(z,\, w)$ for a large z are independent of the shape of Ω in a bounded region.

THEOREM 3.10 ([1]). *Assume the following conditions* 1. *and* 2.

1. *$\Omega \cap X$ is pseudoconvex in X and Ω is regular at ∞.*
2. *At a point in $\Omega \cap X$ sufficiently close to $z = \infty$ the $\psi(z, w)$ is continuous and uniformly continuous with respect to the variables w, that is, for any $\epsilon > 0$, there exist $\delta_\epsilon > 0$ and $R_\epsilon > 0$ for which $\psi(z, w)$ is continuous on $\Omega_{\epsilon, R_\epsilon} := \Omega_\epsilon \cap \{|z| > R_\epsilon\}$ and it satisfies*

$$|\psi(z,\, w) - \psi(z,\, w')| < \epsilon, \qquad ((z,\, w),\, (z,\, w') \in \Omega_{\epsilon, R_\epsilon},\ |w - w'| < \delta_\epsilon).$$

Then we have

$$H^k(\Omega,\, \mathcal{O}_{\hat{X}}^{\exp}) = 0 \qquad (k \neq 0). \tag{23}$$

.

Remark 3.11. *i) Let U (resp. W) be an open subset in $\mathbb{D}^{2n}$ (resp. $\mathbb{C}^m$). If $U \cap \mathbb{C}^n$ and W are pseudoconvex in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, and if U is regular at ∞ in $\mathbb{D}^{2n}$, then $U \times W$ automatically satisfies the condition 2. in the theorem. Therefore we have*

$$H^k(U \times W, \mathcal{O}_{\hat{X}}^{\exp}) = 0 \qquad (k \neq 0).$$

ii) If $n = 1$, the vanishing theorem still holds for an open subset $U \times W \subset \mathbb{D}^2 \times \mathbb{C}^m$ of product type without the regularity of U at ∞. However, if n is greater than one, one cannot expect the vanishing theorem anymore without the regularity condition.

We now give examples.

Example 3.12 ([1]). Assume $n = m = 1$, i.e., $X = \mathbb{C}_z \times \mathbb{C}_w$ and $\hat{X} = \mathbb{D}^2 \times \mathbb{C}_w$. Let $f : X \to \mathbb{C}$ be the holomorphic map defined by $f(z, w) = zw$. Set

$$\begin{aligned} \tilde{\Omega} &:= \{\zeta \in \mathbb{C};\, |\zeta| < 1\} \cup \{\zeta \in \mathbb{C};\, |\arg \zeta| < 1\} \subset \mathbb{C}, \\ \Omega &:= \left(\overline{f^{-1}(\tilde{\Omega})}\right)^{\circ} \subset \hat{X}. \end{aligned}$$

Here the closure and the interior are taken in $\hat{X}$. To understand the shape of Ω clearly, the intersection of Ω and the complex line $\{(z, w) \in \hat{X};\, w = w_0\}$ for $w_0 \in \mathbb{C}_w$ is described below.

$$p_1(\Omega \cap p_2^{-1}(w_0)) = \begin{cases} \left(\overline{\dfrac{1}{w_0}\tilde{\Omega}}\right)^{\circ} \subset \mathbb{D}^2, & (w_0 \neq 0), \\ \\ \mathbb{C} \subset \mathbb{D}^2, & (w_0 = 0). \end{cases}$$

Then Ω satisfies all the conditions of the theorem, and hence, we have $H^k(\Omega, \mathcal{O}_{\hat{X}}^{\exp}) = 0$ $(k \neq 0)$.

Example 3.13 ([1]). Assume $n = 2$ and $m = 0$, i.e., $X = \mathbb{C}^2_{(z_1, z_2)}$ and $\hat{X} = \mathbb{D}^4$. Set

$$\begin{aligned} U &:= \left\{(z_1,\, z_2) \in X;\, |\arg(z_1)| < \frac{\pi}{4},\, |z_2| < |z_1|\right\}, \\ \Omega &:= \left(\overline{U}\right)^{\circ} \setminus \{p_\infty\} \subset \hat{X}, \end{aligned}$$

where p_∞ denotes the point $(1, 0, 0, 0)\infty$ in $S^3\infty \subset \mathbb{D}^4$. Note that $\Omega \cap X = U$ is pseudoconvex in X, while Ω is not regular at ∞. In this case, we have $H^1(\Omega, \mathcal{O}_{\hat{X}}^{\exp}) \neq 0$ which is shown below, and the vanishing theorem does not hold for Ω.

Let $Y = \mathbb{C}^1_z \times \mathbb{C}^1_w$ and $\hat{Y} = \mathbb{D}^2 \times \mathbb{C}^1$, and let us consider the holomorphic map $f : X \setminus \{z_1 = 0\} \to Y$ defined by $f(z_1, z_2) = \left(z_1, \dfrac{z_2}{z_1}\right)$. Set

$$\begin{aligned}
\tilde{U} &:= \left\{(z, w) \in Y;\, |\arg(z)| < \frac{\pi}{4},\, |w| < 1\right\},\\
\tilde{\Omega} &:= \left(\overline{\tilde{U}}\right)^{\circ} \setminus (\{q_\infty\} \times \{0\}) \subset \hat{Y}.
\end{aligned}$$

Here $q_\infty = (1, 0)\infty \in S^1\infty \subset \mathbb{D}^2$. Note that $\tilde{\Omega}$ is an open subset of non-product type in $\mathbb{D}^2 \times \mathbb{C}$. As f gives a biholomorphic map between U and $\tilde{U}$ which extends to a continuous isomorphism between Ω and $\tilde{\Omega}$, we have

$$H^k(\Omega, \mathcal{O}^{\exp}_{\hat{X}}) = H^k(\tilde{\Omega}, \mathcal{O}^{\exp}_{\hat{Y}}), \qquad (k \in \mathbb{N}).$$

Hence it suffices to prove $H^1(\tilde{\Omega}, \mathcal{O}^{\exp}_{\hat{Y}}) \neq 0$. Set

$$V := \left(\overline{\left\{z \in \mathbb{C};\, |\arg(z)| < \frac{\pi}{4}\right\}}\right)^{\circ} \subset \mathbb{D}^2, \qquad W := \{w \in \mathbb{C};\, |w| < 1\}.$$

Noticing $((V \setminus \{q_\infty\}) \times W) \cup (V \times (W \setminus \{0\})) = \tilde{\Omega}$, we have the long exact sequence

$$\begin{aligned}
(24)\quad &\mathcal{O}^{\exp}_{\hat{Y}}((V \setminus \{q_\infty\}) \times W) \oplus \mathcal{O}^{\exp}_{\hat{Y}}(V \times (W \setminus \{0\}))\\
&\qquad \xrightarrow{\iota} \mathcal{O}^{\exp}_{\hat{Y}}((V \setminus \{q_\infty\}) \times (W \setminus \{0\})) \to H^1(\tilde{\Omega}, \mathcal{O}^{\exp}_{\hat{Y}}).
\end{aligned}$$

Suppose $H^1(\tilde{\Omega}, \mathcal{O}^{\exp}_{\hat{Y}}) = 0$. Then ι becomes surjective. It is well known that there exists a holomorphic function $g(z)$ in $\mathcal{O}^{\exp}_{\mathbb{D}^2}(V \setminus \{q_\infty\})$ which does not belong to $\mathcal{O}^{\exp}_{\mathbb{D}^2}(V)$ (for existence of such a holomorphic function, see [15]). Set $h(z, w) := \dfrac{g(z)}{w}$. Then $h(z, w)$ belongs to $\mathcal{O}^{\exp}_{\hat{Y}}((V \setminus \{q_\infty\}) \times (W \setminus \{0\}))$. As ι is surjective, there exist $h_1(z, w) \in \mathcal{O}^{\exp}_{\hat{Y}}((V \setminus \{q_\infty\}) \times W)$ and $h_2(z, w) \in \mathcal{O}^{\exp}_{\hat{Y}}(V \times (W \setminus \{0\}))$ satisfying $h = h_1 + h_2$. Clearly we have

$$\begin{aligned}
2\pi\sqrt{-1}g(z) = \int_C h(z, w)dw &= \int_C (h_1(z, w) + h_2(z, w))dw\\
&= \int_C h_2(z, w)dw,
\end{aligned}$$

where C is a small circle turning around the origin in W. Since $\displaystyle\int_C h_2(z, w)dw$ belongs to $\mathcal{O}^{\exp}_{\mathbb{D}^2}(V)$, we get $g(z) \in \mathcal{O}^{\exp}_{\mathbb{D}^2}(V)$, which contradicts the choice of $g(z)$, i.e., $g(z) \notin \mathcal{O}^{\exp}_{\mathbb{D}^2}(V)$. Therefore we have obtained the conclusion $H^1(\Omega, \mathcal{O}^{\exp}_{\hat{X}}) = H^1(\tilde{\Omega}, \mathcal{O}^{\exp}_{\hat{Y}}) \neq 0$.

4 Laplace hyperfunctions of one variable with holomorphic parameters

Thanks to Theorem 3.10, we can construct the sheaf of Laplace hyperfunctions of one variable with holomorphic parameters. In this section, we give its definition and we construct the sheaf morphism from the sheaf of locally integrable functions with exponential growth to the sheaf of Laplace hyperfunctions. From now on, we consider the case of dimension $n = 1$. Let $N = \mathbb{R} \times \mathbb{C}^m (m \geq 0)$, and let $\overline{N} = \overline{\mathbb{R}} \times \mathbb{C}^m$ be the closure of N in $\hat{X} = \mathbb{D}^2 \times \mathbb{C}^m$.

4.1 Sheaf of Laplace hyperfunctions on $\overline{N}$

The following fact is shown by Theorem 3.10.

