Modulus of continuity and Martin boundary of a cylinder and a cone for p-harmonic functions

(p**-調和関数に関する連続率と筒領域，錐領域の Martin 境界)**

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List of included articles

This doctoral dissertation consists of an introductory part and the following papers:

[I1] T. Itoh, *Modulus of continuity of p-Dirichlet solutions in a metric measure space*, Ann. Acad. Sci. Fenn. Math. **37** (2012), no. 2, 339–355.

[I2] T. Itoh, *Logarithmic Hölder estimates of p-harmonic extension operators in a metric measure space*, Complex analysis and potential theory, CRM Proc. Lecture Notes, vol. 55, Amer. Math. Soc., Providence, RI, 2012, pp. 163–169.

[I3] T. Itoh, *Martin boundary for p-harmonic functions in a cylinder and a cone*, preprint.

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Contents

Chapter 1

Modulus of continuity of p-Dirichlet solutions in a metric measure space

In this chapter we introduce results in papers [I1] and [I2].

1.1 Potential theory in a metric measure space

Let $1 < p < \infty$ and let $X = (X, d, \mu)$ be a complete connected metric measure space endowed with a metric d and a positive complete Borel measure μ such that $0 < \mu(U) < \infty$ for all non-empty bounded open sets U. Let $B(x, r) = \{y \in X : d(x, y) < r\}$ denote the open ball centered at x with radius r. We assume that μ is doubling, i.e., there is a constant $C \geq 1$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for every $x \in X$ and $r > 0$. The integral mean of u over a measurable set E is denoted by

$$\frac{1}{\mu(E)} \int_E u d\mu = \fint_E u d\mu = u_E.$$

In a metric measure space, taking partial derivative is not possible. Therefore the concept of the an upper gradient was introduced in Heinonen-Koskela [12] as a substitute for the usual gradient. We say that a Borel function g on X is an upper gradient of a real-valued function u on X if

$$|u(x) - u(y)| \leq \int_\gamma g ds \tag{1.1.1}$$

for any $x, y \in X$ and all compact rectifiable curves γ joining x and y. If (1.1.1) fails only for a curve family with zero p-modulus (see [12, Definition 2.1]), then g is said to be a p-weak upper gradient of u. We say that g is a minimal p-weak upper gradient of u if $g \leq g'$ μ-almost everywhere for another p-weak upper gradients g' of u. We denote by g_u the minimal p-weak upper gradient of u. The concept of upper gradients gave rise to Newtonian space $N^{1,p}(X)$ which is one of several attempts to define Sobolev spaces on metric measure spaces and perhaps the most fruitful one. Whenever $u \in L^p(X)$, we define the seminorm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients g of u. The Newtonian space on X is the quotient space

$$N^{1,p}(X) = \{u \in L^p(X) : \|u\|_{N^{1,p}(X)} < \infty\}/ \sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$. It is known that every function $u \in N^{1,p}(X)$ has the minimal p-weak upper gradient g_u.

A p-harmonic function can be defined as the continuous minimizer of the variational integral

$$\int g_u^p d\mu.$$

The p-capacity of a subset $E \subset X$ is defined by

$$\mathrm{Cap}_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E. Let $\Omega \subset X$ be a bounded domain with $\mathrm{Cap}_p(X \setminus \Omega) > 0$. Kinnunen-Shanmugalingam [16] proved that p-harmonic functions in Ω satisfy the Harnack inequality and the maximum principle, and are locally Hölder continuous provided X satisfies a $(1, p)$-Poincaré inequality, i.e., there exist constants $\kappa \geq 1$ and $C \geq 1$ such that for all balls $B(x, r) \subset X$, all measurable functions u on X, and all p-weak upper gradients g of u we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq Cr\Big(\fint_{B(x,\kappa r)} g^p \, d\mu\Big)^{1/p}. \tag{1.1.2}$$

From now on, we assume that X admits a $(1, p)$-Poincaré inequality.

The Dirichlet problem for p-harmonic functions was studied by A. Björn, J. Björn and Shanmugalingam ([7], [8], [9] and [21]). In particular, Björn-Björn-Shanmugalingam [9] studied the Perron solution for p-harmonic functions. For a function f on $\partial\Omega$ we denote by $\mathcal{P}_\Omega f$ the Perron solution of f over Ω. A point $\xi \in \partial\Omega$ is said to be a p-regular point (with respect to the p-Dirichlet problem) if

$$\lim_{\Omega \ni x \to \xi} \mathcal{P}_\Omega f(x) = f(\xi)$$

for every $f \in C(\partial\Omega)$. If every boundary point is a p-regular point, then Ω is called p-regular. It is well known that if Ω is p-regular and $f \in C(\partial\Omega)$, then $\mathcal{P}_\Omega f$ is p-harmonic in Ω and continuous in $\overline{\Omega}$. It is natural to raise the following question:

Question 1.1.1. Does improved continuity of a boundary function f guarantee improved continuity of $\mathcal{P}_\Omega f$?

Aikawa-Shanmugalingam [4] studied this question in the context of Hölder continuity. Aikawa [2] investigated this question in the context of general modulus of continuity for the classical setting, i.e., for harmonic functions in a Euclidean domain. The purpose of this chapter is to study this question in the context of general modulus of continuity for p-harmonic functions in a metric measure space.

1.2 Modulus of continuity of p-Dirichlet solutions

Let $\mathcal{M}$ be the family of all positive nondecreasing concave functions ψ on $(0,\infty)$ with $\psi(0) = \lim_{t\to 0}\psi(t) = 0$. We say that f is ψ-Hölder continuous if $|f(x) - f(y)| \le C\psi(d(x,y))$. The modulus of continuity of a uniformly continuous function on any geodesic space is comparable to a certain concave function. See [10, Chapter 2 §6]. Therefore, we have only to check ψ-Hölder continuity for $\psi \in \mathcal{M}$ to study Question 1.1.1 in the context of modulus of continuity.

As a typical example of $\psi \in \mathcal{M}$, we consider $\psi_{\alpha\beta}$ defined by

$$\psi_{\alpha\beta}(t) = \begin{cases} t^{\alpha}(-\log t)^{-\beta} & \text{for } 0 < t < t_0, \\ t_0^{\alpha}(-\log t_0)^{-\beta} & \text{for } t \ge t_0. \end{cases}$$

where either $0 < \alpha < 1$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$; and t_0 is so small that $\psi_{\alpha\beta} \in \mathcal{M}$. In particular we write $\varphi_\alpha = \psi_{\alpha 0}$. If f is φ_α-continuous, then f is α-Hölder continuous in the classical sense.

