Free vibration of unsymmetrically joined shell structures with a closed member

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Free vibration is analyzed for unsymmetrical joined plate or shell structures with a closed member by means of the transfer matrix method. For this purpose, the continuity and equilibrium relations for the displacements and forces at the joints are written with use of the joint matrices. The connection matrix between the closed member and other members at the joints is derived by introducing the structure matrix of the closed member, and the entire structure matrix is obtained by the product of the connection matrix of the closed member and the transfer matrices of other members. This method is applied to a plate structure with an unsymmetrically sited duct and a box-type structure with an aslant interior plate, and the natural frequencies are calculated numerically together with the mode shapes of vibration giving the results.

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INTRODUCTION

This article presents an analysis of the free vibration of unsymmetrical joined plate or shell structures with a closed member which are simply supported at the axial edges. The vibration problems of these joined structures have great importance in many engineering applications, such as in design of machines and structures, and have been studied by many researchers. Hu and Raney,[1] Lashkari and Weingarten,[2] Trompette and Lalanne,[3] and Irie *et al.*[4] studied the free vibration of joined conical-cylindrical shells, and Rose *et al.*[5] studied the elastic-wave propagation in a joined shell structure. Smith and Haft,[6] Hirano and Takahashi,[7–9] and Wu and Cory[10] analyzed the vibration of cylindrical shells with circular plates at the edge or in the intermediate section, and Suzuki *et al.*[11] analyzed a cylindrical vessel. Abrahamson[12] studied the free vibration of a rectangular prismatic shell, Hooker and O'Brien[13] and Ueng and Nickels[14] studied box-type structures, and Irie *et al.*[15,16] studied a three- or a four-lobed cross-sectional shell and also an oblique prismatic shell. Peterson and Boyd[17] analyzed a cylindrical shell partitioned by an interior plate, and Irie *et al.*[18,19] analyzed an interiorly partitioned noncircular cylindrical shell and a longitudinally stiffened prismatic shell. However, all of these articles have been confined to the studies of symmetrical shell structures with respect to one or more central planes, and none has been presented for the unsymmetrical structures reported here.

For the purpose of this study, the continuity relation of the displacements and the equilibrium relation of the forces at the joints are written by using the joining point matrices (joint matrices). The connection matrix between the closed member and other members at the joints is derived by introducing the structure matrix of the closed member. The entire structure matrix is obtained by the product of the connection matrix of the closed member and the transfer matrices of other members, and the frquency equation is derived in terms of the elements of the entire structure matrix.

This method is applied to a joined plate structure with an unsymmetrically sited duct and a rectangular prismatic shell (box-type structure) with an aslant interior plate. Here, the equations of free vibration of a plate are written in a matrix differential equation by using the transfer matrix, the elements of which are determined numerically by quadrature of the equation. The natural frequencies (the eigenvalues of vibration) and the mode shapes are calculated numerically, and the results are presented in some figures.

I. TRANSFER MATRIX AND STRUCTURE MATRIX

Figure 1 shows an unsymmetrical shell structure with a closed member. The Cartesian coordinates (x_i,y_i,z_i) $(i = 1,2,3,4)$ are taken in each member as shown in the figure. For the free vibration of the shell structure simply supported at the axial edges, one can separate variables (see Sec. II). Consequently, the state vector $\{z_i(y)\}$ of each member is generally expressed as

$$\{z_i(y)\} = [T^{(i)}(y)]\{z_i(0)\}, \tag{1}$$

by using the transfer matrix $[T^{(i)}(y)]$ in the y_i direction.[19] The state vector $\{z_i(y)\}$ at an end $y_i = B_i$ can be conveniently partitioned into the displacement vector $\{d_i(B)\}$ and the force vector $\{f_i(B)\}$ as follows:

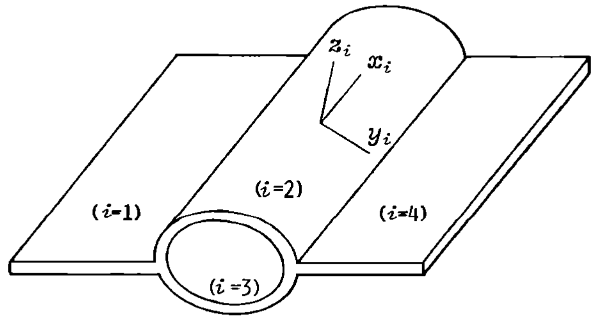

FIG. 1. Joined shell structure with a closed member.