THEOREM 4.1 ([1]). *The closed set $\overline{\mathbb{R}}$ in $\mathbb{D}^2$ is purely* 1- *codimensional relative to the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^2}$. More generally, the closed set $\overline{N}$ in $\hat{X}$ is purely* 1-*codimensional relative to the sheaf $\mathcal{O}^{\exp}_{\hat{X}}$, i.e., $\mathscr{H}^k_{\overline{N}}(\mathcal{O}^{\exp}_{\hat{X}}) = 0$ for $k \neq 1$.*

DEFINITION 4.2. *The sheaf $\mathcal{BO}^{\exp}_{\overline{N}}$ of Laplace hyperfunctions on $\overline{N}$ is defined by*

$$\mathcal{BO}^{\exp}_{\overline{N}} := \mathscr{H}^1_{\overline{N}}(\mathcal{O}^{\exp}_{\hat{X}}) \underset{\mathbb{Z}_{\overline{N}}}{\otimes} \omega_{\overline{N}}. \tag{25}$$

Here $\mathscr{H}^1_{\overline{N}}(\mathcal{O}^{\exp}_{\hat{X}})$ is the first derived sheaf of $\mathcal{O}^{\exp}_{\hat{X}}$ with support in $\overline{N}$, the $\mathbb{Z}_{\overline{N}}$ denotes the constant sheaf on $\overline{N}$ having stalk $\mathbb{Z}$ and $\omega_{\overline{N}}$ denotes the orientation sheaf $\mathscr{H}^1_{\overline{N}}(\mathbb{Z}_{\hat{X}})$ on $\overline{N}$. Especially, in the case of m=0, we define the sheaf $\mathcal{B}^{\exp}_{\overline{\mathbb{R}}}$ of Laplace hyperfunctions of one variable on $\overline{\mathbb{R}}$ by

$$\mathcal{B}^{\exp}_{\overline{\mathbb{R}}} := \mathscr{H}^1_{\overline{\mathbb{R}}}(\mathcal{O}^{\exp}_{\mathbb{D}^2}) \underset{\mathbb{Z}_{\overline{\mathbb{R}}}}{\otimes} \omega_{\overline{\mathbb{R}}}. \tag{26}$$

Since the sheaf $\mathscr{H}^0_{\overline{N}}(\mathcal{O}^{\exp}_{\hat{X}})$ is zero by Theorem 4.1, we find that the presheaf $U \mapsto H^1_{\overline{N}\cap U}(U, \mathcal{O}^{\exp}_{\hat{X}})$ is a sheaf, and is equal to $\mathcal{BO}^{\exp}_{\overline{N}}$. Hence the global sections of the sheaf $\mathcal{BO}^{\exp}_{N}$ can be written in terms of cohomology groups. For an open set $\Omega \subset \overline{\mathbb{R}}$ and a pseudoconvex open subset $T \subset \mathbb{C}^m$, by taking a complex neighborhood V of Ω in $\mathbb{D}^2$, we have

$$\mathcal{BO}^{\exp}_{\overline{N}}(\Omega \times T) = H^1_{\Omega\times T}(V \times T, \mathcal{O}^{\exp}_{\hat{X}}) = \frac{\mathcal{O}^{\exp}_{\hat{X}}((V \setminus \Omega) \times T)}{\mathcal{O}^{\exp}_{\hat{X}}(V \times T)}. \tag{27}$$

According to the excision theorem, we may replace V by any complex open set containing Ω. Similarly we have

$$\Gamma_{[a,\infty]}(\overline{\mathbb{R}},\, \mathcal{B}^{\exp}_{\overline{\mathbb{R}}}) = \frac{\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2 \setminus [a,\, \infty])}{\mathcal{O}^{\exp}_{\mathbb{D}^2}(\mathbb{D}^2)}. \tag{28}$$

Hence the set $\mathcal{B}^{\exp}_{[a,\infty]}$ defined by H. Komatsu coincides with $\Gamma_{[a,\infty]}(\overline{\mathbb{R}}, \mathcal{B}^{\exp}_{\overline{\mathbb{R}}})$ in our framework. Note that the restriction of $\mathcal{B}^{\exp}_{\overline{\mathbb{R}}}$ to $\mathbb{R}$ is isomorphic to the sheaf $\mathcal{B}_{\mathbb{R}}$ of ordinary hyperfunctions because of $\mathcal{O}^{\exp}_{\mathbb{D}^2}|_{\mathbb{C}} = \mathcal{O}_{\mathbb{C}}$.

Now we state the theorem for the flabbiness and the unique continuation property of $\mathcal{BO}^{\exp}_{N}$.

THEOREM 4.3 ([1]). *Let $\Omega_1 \subset \Omega_2 \subset \bar{\mathbb{R}}$ and $W_1 \subset W_2 \subset \mathbb{C}^m$ be open subsets. Then we have*

i) If W_1 is a Stein open subset in $\mathbb{C}^m$, then $\mathcal{BO}^{\exp}_{\overline{N}}(\Omega_2 \times W_1) \to \mathcal{BO}^{\exp}_{\overline{N}}(\Omega_1 \times W_1)$ is surjective, i.e., the sheaf $\mathcal{BO}^{\exp}_{\overline{N}}$ is flabby with respect to the variable of hyperfunction.

ii) If W_1 and W_2 be non-empty connected open subsets in $\mathbb{C}^m$, then $\mathcal{BO}^{\exp}_{\overline{N}}(\Omega_1 \times W_2) \to \mathcal{BO}^{\exp}_{\overline{N}}(\Omega_1 \times W_1)$ is injective, i.e., the sheaf $\mathcal{BO}^{\exp}_{\overline{N}}$ has a unique continuation property with respect to variables of holomorphic parameters.

4.2 Embedding of locally integrable functions with exponential growth

We construct the sheaf morphism from the sheaf of locally integrable functions with exponential growth to the sheaf of Laplace hyperfunctions and show its injectivity.

DEFINITION 4.4. *Let Ω be an open subset in $\overline{\mathbb{R}}$. The set $\mathcal{L}^{\exp}_{\mathrm{loc}}(\Omega)$ of locally integrable functions of exponential type on Ω consists of a locally integrable function $f(x)$ on $\Omega \cap \mathbb{R}$ which satisfies, for any compact set K in Ω,*

$$\int_{K \cap \mathbb{R}} |f(x)| e^{-H_K |x|} dx < \infty \tag{29}$$

with a constant H_K. We denote by $\mathcal{L}^{\exp}_{\mathrm{loc}}$ the associated sheaf on $\overline{\mathbb{R}}$ of the presheaf $\{\mathcal{L}^{\exp}_{\mathrm{loc}}(\Omega)\}_{\Omega}$.

Note that, if $\Omega \subset \mathbb{R}$, the estimate (29) is always satisfied. Hence the restriction of $\mathcal{L}^{\exp}_{\mathrm{loc}}$ to $\mathbb{R}$ is isomorphic to the sheaf $\mathcal{L}^{1}_{\mathrm{loc}}$ of locally integrable functions on $\mathbb{R}$.

Let us construct a sheaf morphism ι from the sheaf $\mathcal{L}^{\exp}_{\mathrm{loc}}$ to the sheaf $\mathcal{B}^{\exp}_{\overline{\mathbb{R}}}$. It suffices to give a morphism $\iota_K : \Gamma_K(\overline{\mathbb{R}},\, \mathcal{L}^{\exp}_{\mathrm{loc}}) \to \Gamma_K(\overline{\mathbb{R}},\, \mathcal{B}^{\exp}_{\overline{\mathbb{R}}})$ for any compact set K in $\overline{\mathbb{R}}$. Moreover, by considering a partition of support, it is enough to give morphisms for any $K \subseteq [0,\, \infty]$ or $K \subseteq [-\infty,\, 0]$.

Let K be a compact set in $[0,\, \infty]$ or $[-\infty,\, 0]$ and let $f \in \Gamma_K(\overline{\mathbb{R}},\, \mathcal{L}^{\exp}_{\mathrm{loc}})$ satisfying (29) for a constant H_K. For an arbitrary constant $A \geq H_K$, we set

$$F^{\pm}(z) := \frac{e^{\pm Az}}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \frac{f(t)e^{-A|t|}}{t-z} dt. \tag{30}$$

As $f(x)e^{-A|x|}$ is integrable on $\mathbb{R}$, the functions $F^{\pm}$ give a holomorphic functions of exponential type on $\mathbb{D}^2 \setminus K$. If $K \subseteq [0,\, \infty]$, we define the morphism $\iota_K : \Gamma_K(\overline{\mathbb{R}},\, \mathcal{L}^{\exp}_{\mathrm{loc}}) \to \Gamma_K(\overline{\mathbb{R}},\, \mathcal{B}^{\exp}_{\overline{\mathbb{R}}})$ by $\iota_K(f) = [F^+]$, where F^+ is given by (30). Note that $\iota_K(f)$ does not depend on a choice of A. As a matter of fact, we have the following equation

$$\begin{aligned} &\frac{e^{Az}}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \frac{f(t)e^{-A|t|}}{t-z} dt - \frac{e^{Bz}}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \frac{f(t)e^{-B|t|}}{t-z} dt \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} f(t)dt \int_{A}^{B} e^{-w(|t|-z)} dw \end{aligned} \tag{31}$$

for constants $B > A \geq H_K$ and the right hand side of (31) is an entire function of exponential type. Hence ι_K is well-defined and clearly satisfies $\operatorname{supp} \iota_K(f) \subset \operatorname{supp} f$. If $K \subseteq [-\infty,\, 0]$, we define ι_K by $\iota_K(f) = [F^-]$ in the similar way as the case of K in $[0,\, \infty]$. Note that, for any compact set K in $\mathbb{R}$, we have

$$\left[F^+\right] = [F^-] = \left[\frac{1}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt\right] \tag{32}$$

in $\Gamma_K(\mathbb{R},\, \mathcal{B}_{\mathbb{R}})$ and $\Gamma_{\{\infty\}}(\overline{\mathbb{R}},\, \mathcal{L}^{\exp}_{\mathrm{loc}}) = \Gamma_{\{-\infty\}}(\overline{\mathbb{R}},\, \mathcal{L}^{\exp}_{\mathrm{loc}}) = \emptyset$. Therefore we can define the morphism ι_K for any compact set K in $\overline{\mathbb{R}}$ (using a partition of support, if necessary). We define the morphism $\iota_c : \Gamma_c(\overline{\mathbb{R}},\, \mathcal{L}^{\exp}_{\mathrm{loc}}) \to \Gamma_c(\overline{\mathbb{R}},\, \mathcal{B}^{\exp}_{\overline{\mathbb{R}}})$ by $\{\iota_K\}_K$. Hence ι_c is extended to the sheaf morphism $\iota : \mathcal{L}^{\exp}_{\mathrm{loc}} \to \mathcal{B}^{\exp}_{\overline{\mathbb{R}}}$ uniquely. The details are as follows; let U be an open subset in $\overline{\mathbb{R}}$. For a locally integrable function of exponential type $f \in \mathcal{L}^{\exp}_{\mathrm{loc}}(U)$, we first decompose it into a locally finite sum of locally integrable functions of exponential type with compact support:

$$f = \sum_{\lambda} f_{\lambda}. \tag{33}$$

Then the morphism $\iota_U : \mathcal{L}_{\rm loc}^{\rm exp}(U) \to \mathcal{B}_{\overline{\mathbb{R}}}^{\rm exp}(U)$ is defined by

$$\iota_U(f) = \sum_{\lambda} \iota_c(f_\lambda). \tag{34}$$

This is defined independently of a choice of a locally finite decomposition (33) of $f \in \mathcal{L}_{\rm loc}^{\rm exp}(U)$.

