THE NOETHERIAN PROPERTIES OF THE RINGS OF DIFFERENTIAL OPERATORS ON CENTRAL 2-ARRANGEMENTS

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Abstract. Whereas Holm proved that the ring of differential operators on a generic hyperplane arrangement is finitely generated as an algebra, the problem of its Noetherian properties is still open. In this article, after proving that the ring of differential operators on a central arrangement is right Noetherian if and only if it is left Noetherian, we prove that the ring of differential operators on a central 2-arrangement is Noetherian. In addition, we prove that its graded ring associated to the order filtration is not Noetherian when the number of the consistuent hyperplanes is greater than 1.

Key Words: Ring of differential operators;Noetherian property; Hyperplane arrangement.

2010 Mathematics Subject Classification: Primary 13N10; Secondary 32S22.

1. Introduction

Let K be a field of characteristic zero. For a commutative K-algebra R, we inductively define K-vector spaces of linear differential operators by

$$\mathscr{D}^0(R) := \{\theta \in \mathrm{End}_K(R) \mid a \in R, \theta a - a\theta = 0\},$$
$$\mathscr{D}^m(R) := \left\{\theta \in \mathrm{End}_K(R) \mid a \in R, \theta a - a\theta \in \mathscr{D}^{m-1}(R)\right\} \quad (m \geq 1).$$

We set $\mathscr{D}(R) := \bigcup_{m\geq 0} \mathscr{D}^m(R)$, and we call $\mathscr{D}(R)$ the ring of differential operators of R. Let $S := K[x_1, \dots, x_n]$ denote the polynomial ring. It is well known that the ring $\mathscr{D}(S)$ of differential operators of S is the n-th Weyl algebra $K[x_1, \dots, x_n]\langle\partial_1, \dots, \partial_n\rangle$ where $\partial_i := \frac{\partial}{\partial x_i}$ (see for example [5, Example 15.1.15] and [5, Corollaty 15.5.6]). We use the multi-index notations, for example, $\partial^{\boldsymbol{\alpha}} := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$ for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. We set $\mathscr{D}^{(m)}(S) := \bigoplus_{|\boldsymbol{\alpha}|=m} S\partial^{\boldsymbol{\alpha}}$ for $m \geq 0$. We regard $\mathscr{D}^{(m)}(S)$ as a left S-module by the left product in the Weyl algebra. Then the Weyl algebra $\mathscr{D}(S)$ is decomposed into the direct sum of left S-modules $\mathscr{D}^{(m)}(S)$ of homogeneous differential operators: $\mathscr{D}(S) = \bigoplus_{m\geq 0} \mathscr{D}^{(m)}(S)$.

There has been a lot of research on finiteness properties of the rings of differential operators. It is well known that $\mathscr{D}(R)$ is Noetherian, if R is a regular domain (see [5, Theorem 15.1.20] and [5, Corollary 15.5.6]). There are some other important classes of algebras such that $\mathscr{D}(R)$ are Noetherian. For example, if R is an integral domain of Krull dimension one, then $\mathscr{D}(R)$ is Noetherian (Muhasky [6] and Smith-Stafford [11]). Saito-Takahashi [10] showed that $\mathscr{D}(R)$ is right Noetherian if R is an affine semigroup algebra. However, $\mathscr{D}(R)$ is not Noetherian in general. Bernstein-Gel'fand-Gel'fand [1] gave an example of a ring of differential operators that is neither Noetherian nor finitely generated.

Let $\mathcal{A} = \{H_i \mid i = 1, \dots, r\}$ be a central (hyperplane) arrangement (i.e., every hyperplane in $\mathcal{A}$ contains the origin) in K^n. Let I be the defining ideal of $\mathcal{A}$. We consider the left S-module $\mathscr{D}^{(m)}(I)$ of differential operators homogeneous of order m that preserve the ideal I. We call $\mathscr{D}^{(m)}(I)$ the modules of $\mathcal{A}$-differential operators. We find many results about the module $\mathscr{D}^{(1)}(I)$ of $\mathcal{A}$-derivations in a rich literature (see for example [8]). In contrast, there are only a few literatures about the modules of $\mathcal{A}$-differential operators of a higher order. Holm [4] proved that the ring of differential operators of the coordinate ring S/I is finitely generated when I is the ideal defining a generic hyperplane arrangement. In this paper, we will prove that $\mathscr{D}(S/I)$ is Noetherian if $n = 2$.

In Section 3, we prove that $\mathscr{D}(S/I)$ is right Noetherian if and only if it is left Noetherian. Thus the Noetherian property of $\mathscr{D}(S/I)$ can be proved by the right or left Noetherian property.

In Section 4, we prove that $\mathscr{D}(S/I)$ is right Noetherian in the case $n = 2$. This is the main result of this article. Let R be a filtered ring, and $\mathcal{F}$ the filtration. If the graded ring associated to the filtration $\mathcal{F}$ of R is right (left) Noetherian, then R is right (left) Noetherian. However, the graded ring associated to the order filtration of $\mathscr{D}(S/I)$ is not Noetherian if $r \geq 2$ (Example 4.17). Hence we cannot take this convenient approach to prove the Noetherian property of $\mathscr{D}(S/I)$. The keys of the proof of the main result are Corollary 4.11 and Lemma 4.14.

There is a well-known basis for the module $\mathscr{D}^{(1)}(I)$ of $\mathcal{A}$-derivation (see for example [8]). Holm [3] studied the module $\mathscr{D}^{(m)}(I)$, and gave its basis for any order m. Let $\mathscr{D}(J)$ denote the subring of $\mathscr{D}(S)$ consisting of the operators preserving an ideal J. Holm [3],[4] showed that $\mathscr{D}(I)$ decomposes into the direct sum of $\mathscr{D}^{(m)}(I)$. For an ideal J, there is a ring isomorphism:

$$\mathscr{D}(S/J) \simeq \mathscr{D}(J)/J\mathscr{D}(S)$$

(see [5, Theorem 15.5.13]). Using these facts, we can write any element of $\mathscr{D}(S/I)$ as a linear combination of bases of the modules of $\mathcal{A}$-differential operators. This expression is useful to prove Corollary 4.11.

We consider a sequence of two-sided ideals of $\mathscr{D}(I)$:

$$I\mathscr{D}(S) = L_r \subseteq L_{r-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = \mathscr{D}(I).$$

We prove that $\mathscr{D}(I)/I\mathscr{D}(S)$ is right Noetherian by proving that each L_{i-1}/L_i is right Noetherian $\mathscr{D}(I)$-module. To show the right Noetherian property of L_{i-1}/L_i, we study a module of lower order operators in L_{i-1}/L_i and that of higher order operators separately.

We prove that the right $\mathscr{D}(I)$-module generated by the higher order operators in L_{i-1}/L_i is Noetherian in Corollary 4.11, and that the module of lower order operators in L_{i-1}/L_i is right Noetherian as a right S-module in Lemma 4.14. In this way, we see that $\mathscr{D}(S/I)$ is Noetherian.

2. Differential operators on a central arrangement

In this section, we fix some notation, and we refer to some facts used in Section 4. Let $\mathcal{A} = \{H_i \mid i = 1, \dots, r\}$ be a central arrangement in K^n. Fix a polynomial p_i defining H_i, and put $Q := p_1 \cdots p_r$. Thus Q is a product of certain homogeneous polynomials of degree 1. We call Q a defining polynomial of $\mathcal{A}$. Let I denote the principal ideal of S generated by Q.