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$$\begin{Bmatrix} d \\ f \end{Bmatrix}_{(B)} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_{(B)} \begin{Bmatrix} d \\ f \end{Bmatrix}_{(0)}, \tag{2}$$

where the super or subscript i is omitted except when necessary. At the joints (the joining lines), the continuity relation of the displacements and the equilibrium relation of the forces, respectively, are expressed as

$$\begin{aligned} &\{d_2(0)\} = [D_{12}]\{d_1(B)\}, \quad \{d_3(0)\} = [D_{13}]\{d_1(B)\}, \\ &\{d_4(0)\} = [D_{24}]\{d_2(B)\}, \quad \{d_4(0)\} = [D_{34}]\{d_3(B)\}, \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\{f_1(B)\} = [F_{21}]\{f_2(0)\} + [F_{31}]\{f_3(0)\}, \\ &\{f_4(0)\} = [F_{24}]\{f_2(B)\} + [F_{34}]\{f_3(B)\}, \end{aligned} \tag{4}$$

by using the joint matrices $[D_{ij}]$ and $[F_{ij}]$.

The substitution of Eqs. (2) and (3) into the first equation of (4) gives the expression

$$\{f_2(0)\} = [G]\{d_1(B)\} + [H]\{f_1(B)\}, \tag{5}$$

where

$$\begin{aligned} &[G] = [P]^{-1}[Q], \quad [H] = [P]^{-1}[F_{21}]^{-1}, \\ &[P] = [I] + [F_{21}]^{-1}[F_{31}][T_{12}^{(3)}]^{-1} \\ &\qquad \times [D_{34}]^{-1}[D_{24}][T_{12}^{(2)}], \\ &[Q] = -[F_{21}]^{-1}[F_{31}][T_{12}^{(3)}]^{-1}([D_{34}]^{-1}[D_{24}] \\ &\qquad \times [T_{11}^{(2)}][D_{12}] - [T_{11}^{(3)}][D_{13}]). \end{aligned} \tag{6}$$

The substitution of Eqs. (2) and (5) into the second equation of (3) yields

$$\{d_4(0)\} = [S_{11}]\{d_1(B)\} + [S_{12}]\{f_1(B)\}, \tag{7}$$

where

$$\begin{aligned} &[S_{11}] = [D_{24}]([T_{11}^{(2)}][D_{12}] + [T_{12}^{(2)}][G]), \\ &[S_{12}] = [D_{24}][T_{12}^{(2)}][H], \end{aligned} \tag{8}$$

and the substitution of Eqs. (2)–(5) into the second equation of (4) also gives

$$\{f_4(0)\} = [S_{21}]\{d_1(B)\} + [S_{22}]\{f_1(B)\}, \tag{9}$$

where

$$\begin{aligned} &[S_{21}] = [F_{24}][T_{21}^{(2)}][D_{12}] \\ &\qquad + [F_{34}][T_{21}^{(3)}][D_{13}] + [R][G], \\ &[S_{22}] = [F_{34}][T_{22}^{(3)}][F_{31}]^{-1} + [R][H], \\ &[R] = [F_{24}][T_{22}^{(2)}] - [F_{34}][T_{22}^{(3)}][F_{31}]^{-1}[F_{21}]. \end{aligned} \tag{10}$$

Therefore, the state vector $\{z_4(0)\}$ can be written as

$$\{z_4(0)\} = [S]\{z_1(B)\}, \tag{11}$$

by using the connection matrix of the closed member

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}. \tag{12}$$

The state vector $\{z_4(y)\}$ is expressed as

$$\begin{aligned} \{z_4(y)\} &= [T^{(4)}(y)][S][T^{(1)}(B)]\{z_1(0)\} \\ &= [\overline{T(y)}]\{z_1(0)\}, \end{aligned} \tag{13}$$

by using the entire structure matrix $[\overline{T(y)}]$ obtained by the product of the connection matrix of the closed member and the transfer matrices of the exterior members.