Let us show that the sheaf morphism $\iota : \mathcal{L}_{\rm loc}^{\rm exp} \to \mathcal{B}_{\overline{\mathbb{R}}}^{\rm exp}$ is injective. For that purpose, we see that ι_K is injective for any compact K in $[0, \infty]$ or $[-\infty, 0]$.

PROPOSITION 4.5 ([19]). *Let K be a compact set in $[0, \infty]$ or $[-\infty, 0]$. Let $f \in \Gamma_K(\overline{\mathbb{R}}, \mathcal{L}_{\rm loc}^{\rm exp})$ and let $F^{\pm}(z)$ be the function defined by (30). Then $F^{\pm}(x+\sqrt{-1}\varepsilon) - F^{\pm}(x-\sqrt{-1}\varepsilon)$ converge to $f(x)$ almost everywhere as $\varepsilon \to 0$.*

By Proposition 4.5, the morphism ι_c satisfies

$$\mathrm{supp}\iota_c(f) = \mathrm{supp} f, \qquad f \in \Gamma_c(\overline{\mathbb{R}}, \mathcal{L}_{\rm loc}^{\rm exp}).$$

Hence we get the following theorem.

THEOREM 4.6. *The sheaf morphism $\iota : \mathcal{L}_{\rm loc}^{\rm exp} \to \mathcal{B}_{\overline{\mathbb{R}}}^{\rm exp}$ is injective and we can regard functions in $\mathcal{L}_{\rm loc}^{\rm exp}$ as Laplace hyperfunctions.*

We also see that the Laplace transformation as a Laplace hyperfunction and an ordinary function coincide on the space of locally integrable functions with exponential growth.

THEOREM 4.7 ([19]). *Let K be a compact subset in $[0, \infty]$ and let $f \in \Gamma_K(\overline{\mathbb{R}}, \mathcal{L}_{\rm loc}^{\rm exp})$. The Laplace transform $\widehat{\iota(f)}$ of the Laplace hyperfunction $\iota(f)$ coincides with the ordinary Laplace transform of f .*

5 The edge of the wedge theorem for the sheaf $\mathcal{O}_{\mathbb{D}^{2n}}^{\rm exp}$ of holomorphic functions of exponential type

The purpose of this section is to establish the edge of the wedge theorem for the sheaf of holomorphic functions of exponential type. The theorem is stated in subsection 5.2.

5.1 Martineau type theorem for holomorphic functions of exponential type

Before going into the proof for the theorem, we prepare several theorems.

LEMMA 5.1. *Let $S \subset \overline{\mathbb{R}}$ be a closed set and $V \subset \mathbb{C}^m (m \geq 0)$ a Stein open set. Assume that $K_1, \ldots, K_{n-1}$ are locally closed sets in $\mathbb{C}$. Then we have*

$$H^k_{S\times K_1\times\cdots\times K_{n-1}\times V}(\mathbb{D}^2\times\mathbb{C}^{n-1}\times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) = 0 \qquad (k \geq n+1). \tag{35}$$

Proof. Take arbitrary open neighborhoods U of S in $\mathbb{D}^2$ and $W_i\,(1 \leq i \leq n-1)$ of K_i in $\mathbb{C}$, respectively. Set

(36)

$$\begin{aligned}
&T := U \times W_1 \times \cdots \times W_{n-1} \times V,\\
&T_0 := (U \setminus S) \times W_1 \times \cdots \times W_j \times \cdots \times W_{n-1} \times V,\\
&T_j := U \times W_1 \times \cdots \times (W_j \setminus K_j) \times \cdots \times W_{n-1} \times V, \quad (j = 1, \ldots, n-1).
\end{aligned}$$

Then the families of those sets $\{T, T_0, \ldots, T_{n-1}\}$ and $\{T_0, \ldots, T_{n-1}\}$ give an open covering of the pair $(T, T \setminus (S \times K_1 \times \cdots \times K_{n-1} \times V))$ of open sets. It follows from Theorem 3.10 and Remark 3.11 that these open sets compose a Leray covering of the pair with respect to the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}$ of holomorphic functions of exponential type on $\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V$. Hence we can compute the cohomology groups (35) by this covering, and the result immediately comes from the fact that the number of open subsets of the covering is $n+1$. $\square$

The following Martineau type theorem for holomorphic functions of exponential type plays an important role in proving the edge of the wedge theorem for the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$.

THEOREM 5.2. (*The Martineau type theorem for holomorphic functions of exponential type*)
Let $S = [a, \infty] (a \in \mathbb{R})$ be a compact set in $\overline{\mathbb{R}}$. Let $K = K_1 \times \cdots \times K_{n-1} \subset L = L_1 \times \cdots \times L_{n-1}$ be a pair of closed polydisc in $\mathbb{C}^{n-1}$, and let $W \subset V \subset \mathbb{C}^m (m \geq 0)$ be a pair of non-empty connected Stein domains. Then the restriction

(37)

$$H^n_{S\times K\times V}(\mathbb{D}^2\times\mathbb{C}^{n-1}\times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) \to H^n_{S\times L\times W}(\mathbb{D}^2\times\mathbb{C}^{n-1}\times W,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})$$

is injective.

Proof. Let $(z,\, w,\, t)$ be the coordinates of $\mathbb{C}_z \times \mathbb{C}_w^{n-1} \times \mathbb{C}_t^m$. Take an open sector U containing S in $\mathbb{D}^2$ whose opening is sufficiently small. Set

(38)

$$\begin{aligned}
T_{S,K,V} &:= (U \setminus S) \times (\mathbb{C} \setminus K_1) \times \cdots \times (\mathbb{C} \setminus K_j) \times \cdots \times (\mathbb{C} \setminus K_{n-1}) \times V, \\
T^{(0)}_{S,K,V} &:= U \times (\mathbb{C} \setminus K_1) \times \cdots \times (\mathbb{C} \setminus K_j) \times \cdots \times (\mathbb{C} \setminus K_{n-1}) \times V, \\
T^{(j)}_{S,K,V} &:= (U \setminus S) \times (\mathbb{C} \setminus K_1) \times \cdots \times \mathbb{C} \times \cdots \times (\mathbb{C} \setminus K_{n-1}) \times V, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j = 1,\, \ldots,\, n-1).
\end{aligned}$$

For $S \times K \times V$, by taking the relative open covering introduced in the proof for Lemma 5.1, we obtain the representation

$$H^n_{S\times K\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) = \frac{\Gamma(T_{S,K,V},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})}{\bigoplus_{j=0}^{n-1} \Gamma(T^{(j)}_{S,K,V},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})}.$$

The sets $T_{S,L,W}$ and $T^{(j)}_{S,L,W}$ are also defined by (38) where K and V are replaced by L and W, respectively. Then the canonical morphism (37) coincides with

$$\iota\, :\, \frac{\Gamma(T_{S,K,V},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})}{\bigoplus_{j=0}^{n-1} \Gamma(T^{(j)}_{S,K,V},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})} \longrightarrow \frac{\Gamma(T_{S,L,W},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})}{\bigoplus_{j=0}^{n-1} \Gamma(T^{(j)}_{S,L,W},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})}. \tag{39}$$

Let us prove that the morphism ι is injective. For an element $F(z,\, w,\, t)$ in $\Gamma(T_{S,K,V},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})$, we define

$$G(z,\, w,\, t) := \frac{1}{(2\pi\sqrt{-1})^{n-1}} \int_{\gamma_1\times\cdots\times\gamma_{n-1}} \frac{F(z,\, \mu,\, t)}{(\mu_1 - w_1)\ldots(\mu_{n-1} - w_{n-1})} d\mu, \tag{40}$$

where each γ_j is an integral path in $\mathbb{C} \setminus K_j$ which encircles K_j with clock direction such that the variable w_j is outside γ_j . Note that $G(z,\, w,\, t)$ is a holomorphic functions of exponential type on $T_{S,K,V}$ by deformation of the integral path. Now let us take an integral path $\tilde{\gamma}_1$ in $\mathbb{C} \setminus L_1$ which encircles L_1 with clock direction such that w_1 and γ_1 are inside $\tilde{\gamma}_1$. Then we have

(41)

$$\begin{aligned}
G(z,\, w,\, t) = &\frac{1}{(2\pi\sqrt{-1})^{n-2}} \int_{\gamma_2\times\cdots\times\gamma_{n-1}} \frac{F(z,\, w_1,\, \mu_2,\, \ldots,\, \mu_{n-1},\, t)}{(\mu_2 - w_2)\ldots(\mu_{n-1} - w_{n-1})} d\mu \\
&+ \frac{1}{(2\pi\sqrt{-1})^{n-1}} \int_{\tilde{\gamma}_1\times\gamma_2\times\cdots\times\gamma_{n-1}} \frac{F(z,\, \mu,\, t)}{(\mu_1 - w_1)\ldots(\mu_{n-1} - w_{n-1})} d\mu.
\end{aligned}$$

We denote by $H_1(z,\, w,\, t)$ the second term on the right hand side of (41). Note that $H_1(z,\, w,\, t)$ is a holomorphic function of exponential type on $T^{(1)}_{S,K,V}$ by

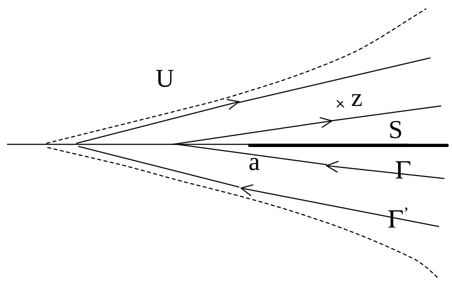

Figure 1.

deformation of the integral path. Let us take integral paths $\tilde{\gamma}_j\,(j \geq 2)$ in $\mathbb{C} \setminus L_j$ in the similar way to $\tilde{\gamma}_1$. Applying the similar deformations for the first term on the right hand side of (41) in the order of $\gamma_2, \ldots, \gamma_{n-1}$, we obtain

$$G(z,\, w,\, t) = F(z,\, w,\, t) + \sum_{j=1}^{n-1} H_j(z,\, w,\, t), \quad H_j \in \Gamma(T^{(j)}_{S,K,V},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}).$$

If we could prove that $G(z,\, w,\, t)$ can be extended to $T^{(0)}_{S,K,V}$ as a holomorphic functions of exponential type when $\iota(F) = 0$ in $H^n_{S\times L\times W}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times W,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})$, then we get the injectivity of the morphism ι.