For any ideal J of S, we define an S-submodule $\mathscr{D}^{(m)}(J)$ of $\mathscr{D}^{(m)}(S)$ and a subring $\mathscr{D}(J)$ of $\mathscr{D}(S)$ by

$$\mathscr{D}^{(m)}(J) := \{\theta \in \mathscr{D}^{(m)}(S) \mid \theta(J) \subseteq J\},$$
$$\mathscr{D}(J) := \{\theta \in \mathscr{D}(S) \mid \theta(J) \subseteq J\}.$$

Among others, Holm [4] proved the following two propositions.

Proposition 2.1 (Proposition 4.3 in [4]). *We have a direct sum*

$$\mathscr{D}(I) = \bigoplus_{m \geq 0} \mathscr{D}^{(m)}(I)$$

as a left S-module.

Proposition 2.2 (Proposition 2.4 in [4]). *Suppose that $f_1, \dots, f_k \in S$ are coprime to one another. Then*

$$\mathscr{D}(\langle f_1 \cdots f_k \rangle) = \bigcap_{i=1}^{k} \mathscr{D}(\langle f_i \rangle).$$

The following is well known (e.g., see [4, Proposition 2.3]).

Proposition 2.3. *Let J be the ideal of S generated by $f_1, \dots, f_k$, and let $\theta \in \mathscr{D}(S)$ be an operator of order $m \geq 1$. Then $\theta \in \mathscr{D}(J)$ if and only if $\theta(x^{\boldsymbol{\alpha}} f_j) \in J$ for $|\boldsymbol{\alpha}| \leq m - 1$ and $j = 1, \dots, k$.*

We use the following lemma in Section 4.

Lemma 2.4. *Let* $\delta \in \sum_{i=1}^{n} K\partial_i$, *and let* $f_1, \dots, f_k$ *be polynomials of degree* 1. *If* $k \le m$, *then*

$$\delta^m f_1 \dots f_k = \sum_{i=0}^{k} [m]_i \Big(\sum_{\substack{\Lambda \subseteq \{1,\dots,k\} \\ \sharp\Lambda = i}} \prod_{j \in \Lambda} \delta(f_j) \prod_{j \notin \Lambda} f_j \Big) \delta^{m-i}$$

$$= \sum_{i=0}^{k} [m]_i \Big(\frac{1}{i!(k-i)!} \sum_{\sigma \in S_k} \delta(f_{\sigma(1)}) \dots \delta(f_{\sigma(i)}) f_{\sigma(i+1)} \cdots f_{\sigma(k)} \Big) \delta^{m-i},$$

where $[m]_0 := 1$ *and* $[m]_i := m(m-1)\cdots(m-i+1)$ *for* $i \ge 1$.

Proof. For any $f \in S$, we see $\delta^\ell f = f\delta^\ell + \ell\delta(f)\delta^{\ell-1}$. We can prove the assertion by induction on k. ☐

For a monomial $x^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}}$ in $\mathscr{D}(S)$, we define its total degree by

$$\operatorname{totdeg}(x^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}}) = |\boldsymbol{\alpha}| - |\boldsymbol{\beta}|. \tag{2.1}$$

For $\theta \in \mathscr{D}(S)$, we define the total degree of θ as the largest total degree of monomials in θ. We consider $\mathscr{D}(S)$ a graded ring by the total degree.

The operator

$$\varepsilon_m := \sum_{|\boldsymbol{\alpha}|=m} \frac{m!}{\boldsymbol{\alpha}!} x^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\alpha}}$$

is called the Euler operator of order m where $\boldsymbol{\alpha}! = (\alpha_1!)\cdots(\alpha_n!)$ for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$. Then ε_1 is the Euler derivation, and $\varepsilon_m = \varepsilon_1(\varepsilon_1 - 1)\cdots(\varepsilon_1 - m + 1)$ [4, Lemma 4.9].

3. Right Noetherian property and left Noetherian property

Let $Q = p_1 \cdots p_r$ be a defining polynomial of a central arrangement $\mathcal{A}$, and let $I = QS$. In this section, we will prove that the ring $\mathscr{D}(S/I)$ of differential operators is right Noetherian if and only if $\mathscr{D}(S/I)$ is left Noetherian. Recall that we have a ring isomorphism $\mathscr{D}(S/I) \simeq \mathscr{D}(I)/I\mathscr{D}(S)$ (see [5, Proposition 15.5.9 (ii)] and [5, Theorem 15.5.13]).

Let $0 \ne h \in S$, and set $J := hS$. We denote by $K(x_1, \dots, x_n)$ the field of fractions of S. Then $\mathscr{D}(S) \cap h\mathscr{D}(S)h^{-1} \subseteq K(x_1, \dots, x_n)\langle \partial_1, \dots, \partial_n \rangle$.

Lemma 3.1. *As a ring,*

$$\mathscr{D}(J) = \mathscr{D}(S) \cap h\mathscr{D}(S)h^{-1}.$$

Proof. Assume that $h\theta h^{-1} \in \mathscr{D}(S)$ with $\theta \in \mathscr{D}(S)$. For any $f \in S$,

$$h\theta h^{-1}(hf) = h\theta(f) \in hS,$$

which means $h\theta h^{-1} \in \mathscr{D}(J)$.

Next we will prove the converse inclusion. Let $\theta \in \mathscr{D}(J)$. Since $h^{-1}\theta h \in K(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$, we can write

$$h^{-1}\theta h = \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}$$

with $f_{\boldsymbol{\alpha}} \in K(x_1, \ldots, x_n)$. We show that $f_{\boldsymbol{\alpha}} \in S$ for all $\boldsymbol{\alpha}$ by induction on $|\boldsymbol{\alpha}|$.

Since

$$f_0 = h^{-1}\theta h(1) = h^{-1}\theta(h) \in h^{-1}hS = S,$$

we have $f_0 \in S$.

Assume that $f_{\boldsymbol{\alpha}} \in S$ for all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| < m$. For $|\boldsymbol{\beta}| = m$,

$$h^{-1}\theta h(x^{\boldsymbol{\beta}}) = \boldsymbol{\beta}! f_{\boldsymbol{\beta}} + \sum_{|\boldsymbol{\alpha}|<m} f_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}(x^{\boldsymbol{\beta}}).$$

Since $\theta \in \mathscr{D}(J)$, we obtain

$$h^{-1}\theta h(x^{\boldsymbol{\beta}}) = h^{-1}\theta(hx^{\boldsymbol{\beta}}) \in h^{-1}hS = S.$$

Then $f_{\boldsymbol{\beta}} \in S$ by the induction hypothesis. Therefore we conclude that $h^{-1}\theta h = \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} \in \mathscr{D}(S)$. □

Define an anti-automorphism ${}^t : \mathscr{D}(S) \longrightarrow \mathscr{D}(S)$ by ${}^t x_i = x_i, {}^t\partial_i = -\partial_i$ for $i = 1, \ldots, n$ (we say that ^{t} is an anti-automorphism if ^{t} is an automorphism as a linear map, and if ${}^t(\theta\eta) = {}^t\eta {}^t\theta$ for any $\theta, \eta \in \mathscr{D}(S)$). It is clear that ${}^t({}^t\theta) = \theta$ for any $\theta \in \mathscr{D}(S)$.