II. MATRIX EQUATION OF PLATE

In this article, the present method is applied to joined plate structures with closed plate members. For a rectangular plate simply supported at the axial edges, one can take

$$\begin{aligned} &u = H\bar{u}\cos(m\pi/L)x, \quad (v,w) = H(\bar{v},\bar{w})\sin(m\pi/L)x, \\ &\psi = (H/a)\bar{\psi}\sin(m\pi/L)x, \\ &M_y = (K/a)\overline{M}_y\sin(m\pi/L)x, \\ &(N_y, S_y) = (K/a^2)(\overline{N}_y, \overline{S}_y)\sin(m\pi/L)x, \\ &N_{yx} = (K/a^2)\overline{N}_{yx}\cos(m\pi/L)x \quad (m = 1,2,...). \end{aligned} \tag{14}$$

Here, u, v, and w are the deflection displacements in the axial, tangential, and normal directions, respectively, ψ is the bending slope, M_y is the bending moment, S_y is the Kelvin–Kirchhoff shearing force, and N_y and N_{yx} are the components of the membrane force. The overbarred quantities $\bar{u},\bar{v}$,... are the respective dimensionless variables, H is the plate thickness, a is the reference length of the structure, L is the axial length, m is the axial half-wavenumber, and $K = EH^3/12(1-\nu^2)$ (E: Young's modulus; ν: Poisson's ratio) is the flexural rigidity.

The equations of vibration for a rectangular plate simply supported at the axial edges are expressed in a matrix differential equation

$$\frac{d}{d\eta}\{z(\eta)\} = [U]\{z(\eta)\}, \tag{15}$$

by use of the state vector

$$\begin{aligned} &\{z(\eta)\} = \{d(\eta) \quad f(\eta)\}^T, \\ &\{d(\eta)\} = \{\bar{u}\,\bar{v}\,\bar{w}\,\bar{\psi}\}^T, \\ &\{f(\eta)\} = \{\overline{M}_y \; -\overline{S}_y \; \overline{N}_y \; -\overline{N}_{yx}\}^T, \end{aligned} \tag{16}$$

which was conveniently employed by Irie *et al.*[19] For simplicity of the analysis, the following dimensionless parameters have been introduced:

$$(h, l, \eta) = (a)^{-1}(H, L, y), \quad \lambda^2 = \rho H a^2 \omega^2 / D. \tag{17}$$

Here, the symbol λ denotes a dimensionless frequency parameter expressed in terms of the mass density ρ, the radian frequency ω, and the extensional rigidity $D = EH/(1-\nu^2)$. The nonzero elements of the coefficient matrix $[U]$ of Eq. (15) are given by

$$\begin{aligned} &U_{12} = -(m\pi/l), \quad U_{18} = -h/6(1-\nu), \\ &U_{21} = \nu(m\pi/l), \quad U_{27} = h/12, \\ &U_{34} = -1, \quad U_{43} = -\nu(m\pi/l)^2, \\ &U_{45} = 1/h, \quad U_{54} = 2(1-\nu)h(m\pi/l)^2, \\ &U_{63} = -(1-\nu^2)h(m\pi/l)^4 + (12/h)\lambda^2, \\ &U_{72} = -(12/h)\lambda^2, \\ &U_{81} = -12(1-\nu^2)(1/h)(m\pi/l)^2 + (12/h)\lambda^2. \end{aligned} \tag{18}$$

The substitution of Eq. (1) into Eq. (15) yields

$$\frac{d}{d\eta}[T(\eta)] = [U][T(\eta)]. \tag{19}$$

The matrix $[T(\eta)]$ is obtained by integrating Eq. (19) numerically with the starting value $[T(0)] = [I]$ (the unit matrix) which is given by taking $\eta = 0$ in Eq. (1). In the numerical calculation, the elements of the transfer matrix are conveniently determined by using the Runge–Kutta–Gill integration method.