Suppose that F satisfies $\iota(F) = 0$ in $H^n_{S\times L\times W}(\mathbb{D}^2\times\mathbb{C}^{n-1}\times W,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})$. Then there exist functions $\{F_j\}_j \subset \oplus_{j=0}^{n-1} \Gamma(T^{(j)}_{S,L,W},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})$ with $F = \sum_j F_j$ on $T_{S,L,W}$. Now we take an arbitrary point $(z,\, w,\, t)$ in $T_{S,K,V}$ and closed sectors Γ and Γ' as Figure 1. Let us denote by D_j and $\tilde{D}_j$ the interior of γ_j and $\tilde{\gamma}_j$, respectively. Take a relatively compact open subset Z in V satisfying $W \cap Z \neq \emptyset$. We may assume $w \in (\tilde{D}_1 \setminus D_1) \times \cdots \times (\tilde{D}_{n-1} \setminus D_{n-1})$ and $t \in Z$. By Cauchy's integral formula we have

(42)

$$G(z,\, w,\, t) = \frac{e^{Az}}{2\pi\sqrt{-1}} \int_{\Gamma} \frac{G(\zeta,\, w,\, t)e^{-A\zeta}}{\zeta - z} d\zeta - \frac{e^{Az}}{2\pi\sqrt{-1}} \int_{\Gamma'} \frac{G(\zeta,\, w,\, t)e^{-A\zeta}}{\zeta - z} d\zeta$$

for a sufficiently large A. Let $G_j(z,\, w,\, t)$ $(j = 0,\, 1)$ be the functions given by the integrals on the right hand side of (42) corresponding to Γ and Γ' in that order. For w and t belonging to a relatively compact open subset in $(\mathbb{C}^{n-1} \setminus L) \cap ((\tilde{D}_1 \setminus D_1) \times \cdots \times (\tilde{D}_{n-1} \setminus D_{n-1}))$ and $W \cap Z$, respectively, we have

(43)

$$
\begin{aligned}
&G_0(z,\, w,\, t)\\
&= \frac{e^{Az}}{(2\pi\sqrt{-1})^n} \sum_{j=0}^{n-1} \int_\Gamma \int_{\gamma_1\times\cdots\times\gamma_{n-1}} \frac{F_j(\zeta,\, \mu,\, t)e^{-A\zeta}}{(\mu_1 - w_1)\dots(\mu_{n-1} - w_{n-1})(\zeta - z)} d\zeta d\mu\\
&= \frac{e^{Az}}{(2\pi\sqrt{-1})^n} \int_\Gamma \int_{\gamma_1\times\cdots\times\gamma_{n-1}} \frac{F_0(\zeta,\, \mu,\, t)e^{-A\zeta}}{(\mu_1 - w_1)\dots(\mu_{n-1} - w_{n-1})(\zeta - z)} d\zeta d\mu.
\end{aligned}
$$

Since $e^{-A\zeta}$ compensates the exponential growth of F_0 at infinity, we have

(44)

$$
\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}} \int_\Gamma \frac{F_0(\zeta,\, \mu,\, t)e^{-A\zeta}}{\zeta - z} d\zeta\\
&= \frac{1}{2\pi\sqrt{-1}} \int_{a+\sqrt{-1}0}^{\infty+\sqrt{-1}0} \frac{F_0(\zeta,\, \mu,\, t)e^{-A\zeta}}{\zeta - z} d\zeta\\
&\qquad\qquad - \frac{1}{2\pi\sqrt{-1}} \int_{a-\sqrt{-1}0}^{\infty-\sqrt{-1}0} \frac{F_0(\zeta,\, \mu,\, t)e^{-A\zeta}}{\zeta - z} d\zeta = 0.
\end{aligned}
$$

This gives $G_0(z,\, w,\, t) \equiv 0$ for $(w,\, t) \in (\tilde{D}_1 \setminus D_1) \times \cdots \times (\tilde{D}_{n-1} \setminus D_{n-1}) \times V$ by the unique continuation property of a holomorphic function, and we obtain $G(z,\, w,\, t) = -G_1(z,\, w,\, t)$. As we can take any paths Γ, Γ', γ_j and $\tilde{\gamma}_j$ $(1 \le j \le n-1)$, we see that $G(z,\, w,\, t)$ can be extended to $T^{(0)}_{S,\, K,\, V}$ as a holomorphic functions of exponential type. □

By the Martineau type theorem we obtain the following results.

Lemma 5.3. *Let $S = [a,\, \infty](a \in \mathbb{R})$ be a compact set in $\overline{\mathbb{R}}$. Let $K = K_1 \times \cdots \times K_{n-1}$ be a closed polydisc in $\mathbb{C}^{n-1}$, and let V be a Stein domain in $\mathbb{C}^m$. Then*

$$
H^k_{S\times K\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) = 0 \qquad (k \neq n). \tag{45}
$$

Proof. It follows from Lemma 5.1 that (45) holds for $k \ge n+1$. Let us prove that the k-th cohomology group vanishes for $k \le n-1$. We use induction on the dimension n. When $n = 1$, the assertion to be proved is

$$
H^0_{S\times V}(\mathbb{D}^2 \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times V}) = \Gamma_{S\times V}(\mathbb{D}^2 \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times V}) = 0,
$$

which is nothing but the uniqueness of an analytic continuation for a usual holomorphic function. Assume that, for $(n-2)$-dimensional cylindrical compact sets, (45) is proved for any Stein domain V. We consider an $(n-1)$-dimensional cylindrical compact set $K = K_1 \times \cdots \times K_{n-1}$. For simplicity, set $\tilde{K} = K_1 \times \cdots \times K_{n-2}$. For the pair $S \times K \times V \subset S \times \tilde{K} \times \mathbb{C} \times V$, we obtain the following exact sequence of cohomology groups.

(46)
$$\begin{aligned}
\cdots &\to H^k_{S\times K\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) \\
&\to H^k_{S\times\tilde{K}\times\mathbb{C}\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-2} \times \mathbb{C} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) \\
&\xrightarrow{\iota} H^k_{S\times\tilde{K}\times(\mathbb{C}\setminus K_{n-1})\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-2} \times (\mathbb{C} \setminus K_{n-1}) \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) \to \cdots
\end{aligned}$$

For $0 \le k \le n-2$, the following cohomology groups vanish by the induction hypothesis:

$$\begin{aligned}
&H^k_{S\times\tilde{K}\times\mathbb{C}\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-2} \times \mathbb{C} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) = 0, \\
&H^k_{S\times\tilde{K}\times(\mathbb{C}\setminus K_{n-1})\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-2} \times (\mathbb{C} \setminus K_{n-1}) \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) = 0.
\end{aligned}$$

Note that the spaces of parameters are $\mathbb{C}\times V$ and $(\mathbb{C}\setminus K_{n-1})\times V$, respectively. Therefore (45) is proved for $0 \le k \le n-2$. According to Theorem 5.2, for $k = n-1$ the morphism ι in (46) is injective. Therefore, together with the fact that the preceding term vanishes, we conclude that $H^{n-1}_{S\times K\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) = 0$. □

Corollary 5.4. *Let $S = [a, \infty]$ $(a \in \mathbb{R})$ be a compact set in $\overline{\mathbb{R}}$. Let $K = K_1 \times \cdots \times K_{n-1} \subset L = L_1 \times \cdots \times L_{n-1}$ be a pair of closed polydisc in $\mathbb{C}^{n-1}$, and let $V \subset \mathbb{C}^m$ $(m \ge 0)$ be a Stein domain. Then*

$$H^k_{S\times(L\setminus K)\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) = 0 \qquad (k \neq n). \tag{47}$$

Proof. Consider the fundamental long exact sequence of cohomology groups

$$\begin{aligned}
\cdots &\to H^k_{S\times L\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) \\
&\to H^k_{S\times(L\setminus K)\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) \\
&\to H^{k+1}_{S\times K\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) \\
&\to H^{k+1}_{S\times L\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) \to \cdots
\end{aligned} \tag{48}$$

for the pair $S \times K \times V \subset S \times L \times V$. For $0 \le k \le n-2$, we have (47) by Lemma 5.3, and for $k \ge n+1$, this follows from Lemma 5.1. Finally, for $k = n-1$, this follows from the facts that the preceding term $H^{n-1}_{S\times L\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})$ vanishes, and that the morphism

$$\begin{aligned}
H^n_{S\times K\times V}&(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}) \\
&\to H^n_{S\times L\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V})
\end{aligned} \tag{49}$$

is injective by Theorem 5.2. Hence we have the assertion. □

This result can be generalized to the pair of two analytic polyhedra by making use of Oka's embedding. Now we recall the definition of the analytic polyhedron and prepare two lemmas to prove the similar theorem for analytic polyhedra.

Definition 5.5. *Let U be a domain. A compact subset D in U defined by*

$$\{z \in \mathbb{C}^n ; \mid F_1(z) \mid \leq 1, \ldots, \mid F_N(z) \mid \leq 1\}, \tag{50}$$

with some finitely many $F_1, \ldots, F_N \in \mathcal{O}_{\mathbb{C}^n}$ is called an analytic polyhedron of U.

Lemma 5.6. ([*4*], *Corollary* 5.3.7) *Let $0 \leftarrow \mathcal{F} \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \cdots \leftarrow \mathcal{L}_n \leftarrow 0$ be an exact sequence of sheaves on a topological space on X, and $S \subset X$ a locally closed set. If*

$$H^k_S(X, \mathcal{L}_j) = 0 \qquad (r + j \leq k \leq N + j, \quad j = 0, \ldots, n),$$

then

$$H^k_S(X, \mathcal{F}) = 0 \qquad (r \leq k \leq N).$$

Lemma 5.7. ([*18*], *Proposition B*.4.2) *Let M be a module, and let $\phi_1, \ldots, \phi_p$ be a family of commuting endomorphism of M. Let $M^{\cdot}$ be a Koszul complex associated to the sequence $(\phi_1, \ldots, \phi_p)$. Assume for each $1 \leq j \leq p$, ϕ_j is injective as an endomorphism of the module $\dfrac{M}{\sum_{i=1}^{j-1} \phi_i(M)}$. Then $H^j(M^{\cdot}) = 0$ for $j \neq p$, and $H^p(M^{\cdot}) \simeq \dfrac{M}{\sum_{i=1}^{p} \phi_i(M)}$.*

Theorem 5.8. *Let $S = [a, \infty](a \in \mathbb{R})$ be a compact set in $\overline{\mathbb{R}}$. Let K and L be two compact analytic polyhedra in $\mathbb{C}^{n-1}$, and let V be a Stein domain in $\mathbb{C}^m$. Then*

$$H^k_{S \times (L \setminus K) \times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V, \mathcal{O}^{\exp}_{\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V}) = 0 \qquad (0 \leq k \leq n-1). \tag{51}$$