For $\theta \in \mathscr{D}(J)$, put $\theta^* := h\, {}^t\theta h^{-1}$. Then

$$\begin{aligned}
(\mathscr{D}(J))^* &= (\mathscr{D}(S) \cap h\mathscr{D}(S)h^{-1})^* \\
&= h\, {}^t(\mathscr{D}(S) \cap h\mathscr{D}(S)h^{-1})h^{-1} \\
&= h\, {}^t\mathscr{D}(S)h^{-1} \cap {}^t\mathscr{D}(S) \\
&= h\mathscr{D}(S)h^{-1} \cap \mathscr{D}(S) \\
&= \mathscr{D}(J)
\end{aligned}$$

by Lemma 3.1. Thus

$$^* : \mathscr{D}(J) \longrightarrow \mathscr{D}(J) \tag{3.1}$$

is an anti-automorphism. If $h\theta \in J\mathscr{D}(S)$, then

$$(h\theta)^* = h\, {}^t(h\theta)h^{-1} = h\, {}^t\theta hh^{-1} = h\, {}^t\theta \in J\mathscr{D}(S).$$

It is clear that $\theta = (\theta^*)^*$ for any $\theta \in \mathscr{D}(J)$. Hence we have $(J\mathscr{D}(S))^* = J\mathscr{D}(S)$. Therefore the anti-automorphism * induces an anti-automorphism

$$^* : \mathscr{D}(J)/J\mathscr{D}(S) \longrightarrow \mathscr{D}(J)/J\mathscr{D}(S).$$

The following is clear from the existence of the anti-automorphism *.

Theorem 3.2. *The ring $\mathscr{D}(J)/J\mathscr{D}(S)$ is right Noetherian if and only if $\mathscr{D}(J)/J\mathscr{D}(S)$ is left Noetherian.*

Corollary 3.3. *Let I be the defining ideal of a central arrangement. Then the ring $\mathscr{D}(I)/I\mathscr{D}(S)$ is right Noetherian if and only if $\mathscr{D}(I)/I\mathscr{D}(S)$ is left Noetherian.*

Remark 3.4. *By the anti-isomorphism* (3.1)*, it is also true that the ring $\mathscr{D}(J)$* (also *$\mathscr{D}(I)$) is right Noetherian if and only if is left Noetherian.*

It is known that some finiteness properties of rings of differential operators on irreducible affine algebraic varieties over an algebraically closed field (see [11, Theorem 2.5], [11, Proposition 7.3] and [11, Theorem 7.5]), whereas varieties of central hyperplane arrangements are reducible. Thus we cannot apply the results in [11].

4. The case $n = 2$

In this section, let $n = 2$ and $S = K[x, y]$. We will prove that the ring $\mathscr{D}(S/I) \simeq \mathscr{D}(I)/I\mathscr{D}(S)$ of differential operators is Noetherian. We will also prove that, in contrast, the graded ring $\operatorname{Gr}\mathscr{D}(S/I)$ associated to the order filtration is not Noetherian when $r \geq 2$.

Put $P_i := \frac{Q}{p_i}$ for $i = 1, \dots, r$, and define

$$\delta_i := \begin{cases} \partial_y & \text{if } p_i = ax \quad (a \in K^\times) \\ \partial_x + a_i\partial_y & \text{if } p_i = a(y - a_i x) \quad (a \in K^\times). \end{cases}$$

Then $\delta_i(p_j) = 0$ if and only if $i = j$.

Proposition 4.1 (Paper III, Proposition 6.7 in [3], Proposition 4.14 in [12])**.** *For any $m \geq 1$, $\mathscr{D}^{(m)}(I)$ is a free left S-module with basis*

$$\begin{aligned} &\{\varepsilon_m, P_1\delta_1^m, \dots, P_m\delta_m^m\} \text{ if } m < r-1, \\ &\{P_1\delta_1^m, \dots, P_r\delta_r^m\} \text{ if } m = r-1, \\ &\{P_1\delta_1^m, \dots, P_r\delta_r^m, Q\eta_{r+1}^{(m)}, \dots, Q\eta_{m+1}^{(m)}\} \text{ if } m > r-1, \end{aligned}$$

where the set $\{\delta_1^m, \dots, \delta_r^m, \eta_{r+1}^{(m)}, \dots, \eta_{m+1}^{(m)}\}$ forms a K-basis for $\sum_{|\boldsymbol{\alpha}|=m} K\partial^{\boldsymbol{\alpha}}$ if $m > r-1$.

By Proposition 2.1, we have

$$\begin{aligned} \mathscr{D}(I) = S \oplus &\left(\bigoplus_{m=1}^{r-2} \left(S\varepsilon_m \oplus SP_1\delta_1^m \oplus \cdots \oplus SP_m\delta_m^m\right)\right) \\ &\oplus \left(\bigoplus_{m\geq r-1} \left(SP_1\delta_1^m \oplus \cdots \oplus SP_r\delta_r^m \oplus SQ\eta_{r+1}^{(m)} \oplus \cdots \oplus SQ\eta_{m+1}^{(m)}\right)\right). \end{aligned}$$

For $i = 1, \dots, r$, define an additive group

$$L_i := \mathscr{D}(I) \cap (p_1 \cdots p_i)\mathscr{D}(S).$$

Proposition 4.2. *For $i = 1, \dots, r$, the additive group L_i is a two-sided ideal of $\mathscr{D}(I)$.*

Proof. It is clear that L_i is a right ideal of $\mathscr{D}(I)$.

To prove that L_i is a left ideal of $\mathscr{D}(I)$, by Proposition 2.1, we only need to prove that $\mathscr{D}^{(m)}(I)L_i \subseteq L_i$ for $m \geq 0$. Fix $\theta_m \in \mathscr{D}^{(m)}(I)$. For any $j = 1, \dots, i$, there exist $\eta_\ell \in D^{(\ell)}(S)$ such that

$$\theta_m p_j = \eta_0 + \cdots + \eta_m. \tag{4.1}$$

We prove that $\eta_\ell \in p_j \bigcap_{i' \neq j} \mathscr{D}^{(\ell)}(p_{i'}S) \subseteq \mathscr{D}^{(\ell)}(I)$ for $0 \leq \ell \leq m$ by induction on ℓ.

In the case $\ell = 0$, let (4.1) act on 1. Then

$$p_j S \ni \theta_m(p_j) = \eta_0$$

because $\theta_m \in \mathscr{D}^{(\ell)}(p_j S)$ by Proposition 2.2. If $\ell \geq 1$, then it follows from the induction hypothesis that $\eta_\ell(x^{\boldsymbol{\alpha}}) \in p_j S$ for any $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = \ell$ since

$$p_j S \ni \theta_m(p_j x^{\boldsymbol{\alpha}}) = \eta_0(x^{\boldsymbol{\alpha}}) + \cdots + \eta_{\ell-1}(x^{\boldsymbol{\alpha}}) + \eta_\ell(x^{\boldsymbol{\alpha}}).$$

Therefore $\eta_\ell \in p_j \mathscr{D}^{(\ell)}(S)$. Write $\eta_\ell = p_j \eta'_\ell$. For any $i' \neq j$ and $|\boldsymbol{\alpha}| = \ell - 1$, it also follows from the induction hypothesis that $p_j \eta'_\ell(p_{i'} x^{\boldsymbol{\alpha}}) = \eta_\ell(p_{i'} x^{\boldsymbol{\alpha}}) \in p_{i'} S$ since

$$p_{i'} S \ni \theta_m(p_j p_{i'} x^{\boldsymbol{\alpha}}) = \eta_0(p_{i'} x^{\boldsymbol{\alpha}}) + \cdots + \eta_{\ell-1}(p_{i'} x^{\boldsymbol{\alpha}}) + \eta_\ell(p_{i'} x^{\boldsymbol{\alpha}}).$$

Since p_j and $p_{i'}$ are coprime, we see that $\eta'_\ell(p_{i'} x^{\boldsymbol{\alpha}}) \in p_{i'} S$. So $\eta'_\ell \in \mathscr{D}(p_{i'} S)$ by Proposition 2.3, and $\eta_\ell \in p_j \bigcap_{i' \neq j} \mathscr{D}^{(\ell)}(p_{i'} S)$. Thus $\theta_m p_j \in p_j \bigcap_{i' \neq j} \mathscr{D}(p_{i'} S)$. Then we conclude that

$$\mathscr{D}(I) p_1 \cdots p_i \mathscr{D}(S) \subseteq p_1 \cdots p_i \mathscr{D}(S).$$

□

By Proposition 2.1, L_i is decomposed as follows:

$$L_i = \bigoplus_{m \geq 0} L_i^{(m)},$$

where $L_i^{(m)} := \mathscr{D}^{(m)}(I) \cap (p_1 \cdots p_i) \mathscr{D}^{(m)}(S)$. We consider a sequence

$$I\mathscr{D}(S) = L_r \subseteq L_{r-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = \mathscr{D}(I) \tag{4.2}$$

of two-sided ideals of $\mathscr{D}(I)$. If a right $\mathscr{D}(I)$-module L_{i-1}/L_i is Noetherian for any i, then $\mathscr{D}(I)/I\mathscr{D}(S)$ is a right Noetherian ring. Now we fix i, and we will prove that L_{i-1}/L_i is right Noetherian.