Let $\psi \in \mathcal{M}$ and $E \subset X$. We consider the family $\Lambda_\psi(E)$ of all bounded continuous functions f on E with norm

$$\|f\|_{\psi,E} = \sup_{x\in E}|f(x)| + \sup_{\substack{x,y\in E\\ x\ne y}} \frac{|f(x)-f(y)|}{\psi(d(x,y))} < \infty.$$

We define the operator norm

$$\|\mathcal{P}_\Omega\|_\psi = \sup_{\substack{f\in\Lambda_\psi(\partial\Omega)\\ \|f\|_{\psi,\partial\Omega}\ne 0}} \frac{\|\mathcal{P}_\Omega f\|_{\psi,\Omega}}{\|f\|_{\psi,\partial\Omega}}.$$

Observe that ψ-Hölder continuity of a boundary function f ensures ψ-Hölder continuity of $\mathcal{P}_\Omega f$ if and only if $\|\mathcal{P}_\Omega\|_\psi < \infty$.

Aikawa [2] characterized the family of Euclidean domains Ω such that $\|\mathcal{P}_\Omega\|_\psi < \infty$ for $\psi \in \mathcal{M}$ in the context of harmonic functions. We consider the same problem in the context of p-harmonic functions in a metric measure space. It is known that there exists $\alpha_0 \in (0,1]$ such that every p-harmonic function in any domain Ω is locally α_0-Hölder continuous in Ω (see [16, Theorem 5.2]). Hence, $\|\mathcal{P}_\Omega\|_\psi < \infty$ can hold only for $\psi \in \mathcal{M}$, in some sense, bigger than the function $\varphi_{\alpha_0}(t) = t^{\alpha_0}$.

Let $\psi, \varphi \in \mathcal{M}$. We say that $\varphi \lesssim \psi$ if there are $r_0 > 0$ and $C > 0$ such that

$$\frac{\varphi(s)}{\varphi(r)} \le C\frac{\psi(s)}{\psi(r)} \quad \text{for } 0 < s < r < r_0.$$

Let $\mathcal{M}_0$ be the family of all $\psi \in \mathcal{M}$ with $t^{\alpha_0} \lesssim \psi(t)$. For example, if either $0 < \alpha < \alpha_0$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$, then $\psi_{\alpha\beta} \in \mathcal{M}_0$. But if $\alpha = \alpha_0$ and $\beta < 0$, then $\psi_{\alpha_0\beta} \notin \mathcal{M}_0$. Hence we see that $\mathcal{M}_0 \subsetneq \mathcal{M}$. Our results will be given for $\psi \in \mathcal{M}_0$.

Let U be an open set in X and let E be a Borel set in ∂U. A p-harmonic measure can be defined as the upper Perron solution of the characteristic function χ_E. We denote by $\omega_p(x, E, U)$ the p-harmonic measure evaluated at x of E in U. Note that the p-harmonic *measure* is not a measure, i.e., the p-harmonic measure is not additive. We define two decay properties for p-harmonic measures. We say that Ω enjoys the Local Harmonic Measure Decay property with ψ (abbreviated to the LHMD(ψ) property) if there are positive constants C_1 and r_0 depending only Ω and ψ such that

$$\omega_p(x, \Omega \cap \partial B(a,r), \Omega \cap B(a,r)) \le C_1 \frac{\psi(d(x,a))}{\psi(r)} \quad \text{for } x \in \Omega \cap B(a,r), \tag{1.2.1}$$

whenever a $\in \partial\Omega$ and $0 < r < r_0$. See Figure 1.1. We say that Ω enjoys the Global Harmonic Measure Decay property with ψ (abbreviated to the GHMD(ψ) property) if there are positive constants C_2 and r_0 depending only Ω and ψ such that

$$\omega_p(x, \partial\Omega \setminus B(a,r), \Omega) \le C_2 \frac{\psi(d(x,a))}{\psi(r)} \quad \text{for } x \in \Omega \cap B(a,r), \tag{1.2.2}$$

whenever a $\in \partial\Omega$ and $0 < r < r_0$. See Figure 1.2. By the comparison principle (see [15, Theorem 7.2]) it is easy to see that (1.2.1) implies (1.2.2).

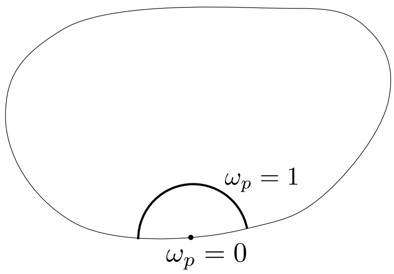

Figure 1.1: LHMD(ψ)

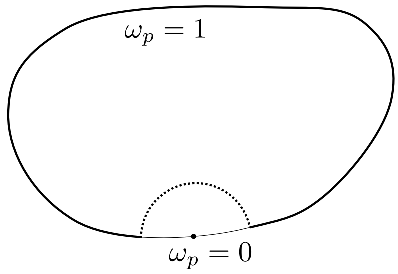

Figure 1.2: GHMD(ψ)

Without loss of generality, we may assume that Ω is a bounded p-regular domain (see [4, Proposition 2.1]). For $a \in \partial\Omega$ we define a test function $\tau_{a,\psi}$ on $\partial\Omega$ by

$$\tau_{a,\psi}(\xi) = \psi(d(a,\xi)) \quad \text{for } \xi \in \partial\Omega.$$

Then we have the following theorem.

Theorem 1.2.1. *Let $\psi \in \mathcal{M}_0$ and let Ω be a bounded p-regular domain. Consider the following conditions:*

(i) $\|\mathcal{P}_\Omega\|_\psi < \infty$.

(ii) *There is a constant C such that*

$$\mathcal{P}_\Omega \tau_{a,\psi} \le C\psi(d(x,a)) \quad \textit{for } x \in \Omega,$$

whenever $a \in \partial\Omega$.