Proof. For simplicity, we omit the symbol V for the space of parameters. By replacing K by $K \cap L$, we can assume $K \subset L$. Hence, by the definition of the analytic polyhedron, there are holomorphic functions $F_1, \ldots, F_{k'}, \ldots, F_k$ on $\mathbb{C}^{n-1}$ such that L and K can be expressed as

$$L = \left\{ w \in \mathbb{C}^{n-1} ; |F_i(w)| \leq 1, \quad i = 1, 2, \ldots, k' \right\}$$

and

$$K = \left\{ w \in \mathbb{C}^{n-1} ; |F_i(w)| \leq 1, \quad i = 1, 2, \ldots, k', \ldots, k \right\}.$$

Since L is bounded, we assume that L is contained in the polydisc of the radius r. Further, choose a real number

$$R := \max\left\{1,\ \max_{k'+1 \le j \le k}\ \sup_{w \in L}|F_j(w)|,\ r\right\}.$$

Let us consider the closed embedding $\Psi : \mathbb{D}_z^2 \times \mathbb{C}_w^{n-1} \to \mathbb{D}_z^2 \times \mathbb{C}_{w,\,\tilde{w}}^{(n-1)+k}$ defined by

$$\Psi(z,\, w) = (z,\, w,\, F_1(w),\, \ldots,\, F_{k'}(w),\, \ldots,\, F_k(w)).$$

We also define $\tilde{L}$ and $\tilde{K}$ in $\mathbb{C}^{(n-1)+k}$ by

$$\tilde{L} := \left\{\ (w,\, \tilde{w}) \in \mathbb{C}^{(n-1)+k}\,;\ \begin{array}{l} |\, w_1 \,|\le R,\, \ldots,\, |w_{n-1}| \le R, \\ |\, \tilde{w}_1 \,|\le 1,\, \ldots,\, |\, \tilde{w}_{k'} \,|\le 1, \\ |\, \tilde{w}_{k'+1} \,|\le R, \ldots,\, |\, \tilde{w}_k \,|\le R \end{array} \right\}$$

and

$$\tilde{K} := \left\{\ (w,\, \tilde{w}) \in \mathbb{C}^{(n-1)+k}\,;\ \begin{array}{l} |\, w_1 \,|\le R,\, \ldots,\, |w_{n-1}| \le R, \\ |\, \tilde{w}_1 \,|\le 1,\, \ldots,\, |\, \tilde{w}_{k'} \,|\le 1, \\ |\, \tilde{w}_{k'+1} \,|\le 1, \ldots,\, |\, \tilde{w}_k \,|\le 1 \end{array} \right\}.$$

Noticing $S \times (L \setminus K) = \Psi^{-1}(S \times (\tilde{L} \setminus \tilde{K}))$, we have

(52)
$$H^p_{S\times(L\setminus K)}(\mathbb{D}^2 \times \mathbb{C}^{n-1},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) \cong H^p_{S\times(\tilde{L}\setminus\tilde{K})}(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k},\, \Psi_*\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}).$$

On the other hand, for the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{(n-1)+k}}$ of holomorphic functions of exponential type on $\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k}$, we have, from Corollary 5.4,

$$(53)\quad H^p_{S\times(\tilde{L}\setminus\tilde{K})}(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{(n-1)+k}}) = 0 \qquad (0 \le p \le (n-1)+k).$$

Hence, if there exists an inverse resolution of the sheaf $\Psi_*\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}$ by the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{(n-1)+k}}$ of length k, it follows from Lemma 5.6 that (51) holds. Let $\phi_1,\, \ldots,\, \phi_k$ be a family of commuting endomorphisms of $\Gamma(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{(n-1)+k}})$ defined by

$$\phi_j(f) = f(z,\, w,\, \tilde{w})(\tilde{w}_j - F_j(w)) \quad (1 \le j \le k),$$

and let $e_1,\, \ldots,\, e_k$ be the canonical basis of $\mathbb{Z}^k$. For an ordered subset $I := (i_1,\, \ldots,\, i_j)$ of $\{1,\, \ldots,\, k\}$, we define the element of $\overset{j}{\wedge}(\mathbb{Z}^k)$ by $e_I := e_{i_1} \wedge \cdots \wedge e_{i_j}$. We set

$$M^{(j)} := M \underset{\mathbb{Z}}{\otimes} \overset{j}{\wedge}(\mathbb{Z}^k), \qquad M := \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{(n-1)+k}},$$

and we define the differential d from $M^{(j)}$ to $M^{(j+1)}$ by letting, for an element fe_I in $M^{(j)}$,

$$d(fe_I) := \sum_{i=1}^{k} \phi_i(f) e_i \wedge e_I.$$

The commutativity of the operators ϕ_j clearly implies $d \circ d = 0$. Hence we obtain a Koszul complex $\mathcal{M}$ associated to the sequence $(\phi_1, \ldots, \phi_k)$:

$$0 \to M^{(0)} \xrightarrow{d} M^{(1)} \to \ldots \xrightarrow{d} M^{(k)} \to 0. \tag{54}$$

Let us show that this complex is an inverse resolution of the sheaf $\Psi_* \mathcal{O}^{\exp}_{\mathbb{D}^2 \times \mathbb{C}^{n-1}}$ by the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k}}$ of length k. For the proof of this fact, we need the following lemma.

LEMMA 5.9. *For each $1 \le j \le k$ and $p \in \mathbb{D}^2 \times \mathbb{C}^{(n-1)+k}$, the morphism ϕ_j is injective as an endomorphism of the module* $\dfrac{M_p}{\sum_{i=1}^{j-1} \phi_i(M_p)}$.

Proof. We first prove the lemma in the case of $p \in \Psi(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k})$. Suppose that $f \in M_p$ satisfies $\phi_j(f) = 0$ in $\dfrac{M_p}{\sum_{i=1}^{j-1} \phi_i(M_p)}$. Then there exist $g_1, \ldots, g_{j-1} \in M_p$ such that $\phi_j(f) = \sum_{i=1}^{j-1} \phi_i(g_i)$ holds. Therefore we have

$$f(z, w, \tilde{w})(\tilde{w}_j - F_j(w)) = \sum_{i=1}^{j-1} g_i(z, w, \tilde{w})(\tilde{w}_i - F_i(w)) \tag{55}$$

on a neighborhood of p. Setting $\tilde{w}_j = F_j(w)$ in (55), we obtain

$$\sum_{i=1}^{j-1} g_i(z, w, \tilde{w}_1, \ldots, \tilde{w}_{j-1}, F_j(w), \tilde{w}_{j+1}, \ldots, \tilde{w}_k)(\tilde{w}_i - F_i(w)) = 0. \tag{56}$$

For $1 \le i \le j-1$, we put

$$h_i(z, w, \tilde{w}) := \frac{g_i(z, w, \tilde{w}) - g_i(z, w, \tilde{w}_1, \ldots, \tilde{w}_{j-1}, F_j(w), \tilde{w}_{j+1}, \ldots, \tilde{w}_k)}{\tilde{w}_j - F_j(w)}.$$

Then h_i belongs to M_p as $p \in \Psi(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k})$. By (55) and (56), we have

$$\begin{aligned}
\sum_{i=1}^{j-1} \phi_i(h_i(z, w, \tilde{w})) &= \sum_{i=1}^{j-1} h_i(z, w, \tilde{w})(\tilde{w}_i - F_i(w)) \\
&= \sum_{i=1}^{j-1} \frac{g_i(z, w, \tilde{w})}{\tilde{w}_j - F_j(w)}(\tilde{w}_i - F_i(w)) \\
&= \frac{f(z, w, \tilde{w})(\tilde{w}_j - F_j(w))}{\tilde{w}_j - F_j(w)} = f(z, w, \tilde{w})
\end{aligned}$$

on a neighborhood of p. This implies that ϕ_j is injective as an endomorphism of $\dfrac{M_p}{\sum_{i=1}^{j-1}\phi_i(M_p)}$ for each $p \in \Psi(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k})$. Next, we show the injectivity of ϕ_j in the case $p \notin \Psi(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k})$. If $\tilde{w}_j \neq F_j(w)$ holds on a neighborhood of p, by (55) we have

$$f(z,\, w,\, \tilde{w}) = \sum_{i=1}^{j-1} \frac{g_i(z,\, w,\, \tilde{w})}{\tilde{w}_j - F_j(w)}(\tilde{w}_i - F_i(w)) = \sum_{i=1}^{j-1} \phi_i\left(\frac{g_i(z,\, w,\, \tilde{w})}{\tilde{w}_j - F_j(w)}\right). \tag{57}$$

Otherwise, as is shown in the case of $p \in \Psi(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k})$, we see the injectivity of ϕ_j. Therefore the claim of the lemma holds on $\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k}$. $\square$

Now we prove that the complex $\mathcal{M}$ is the inverse resolution of the sheaf $\Psi_*\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}$ by the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{(n-1)+k}}$ of length k.

Lemma 5.10. *The following complex of the sheaves on $\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k}$ is exact:*

$$0 \to M^{(0)} \xrightarrow{d} M^{(1)} \to \dots \xrightarrow{d} M^{(k)} \to \Psi_*\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}} \to 0. \tag{58}$$

Proof. Let us show that for each $p \in \mathbb{D}^2 \times \mathbb{C}^{(n-1)+k}$ the sequence of the stalks

$$0 \to M^{(0)}_p \xrightarrow{d} M^{(1)}_p \to \dots \xrightarrow{d} M^{(k)}_p \to (\Psi_*\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}})_p \to 0 \tag{59}$$

is exact. Note that we have

$$(\Psi_*\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}})_p = \begin{cases} 0 & (p \notin \Psi(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k})), \\ (\mathcal{O}^{\exp}_{\Psi(\mathbb{D}^2\times\mathbb{C}^{n-1})})_p & (p \in \Psi(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k})). \end{cases}$$

By Lemma 5.9, we can apply Lemma 5.7 to the complex $\mathcal{M}_p$. Hence we obtain

$$H^j(\mathcal{M}_p) = \begin{cases} 0, & (j \neq k), \\ \operatorname{Coker}(M_p^{(k-1)} \to M_p^{(k)}) \simeq \dfrac{M_p}{\sum_{i=1}^{k}\phi_i(M_p)}, & (j = k). \end{cases}$$

Therefore the exactness of the sequence (59) have been obtained for $p \notin \Psi(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k})$. For $p \in \Psi(\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k})$, the restriction mapping $\rho : M_p \to (\mathcal{O}^{\exp}_{\Psi(\mathbb{D}^2\times\mathbb{C}^{n-1})})_p$ turns out to be the substitution $\tilde{w} = (F_1(w),\, \dots,\, F_k(w))$. Hence ρ is surjective. Furthermore, by considering the Taylor expansion at p, its kernel consists of the germs of type $\sum_{i=1}^k g_i(z,\, w,\, \tilde{w})(\tilde{w}_i - F_i(w))$. We thus obtain the exact sequence $0 \to \sum_{i=1}^k \phi_i(M_p) \to M_p \to (\mathscr{O}^{\mathrm{exp}}_{\Psi(\mathbb{D}^2\times\mathbb{C}^{n-1})})_p \to 0$. This implies that $\dfrac{M_p}{\sum_{i=1}^k \phi_i(M_p)}$ is isomorphic to $(\mathscr{O}^{\mathrm{exp}}_{\Psi(\mathbb{D}^2\times\mathbb{C}^{n-1})})_p$. Therefore we get the exactness of (59) on $\mathbb{D}^2 \times \mathbb{C}^{(n-1)+k}$. $\square$

This completes the proof of Theorem 5.8. $\square$

5.2 The edge of the wedge theorem for the sheaf of holomorphic functions of exponential type

Now we are ready to show the edge of the wedge theorem for the sheaf $\mathscr{O}^{\mathrm{exp}}_{\mathbb{D}^{2n}}$.