As a left S-module,

$$L_{i-1}/L_i = \bigoplus_{m \geq 0} \left(L_{i-1}^{(m)} + L_i / L_i \right) \simeq \bigoplus_{m \geq 0} \left(L_{i-1}^{(m)} / L_i^{(m)} \right).$$

Put

$$\big(L_{i-1}/L_i\big)^{<r-1} := \bigoplus_{m<r-1} \big(L_{i-1}^{(m)}/L_i^{(m)}\big),$$
$$\big(L_{i-1}/L_i\big)^{\geq r-1} := \bigoplus_{m\geq r-1} \big(L_{i-1}^{(m)}/L_i^{(m)}\big).$$

Then L_{i-1}/L_i is decomposed as a left S-module:

$$L_{i-1}/L_i = \big(L_{i-1}/L_i\big)^{<r-1} \oplus \big(L_{i-1}/L_i\big)^{\geq r-1}. \tag{4.3}$$

We will study $\big(L_{i-1}/L_i\big)^{<r-1}$ and $\big(L_{i-1}/L_i\big)^{\geq r-1}$ separately.

First we argue the part of order $\geq r-1$.

Lemma 4.3. *Assume that $m \geq r-1$. As a left S-module,*

$$L_i^{(m)} = SQ\delta_1^m \oplus \cdots \oplus SQ\delta_i^m \oplus SP_{i+1}\delta_{i+1}^m \oplus \cdots \oplus SP_r\delta_r^m \oplus SQ\eta_{r+1}^{(m)} \oplus \cdots \oplus SQ\eta_{m+1}^{(m)}.$$

Proof. Recall that $P_i = \frac{Q}{p_i}$. We see the assertion by Proposition 4.1 and the definition of L_i. □

Proposition 4.4. *For $m \geq 0$, we have*

$$L_i^{(m)} \cap SP_i\delta_i^m = SQ\delta_i^m \subseteq L_{i-1}^{(m)}$$

as a left S-module. Hence

$$\big(L_{i-1}/L_i\big)^{\geq r-1} = \bigoplus_{m\geq r-1} \big(SP_i\delta_i^m + L_i^{(m)}/L_i^{(m)}\big) \simeq \bigoplus_{m\geq r-1} \big(SP_i\delta_i^m/SQ\delta_i^m\big)$$

as a left S-module.

Proof. By Lemma 4.3, $L_{i-1}^{(m)} = SP_i\delta_i^m + L_i^{(m)}$ for $m \geq r-1$. Then as a left S-module

$$\big(L_{i-1}/L_i\big)^{\geq r-1} = \bigoplus_{m\geq r-1} \big(SP_i\delta_i^m + L_i^{(m)}/L_i^{(m)}\big) \simeq \bigoplus_{m\geq r-1} \big(SP_i\delta_i^m/L_i^{(m)} \cap SP_i\delta_i^m\big).$$

It remains to prove that

$$L_i^{(m)} \cap SP_i\delta_i^m = SQ\delta_i^m \subseteq L_{i-1}^{(m)}$$

for $m \geq 0$. It is clear that $SQ\delta_i^m \subseteq L_i^{(m)} \cap SP_i\delta_i^m$. Conversely, suppose that $fP_i\delta_i^m \in L_i^{(m)}$ with $f \in S$. Then $fP_i\delta_i^m \in p_1 \cdots p_i \mathscr{D}^{(m)}(S)$. Since the polynomials $p_i, \ldots, p_r$ are coprime to one another, we have $f \in p_iS$. Thus $L_i^{(m)} \cap SP_i\delta_i^m \subseteq SQ\delta_i^m$. □

We define a left S-module

$$E_i := \bigoplus_{m\geq 0} \big(SP_i\delta_i^m + L_i^{(m)}/L_i^{(m)}\big) \simeq \bigoplus_{m\geq 0} \big(SP_i\delta_i^m/SQ\delta_i^m\big). \tag{4.4}$$

Note that $(SP_i + L_i^{(0)})/L_i^{(0)} \simeq SP_i/SP_i \cap L_i^{(0)} = SP_i/SQ$. By Proposition 4.4, we may identify $\big(L_{i-1}/L_i\big)^{\geq r-1}$ with the S-submodule of E_i of order $m \geq r-1$. It is clear that $\delta_i p_i = p_i \delta_i$ for $i = 1, \ldots, r$. For $g \in S$, we have

$$P_i \delta_i^m (Qg) = Q \delta_i^m (\frac{Q}{p_i} g) \in QS.$$

Proposition 4.1 says already that $P_i \delta_i^m \in \mathscr{D}^{(m)}(I)$. Since there is so much notation, we should remind us that $P_i \delta_i^m \in (p_1 \cdots p_{i-1})\mathscr{D}(S)$, and so $P_i \delta_i^m \in L_{i-1}$. Hence E_i is a left S-submodule of L_{i-1}/L_i. Moreover, the following proposition is true:

Proposition 4.5. *The module E_i is a right $\mathscr{D}(I)$-submodule of L_{i-1}/L_i.*

Proof. We only need to check the right multiplication by the elements of S and the bases for $\mathscr{D}(I)$ in Proposition 4.1.

Let $m \geq 1$. For $g \in S$, we have

$$\delta_i^m \cdot g \in S + \sum_{\ell=1}^{m} S \delta_i^\ell,$$

and hence $P_i \delta_i^m \cdot S \subseteq E_i$.

We show that E_i is closed under the right action of the elements of bases for $\mathscr{D}(I)$. We only need to check the right multiplication by the elements $P_i \delta_i^\ell, P_j \delta_j^m (j \neq i), \varepsilon_\ell, Q\eta_j^{(\ell)}$. For $m \geq 1$, we have

$$\begin{aligned}
P_i \delta_i^m \cdot P_i \delta_i^\ell &= P_i (\delta_i^m \cdot P_i) \delta_i^\ell \in \bigoplus_{m \geq 0} \big(SP_i \delta_i^m + L_i^{(m)}\big), \\
P_i \delta_i^m \cdot P_j \delta_j^\ell &= Q \delta_i^m \cdot \frac{P_j}{p_i} \delta_j^l \in \mathscr{D}(I) \cap (p_1 \cdots p_i)\mathscr{D}(S) = L_i, \\
P_i \delta_i^m \cdot Q\eta_j^{(\ell)} &= Q \delta_i^m \cdot \frac{Q}{p_i} \eta_j^{(\ell)} \in \mathscr{D}(I) \cap (p_1 \cdots p_i)\mathscr{D}(S) = L_i
\end{aligned}$$

from the inclusion $P_i \delta_i^m \cdot S \subseteq E_i$. It remains to show that E_i is closed under the right multiplication by $\varepsilon_\ell = \varepsilon_1(\varepsilon_1 - 1)\cdots(\varepsilon_1 - \ell + 1)$. We consider the Euler derivation ε_1. We may assume $p_i = y - ax$ $(a \in K^\times)$. Recall $\delta_i = a^{-1}\partial_x + \partial_y$. Since

$$\begin{aligned}
\varepsilon_1 &= x\partial_x + y\partial_y \\
&= a^{-1}ax\partial_x + y\partial_y + a^{-1}y\partial_x - a^{-1}y\partial_x \\
&= a^{-1}(ax - y)\partial_x + y(\partial_y + a^{-1}\partial_x) \\
&= -a^{-1}p_i\partial_x + y\delta_i,
\end{aligned}$$

we have, for any $m \geq 0$,

$$P_i \delta_i^m \cdot \varepsilon_1 = P_i \delta_i^m \cdot (-a^{-1}p_i\partial_x + y\delta_i) = -a^{-1}Q\delta_i^m\partial_x + yP_i\delta_i^{m+1} + mP_i\delta_i^m.$$