(iii) *Ω satisfies the GHMD(ψ) property.*

(iv) *Ω satisfies the LHMD(ψ) property.*

Then we have

$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)} \impliedby \text{(iv)}.$$

The remaining implications in Theorem 1.2.1 are of interest. In [I1] we gave the equivalence (iii) $\Longleftrightarrow$ (iv) under additional assumptions on X and $\psi \in \mathcal{M}_0$. As was observed in [4, Remark 2.4], the implication (iv) $\Longrightarrow$ (i) does not necessarily hold in general. However, we prove that a condition slightly stronger than (iv) implies (i).

Theorem 1.2.2. *Let $\psi, \psi_1 \in \mathcal{M}_0$. Let $\psi_2 = \psi_1/\psi$. Suppose that $\lim_{r\to 0} \psi_2(r) = 0$ and there are constants $0 < C_3 < 1$ and $r_0 > 0$ such that ψ_2 is increasing on $(0, r_0)$ and*

$$\sup_{0<\rho<r\le r_0} \left\{ \frac{\psi(r)}{\psi(\rho)} : \frac{\psi_2(\rho)}{\psi_2(r)} = C_3 \right\} < \infty. \tag{1.2.3}$$

If Ω satisfies the LHMD(ψ_1) property, then $\|\mathcal{P}_\Omega\|_\psi < \infty$.

Condition (1.2.3) looks rather complicated. We have a simple condition.

Corollary 1.2.3. *Let $\psi, \psi_1 \in \mathcal{M}_0$. Let $\psi_2 = \psi_1/\psi$. Suppose that there are constants $0 < C_4 < 1$ and $r_0 > 0$ such that ψ_2 is increasing on $(0, r_0)$ and*

$$\inf_{0<r\le r_0} \frac{\psi_2(r)}{\psi_2(C_4 r)} > 1. \tag{1.2.4}$$

If Ω satisfies the LHMD(ψ_1) property, then $\|\mathcal{P}_\Omega\|_\psi < \infty$.

Theorem 1.2.2 and Corollary 1.2.3 are main results of this chapter. They give several corollaries for $\psi_{\alpha\beta}$.

Corollary 1.2.4. *Let Ω be a bounded p-regular domain. Consider the following conditions:*

(i) $0 < \alpha < \alpha' < \alpha_0$ *and* $\beta, \beta' \in \mathbb{R}$.

(ii) $0 = \alpha < \alpha' < \alpha_0$ *and* $\beta > 0, \beta' \in \mathbb{R}$.

(iii) $\alpha = \alpha' = 0$ *and* $0 < \beta < \beta'$.

Assume that either (i), (ii), or (iii) holds. If Ω satisfies the LHMD($\psi_{\alpha'\beta'}$) property, then $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$.

Let $E \subset U \subset X$. We define the *relative p-capacity* of E in U by

$$\mathrm{Cap}_p(E, U) = \inf_u \int_U g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on E and $\mathrm{Cap}_p(\{x \in X \setminus U : u(x) \ne 0\}) = 0$. We say that $E \subset X$ is uniformly p-fat (or satisfies the p-capacity density condition) if there are constants $C > 0$ and $r_0 > 0$ such that

$$\frac{\mathrm{Cap}_p(E \cap B(a, r), B(a, 2r))}{\mathrm{Cap}_p(B(a, r), B(a, 2r))} \ge C, \tag{1.2.5}$$

whenever $a \in E$ and $0 < r < r_0$. The uniform p-fatness of the complement of a domain Ω is closely related to the condition $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$. For $\alpha > 0$ we obtain the following corollary.

Corollary 1.2.5. *Let Ω be a bounded p-regular domain. If $X \setminus \Omega$ is uniformly p-fat, then there is a constant $0 < \alpha_1 \leq \alpha_0$ such that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for $0 < \alpha < \alpha_1$ and $\beta \in \mathbb{R}$. Conversely, if $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for some $0 < \alpha < \alpha_0$ and $\beta \in \mathbb{R}$, then $X \setminus \Omega$ is uniformly p-fat, provided that there is a constant $Q \geq p$ such that X is Ahlfors Q-regular, i.e.,*

$$C^{-1} r^Q \leq \mu(B(x, r)) \leq C r^Q$$

for every $x \in X$ and $r > 0$.

Remark 1.2.6. Aikawa and Shanmugalingam [4] showed the case $\beta = 0$ of Corollary 1.2.5.

For $\alpha = 0$ we obtain the following corollary.

Corollary 1.2.7. *If $X \setminus \Omega$ is uniformly p-fat, then $\|\mathcal{P}_\Omega\|_{\psi_{0\beta}} < \infty$ for every $\beta > 0$.*

Chapter 2

Martin boundary for p-harmonic functions in a cylinder and a cone in $\mathbb{R}^n$

In this chapter we introduce results in [I3]. We restrict ourselves to $\mathbb{R}^n$ to have delicate arguments for p-harmonic functions.

2.1 Martin boundary theory for harmonic functions

Let us recall the classical Martin boundary theory for harmonic functions. Let D be an arbitrary domain with Green function $G(x, y)$. Martin [20] introduced the Martin boundary Δ as the smallest ideal boundary for which $G(x, y)/G(x_0, y)$ has a continuous extension $K(x, y)$. An ideal boundary point y is called minimal if $K(\cdot, y)$ is a minimal harmonic function; that is, every harmonic functions h in D with $0 \le h \le K(\cdot, y)$ is a constant multiple of $K(\cdot, y)$. The set of all minimal Martin boundary points is called the minimal Martin boundary Δ_1. Martin proved that if u is a positive harmonic function in D, then there exists a measure μ_u on Δ, uniquely determined by u, such that $\mu_u(\Delta \setminus \Delta_1) = 0$ and

$$u(x) = \int_\Delta K(x, y) d\mu_u(y).$$

The identification of the (minimal) Martin boundary for specific domains is of great interest. There are a number of works on this topic. Hunt-Wheeden [13] gave the first cornerstone. They showed that the Martin boundary of a Lipschitz domain D is homeomorphic to the Euclidean boundary ∂D and every boundary point is minimal. They said that a positive harmonic function u on D is a *kernel function* in D at a boundary point $w \in \partial D$ if u has continuous boundary values 0 on $\partial D \setminus \{w\}$ and $u(x_0) = 1$ ([13, p.507]). They proved that every boundary point has a unique kernel function. This is crucial for the identification of the Martin boundary.