THEOREM 5.11. *The closed set $\overline{\mathbb{R}^n} \subset \mathbb{D}^{2n}$ is purely n-codimensional relative to the sheaf $\mathscr{O}^{\mathrm{exp}}_{\mathbb{D}^{2n}}$, i.e.,*

$$\mathscr{H}^k_{\overline{\mathbb{R}^n}}(\mathscr{O}^{\mathrm{exp}}_{\mathbb{D}^{2n}}) = 0 \qquad (k \neq n). \tag{60}$$

Proof. It suffices to compute the stalks of $\mathscr{H}^k_{\overline{\mathbb{R}^n}}(\mathscr{O}^{\mathrm{exp}}_{\mathbb{D}^{2n}})$ at $p_\infty := (+\infty,\, 0,\, \dots,\, 0) \in S^{2n-1}\infty$, as $\mathbb{R}^n$ is purely n-codimensional relative to the sheaf $\mathscr{O}_{\mathbb{C}^n}$ of holomorphic functions on $\mathbb{C}^n$. We set

$$U_\epsilon := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n;\, |\arg z_1| < \epsilon,\ |z_1| > \frac{1}{\epsilon},\ \ |z_i| < \epsilon |z_1| \quad (2 \le i \le n) \right\},$$
$$A_\epsilon := \overline{U_\epsilon}^{\circ} \subset \mathbb{D}^{2n}$$

for any $\epsilon > 0$. Here the closure and interior are taken in $\mathbb{D}^{2n}$. Note that $\{A_\epsilon\}_{\epsilon>0}$ is a fundamental system of neighborhoods of p_∞ in $\mathbb{D}^{2n}$. Hence we have

$$\mathscr{H}^k_{\overline{\mathbb{R}^n}}(\mathscr{O}^{\mathrm{exp}}_{\mathbb{D}^{2n}})_{p_\infty} = \varinjlim_{\epsilon\downarrow 0} H^k_{A_\epsilon \cap \overline{\mathbb{R}^n}}(A_\epsilon,\, \mathscr{O}^{\mathrm{exp}}_{\mathbb{D}^{2n}}).$$

We also set

$$V_\epsilon := \left\{ z \in \mathbb{C};\, |\arg z| < \epsilon,\ |z| > \frac{1}{\epsilon} \right\},$$
$$W_\epsilon := \left\{ (w_1,\, \dots,\, w_{n-1}) \in \mathbb{C}^{n-1};\, |w_i| < \epsilon \quad (1 \le i \le n-1) \right\},$$
$$B_\epsilon := \overline{V_\epsilon}^{\circ} \times W_\epsilon \subset \mathbb{D}^2 \times \mathbb{C}^{n-1}$$

for any $\epsilon > 0$. Then $\{B_\epsilon\}_{\epsilon>0}$ is a fundamental system of neighborhoods of $q_\infty := ((+\infty,\, 0),\, 0) \in S^1\infty \times \mathbb{C}^{n-1}$ in $\mathbb{D}^2 \times \mathbb{C}^{n-1}$. Let us consider the holomorphic map $\Phi : \mathbb{C}^n \setminus \{z_1 = 0\} \to \mathbb{C} \times \mathbb{C}^{n-1}$ defined by

$$\Phi(z_1,\, \dots,\, z_n) = \left(z_1,\, \frac{z_2}{z_1},\, \dots,\, \frac{z_n}{z_1} \right).$$

Then Φ gives a biholomorphic map between U_ϵ and $V_\epsilon \times W_\epsilon$. As $S^{2n-1} \setminus \{x_1 = 0\}$ and $S^1 \times \mathbb{R}^{2n-2}$ are isomorphic by the correspondence

$$(x_1,\, x_2,\, \ldots,\, x_{2n}) \mapsto \left(\frac{x_1}{\sqrt{x_1^2+x_2^2}},\, \frac{x_2}{\sqrt{x_1^2+x_2^2}}\right) \times \left(\frac{x_3}{\sqrt{x_1^2+x_2^2}},\, \ldots,\, \frac{x_{2n}}{\sqrt{x_1^2+x_2^2}}\right),$$

$$\left(\frac{x_1}{\sqrt{1+|y|^2}},\, \frac{x_2}{\sqrt{1+|y|^2}},\, \frac{y}{\sqrt{1+|y|^2}}\right) \hookleftarrow (x_1,\, x_2) \times (y),$$

the map Φ extends to a continuous isomorphism between A_ϵ and B_ϵ. Therefore we obtain

$$\mathscr{H}^k_{\mathbb{R}^n}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}})_{p_\infty} \cong \mathscr{H}^k_{\mathbb{R}\times\mathbb{R}^{n-1}}(\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}})_{q_\infty}. \tag{61}$$

Hence it suffices to prove that the right hand side of (61) vanishes for $k \neq n$. Let us take two points a, b satisfying $\epsilon^{-1} < a < b < \infty$. We put $S_1 = (\epsilon^{-1},\, b]$ and $S_2 = [a,\, \infty]$. We have the long exact sequence of cohomology groups

$$\begin{aligned} \cdots &\to H^k_{(S_1\cap S_2)\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) \\ &\to \bigoplus_{j=1,\,2} H^k_{S_j\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) \\ &\to H^k_{(S_1\cup S_2)\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) \to \cdots . \end{aligned} \tag{62}$$

Noticing that $\mathbb{R}^n$ is purely n-codimensional relative to the sheaf $\mathcal{O}_{\mathbb{C}^n}$ and the sheaf $\mathscr{H}^n_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})$ of hyperfunctions is flabby, that we obtain

$$\begin{aligned} &H^k_{(S_1\cap S_2)\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) = 0 \quad (k \neq n), \\ &H^k_{S_1\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) = 0 \quad (k \neq n). \end{aligned} \tag{63}$$

Hence, by the long exact sequence of cohomology groups (62), this implies

$$H^k_{(S_1\cup S_2)\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) = H^k_{S_2\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) \qquad (k \neq n-1,\, n). \tag{64}$$

We also have the following exact sequence by (62).

$$\begin{aligned} 0 &\to H^{n-1}_{S_2\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) \\ &\to H^{n-1}_{(S_1\cup S_2)\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) \\ &\to \Gamma_{(S_1\cap S_2)\times(W_\epsilon\times\mathbb{R}^{n-1})}(\mathbb{C}^n,\, \mathscr{H}^n_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})) \\ &\xrightarrow{\iota} \Gamma_{S_1\times(W_\epsilon\cap\mathbb{R}^{n-1})}(\mathbb{C}^n,\, \mathscr{H}^n_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})) \oplus H^n_{S_2\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}). \end{aligned} \tag{65}$$

As the morphism ι in (65) is injective, if $H^{n-1}_{S_2\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}})$ vanishes, we have $H^{n-1}_{(S_1\cup S_2)\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) = 0$. Therefore, together with (64), If we could prove that

$$H^k_{S_2\times(W_\epsilon\cap\mathbb{R}^{n-1})}(\mathbb{D}^2\times W_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) = 0 \qquad (k \neq n) \tag{66}$$

for a complex neighborhood W_ϵ of the origin in $\mathbb{C}^{n-1}$, then the assertion follows. For $k \geq n+1$, (66) holds by Lemma 5.1. Hence let us show that (66) holds for $0 \leq k \leq n-1$. Using the entire function on $\mathbb{C}^{n-1}$

$$f_\epsilon(w) = \epsilon^2 - (w_1^2 + \cdots + w_{n-1}^2)$$

for $\epsilon > 0$, we put

$$\begin{aligned} &L_\epsilon = \left\{w \in \mathbb{C}^{n-1}\,;\ |w_1| \leq \epsilon,\, \ldots,\, |w_{n-1}| \leq \epsilon\right\} \cap \mathbb{R}^{n-1},\\ &M_\epsilon = L_\epsilon \cap \{\mathrm{Re}(f_\epsilon(w)) \leq 0\}\,. \end{aligned} \tag{67}$$

Note that L_ϵ and M_ϵ are closed analytic polyhedra because $\mathbb{R}$ can be expressed as $\mathbb{R} = \{w \in \mathbb{C}; |e^{\sqrt{-1}w}| \leq 1,\, |e^{-\sqrt{-1}w}| \leq 1\}$ and $\mathrm{Re}(f_\epsilon(w)) \leq 0$ is equivalent to $|e^{f_\epsilon(w)}| \leq 1$. Now

$$\tilde{W}_\epsilon = \left\{(w_1,\, \ldots,\, w_{n-1}) \in \mathbb{C}^{n-1}\,;\ |w_i| < \epsilon,\ \mathrm{Re}(f_\epsilon(w)) > 0\right\}$$

is clearly a complex neighborhood of the origin in $\mathbb{C}^{n-1}$ and

$$L_\epsilon \setminus M_\epsilon = \tilde{W}_\epsilon \cap \mathbb{R}^{n-1}.$$

Hence by Theorem 5.11 , we obtain

$$\begin{aligned} H^k_{S_2\times(\tilde{W}_\epsilon\cap\mathbb{R}^{n-1})}(\mathbb{D}^2\times\mathbb{C}^{n-1},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) &= H^k_{S_2\times(L_\epsilon\setminus M_\epsilon)}(\mathbb{D}^2\times\mathbb{C}^{n-1},\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}})\\ &= 0 \qquad (0 \leq k \leq n-1). \end{aligned} \tag{68}$$