We see that $-a^{-1}Q\delta_i^m\partial_x \in L_i$, and that the remaining terms belong to $SP_i\delta_i^{m+1}$ and $SP_i\delta_i^m$, respectively. It follows that

$$P_i\delta_i^m \cdot \varepsilon_\ell \in \bigoplus_{m\geq 0} \left(SP_i\delta_i^m + L_i^{(m)}\right).$$

Hence $E_i \cdot \varepsilon_\ell \subseteq E_i$. This completes the assertion. $\square$

As a left S-module,

$$\left(L_{i-1}/L_i\right)^{\geq r-1} \subseteq E_i.$$

The right $\mathscr{D}(I)$-module generated by $\left(L_{i-1}/L_i\right)^{\geq r-1}$ is a $\mathscr{D}(I)$-submodule of E_i by Proposition 4.5:

$$\left(L_{i-1}/L_i\right)^{\geq r-1} \cdot \mathscr{D}(I) \subseteq E_i.$$

If we prove that E_i is a right Noetherian $\mathscr{D}(I)$-module, then $\left(L_{i-1}/L_i\right)^{\geq r-1} \cdot \mathscr{D}(I)$ is Noetherian as a $\mathscr{D}(I)$-module. We will prove that E_i is a right Noetherian $\mathscr{D}(I)$-module.

We define a left action of S/p_iS on E_i by

$$\overline{f} \cdot \overline{\theta} = \overline{f\theta}$$

for $\overline{f} \in S/p_iS$ and $\overline{\theta} \in E_i$. This is well-defined, since

$$f\theta - g\theta' = \frac{(f-g)(\theta+\theta')}{2} + \frac{(f+g)(\theta-\theta')}{2} \in L_i$$

for $f, g \in S$ and $\theta, \theta' \in \bigoplus_{m\geq 0}\left(SP_i\delta_i^m + L_i^{(m)}\right)$ with $f - g \in p_iS$ and $\theta - \theta' \in L_i$. Thus E_i is a left S/p_iS-module. We may assume that $p_i = y - ax$ with $a \neq 0$. Then E_i is a K-vector space with a basis $\left\{\overline{y}^\alpha \cdot \overline{P_i\delta_i^m} \mid \alpha \in \mathbb{N}, m \geq 0\right\}$.

Define an exponent by

$$\exp(\overline{y}^\alpha \cdot \overline{P_i\delta_i^m}) := (\alpha + r - 1, m)$$

for an element of the basis above. We call $\overline{y}^\alpha \cdot \overline{P_i\delta_i^m}$ a monomial of E_i. Let $\overline{\theta_1}$ and $\overline{\theta_2}$ be two monomials of E_i with $\exp(\overline{\theta_1}) = (\alpha_1, m_1)$ and $\exp(\overline{\theta_2}) = (\alpha_2, m_2)$. We define a total order in the set of exponents of monomials by

$$\exp(\overline{\theta_1}) < \exp(\overline{\theta_2}),$$

if $m_1 < m_2$, or if $m_1 = m_2$ and $\alpha_1 < \alpha_2$. For $\overline{\theta} \in E_i$, write $\overline{\theta}$ as a linear combination of monomials. Then we define an exponent of $\overline{\theta}$ as the largest exponent of a monomial in $\overline{\theta}$ with a nonzero coefficient, and we denote it by $\exp(\overline{\theta})$. For a subset X of E_i, set

$$\operatorname{Exp}(X) := \left\{\exp(\overline{\theta}) \mid \overline{\theta} \in X\right\}.$$

Throughout the remaining of this section, we write $\theta \in E_i$ instead of $\overline{\theta}$ for simplicity.

Lemma 4.6. *Let M_1, M_2 be right $\mathscr{D}(I)$-submodules of E_i. If $M_1 \subseteq M_2$ and* $\mathrm{Exp}(M_1) = \mathrm{Exp}(M_2)$*, then*

$$M_1 = M_2.$$

Proof. Suppose that $M_1 \subsetneq M_2$. We can take an element $\theta \in M_2 \setminus M_1$ such that $\exp(\theta)$ is the smallest exponent in $M_2 \setminus M_1$.

Since $\exp(\theta) \in \mathrm{Exp}(M_2) = \mathrm{Exp}(M_1)$, there exists $\eta \in M_1$ such that $\exp(\eta) = \exp(\theta)$. Then

$$\exp(\theta - c\eta) < \exp(\theta)$$

for some $c \in K^\times$. We have $\theta - c\eta \in M_2 \setminus M_1$ since $\theta \notin M_1$. This is a contradiction to the minimality. $\square$

Lemma 4.7. *Let $M \neq 0$ be a right $\mathscr{D}(I)$-submodule of E_i. If $(k, m) \in \mathrm{Exp}(M)$, then*

$$\{(k + a, m), (k + b, m + m') \mid a \geq 0, b \geq r - 1, m' \geq 1\} \subset \mathrm{Exp}(M).$$

Proof. By the assumption, there exists $\theta \in M$ such that $\exp(\theta) = (k, m)$. Put $\alpha := k-r+1$, and write $\theta = y^\alpha P_i \delta_i^m + \theta'$ with $\exp(\theta') < \exp\left(y^\alpha P_i \delta_i^m\right)$. The multiplication $\theta \cdot y^a$ belongs to M, since $S \subseteq \mathscr{D}(I)$. Thus we see that $(k + a, m) \in \mathrm{Exp}(M)$ for all $a \geq 0$.

Fix $1 \leq j \neq i \leq r$, $b \geq r - 1$, and $m' \geq 1$. We can write

$$y^\alpha P_i \delta_i^m \cdot p_j^{b-r+1} P_i \delta_i^{m'} = y^\alpha (p_j^{b-r+1} P_i) P_i \delta_i^{m+m'} + \eta.$$

for some $\eta \in E_i$ with $\exp(\eta) < \exp\left(y^\alpha (p_j^{b-r+1} P_i) P_i \delta_i^{m+m'}\right)$. Since $p_j^{b-r+1} P_i \notin p_i S$, we see that $\exp(\theta \cdot p_j^{b-r+1} P_i \delta_i^{m'}) = (k+b, m+m')$. Therefore $(k+b, m+m') \in \mathrm{Exp}(M)$ since $\theta \cdot p_j^{b-r+1} P_i \delta_i^{m'} \in M$. $\square$

Now we induce the total degree (2.1) of $\mathscr{D}(S)$ to those of $\mathscr{D}(I)$ and E_i. Then E_i becomes a graded $\mathscr{D}(I)$-module by the total degree. For monomials of E_i, we denote the total degree by

$$\mathrm{totdeg}(y^\alpha) = \alpha,\ \mathrm{totdeg}(y^{\alpha'} \cdot P_i \delta_i^m) = \alpha' + r - 1 - m.$$

Let M be a right graded $\mathscr{D}(I)$-submodule of E_i. Set $X_j := \{\ell \mid (j, \ell) \in \mathrm{Exp}(M)\}$. From Lemma 4.7, there exists the smallest integer j with $\sharp X_j = \infty$. Put $s := s_M := \min\{j \mid \sharp X_j = \infty\}$, and set $M_s := \{\theta \in M \mid \exp(\theta) = (s, \ell) \text{ for some } \ell\}$. Then it is clear that $s \geq r - 1$.