III. NUMERICAL EXAMPLES AND DISCUSSION

By the application of the method, two examples are presented for joined plate structures.

A. Example 1: Plate structure with an unsymmetrically sited duct

Figure 2 shows a joined plate structure with an unsymmetrically sited duct which is clamped at the edges of the exterior members. With the breadths or heights of each member denoted by B_i $(i = 1,2,3,4)$, the numbers $i = 1,3,4$ and $ij = 21,22,23$ are taken for each member as shown in the figure. With the radius a of a circle with the same circumferential length as the closed member length taken as the reference length, the breadth ratio $b_i = B_i/a$ is introduced here. The state vector $\{z_{2,j+1}(0)\}$ is written as

$$\{z_{2,j+1}(0)\} = [J]\{z_{2j}(b)\} \quad (j = 1,2), \tag{20}$$

by using the point matrix $[J]$ between two plates perpendicular to each other

$$[J] = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad [A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{21}$$

Therefore, the structure matrix $[T^{(2)}(b)]$ is obtained by the product of the transfer matrices of the closed members $(ij = 21,22,23)$ and the point matrix as follows:

$$[T^{(2)}(b)] = [T^{(23)}(b)][J][T^{(22)}(b)][J][T^{(21)}(b)]. \tag{22}$$

In this case, the joint matrices $[D_{ij}]$ and $[F_{ij}]$ are given by

$$[D_{12}] = [D_{24}] = [F_{24}] = [A]^{-1}, \quad [F_{21}] = [A],$$
$$[D_{13}] = [D_{34}] = [F_{31}] = [F_{34}] = [I]. \tag{23}$$

The boundary conditions at the clamped edges of the exterior members are expressed as

$$\bar{u} = \bar{v} = \bar{w} = \bar{\psi} = 0, \quad \text{at } \eta_1 = 0 \text{ and } \eta_4 = b_4. \tag{24}$$

The substitution of Eq. (24) into Eq. (13) derives, the frequency equation of the system

$$\begin{bmatrix} T_{15} & T_{16} & T_{17} & T_{18} \\ T_{25} & T_{26} & T_{27} & T_{28} \\ T_{35} & T_{36} & T_{37} & T_{38} \\ T_{45} & T_{46} & T_{47} & T_{48} \end{bmatrix}_{(b_4)} \begin{Bmatrix} \bar{M}_y \\ -\bar{S}_y \\ \bar{N}_y \\ -\bar{N}_{yx} \end{Bmatrix}^{(1)}_{(0)} = 0. \tag{25}$$

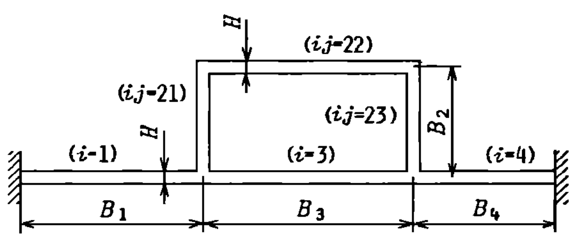

FIG. 2. Joined plate structure with an unsymmetrically sited duct.