This completes the proof. $\square$

Theorem 5.12. *The boundary $\partial\mathbb{R}^n$ of $\mathbb{R}^n$ in $\mathbb{D}^{2n}$ is purely n-codimensional relative to the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$, i.e.,*

$$\mathscr{H}^k_{\partial\mathbb{R}^n}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) = 0, \qquad (k \neq n) \tag{69}$$

Proof. It suffices to compute the stalks of $\mathscr{H}^k_{\partial\mathbb{R}^n}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}})$ at $p_\infty = (+\infty,\, 0,\, \ldots,\, 0) \in \partial\mathbb{R}^n \subset \mathbb{D}^{2n}$. Let us consider the fundamental system $\{A_\epsilon\}_{\epsilon>0}$ of neighborhoods of the point p_∞, the one $\{B_\epsilon\}_{\epsilon>0}$ of neighborhoods of the point $q_\infty = ((+\infty,\, 0),\, 0) \in S^1\infty \times \mathbb{C}^{n-1}$ and the holomorphic map Φ in the proof of Theorem 5.11. As Φ gives a biholomorphic map between $S^{n-1}\infty \cap A_\epsilon$ and $(S^0\infty \times \mathbb{R}^{n-1}) \cap B_\epsilon$, we have

$$\begin{aligned}\mathscr{H}^k_{\partial\mathbb{R}^n}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}})_{p_\infty} &\cong \mathscr{H}^k_{\partial\mathbb{R}\times\mathbb{R}^{n-1}}(\mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}})_{q_\infty} \\ &= \varinjlim_{\epsilon\downarrow 0} H^k_{\{\infty\}\times(W_\epsilon\cap\mathbb{R}^{n-1})}(B_\epsilon,\, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}).\end{aligned} \tag{70}$$

Hence it follows from (66) that the purely n-codimensionality of $\partial\mathbb{R}^n$ relative to the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$. □

6 Sheaf of Laplace hyperfunctions in several variables

As an application of Theorem 5.11 established in the previous section, we construct cohomologically the sheaf of Laplace hyperfunctions on $\overline{\mathbb{R}^n}$. In this section, we give the sheaf of Laplace hyperfunctions on $\overline{\mathbb{R}^n}$ and we show that the sheaf of real analytic functions of exponential type is a subsheaf of the sheaf of Laplace hyperfunctions.

Definition 6.1. *The sheaf of Laplace hyperfunctions on* $\overline{\mathbb{R}^n}$ *is defined by*

$$\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}} := \mathscr{H}^n_{\overline{\mathbb{R}^n}}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) \underset{\mathbb{Z}_{\overline{\mathbb{R}^n}}}{\otimes} \omega_{\overline{\mathbb{R}^n}} \tag{71}$$

where $\mathbb{Z}_{\overline{\mathbb{R}^n}}$ *denotes the constant sheaf on* $\overline{\mathbb{R}^n}$ *having stalk* $\mathbb{Z}$ *and* $\omega_{\overline{\mathbb{R}^n}}$ *denotes the orientation sheaf* $\mathscr{H}^n_{\overline{\mathbb{R}^n}}(\mathbb{Z}_{\mathbb{D}^{2n}})$ *on* $\overline{\mathbb{R}^n}$.

Since the sheaves $\mathscr{H}^k_{\overline{\mathbb{R}^n}}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}})$ are zero for $k < n$ by Theorem 5.11, we find that the presheaf $U \mapsto H^n_{\overline{\mathbb{R}^n}\cap U}(U,\, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}})$ is a sheaf, and is equal to $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$. Hence the global sections of the sheaf $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$ of Laplace hyperfunctions can be written in terms of cohomology groups. For an open set $\Omega \subset \overline{\mathbb{R}^n}$, by taking a complex neighborhood U of Ω in $\mathbb{D}^{2n}$, we have

$$\Gamma(\Omega,\, \mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}) = H^n_\Omega(U,\, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}).$$

Note that the above representation does not depend on a choice of the complex neighborhood U.

Definition 6.2. *Let* $i : \overline{\mathbb{R}^n} \to \mathbb{D}^{2n}$ *be the natural embedding. The sheaf of real analytic functions of exponential type on* $\overline{\mathbb{R}^n}$ *is defined by*

$$\mathcal{A}^{\exp}_{\overline{\mathbb{R}^n}} := i^{-1}\mathcal{O}^{\exp}_{\mathbb{D}^{2n}} = \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}|_{\overline{\mathbb{R}^n}}. \tag{72}$$

We show that real analytic functions of exponential type are regarded as Laplace hyperfunctions. There is a natural morphism

$$i^{-1}\mathcal{O}^{\exp}_{\mathbb{D}^{2n}} \otimes \omega_{\overline{\mathbb{R}^n}}[-n] \longrightarrow i^{!}\mathcal{O}^{\exp}_{\mathbb{D}^{2n}} \simeq i^{-1}R\Gamma_{\overline{\mathbb{R}^n}}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}).$$

Using the shift functor $[n]$ and the functor $(\cdot) \otimes \omega_{\overline{\mathbb{R}^n}}$, we get the sheaf morphism $\phi : \mathcal{A}^{\exp}_{\overline{\mathbb{R}^n}} \to \mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$. We denote by $\mathcal{A}_{\mathbb{R}^n}$ the sheaf of real analytic functions on $\mathbb{R}^n$. Let $j : \mathbb{R}^n \to \overline{\mathbb{R}^n}$ be a natural embedding, and let $\psi : \mathcal{A}_{\mathbb{R}^n} \to \mathcal{B}_{\mathbb{R}^n}$. Now we consider the commutative diagram of sheaf morphisms:

$$\begin{tikzcd} \mathcal{A}^{\exp}_{\overline{\mathbb{R}^n}} \arrow[r, "\phi"] \arrow[d, "\alpha"'] & \mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}} \arrow[d, "\beta"] \\ j_*\mathcal{A}_{\mathbb{R}^n} \arrow[r, "j_*\psi"] & j_*\mathcal{B}_{\mathbb{R}^n} \end{tikzcd}.$$

As α and $j_*\psi$ are injective, we have the following theorem.

Theorem 6.3. *The sheaf morphism $\mathcal{A}^{\exp}_{\overline{\mathbb{R}^n}} \to \mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$ is injective and we can regard real analytic functions of exponential type as Laplace hyperfunctions.*

Theorem 6.4. *The sheaf morphism $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}} \to j_*\mathcal{B}_{\mathbb{R}^n}$ is surjective.*

Proof. Let $\Omega \subset \overline{\mathbb{R}^n}$ be an open set, and let $U \subset \mathbb{D}^{2n}$ be a complex neighborhood of Ω. Consider the following exact sequence of cohomology groups:

$$H^n_{\overline{\mathbb{R}^n}\cap\Omega}(U, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) \to H^n_{\mathbb{R}^n\cap\Omega}(U, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) \to H^{n+1}_{\partial\mathbb{R}^n\cap\Omega}(U, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}).$$

It follows from Theorem 5.12 that we have $H^{n+1}_{\partial\mathbb{R}^n\cap\Omega}(U, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) = 0$. Hence, by taking inductive limit with respect to $\Omega \subset \overline{\mathbb{R}^n}$ of the above exact sequence of cohomology groups, we have the following exact sequence of sheaves

$$\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}} \to j_*\mathcal{B}_{\mathbb{R}^n} \to 0.$$

This implies that the morphism $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}} \to j_*\mathcal{B}_{\mathbb{R}^n}$ is surjective. $\square$

7 Softness of the sheaf $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$

In this section we prove the softness of the sheaf $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$. For that purpose we prepare some propositions.

Lemma 7.1. *Let S be a locally closed set in $\overline{\mathbb{R}^n}$. Then we have*

$$H^k_S(\mathbb{D}^{2n}, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) = 0 \qquad (k \neq n, n+1). \tag{73}$$

Proof. For $0 \le k \le n-1$, Theorem 5.11 implies

$$H^k_S(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = \Gamma_S(\mathbb{D}^{2n},\, \mathscr{H}^k_{\mathbb{R}^n}(\mathscr{O}^{\exp}_{\mathbb{D}^{2n}})) = 0.$$

On the other hand, for $k \ge n+2$, (73) follows from that flabby dim $\mathscr{O}^{\exp}_{\mathbb{D}^{2n}} \le n+1$. □

We introduce some notations which are needed for the subsequent proposition. For $1 \le j \le n$, we define the holomorphic map $\Phi_j : \mathbb{C}^n \setminus \{z_j = 0\} \to \mathbb{C} \times \mathbb{C}^{n-1}$ by

$$\Phi_j(z_1,\, \dots, z_j,\, \dots,\, z_n) = \left(z_j,\, \frac{z_1}{z_j},\, \dots,\, \frac{z_{j-1}}{z_j},\, \frac{z_{j+1}}{z_j},\, \dots,\, \frac{z_n}{z_j}\right).$$

Definition 7.2. *Let $K \subset \overline{\mathbb{R}^n}$ and $S \subset \overline{\mathbb{R}}$ be closed sets, and let $L_1, \cdots, L_{n-1}$ be closed sets in $\mathbb{C}$. We say that K is a Φ_j-closed subset in $\overline{\mathbb{R}^n}$ if Φ_j gives a biholomorphic map between $K \cap \mathbb{R}^n$ and $(S \cap \mathbb{R}) \times L_1 \times \cdots \times L_{n-1}$ which extends to a continuous isomorphism between K and $S \times L_1 \times \cdots \times L_{n-1}$.*

Proposition 7.3. *Let $\partial\mathbb{R}^n$ be a boundary of $\mathbb{R}^n$ in $\mathbb{D}^{2n}$. Then we have*

$$H^k_{\partial\mathbb{R}^n}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = 0 \qquad (k \ne n). \tag{74}$$

Proof. For $k \ne n, n+1$, taking Lemma 7.1 into account, we have (74). Let us show $H^{n+1}_{\partial\mathbb{R}^n}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = 0$. For a sufficiently large ϵ, we set

$$\begin{aligned}
K_j^+ &:= \{(z_1,\, \dots,\, z_n) \in \mathbb{C}^n;\, |\arg z_j| \le \epsilon,\ \mathrm{Re} z_j \ge \epsilon,\quad |z_k| \le \epsilon |z_j| \quad (k \ne j)\},\\
K_j^- &:= \{(z_1,\, \dots,\, z_n) \in \mathbb{C}^n;\, |\arg z_j - \pi| \le \epsilon,\ \mathrm{Re} z_j \le -\epsilon,\ |z_k| \le \epsilon |z_j|\ (k \ne j)\},\\
L_j^* &:= \overline{K_j^*} \cap \partial\mathbb{R}^n \quad (* = +,\, -,\quad 1 \le j \le n).
\end{aligned}$$