Let $\theta_m \in M$ be a homogeneous operator satisfying $\exp(\theta_m) = (s, m)$ with $m \geq s$. Since $m - s + r - 1 > 0$, we can write

$$\theta_m = \sum_{\ell=0}^{s-r+1} a_\ell y^{s-r+1-\ell} P_i \delta_i^{m-\ell} \qquad (a_\ell \in K). \tag{4.5}$$

We may assume that $a_0 = 1$. Set $\Omega := \{1, \dots, i-1, i+1, \dots, r\}$. For $0 \le \ell \le s-r+1$, we write

$$\begin{aligned}
&\delta_i^{m-\ell} P_i \\
&= \sum_{\ell'=0}^{r-1} [m-\ell]_{\ell'} \left(\frac{1}{\ell'!(r-1-\ell')!} \sum_{\sigma \in S^\Omega} \delta(p_{\sigma(1)}) \cdots \delta(p_{\sigma(\ell')}) p_{\sigma(\ell'+1)} \cdots p_{\sigma(r)} \right) \delta_i^{m-\ell-\ell'} \\
&\equiv \sum_{\ell'=0}^{r-1} [m-\ell]_{\ell'} d_{\ell'} y^{r-1-\ell'} \delta_i^{m-\ell-\ell'} \qquad\qquad (\mathrm{mod}\ p_i D(S)),
\end{aligned}$$

for some $d_{\ell'} \in K$ by Lemma 2.4. It should be argued that $d_0 \neq 0$ and $d_{r-1} \neq 0$. We can write

$$\delta_i^{m-\ell} P_i = \sum_{\ell'=0}^{r-1} f_{\ell'} \delta_i^{m-\ell-\ell'}$$

in the Weyl algebra.

First we argue $d_0 \neq 0$. The polynomial coefficient f_0 of $\delta_i^{m-\ell}$ is the polynomial P_i, and $P_i \not\equiv 0 \pmod{p_i}$ since $p_1, \dots, p_r$ are coprime to one another. By the definition d_0, we have $P_i \equiv d_0 y^{r-1} \pmod{p_i}$. This implies $d_0 \neq 0$. Next we argue $d_{r-1} \neq 0$, The coefficient f_{r-1} of $\delta_i^{m-\ell-r+1}$ is equal to $\delta_i(p_1) \cdots \delta_i(\hat{p_i}) \cdots \delta_i(p_r)$. Since $\delta_i(p_j) = 0$ if and only if $i = j$, we have $\delta_i(p_1) \cdots \delta_i(\hat{p_i}) \cdots \delta_i(p_r) \neq 0$. This means $d_{r-1} \neq 0$.

Therefore we obtain $d_0 \neq 0$ and $d_{r-1} \neq 0$. Then

$$\begin{aligned}
\theta_m \cdot P_i \delta_i^{m'} &= \sum_{\ell=0}^{s-r+1} a_\ell y^{s-r+1-\ell} P_i (\delta_i^{m-\ell} P_i) \delta_i^{m'} \\
&= \sum_{\ell=0}^{s-r+1} \sum_{\ell'=0}^{r-1} a_\ell [m-\ell]_{\ell'} d_{\ell'} y^{k-\ell-\ell'} P_i \delta_i^{m-\ell-\ell'} \\
&= \sum_{t=0}^{s} c_t y^{s-t} P_i \delta_i^{m+m'-t}, \qquad (4.6)
\end{aligned}$$

where

$$c_t := \sum_{\substack{0 \le \ell \le s-r+1 \\ 0 \le \ell' \le r-1 \\ \ell+\ell'=t}} a_\ell [m-\ell]_{\ell'} d_{\ell'} \qquad (4.7)$$

for $0 \le t \le s$. We remark that c_t does not depend on m'. Put $m_0 := \max\{\ell \mid (s-1, \ell) \in \mathrm{Exp}(M)\} + s$.

Lemma 4.8. *For $1 \leq j \leq r$, there exist operators $\theta_{m_1}, \ldots, \theta_{m_j} \in M_s$ such that*

$$\operatorname{rank}\begin{pmatrix} c_0^{(1)} & \cdots & c_0^{(j)} \\ \vdots & & \vdots \\ c_{r-1}^{(1)} & \cdots & c_{r-1}^{(j)} \end{pmatrix} = j, \tag{4.8}$$

and $m_0 < m_1 < \cdots < m_j$, where $c_t^{(j)}$ for θ_{m_j} has been defined in (4.7).

Proof. We prove the assertion by induction. It is clear in the case $j = 1$.

Let $1 < j < r$. Assume that there exist $\theta_{m_1}, \ldots, \theta_{m_j} \in M_s$ $(m_1 < \cdots < m_j)$ satisfying the condition (4.8).

For $m > m_j$, put a vector

$$\boldsymbol{w} := \left(y^s P_i \delta_i^{m+m'}, y^{s-1} P_i \delta_i^{m+m'-1}, \ldots, P_i \delta_i^{m+m'-s}\right),$$

and put an $(s+1) \times (s-r+j+2)$ matrix

$$A := \begin{pmatrix} c_0^{(1)} & \cdots & c_0^{(j)} & d_0 & 0 & 0 & \cdots & 0 \\ c_1^{(1)} & \cdots & c_1^{(j)} & [m]_1 d_1 & d_0 & 0 & \cdots & 0 \\ c_2^{(1)} & \cdots & c_2^{(r-1)} & [m]_2 d_2 & [m-1]_1 d_1 & d_0 & \ddots & 0 \\ \vdots & & \vdots & \vdots & & \ddots & \ddots & \vdots \\ c_{r-2}^{(1)} & \cdots & c_{r-2}^{(j)} & [m]_{r-2} d_{r-2} & & & & 0 \\ c_{r-1}^{(1)} & \cdots & c_{r-1}^{(j)} & [m]_{r-1} d_{r-1} & [m-1]_{r-2} d_{r-2} & & & d_0 \\ \vdots & & \vdots & 0 & [m-1]_{r-1} d_{r-1} & & & \vdots \\ \vdots & & \vdots & 0 & 0 & \ddots & & \vdots \\ \vdots & & \vdots & \vdots & & \ddots & \ddots & \vdots \\ c_s^{(1)} & \cdots & c_s^{(j)} & 0 & \cdots & \cdots & 0 & [m-s+r-1]_{r-1} d_{r-1} \end{pmatrix}.$$

We consider m as a variable. By the induction hypothesis, there exists a nonzero j-minor of the matrix in (4.8). We denote by B the matrix of this j-minor. We take the lowest $s-r$ rows of A and j rows from the remaining $r-1$ rows of A so that we get the $(s-r+j+2)$-minor C whose matrix contains the matrix B. The coefficient of the leading term of C is the determinant of B. Thus C is not zero as a polynomial in variable m, and hence the solutions of $C = 0$ is finite. Because of this, the number of m with $\operatorname{rank}(A) < s-r+j+2$ is finite. Hence we can take a positive integer $m > m_j$ such that $\exp(\theta_m) \in M_s$, and $\operatorname{rank}(A) = s-r+j+2$.