In the linear case Kemper [14] proved that the uniqueness of kernel functions follows from the scale invariant boundary Harnack principle (see also [1]). For the reader's convenience we give a short proof below. By $\mathcal{H}(w)$ we denote the family of all kernel functions at w with reference point x_0. Then the scale invariant boundary Harnack principle implies that there exists a constant $C \ge 1$ such that

$$C^{-1}u(x) \le v(x) \le Cu(x) \quad \text{for all } u, v \in \mathcal{H}(w) \text{ and } x \in D. \tag{2.1.1}$$

Theorem A. If (2.1.1) holds, then $\mathcal{H}(w)$ is a singleton.

Proof. Let

$$C_0 = \sup_{u,v \in \mathcal{H}(w),\ x \in D} \frac{u(x)}{v(x)}.$$

Then $1 \le C_0 < \infty$ by (2.1.1). It is sufficient to show $C_0 = 1$. Suppose $C_0 > 1$. Take $u, v \in \mathcal{H}(w)$. By the linearity of harmonicity $v_1 = (C_0 v - u)/(C_0 - 1)$ is a positive harmonic function with the same boundary values as u and v such that $v_1(x_0) = (C_0 v(x_0) - u(x_0))/(C_0 - 1) = 1$. Hence $v_1 \in \mathcal{H}(w)$, and so $u \le C_0 v_1 = C_0(C_0 v - u)/(C_0 - 1)$, which implies

$$\frac{u}{v} \le \frac{C_0^2}{2C_0 - 1} < C_0 \quad \text{on } D.$$

This contradicts the definition of C_0. □

It is natural to extend the notion of kernel functions to p-harmonic functions. We study p-harmonic kernel functions in a cylinder and a cone.

2.2 p-harmonic kernel functions in a cylinder and a cone

A point $x \in \mathbb{R}^n$ is denoted by (x', x_n) with $x' = (x_1, \dots, x_{n-1})$. We denote a point $x \in \mathbb{R}^n \setminus \{0\}$ by (r, σ) with $r = |x|$ and $\sigma = x/|x|$. We let ∂E be the boundary of a set E in $\mathbb{R}^n$. Let $B(x, r)$ be the open ball with center x and radius r.

Let $1 < p < \infty$. Let $D \subset \mathbb{R}^n$ be a domain. We say that u is a p-harmonic function in D if $u \in W^{1,p}_{loc}(D)$ is continuous and satisfies the p-Laplace equation $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in D in the weak sense; that is, whenever D' is a relatively compact subdomain of D and $\varphi \in W^{1,p}_0(D')$, we have

$$\int_{D'} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = 0.$$

If $p = 2$, then the p-Laplace equation reduces to the classical linear equation $\triangle u = 0$. The set of all positive p-harmonic functions in D is denoted by $\mathcal{H}_+(D)$.

Let D be an arbitrary domain with compactification D^*. We write $\partial^* D$ for the ideal boundary $D^* \setminus D$. We say that $u \in \mathcal{H}_+(D)$ is a p-harmonic kernel function in D at $w \in \partial^* D$ with reference point x_0 if u has continuous boundary values 0 on $\partial^* D \setminus \{w\}$ and $u(x_0) = 1$. If each $w \in \partial^* D$ corresponds to a unique p-harmonic kernel function, we say that the p-Martin boundary of D is homeomorphic to $\partial^* D$.

Let Ω be a $C^{2,\alpha}$-domain in $\mathbb{R}^{n-1}$. The domain $\Omega \times \mathbb{R} = \{(x', x_n) : x' \in \Omega, x_n \in \mathbb{R}\}$ is said to be a cylinder generated by Ω. We compactify $\Omega \times \mathbb{R}$ by adding the topological boundary and the ideal boundary $\{+\infty, -\infty\}$, where $\pm\infty$ corresponds to the limit as $x_n \to \pm\infty$, respectively.

We investigate p-harmonic kernel functions in $\Omega \times \mathbb{R}$ at $\pm\infty$ with the aid of a translation operator similar to the stretching operator used by Tolksdorf [22] in his study on p-harmonic functions in a cone. We let

$$\mathcal{H}_+^{+\infty}(\Omega \times \mathbb{R}) = \{u \in \mathcal{H}_+(\Omega \times \mathbb{R}) : u = 0 \text{ on } \partial(\Omega \times \mathbb{R}) \cup \{-\infty\}\},$$

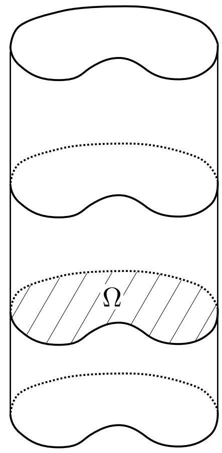

Figure 2.1: The cylinder $\Omega \times \mathbb{R}$

where $u = 0$ at $-\infty$ means $\lim_{x_n \to -\infty} u(x) = 0$. Similarly, we let

$$\mathcal{H}_+^{-\infty}(\Omega \times \mathbb{R}) = \{u \in \mathcal{H}_+(\Omega \times \mathbb{R}) : u = 0 \text{ on } \partial(\Omega \times \mathbb{R}) \cup \{\infty\}\}.$$

By definition $u \in \mathcal{H}_+(\Omega \times \mathbb{R})$ is a p-harmonic kernel function at $+\infty$ (resp. $-\infty$) if and only if $u(x_0) = 1$ and $u \in \mathcal{H}_+^{+\infty}(\Omega \times \mathbb{R})$ (resp. $u \in \mathcal{H}_+^{-\infty}(\Omega \times \mathbb{R})$). The following theorem shows that $+\infty$ and $-\infty$ have a unique p-harmonic kernel function.

Theorem 2.2.1. *There exist a positive constant λ and a function $f(x')$ of $x' \in \Omega$, depending only on p, n and Ω, such that*

$$\mathcal{H}_+^{+\infty}(\Omega \times \mathbb{R}) = \{C \exp(\lambda x_n) f(x') : C > 0\}, \tag{2.2.1}$$

$$\mathcal{H}_+^{-\infty}(\Omega \times \mathbb{R}) = \{C \exp(-\lambda x_n) f(x') : C > 0\}. \tag{2.2.2}$$

Since $\Omega \times \mathbb{R}$ is locally a $C^{2,\alpha}$-domain in $\mathbb{R}^n$, every boundary point in $\partial(\Omega \times \mathbb{R})$ has a unique p-harmonic kernel function, in view of Lewis-Nyström [18]. So, we have the following corollary.