Here the closer $\overline{K_j^*}$ is taken in $\mathbb{D}^{2n}$. Then the family of those closed sets $\{L_1^+,\, L_1^-,\, \dots,\, L_n^+,\, L_n^-\}$ gives a closed covering of $\partial\mathbb{R}^n$. As Φ_j gives a continuous isomorphism between $L_j^\pm$ and $\{\pm\infty\} \times \{|x_1| \le \epsilon\} \times \cdots \times \{|x_{j-1}| \le \epsilon\} \times \{|x_{j+1}| \le \epsilon\} \times \cdots \times \{|x_n| \le \epsilon\}$, we have

$$H^k_{L_j^\pm}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = H^k_{\{\pm\infty\}\times\{|x_1|\le\epsilon\}\times\cdots\times\{|x_n|\le\epsilon\}}(\mathbb{D}^2 \times \mathbb{C}^{n-1},\, \mathscr{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}}) \tag{75}$$
$$(1 \le j \le n,\, k \in \mathbb{Z}).$$

Then the right hand side of (75) vanishes for $k \ge n+1$ by Lemma 5.1. In particular, we obtain $H^{n+1}_{L_j^\pm}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = 0$. So we use induction on the number $2n$ of closed sets L_j^* comprising $\partial\mathbb{R}^n$ to see $H^{n+1}_{\partial\mathbb{R}^n}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = 0$. It suffices to prove that $H^{n+1}_{L_1^+\cup L_2^+}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = 0$ from $H^{n+1}_{L_1^+}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = 0$

and $H^{n+1}_{L_2^+}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = 0$. We consider the following exact sequence of cohomology groups

$$(76)\qquad \begin{aligned} \cdots &\to H^{n+1}_{L_1^+}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) \oplus H^{n+1}_{L_2^+}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) \\ &\to H^{n+1}_{L_1^+\cup L_2^+}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) \\ &\to H^{n+2}_{L_1^+\cap L_2^+}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) \to \cdots . \end{aligned}$$

As $L_1^+ \cap L_2^+$ is Φ_1-closed or Φ_2-closed, we obtain $H^{n+2}_{L_1^+\cap L_2^+}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = 0$ by Lemma 5.1 . This implies $H^{n+1}_{L_1^+\cup L_2^+}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = 0$. □

As an immediate consequence of the proposition, we obtain the following corollary.

Corollary 7.4. *Let L be a finite union of Φ_j-closed subsets in $\overline{\mathbb{R}^n}$. Then we have*

$$(77)\qquad H^k_L(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) = 0 \qquad (k \neq n).$$

By Theorem 5.12 given in the previous section, we obtain the following proposition.

Proposition 7.5. *The sheaf $\mathscr{H}^n_{\partial\mathbb{R}^n}(\mathscr{O}^{\exp}_{\mathbb{D}^{2n}})$ is soft on $\partial\mathbb{R}^n$.*

Proof. Let us show that every section on an arbitrary closed set $K \subset \partial\mathbb{R}^n$ can be extended to the whole space, i.e., the restriction map $\Gamma(\partial\mathbb{R}^n,\, \mathscr{H}^n_{\partial\mathbb{R}^n}(\mathscr{O}^{\exp}_{\mathbb{D}^{2n}}))$ $\to \Gamma(K,\, \mathscr{H}^n_{\partial\mathbb{R}^n}(\mathscr{O}^{\exp}_{\mathbb{D}^{2n}}))$ is surjective. As $\partial\mathbb{R}^n$ is a paracompact Hausdorff topological space, we have

$$\Gamma(K,\, \mathscr{H}^n_{\partial\mathbb{R}^n}(\mathscr{O}^{\exp}_{\mathbb{D}^{2n}})) = \varinjlim_{\Omega\supset K} \Gamma(\Omega,\, \mathscr{H}^n_{\partial\mathbb{R}^n}(\mathscr{O}^{\exp}_{\mathbb{D}^{2n}})).$$

Here the limit is taken with respect to all open subsets on $\partial\mathbb{R}^n$ containing K. Therefore every element of $\Gamma(K,\, \mathscr{H}^n_{\partial\mathbb{R}^n}(\mathscr{O}^{\exp}_{\mathbb{D}^{2n}}))$ can be first extended to an open neighborhood Ω of K on $\partial\mathbb{R}^n$. Let us take a finite closed covering of $\partial\mathbb{R}^n \setminus \Omega$ which satisfies the condition of Corollary 7.4. Note that, by Theorem 5.12, we have the following representation of the global sections of $\mathscr{H}^n_{\partial\mathbb{R}^n}(\mathscr{O}^{\exp}_{\mathbb{D}^{2n}})$ on an open set Ω :

$$\Gamma(\Omega,\, \mathscr{H}^n_{\partial\mathbb{R}^n}(\mathscr{O}^{\exp}_{\mathbb{D}^{2n}})) = H^n_\Omega(U,\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}),$$

where U is an arbitrary open neighborhood of Ω in $\mathbb{D}^{2n}$. Let us consider the long exact sequence of cohomology groups

$$(78)\quad \cdots \to H^n_{\partial\mathbb{R}^n}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) \to H^n_\Omega(U,\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) \to H^{n+1}_{\partial\mathbb{R}^n\setminus\Omega}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}}) \to \cdots .$$

Then $H^{n+1}_{\partial\mathbb{R}^n\setminus\Omega}(\mathbb{D}^{2n},\, \mathscr{O}^{\exp}_{\mathbb{D}^{2n}})$ vanishes by Corollary 7.4. This implies that the sheaf $\mathscr{H}^n_{\partial\mathbb{R}^n}(\mathscr{O}^{\exp}_{\mathbb{D}^{2n}})$ is soft on $\partial\mathbb{R}^n$. □

Theorem 7.6. *The sheaf* $\mathscr{H}^n_{\overline{\mathbb{R}^n}}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}})$ *is soft on* $\overline{\mathbb{R}^n}$.

Proof. Let $j : \mathbb{R}^n \to \overline{\mathbb{R}^n}$ be the embedding. For an open set $\Omega \subset \overline{\mathbb{R}^n}$ and a complex neighborhood $U \subset \mathbb{D}^{2n}$, we consider the long exact sequence of cohomology groups

$$\cdots \to H^k_{\partial\mathbb{R}^n \cap \Omega}(U, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) \to H^k_{\overline{\mathbb{R}^n} \cap \Omega}(U, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) \to H^k_{\mathbb{R}^n \cap \Omega}(U, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) \to \cdots$$

for the pair $\partial\mathbb{R}^n \cap \Omega \subset \overline{\mathbb{R}^n} \cap \Omega$. It follows from the flabbiness of the sheaf of usually hyperfunctions and Theorem 5.12 that we have

$$H^{n-1}_{\mathbb{R}^n \cap \Omega}(U, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) = 0, \qquad H^{n+1}_{\partial\mathbb{R}^n \cap \Omega}(U, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) = 0.$$

Hence, by taking inductive limit with respect to $\Omega \subset \overline{\mathbb{R}^n}$ of the above long exact sequence of cohomology groups, we have the following exact sequence of sheaves

$$0 \to \mathscr{H}^n_{\partial\mathbb{R}^n}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) \to \mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}} \to j_*\mathcal{B}_{\mathbb{R}^n} \to 0.$$

Since the sheaves $\mathscr{H}^n_{\partial\mathbb{R}^n}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}})$ and $j_*\mathcal{B}_{\mathbb{R}^n}$ are soft, the softness of $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$ follows from the above exact sequence. □

References

[1] N. Honda and K. Umeta, On the sheaf of Laplace hyperfunctions with holomorphic parameters, *Journal of Mathematical Sciences the University of Tokyo*, **19** (2012), 559-586.

[2] N. Honda, T. Murata, The cohomology vanishing theorem for the Cauchy-Riemann complex of tempered ultra-distributions, and Gevrey growth, in preprint.

[3] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, 3rd edition, North-Holland, Amsterdam (1990), 89.

[4] A. Kaneko, Introduction to Hyperfunctions, Kluwer Academic Publishers, (1988).

[5] M. Kashiwara, P. Schapira, Microlocal study of sheaves, Astérisque **128** (1985).

[6] T. Kawai, On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients, J. Fac. Sci. Univ. Tokyo, **17** (1970), 467-517.

[7] H. Komatsu, Laplace transforms of hyperfunctions: a new foundation of the Heaviside calculus, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 805-820.

[8] ____, Laplace transforms of hyperfunctions: another foundation of the Heaviside operational calculus, Generalized Functions, Convergence Structures, and Their Applications (Proc. Internat. Conf., Dubrovnik, 1987; B. Stanković, editor), Plenum Press, New York (1988), 57-70.

[9] ____, Operational calculus, hyperfunctions and ultradistributions, Algebraic Analysis (M. Sato Sixtieth Birthday Vols.), Vol. I, Academic Press, New York (1988), 357-372.

[10] ____, Operational calculus and semi-groups of operators, Functional Analysis and Related Topics (Proc. Internat. Conf. in Memory of K. Yoshida, kyoto, 1991), Lecture Notes in Math., vol. 1540, Springer-Verlag, Berlin (1993), 213-234.

[11] ____, Multipliers for Laplace hyperfunctions - A Justification of Heaviside's rules, Proceedings of the Steklov Institute of Mathematics, **203** (1994), 323-333.

[12] ____, Solution of differential equations by means of Laplace hyperfunctions, in Structure of Solutions of Differential Equations, World Scientific, Singapore, (1996), 227-252.

[13] ____, An introduction to ultra-distributions, Iwanami *Kiso Sûgaku* (1978) in Japanese.

[14] J. Mikusiński, Operational Calculus, Pergamon Press, 1959, 2nd Ed., (1983).

[15] M. Morimoto, K. Yoshino, Some examples of analytic functionals with carrier at the infinity, Proc. Japan Acad. **56** (1980), 357-361.

[16] G. Pólya, *Untersuchungen über Lücken und Singularitäten von Pntenzreihen*, Math. Z. **29** (1929), 549-640.

[17] Y. Saburi, Vanishing theorem of cohomology groups with values in the sheaves $\mathcal{O}_{\mathrm{inc},\phi}$ and $\mathcal{O}_{\mathrm{dec}}$, Tokyo J. Math. vol. **5** (1982), 225-248.

[18] P. Schapira, Microdifferential Systems in the Conplex Domain, Grundleherm der Math., vol.**269**, Springer(1985).

[19] K. Umeta, Vanishing theorem for holomorphic functions of exponential type and Laplace hyperfunctions, RIMS Kôkyûroku Bessatsu B40(2013),137-145.