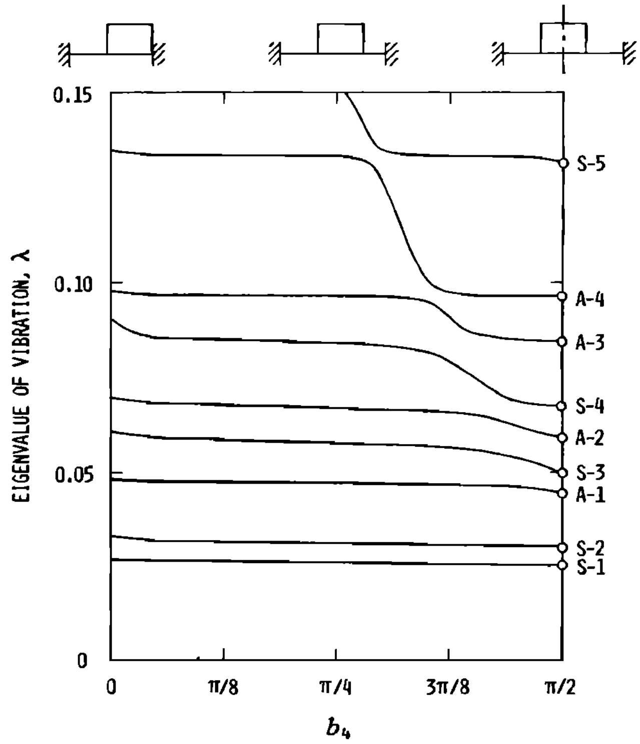

FIG. 3. Eigenvalues of vibration λ versus breadth ratio b_4 for joined plate structures with an unsymmetrically sited duct, for which $b_1 = \pi/2$, $b_2 = 2\pi/5$, $b_3 = 3\pi/5$, $h = 0.02$, $l = 4$, $\nu = 0.3$, $m = 1$.

The natural frequencies of the system are determined by calculating the eigenvalues λ of Eq. (25), and the mode shapes of vibration are determined by calculating the eigenvectors corresponding to the eigenvalues.

Figure 3 shows the eigenvalues of vibration λ versus the breadth ratio b_4 $(= B_4/a)$ for the structures. With an increase of the ratio b_4, the eigenvalues of lower mode vibrations decrease slightly, while the values of higher mode vibrations decrease remarkably, changing the mode shapes within the range of $b_4 = \pi/4 - 3\pi/8$. When the structure is symmetrical with respect to the central plane $(b_4 = \pi/2)$, symmetrical (S-type) and antisymmetrical (A-type) vibrations arise in the system. The eigenvalues of these vibrations are shown by the circled points on the ordinate at $b_4 = \pi/2$.

Figure 4 shows the eigenvalues of vibration λ versus the breadth ratio b_3/b_2 $(= B_3/B_2)$ of the closed members. With the variation of the ratio b_3/b_2, the eigencurves come close to each other changing in a wavelike manner. In this case, the mode shapes change each other, but any frequency crossings do not occur.

For verification of the method, the eigenvalues of corresponding modes of the system, which has the free edges of the exterior members, are compared with those of a square duct $(b_1 = b_4 = 0)$ of Azimi *et al.*,[20] which are obtained by the receptance method in Table I. In this case, the frequency equation of the system is derived as follows:

$$\begin{bmatrix} T_{51} & T_{52} & T_{53} & T_{54} \\ T_{61} & T_{62} & T_{63} & T_{64} \\ T_{71} & T_{72} & T_{73} & T_{74} \\ T_{81} & T_{82} & T_{83} & T_{84} \end{bmatrix}_{(b_4)} \begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \\ \bar{\psi} \end{Bmatrix}^{(1)}_{(0)} = 0. \tag{26}$$