We write $\theta_m = \sum_{\ell=0}^{s-r+1} a_\ell y^{s-r+1-\ell} P_i \delta_i^{m-\ell}$ in the same way as in (4.5). Put

$$\boldsymbol{v} := (\lambda_1, \ldots, \lambda_j, \lambda_{j+1} a_0, \ldots, \lambda_{j+1} a_{s-r+1}).$$

Then

$$\boldsymbol{w} A\, {}^t\boldsymbol{v} = \lambda_1 \theta_{m_1} \cdot P_i \delta_i^{m_1'} + \cdots + \lambda_j \theta_{m_j} \cdot P_i \delta_i^{m_j'} + \lambda_{j+1} \theta_m \cdot P_i \delta_i^{m'}$$

with $m_1 + m'_1 = \cdots = m_j + m'_j = m + m'$. If $\boldsymbol{w}A\,{}^t\boldsymbol{v} = 0$, then $\boldsymbol{v} = 0$ since $\operatorname{rank}(A) = s-r+j+2$. Therefore $\{\theta_{m_1} \cdot P_i\delta_i^{m'_1}, \dots, \theta_{m_j} \cdot P_i\delta_i^{m'_j}, \theta_m \cdot P_i\delta_i^{m'}\}$ is linearly independent over K.

Put $\theta_{m_{j+1}} := \theta_m$, and suppose that

$$\operatorname{rank}\begin{pmatrix} c_0^{(1)} & \cdots & c_0^{(j+1)} \\ \vdots & & \vdots \\ c_{r-1}^{(1)} & \cdots & c_{r-1}^{(j+1)} \end{pmatrix} < j+1.$$

Then there exists $(\lambda_1, \dots, \lambda_{j+1}) \in K^{j+1} \setminus \{\mathbf{0}\}$ such that

$$\lambda_1\,{}^t(c_0^{(1)}, \dots, c_{r-1}^{(1)}) + \cdots + \lambda_{j+1}\,{}^t(c_0^{(j+1)}, \dots, c_{r-1}^{(j+1)}) = \mathbf{0}. \tag{4.9}$$

Since $\{\theta_{m_1} \cdot P_i\delta_i^{m'_1}, \dots, \theta_{m_{j+1}} \cdot P_i\delta_i^{m'_{j+1}}\}$ is linearly independent, we have

$$\sum_{k=0}^{j+1} \lambda_k\theta_{m_k} \cdot P_i\delta_i^{m'_k} \neq 0. \tag{4.10}$$

Hence we can write $\exp\left(\sum_{k=0}^{j+1} \lambda_k\theta_{m_k} \cdot P_i\delta_i^{m'_k}\right) = (\alpha, \beta)$ for some $\alpha < s$ and $\beta > m_{j+1} + m'_{j+1} - s > m_0 - s = \max\{\ell \mid (s-1, \ell) \in \operatorname{Exp}(M)\}$ by (4.9) and (4.10). This is a contradiction. $\square$

Let M be a right graded $\mathscr{D}(I)$-submodule of E_i. For a nonnegative integer ℓ with $(k, \ell) \in \operatorname{Exp}(M)$ for some k, we define an integer t_ℓ by

$$t_\ell := \min\{k \mid (k, \ell) \in \operatorname{Exp}(M)\}.$$

By Lemma 4.8, there exist operators $\theta_{m_1}, \dots, \theta_{m_r} \in M_s$ satisfying the condition (4.8). We denote by N the right submodule of M generated by the operators $\theta_{m_1}, \dots, \theta_{m_r}$.

Lemma 4.9. *There exists a positive integer n_0 such that, for any $m \geq n_0$,*

$$(s, m) \in \operatorname{Exp}(N) \text{ and } t_m = s.$$

Proof. By Lemma 4.8, there exist $\theta_{m_1}, \dots, \theta_{m_r} \in M_s$ such that

$$\operatorname{rank}\begin{pmatrix} c_0^{(1)} & \cdots & c_0^{(r)} \\ \vdots & & \vdots \\ c_{r-1}^{(1)} & \cdots & c_{r-1}^{(r)} \end{pmatrix} = r, \quad \text{and} \quad \operatorname{rank}\begin{pmatrix} c_0^{(1)} & \cdots & c_0^{(r)} \\ \vdots & & \vdots \\ c_{r-2}^{(1)} & \cdots & c_{r-2}^{(r)} \end{pmatrix} < r.$$

Then there exists a nonzero vector $(\lambda_1, \dots, \lambda_r) \in K^r \setminus \{\mathbf{0}\}$ such that

$$\lambda_1\,{}^t(c_0^{(1)}, \dots, c_{r-2}^{(1)}) + \cdots + \lambda_r\,{}^t(c_0^{(r)}, \dots, c_{r-2}^{(r)}) = \mathbf{0},$$

and

$$\lambda_1 c_{r-1}^{(1)} + \cdots + \lambda_r c_{r-1}^{(r)} \neq 0.$$

Put $\theta := \sum_{k=0}^{r} \lambda_k \theta_{m_k} \cdot P_i \delta_i^{m'_k} \in N$ with $m_1 + m'_1 = \cdots = m_r + m'_r$. It follows that $\exp(\theta) = (s, m_r + m'_r - r + 1)$. Put $n_0 = m_r - r + 1$, and put $m'_r = m - n_0$ for any $m \geq n_0$. Thus

$$(s, m) = (s, m_r + m'_r - r + 1) = \exp(\theta) \in \operatorname{Exp}(N).$$

It remains to prove that $t_m = s$. We have $t_m \geq s$ since $m \geq n_0 \geq m_0 - s = \max\{\ell \mid (s-1, \ell) \in \operatorname{Exp}(M)\}$. Conversely we have $t_m \leq s$ since $(s, m) \in \operatorname{Exp}(M)$, as required. □

Let R be a graded ring. A right graded R-module M is said to be right gr-Noetherian, if M satisfies the ascending chain condition for graded submodules of M. It is straightforward to verify that M is right gr-Noetherian if and only if each graded submodule of M is finitely generated.

Proposition 4.10. *The right $\mathscr{D}(I)$-module E_i is right Noetherian.*

Proof. Recall that $\mathscr{D}(I)$ is a graded ring by the total degree, and that E_i is a graded $\mathscr{D}(I)$-module. By [7, Theorem II.3.5], it is enough to prove that E_i is right gr-Noetherian. Let M be a right graded $\mathscr{D}(I)$-submodule of E_i. We will prove that M is finitely generated.

Let n_0 be the integer satsfying Lemma 4.9. Set

$$G := \{(t_\ell, \ell) \mid \ell < n_0 \text{ and } (k, \ell) \in \operatorname{Exp}(M) \text{ for some } k\}.$$

Then G is a finite set. Fix an operator $\theta_{(t_\ell, \ell)} \in M$ for $(t_\ell, \ell) \in G$, and set

$$\overline{G} := \left\{\theta_{(t_\ell, \ell)} \in M \mid (t_\ell, \ell) \in G\right\}.$$

Then $\overline{G}$ is also a finite set. We denote by M' the right $\mathscr{D}(I)$-module generated by $\overline{G}$ and N. Then M' is finitely generated and $M' \subseteq M$.

Let $(k, m) \in \operatorname{Exp}(M)$, then $k \geq t_m$. If $m < n_0$, then $(t_m, m) \in G \subsetneq \operatorname{Exp}(M')$ by the definitions of t_m and G. We have $(k, m) = (t_m + k - t_m, m) \in \operatorname{Exp}(M')$ by Lemma 4.7.

If $m \geq n_0$, then $(s, m) \in \operatorname{Exp}(M')$ by Lemma 4.9. It follows from Lemma 4.7 that $(k, m) = (s + k - s, m) \in \operatorname{Exp}(M')$. Hence $\operatorname{Exp}(M') = \operatorname{Exp}(M)$. The assertion follows from Lemma 4.6. □

Corollary 4.11. *The right $\mathscr{D}(I)$-module*

$$(L_{i-1}/L_i)^{\geq r-1} \cdot \mathscr{D}(I) = \left(L_{i-1}^{\geq r-1} \cdot \mathscr{D}(I) + L_i/L_i\right)$$

is right Noetherian.