Corollary 2.2.2. *The p-Martin boundary of $\Omega \times \mathbb{R}$ is homeomorphic to $\partial(\Omega \times \mathbb{R}) \cup \{-\infty, +\infty\}$.*

Remark 2.2.3. Lewis-Nyström obtained the uniqueness of p-harmonic kernel functions by using their scale invariant Harnack principle for Lipschitz domains and starlike Lipschitz ring domains ([17]) and a very delicate argument. If $p \neq 2$, then the proof of Theorem A fails, as $v_1 = (C_0 v - u)/(C_0 - 1)$ need not be p-harmonic even if u and v are p-harmonic. Unlike the linear case, the scale invariant boundary Harnack principle is not enough to deduce the uniqueness of p-harmonic kernel functions. This is the reason why the domains in [18] are restricted to C^1 or convex. To avoid such difficulties, we restrict ourselves to $C^{2,\alpha}$-domains in this chapter. In this case the scale invariant boundary Harnack principle can be proved rather easily. See [3, Theorem 1.2].

In case $n = 2$, we can explicitly calculate λ and f.

Theorem 2.2.4. *Let $n = 2$ and $\Omega = (0, L)$ with $0 < L < \infty$. Then*

$$\lambda = \frac{p\pi}{2(p-1)L},$$

and $f(x_1)$ has a parametric representation given by

$$\begin{cases} f(s) = \exp\Big(\dfrac{-(p-2)\sin^2 s}{p-1}\Big)\sin s, \\ x_1(s) = \dfrac{1}{\lambda}\Big(\dfrac{p}{2(p-1)}s + \dfrac{p-2}{4(p-1)}\sin 2s\Big). \end{cases} \tag{2.2.3}$$

Next we consider p-harmonic kernel functions in a cone. Let Σ be a $C^{2,\alpha}$-domain in the unit sphere. The domain $\Gamma = \{(r, \sigma) : 0 < r < \infty, \sigma \in \Sigma\}$ is said to be a cone generated by Σ. We compactify Γ by adding the topological boundary and the ideal boundary $\{\infty\}$, where ∞ is the point at infinity.

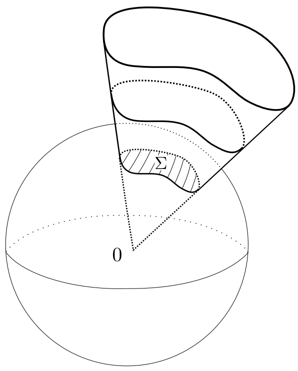

Figure 2.2: The cone Γ

We study p-harmonic kernel functions in Γ at ∞ and 0 with the aid of the stretching operator used by Tolksdorf [22]. We let

$$\mathcal{H}_+^\infty(\Gamma) = \{u \in \mathcal{H}_+(\Gamma) : u = 0 \text{ on } \partial\Gamma\},$$
$$\mathcal{H}_+^0(\Gamma) = \{u \in \mathcal{H}_+(\Gamma) : u = 0 \text{ on } (\partial\Gamma \cup \{\infty\}) \setminus \{0\}\},$$

where $u = 0$ on ∞ means $\lim_{|x|\to\infty} u(x) = 0$. By definition $u \in \mathcal{H}_+(\Gamma)$ is a p-harmonic kernel function at ∞ (resp. 0) if and only if $u(x_0) = 1$ and $u \in \mathcal{H}_+^\infty(\Gamma)$ (resp. $u \in \mathcal{H}_+^0(\Gamma)$). The following theorems show that ∞ and 0 have a unique p-harmonic kernel function.

Theorem 2.2.5. *There exist a positive constant μ and a function $g(\sigma)$ of $\sigma \in \Sigma$, depending only on p, n and Σ, such that*

$$\mathcal{H}_+^\infty(\Gamma) = \{Cr^\mu g(\sigma) : C > 0\}. \tag{2.2.4}$$

Theorem 2.2.6. *There exist a positive constant ν and a function $h(\sigma)$ of $\sigma \in \Sigma$, depending only on p, n and Σ, such that*

$$\mathcal{H}_+^0(\Gamma) = \{Cr^{-\nu} h(\sigma) : C > 0\}.$$

Corollary 2.2.7. *The p-Martin boundary of Γ is homeomorphic to $\partial\Gamma \cup \{\infty\}$.*

Remark 2.2.8. Tolksdorf [22] studied functions $u \in \mathcal{H}_+^\infty(\Gamma)$ satisfying the doubling condition:

$$\sup_{\Gamma\cap B(0,2R)} u \le C \sup_{\Gamma\cap B(0,R)} u \quad \text{for } R \ge 1, \tag{2.2.5}$$

with a constant $C \ge 1$ depending only on u. The set of all $u \in \mathcal{H}_+^\infty(\Gamma)$ satisfying (2.2.5) is denoted by $\widetilde{\mathcal{H}_+^\infty}(\Gamma)$. By applying the stretching operator, he gave a characterization of $\widetilde{\mathcal{H}_+^\infty}(\Gamma)$ similar to (2.2.4). Theorem 2.2.5 implies that the doubling condition (2.2.5) is superfluous, that is, $\widetilde{\mathcal{H}_+^\infty}(\Gamma) = \mathcal{H}_+^\infty(\Gamma)$.

In case $n = 2$, we can explicitly calculate μ, ν, g and h. Our method goes back to Aronsson [5], who studied p-harmonic functions in the whole plane $\mathbb{R}^2 \setminus \{0\}$ of the form $u(r,\sigma) = r^k F(\sigma)$ and gave a representation of F depending on k. Although he assumed $2 < p < \infty$, his technique is appliable for $1 < p \le 2$.