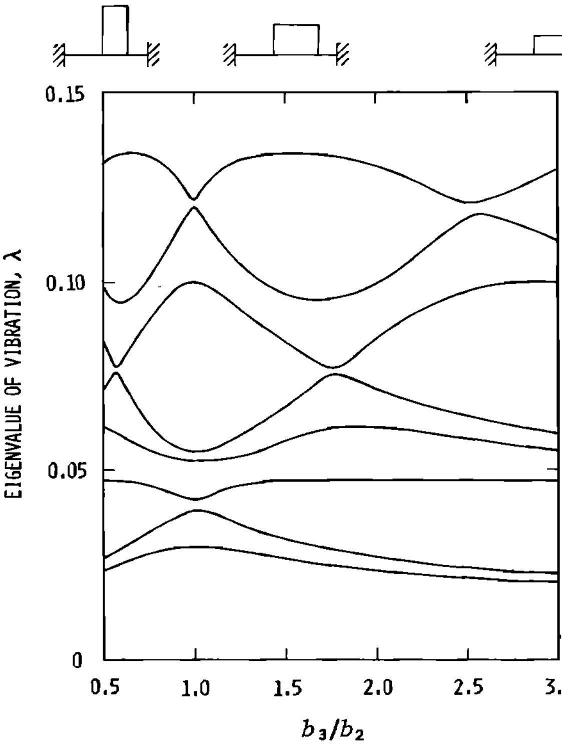

FIG. 4. Eigenvalues of vibration λ versus breadth ratio b_3/b_2 for joined plate structures with an unsymmetrically sited duct, for which $b_1=\pi/2$, $b_2+b_3=\pi$, $b_4=\pi/4$, $h=0.02$, $l=4$, $\nu=0.3$, $m=1$.

The values of $b_1=b_4=\pi/20$ are in good agreement with the results of other authors.

B. Example 2: Rectangular prismatic shell with an aslant interior plate

Figure 5 shows a rectangular prismatic shell (box-type structure) with an aslant interior plate. With the breadths or heights of the members denoted by B_i $(i=1,2,3,4)$, and the slant angle by α, the numbers $i=3,4$ and $ij=11,12,21,22,23$ are taken in each member as shown in the figure. In this case, the radius a of a circle with the same circumferential length as the tangential one of the box is conveniently taken as the reference length. The structure matrix $[T^{(2)}(b)]$ is also given by Eq. (22), and the structure matrix $[T^{(1)}(b)]$ is derived in the same manner. The joint matrices are written as

$$\begin{aligned}&[D_{12}]=[D_{24}]=[F_{21}]=[F_{24}]=[I],\\&[D_{13}]=[A_1],\quad [F_{31}]=[A_1]^{-1},\\&[D_{34}]=[F_{34}]=[A_2],\end{aligned}\tag{27}$$

where

TABLE I. Comparison of eigenvalues of vibration λ of joined plate structure, for which $b_2=b_3=\pi/2$, $h=0.02$, $l=\pi/2$, $\nu=0.3$, $m=1$.

$b_1=b_4$	S-1	A-1	S-2	A-2	S-3
$\pi/4$	0.0428	0.0590	0.0677	0.1203	0.1395
$\pi/20$	0.0468	0.0558	0.0677	0.1159	0.1372
0[a]	0.0462	0.0553	0.0677	0.1155	0.1372

[a] See Ref. 20.

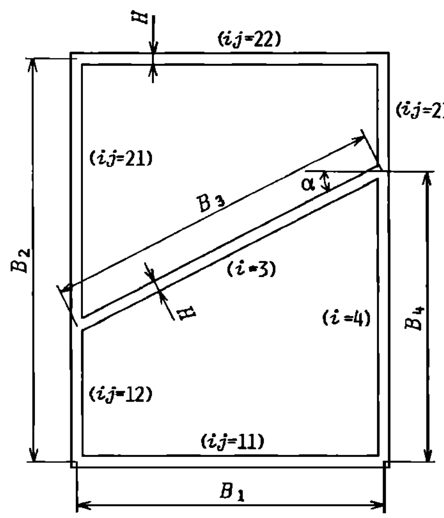

FIG. 5. Rectangular prismatic shell with an aslant interior plate.

$$[A_1]=\begin{bmatrix}1&0&0&0\\0&\mathrm{s}&-\mathrm{c}&0\\0&\mathrm{c}&\mathrm{s}&0\\0&0&0&1\end{bmatrix},$$
$$[A_2]=\begin{bmatrix}1&0&0&0\\0&-\mathrm{s}&-\mathrm{c}&0\\0&\mathrm{c}&-\mathrm{s}&0\\0&0&0&1\end{bmatrix}.\tag{28}$$