Next we study the S-module $\left(L_{i-1}/L_i\right)^{<r-1}$.

Lemma 4.12. *The K-vector space*

$$L_i^{<r-1} := \bigoplus_{m<r-1} L_i^{(m)}$$

is a right S-module.

Proof. Suppose that $0 \leq m < r-1$. Let $\theta \in L_i^{(m)} \subseteq \mathscr{D}(I)$. For $f \in S$,

$$\theta f(QS) = \theta(QfS) \subseteq I.$$

Thus $\theta f \in \mathscr{D}(I)$. It follows from Proposition 2.1 that $\theta f \in \bigoplus_{\ell=0}^{m} \mathscr{D}^{(\ell)}(I)$. The operator θf is divisible by the polynomial $p_1 \cdots p_i$ since $\theta \in p_1 \cdots p_i \mathscr{D}^{(m)}(S)$. Thus each homogeneous component of θf is divisible by $p_1 \cdots p_i$. It follows that

$$\theta f \in \bigoplus_{\ell=0}^{m} \left(\mathscr{D}^{(\ell)}(I) \cap (p_1 \cdots p_i)\mathscr{D}^{(\ell)}(S)\right) = \bigoplus_{\ell=0}^{m} L_i^{(\ell)}.$$

Hence $L_i^{<r-1} \cdot S \subseteq L_i^{<r-1}$. □

The following holds in general.

Proposition 4.13. *As a vector space,*

$$\bigoplus_{|\boldsymbol{\alpha}|<r-1} S\partial^{\boldsymbol{\alpha}} = \bigoplus_{|\boldsymbol{\alpha}|<r-1} \partial^{\boldsymbol{\alpha}} S.$$

Define a right S-module $\mathscr{D}(S)^{<r-1} := \bigoplus_{|\boldsymbol{\alpha}|<r-1} \partial^{\boldsymbol{\alpha}} S$. Then $\mathscr{D}(S)^{<r-1}$ is the module of differential operators of order less than $r-1$ by Proposition 4.13. By Lemma 4.12, we have the inclusion of right S-modules:

$$L_i^{<r-1} \subseteq \mathscr{D}(S)^{<r-1}.$$

Lemma 4.14. *The right S-module $\left(L_{i-1}/L_i\right)^{<r-1}$ is Noetherian.*

Proof. Since $\mathscr{D}(S)^{<r-1}$ is a finitely generated right S-module, $\mathscr{D}(S)^{<r-1}$ is Noetherian as a right S-module. Hence the subquotient $\left(L_{i-1}/L_i\right)^{<r-1} = \left(L_{i-1}^{<r-1} + L_i/L_i\right)$ of $\mathscr{D}(S)^{<r-1}$ is Noetherian as a right S-module. □

Lemma 4.15. *The right $\mathscr{D}(I)$-module L_{i-1}/L_i is Noetherian.*

Proof. By Corollary 4.11, $N := (L_{i-1}/L_i)^{\geq r-1} \cdot \mathscr{D}(I)$ is Noetherian as a right $\mathscr{D}(I)$-module. Consider the factor $N' := (L_{i-1}/L_i)/N$. It is clear that as a right S-module, N' is a factor of $(L_{i-1}/L_i)^{<r-1}$. Thus N' is Noetherian as a right S-module as so certainly as a right $\mathscr{D}(I)$-module. By [2, Proposition 1.2], L_{i-1}/L_i is Noetherian as a right $\mathscr{D}(I)$-module. □

Theorem 4.16. *The ring $\mathscr{D}(S/I) \simeq \mathscr{D}(I)/I\mathscr{D}(S)$ is Noetherian (i.e., $\mathscr{D}(S/I)$ is right Noetherian and left Noetherian).*

Proof. By Lemma 4.15 and by considering the sequence (4.2), we see that the ring $\mathscr{D}(I)/I\mathscr{D}(S)$ is right Noetherian. Therefore, by Corollary 3.3, we conclude that the ring $\mathscr{D}(S/I) \simeq \mathscr{D}(I)/I\mathscr{D}(S)$ is Noetherian. □

It is known that idealisers in the second Wely algebra may or may not be Noetherian ([9, Theorem 2]). However, the ideal I dose not satisfy the hypothesis of [9, Theorem 2]. The Noetherian property of the idealiser $\mathscr{D}(I)$ is still open.

In the rest of this section, we give an example of a family of Noetherian rings whose graded rings associated to the order filtration are not Noetherian.

By Proposition 2.1, we can decompose $\mathscr{D}(I)/I\mathscr{D}(S)$ into the direct sum

$$\mathscr{D}(I)/I\mathscr{D}(S) = \bigoplus_{m\geq 0} \left(\mathscr{D}^{(m)}(I)/I\mathscr{D}^{(m)}(S)\right)$$

as a left S-module. The order filtration of $\mathscr{D}(I)/I\mathscr{D}(S)$ is the filtration $\mathcal{F} = \{F_m\}_{m\geq 0}$ defined by

$$F_m = \bigoplus_{\ell\leq m} \left(\mathscr{D}^{(\ell)}(I)/I\mathscr{D}^{(\ell)}(S)\right).$$

We denote by S_j the K-vector subspace of S spanned by the monomials of degree j. An element $\theta = \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}}\partial^{\boldsymbol{\alpha}} \in \mathscr{D}(S)$ is of polynomial degree k, if k is the smallest integer such that $f_{\boldsymbol{\alpha}} \in \bigoplus_{j=0}^{k} S_j$ for all $\boldsymbol{\alpha}$ with nonzero $f_{\boldsymbol{\alpha}}$.

Example 4.17. *Let $S = k[x,y]$ be the polynomial ring, and let I be the ideal generated by the polynomial $Q = p_1\cdots p_r$ $(r\geq 2)$ defining a central arrangement.*

The graded ring $\operatorname{Gr}\mathscr{D}(S/I)$ associated to the order filtration is a commutative ring. Let $\overline{\theta}$ be the image of $\theta \in \mathscr{D}(S/I)$ in $\operatorname{Gr}\mathscr{D}(S/I)$. We consider the ideal $M := \langle \overline{P_1\delta_1^m} \mid m \geq 1\rangle$ of $\operatorname{Gr}\mathscr{D}(S/I)$.

Assume that M is finitely generated with generators $\eta_1,\dots,\eta_\ell$. Then there exists a positive integer m such that

$$M = \langle \eta_1,\dots,\eta_\ell\rangle \subseteq \langle \overline{P_1\delta_1},\dots,\overline{P_1\delta_1^{m-1}}\rangle.$$

Since $\overline{P_1\delta_1^m} \in M$, we can write

$$\overline{P_1\delta_1^m} = \overline{P_1\delta_1}\cdot\overline{\theta_1} + \cdots + \overline{P_1\delta_1^{m-1}}\cdot\overline{\theta_{m-1}} \tag{4.11}$$

for some $\theta_1,\dots,\theta_{m-1} \in \mathscr{D}(I)$.

If $\theta \in \mathscr{D}(I)$ with $\operatorname{ord}(\theta) \leq 1$, then the polynomial degree of θ is greater than or equal to 1 by Proposition 4.1. Since the order of the LHS of (4.11) equals m, there exists at least one θ_j such that the order of θ_j is greater than or equal to 1. Thus the polynomial degree of the RHS of (4.11) is greater than $r-1$. However, the polynomial degree of the LHS of (4.11) is exactly $r-1$. This is a contradiction.

Therefore M is not finitely generated, and thus we have proved that $\operatorname{Gr}\mathscr{D}(S/I)$ is not Noetherian.

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