We introduce the spherical coordinates (r,θ) in $\mathbb{R}^2$ which are related to the coordinates $(x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$ by

$$x_1 = r\sin\theta, \quad x_2 = r\cos\theta,$$

where $0 < r < \infty$, $-\pi \le \theta < \pi$. For $0 < \phi < \pi$, we let

$$\Gamma_\phi = \{(r,\theta) : |\theta| < \phi\}.$$

For simplicity, we let

$$\kappa = \frac{p-2}{p-1}.$$

Proposition 2.2.9. *Let $n = 2$ and $\Gamma = \Gamma_\phi$. Then*

$$\mu = \frac{2\pi^2 - \kappa(\pi-2\phi)^2 + (\pi-2\phi)\sqrt{4\pi^2(1-\kappa) + \kappa^2(\pi-2\phi)^2}}{8(\pi-\phi)\phi},$$

and $g(\theta)$ has a parametric representation given by

$$\begin{cases} g(s) = \left(1 - \dfrac{\kappa}{\mu}\cos^2 s\right)^{(\mu-1)/2} \cos s, \\ \theta(s) = s + \dfrac{1-\mu}{\sqrt{\mu(\mu-\kappa)}} \arctan\left(\sqrt{\dfrac{\mu}{\mu-\kappa}} \tan s\right). \end{cases}$$

Proposition 2.2.10. *Let $n = 2$ and $\Gamma = \Gamma_\phi$. Then*

$$\nu = \frac{2\pi^2 - \kappa(\pi+2\phi)^2 + (\pi+2\phi)\sqrt{4\pi^2(1-\kappa) + \kappa^2(\pi+2\phi)^2}}{8(\pi+\phi)\phi},$$

and $h(\theta)$ has a parametric representation given by

$$\begin{cases} h(t) = \left(1 + \dfrac{\kappa}{\nu}\cos^2 t\right)^{(-\nu-1)/2} \cos t, \\ \theta(t) = t - \dfrac{1+\nu}{\sqrt{\nu(\nu+\kappa)}} \arctan\left(\sqrt{\dfrac{\nu}{\nu+\kappa}} \tan t\right). \end{cases}$$

Remark 2.2.11. Dobrowolski [11] gave μ but not g. Lundström-Vasilis [19] calculated ν and h for case $p > 2$ in the same way as in the proof of Proposition 2.2.10. On the other hand, for case $1 < p < 2$, they considered $p/(p-1)$-harmonic stream functions and so they obtained the explicit representation of ν and h. See [6] for details of stream functions.

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Chapter 3

Appendix

[I1] T. Itoh, *Modulus of continuity of p-Dirichlet solutions in a metric measure space*, Ann. Acad. Sci. Fenn. Math. **37** (2012), no. 2, 339–355.

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[I1]

Modulus of continuity of p-Dirichlet solutions in a metric measure space

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MODULUS OF CONTINUITY OF p-DIRICHLET SOLUTIONS IN A METRIC MEASURE SPACE

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Abstract. Let $1 < p < \infty$ and let X be a metric measure space with a doubling measure and a $(1,p)$-Poincaré inequality. Let Ω be a bounded domain in X. For a function f on $\partial\Omega$ we denote by $\mathcal{P}_\Omega f$ the p-Dirichlet solution of f over Ω. It is well known that if Ω is p-regular and $f \in C(\partial\Omega)$, then $\mathcal{P}_\Omega f$ is p-harmonic in Ω and continuous in $\overline{\Omega}$. We characterize the family of domains Ω such that improved continuity of boundary functions f ensures improved continuity of $\mathcal{P}_\Omega f$. We specify such improved continuity if X is Ahlfors regular and $X \setminus \Omega$ is uniformly p-fat.

1. Introduction

Let $X = (X, d, \mu)$ be a complete connected metric measure space endowed with a metric d and a positive complete Borel measure μ such that $0 < \mu(U) < \infty$ for all non-empty bounded open sets U.

By the symbol C we denote an absolute positive constant whose value is unimportant and may change from line to line. Let $B(x,r) = \{y \in X : d(x,y) < r\}$ denote the open ball centered at x with radius r. We assume that μ is doubling, i.e., there is a constant $C \geq 1$ such that $\mu(B(x,2r)) \leq C\mu(B(x,r))$ for every $x \in X$ and $r > 0$. Let $1 < p < \infty$. We assume that X admits a $(1,p)$-Poincaré inequality.

We denote by Cap_p the p-capacity defined on X (Definition 2.5). Let $\Omega \subset X$ be a bounded domain with $\mathrm{Cap}_p(X \setminus \Omega) > 0$. For a function f on $\partial\Omega$ we donate by $\mathcal{P}_\Omega f$ the p-Perron solution of f over Ω. A point $\xi \in \partial\Omega$ is said to be a *p-regular point* (with respect to the p-Dirichlet problem) if

$$\lim_{\Omega \ni x \to \xi} \mathcal{P}_\Omega f(x) = f(\xi)$$

for every $f \in C(\partial\Omega)$. If every boundary point is a p-regular point, then Ω is called *p-regular*. It is well known that if Ω is p-regular and $f \in C(\partial\Omega)$, then $\mathcal{P}_\Omega f$ is p-harmonic in Ω and continuous in $\overline{\Omega}$. It is natural to raise the following question:

Question 1.1. Does improved continuity of a boundary function f guarantee improved continuity of $\mathcal{P}_\Omega f$?

Aikawa and Shanmugalingam [3] studied this question in the context of Hölder continuity. Aikawa [2] investigated this question in the context of general modulus of continuity for the classical setting, i.e., for harmonic functions in a Euclidean domain.

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The purpose of this paper is to study this question in the context of general modulus of continuity in a metric measure space.

Let $\mathcal{M}$ be the family of all positive nondecreasing concave functions ψ on $(0,\infty)$ with $\psi(0)=\lim_{t\to 0}\psi(t)=0$. We say that f is *ψ-Hölder continuous* if $|f(x)-f(y)|\le C\psi(d(x,y))$. The modulus of continuity of a uniformly continuous function on any geodesic space is comparable to a certain concave function. See [5, Chapter 2 §6] and Propositions 2.13 and 2.14. The author would like to thank Kuroda for drawing his attention to [5]. Therefore, we have only to check ψ-Hölder continuity for $\psi\in\mathcal{M}$ to study Question 1.1 in the context of modulus of continuity.

As a typical example of $\psi\in\mathcal{M}$ we consider $\psi_{\alpha\beta}$ defined by

$$\psi_{\alpha\beta}(t)=\begin{cases} t^{\alpha}(-\log t)^{-\beta} & \text{for } 0<t<t_0,\\ t_0^{\alpha}(-\log t_0)^{-\beta} & \text{for } t\ge t_0.\end{cases}$$

where either $0<\alpha<1$ and $\beta\in\mathbf{R}$ or $\alpha=0$ and $\beta>0$; and t_0 is so small that $\psi_{\alpha\beta}\in\mathcal{M}$. In particular, we write $\varphi_\alpha=\psi_{\alpha 0}$, and we say that f is α-Hölder continuous if f is φ_α-continuous.