Here, the symbols s and c denote $\sin\alpha$ and $\cos\alpha$, respectively. The joining relation between the first $(i=1)$ and the fourth $(i=4)$ members is written as

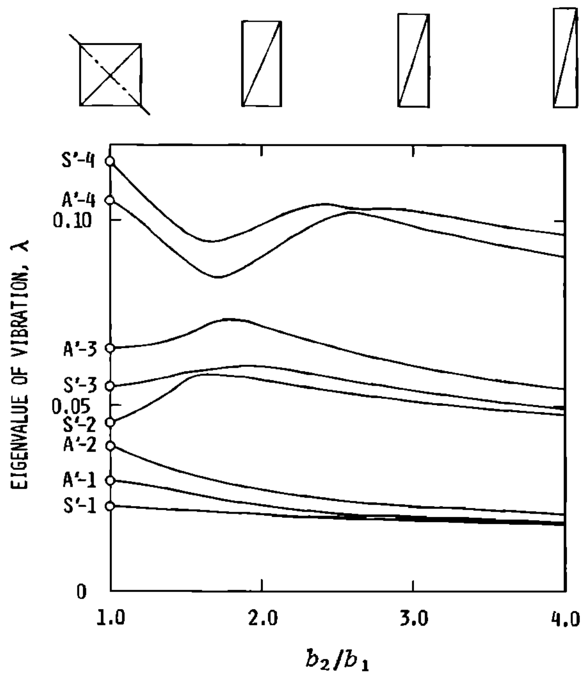

FIG. 6. Eigenvalues of vibration λ versus breadth ratio b_2/b_1 for rectangular prismatic shells with a diagonal interior plate, for which $b_1+b_2=\pi$, $h=0.02$, $l=4$, $\nu=0.3$, $m=1$.

$$\{z_1(0)\} = [J]\{z_4(b)\}, \quad (29)$$

by using the point matrix $[J]$. The substitution of Eq. (29) into Eq. (13) derives the frequency equation of the system

$$([\overline{T(b_4)}] - [J]^{-1})\{z_1(0)\} = 0. \quad (30)$$

Figure 6 shows the eigenvalues of vibration λ versus the aspect ratio b_2/b_1 ($= B_2/B_1$) of the box with a diagonal interior plate. With an increase of the ratio b_2/b_1, the eigenvalues of lower mode vibrations decrease monotonically, while the eigencurves of higher mode vibrations change in a wavelike manner. The box of $b_2/b_1 = 1$ is symmetrical with respect to a central plane perpendicular to the interior plate. In this case, symmetrical (S′-type) and antisymmetrical (A′-type) vibrations also arise in the structure, for which the mode types are shown on the ordinate at $b_2/b_1 = 1$.

Figure 7 shows the eigenvalues of vibration λ versus the breadth ratio $b_{12} = b_{23}$ ($B_{12}/a = B_{23}/a$). With an increase of the ratio $b_{12} = b_{23}$, the eigenvalues of lower mode vibrations increase monotonically, while those of higher mode vibrations take maximum values near $b_{12} = b_{23} = 2\pi/10$. When the ratio $b_{12} = b_{23}$ is equal to $3\pi/10$, the interior plate is parallel to two plates of the box, and the structure becomes symmetrical with respect to the central plane perpendicular to the interior plate. The eigenvalues of symmetrical and antisymmetrical vibrations arising in the system are shown by the circled points on the ordinate at $b_{12} = b_{23} = 3\pi/10$.

Figure 8 shows the mode shapes of the box-type structure with an aslant interior plate. The thick lines show the composition of the tangential and normal displacements, where the maximum normal displacements are taken to have unit value. Though coupled vibrations usually arise in the box and interior plate, a few weakly coupled vibrations arise there.