Let $\psi\in\mathcal{M}$ and $E\subset X$. We consider the family $\Lambda_\psi(E)$ of all bounded continuous functions f on E with norm

$$\|f\|_{\psi,E}=\sup_{x\in E}|f(x)|+\sup_{\substack{x,y\in E\\ x\neq y}}\frac{|f(x)-f(y)|}{\psi(d(x,y))}<\infty.$$

We define the operator norm

$$\|\mathcal{P}_\Omega\|_\psi=\sup_{\substack{f\in\Lambda_\psi(\partial\Omega)\\ \|f\|_{\psi,\partial\Omega}\neq 0}}\frac{\|\mathcal{P}_\Omega f\|_{\psi,\Omega}}{\|f\|_{\psi,\partial\Omega}}.$$

Observe that ψ-Hölder continuity of a boundary function f ensures ψ-Hölder continuity of $\mathcal{P}_\Omega f$ if and only if $\|\mathcal{P}_\Omega\|_\psi<\infty$.

Aikawa [2] characterized the family of Euclidean domains Ω such that $\|\mathcal{P}_\Omega\|_\psi<\infty$ for $\psi\in\mathcal{M}$ in context of harmonic functions. We consider the same problem in the context of p-harmonic functions in a metric measure space. It is known that there exists $\alpha_0\in(0,1]$ such that every p-harmonic function in any domain Ω is locally α_0-Hölder continuous in Ω (see [10]). Hence, $\|\mathcal{P}_\Omega\|_\psi<\infty$ can hold only for $\psi\in\mathcal{M}$, in some sense, bigger than the function $\varphi_{\alpha_0}(t)=t^{\alpha_0}$.

Let ψ, $\varphi\in\mathcal{M}$. We say that $\varphi\precsim\psi$ if there are $r_0>0$ and $C>0$ such that

$$\frac{\varphi(s)}{\varphi(r)}\le C\frac{\psi(s)}{\psi(r)}\quad\text{for } 0<s<r<r_0.$$

Let $\mathcal{M}_0$ be the family of all $\psi\in\mathcal{M}$ with $t^{\alpha_0}\precsim\psi(t)$. For example, if either $0<\alpha<\alpha_0$ and $\beta\in\mathbf{R}$ or $\alpha=0$ and $\beta>0$, then $\psi_{\alpha\beta}\in\mathcal{M}_0$. But if $\alpha=\alpha_0$ and $\beta<0$, then $\psi_{\alpha_0\beta}\notin\mathcal{M}_0$. Hence we see that $\mathcal{M}_0\subsetneq\mathcal{M}$. Our results will be given for $\psi\in\mathcal{M}_0$.

Let U be an open set in X and let E be a Borel set in ∂U. We denote by $\omega_p(x,E,U)$ the p-harmonic measure evaluated at x of E in U. Note that the p-harmonic *measure* is not a measure, i.e., the p-harmonic measure is not additive. We define two decay properties for p-harmonic measures. We say that Ω enjoys the *Local Harmonic Measure Decay property* with ψ (abbreviated to the LHMD(ψ) property) if there are positive constants C_1 and r_0 depending only Ω and ψ such that

$$\omega_p(x, \Omega \cap \partial B(a,r), \Omega \cap B(a,r)) \le C_1 \frac{\psi(d(x,a))}{\psi(r)} \quad \text{for } x \in \Omega \cap B(a,r), \tag{1.1}$$

whenever a $\in \partial\Omega$ and $0 < r < r_0$. We say that Ω enjoys the *Global Harmonic Measure Decay property* with ψ (abbreviated to the GHMD(ψ) property) if there are positive constants C_2 and r_0 depending only Ω and ψ such that

$$\omega_p(x, \partial\Omega \setminus B(a,r), \Omega) \le C_2 \frac{\psi(d(x,a))}{\psi(r)} \quad \text{for } x \in \Omega \cap B(a,r), \tag{1.2}$$

whenever a $\in \partial\Omega$ and $0 < r < r_0$. By the comparison principle (see [9, Theorem 7.2]) it is easy to see that (1.1) implies (1.2).

Without loss of generality, we may assume that Ω is a bounded p-regular domain (see [3, Proposition 2.1]). For $a \in \partial\Omega$ we define a test function $\tau_{a,\psi}$ on $\partial\Omega$ by

$$\tau_{a,\psi}(\xi) = \psi(d(a,\xi)) \quad \text{for } \xi \in \partial\Omega.$$

Then we have the following theorem.

Theorem 1.2. *Let $\psi \in \mathcal{M}_0$ and let Ω be a bounded p-regular domain. Consider the following conditions:*

(i) $\|\mathcal{P}_\Omega\|_\psi < \infty$.
(ii) *There is a constant C such that*

$$\mathcal{P}_\Omega \tau_{a,\psi}(x) \le C\psi(d(x,a)) \quad \textit{for } x \in \Omega,$$

whenever $a \in \partial\Omega$.
(iii) *Ω satisfies the GHMD(ψ) property.*
(iv) *Ω satisfies the LHMD(ψ) property.*

Then we have

$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)} \impliedby \text{(iv)}.$$

The remaining implications in Theorem 1.2 are of interest. Theorem 4.1 in Section 4 will give the equivalence (iii) $\iff$ (iv) under additional assumptions on X and $\psi \in \mathcal{M}_0$. As was observed in [3, Remark 2.4], the implication (iv) $\implies$ (i) does not hold. However, we prove that a condition slightly stronger than (iv) implies (i).

Theorem 1.3. *Let $\psi, \psi_1 \in \mathcal{M}_0$. Let $\psi_2 = \psi_1/\psi$. Suppose that $\lim_{r\to 0} \psi_2(r) = 0$ and there are constants $0 < C_3 < 1$ and $r_0 > 0$ such that ψ_2 is increasing on $(0, r_0)$ and*

$$\sup_{0<\rho<r\le r_0} \left\{ \frac{\psi(r)}{\psi(\rho)} : \frac{\psi_2(\rho)}{\psi_2(r)} = C_3 \right\} < \infty. \tag{1.3}$$

If Ω satisfies the LHMD(ψ_1) property, then $\|\mathcal{P}_\Omega\|_\psi < \infty$.