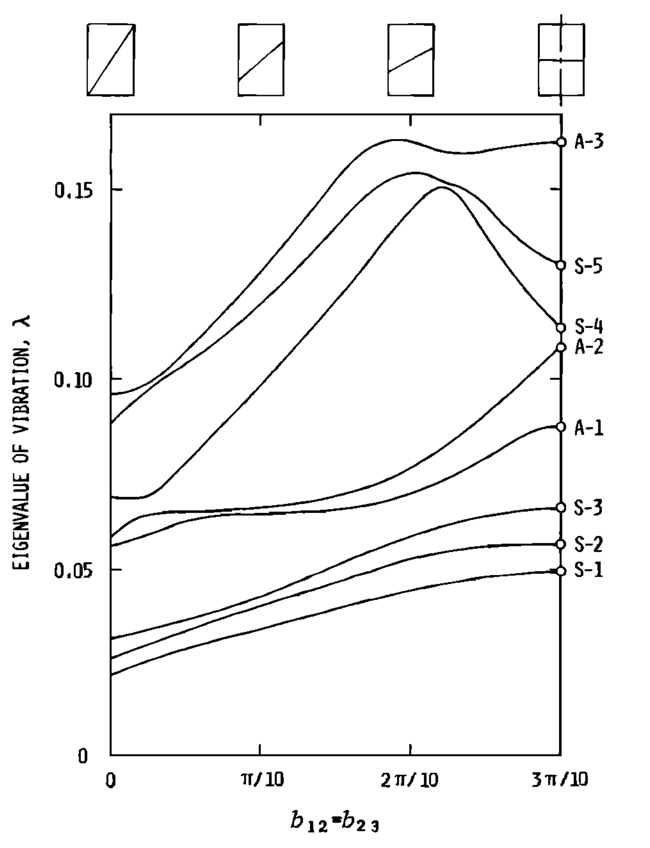

FIG. 7. Eigenvalues of vibration λ versus breadth ratio $b_{12} = b_{23}$ for rectangular prismatic shells with an aslant interior plate, for which $b_1 = 2\pi/5$, $b_2 = 3\pi/5$, $h = 0.02$, $l = 4$, $\nu = 0.3$, $m = 1$.

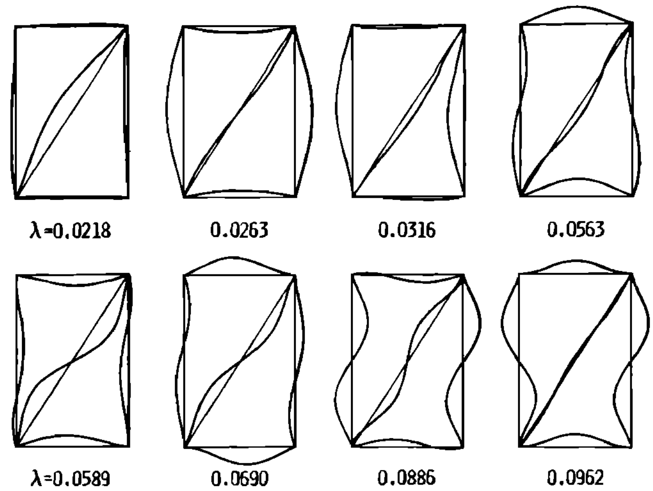

FIG. 8. Mode shapes of vibration of a rectangular prismatic shell with a diagonal interior plate, for which $b_1 = 2\pi/5$, $b_2 = 3\pi/5$, $h = 0.02$, $l = 4$, $\nu = 0.3$, $m = 1$.

IV. CONCLUSIONS

The free vibration of unsymmetrical joined plate or shell structures with a closed member has been studied by use of the transfer matrix integration method. By the introduction of the connection matrix of a closed member, the entire structure matrix has been obtained by the product of the connection matrix of the closed member and the transfer matrices of other members. This method has been applied to a plate structure with an unsymmetrically sited duct and a box-type structure with an aslant interior plate, and the vibration characteristics of them have been clarified quantitatively.

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