Condition (1.3) looks rather complicated. We have a simple condition.

Corollary 1.4. *Let $\psi, \psi_1 \in \mathcal{M}_0$. Let $\psi_2 = \psi_1/\psi$. Suppose that there are constants $0 < C_4 < 1$ and $r_0 > 0$ such that ψ is increasing on $(0, r_0)$ and*

$$\inf_{0<r\le r_0} \frac{\psi_2(r)}{\psi_2(C_4 r)} > 1. \tag{1.4}$$

If Ω satisfies the LHMD(ψ_1) property, then $\|\mathcal{P}_\Omega\|_\psi < \infty$.

Theorem 1.3 and Corollary 1.4 are main results of this paper. They give several corollaries for $\psi_{\alpha\beta}$.

Corollary 1.5. *Let Ω be a bounded p-regular domain. Consider the following conditions:*

(i) $0 < \alpha < \alpha' < \alpha_0$ *and* $\beta, \beta' \in \mathbf{R}$.
(ii) $0 = \alpha < \alpha' < \alpha_0$ *and* $\beta > 0, \beta' \in \mathbf{R}$.
(iii) $\alpha = \alpha' = 0$ *and* $0 < \beta < \beta'$.

Assume that either (i), (ii), *or* (iii) *holds. If Ω satisfies the LHMD($\psi_{\alpha'\beta'}$) property, then $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$.*

We say that $E \subset X$ is *uniformly p-fat* or satisfies the *p-capacity density condition* if there are constants $C > 0$ and $r_0 > 0$ such that

$$\frac{\operatorname{Cap}_p(E \cap B(a,r), B(a,2r))}{\operatorname{Cap}_p(B(a,r), B(a,2r))} \geq C, \tag{1.5}$$

whenever $a \in E$ and $0 < r < r_0$. The uniform p-fatness of the complement of a domain Ω is closely related to the condition $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$. For $\alpha > 0$ we obtain the following corollary.

Corollary 1.6. *Let Ω be a bounded p-regular domain. If $X \setminus \Omega$ is uniformly p-fat, then there is a constant $0 < \alpha_1 \leq \alpha_0$ such that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for $0 < \alpha < \alpha_1$ and $\beta \in \mathbf{R}$. Conversely, if $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for some $0 < \alpha < \alpha_0$ and $\beta \in \mathbf{R}$, then $X \setminus \Omega$ is uniformly p-fat, provided that there is a constant $Q \geq p$ such that X is Ahlfors Q-regular, i.e.,*

$$C^{-1} r^Q \leq \mu(B(x,r)) \leq C r^Q$$

for every $x \in X$ and $r > 0$.

Aikawa and Shanmugalingam [3] showed the case $\beta = 0$ of Corollary 1.6. For $\alpha = 0$ we obtain the following corollary.

Corollary 1.7. *If $X \setminus \Omega$ is uniformly p-fat, then $\|\mathcal{P}_\Omega\|_{\psi_{0\beta}} < \infty$ for every $\beta > 0$.*

The plan of this paper is as follows. In the next section we shall define notions of p-harmonicity, p-Dirichlet problem, p-capacity, and p-harmonic measure, and we shall observe some properties for $\mathcal{M}$. In Section 3 we shall show Theorem 1.2. In Section 4 we shall prove that Ω satisfies the LHMD(ψ) property if and only if Ω satisfies the GHMD(ψ) property under certain additional assumptions. The proof of Theorem 1.3 and Corollary 1.4 will be given in Section 5. Finally, we shall give the proof of Corollaries 1.5, 1.6, and 1.7.

2. Preliminaries

In this section we introduce notions of p-harmonicity, p-Dirichlet problem, p-capacity, and p-harmonic measure; for details we refer to [3], and we observe some properties for $\mathcal{M}$.

The integral mean of u over a measurable set E is denoted by

$$\frac{1}{\mu(E)} \int_E u \, d\mu = \fint_E u \, d\mu = u_E.$$

Definition 2.1. We say that a Borel function g on X is an *upper gradient* of a real-valued function u on X if

$$|u(x) - u(y)| \le \int_\gamma g\, ds \tag{2.1}$$

for any $x, y \in X$ and all compact rectifiable curves γ joining x and y. If (2.1) fails only for a curve family with zero p-modulus (see [7, Definition 2.1]), then g is said to be a *p-weak upper gradient* of u. We say that g is a *minimal p-weak upper gradient* of u if $g \le g'$ μ-almost everywhere for another p-weak upper gradients g' of u. We denote by g_u a minimal p-weak upper gradient of u.

Definition 2.2. Let $u \in L^p(X)$. We define the seminorm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all p-weak upper gradients g of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u \in L^p(X) \colon \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

Remark 2.3. The Newtonian space $N^{1,p}(X)$ with the norm $\|\cdot\|_{N^{1,p}(X)}$ is a Banach space. Every function $u \in N^{1,p}(X)$ has the minimal p-weak upper gradient g_u.

Definition 2.4. We say that X admits a $(1,p)$*-Poincaré inequality* if there are constants $\kappa \ge 1$ and $C \ge 1$ such that for all balls $B(x,r) \subset X$, all measurable functions u on X, and all p-weak upper gradients g of u we have

$$⨍_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le Cr\biggl(⨍_{B(x,\kappa r)} g^p\, d\mu\biggr)^{1/p}. \tag{2.2}$$

A consequence of the $(1,p)$-Poincaré inequality is the following p-Sobolev inequality (see [10, Lemma 2.1]): if $0 < \gamma < 1$ and $\mu(\{z \in B(x,R)\colon |u(z)| > 0\}) \le \gamma\mu(B(x,R))$, then there exists a positive constant C_γ depending only on γ such that

$$\biggl(⨍_{B(x,R)} |u|^p\, d\mu\biggr)^{1/p} \le C_\gamma R\biggl(⨍_{B(x,\kappa R)} g_u^p\, d\mu\biggr)^{1/p}. \tag{2.3}$$

If X admits a $(1,p)$-Poincaré inequality, then X admits a $(1,q)$-Poincaré inequality for every $q \ge p$ by Hölder's inequality. Keith and Zhong [8] showed that if X is proper (that is, closed and bounded subsets of X are compact) and X admits a $(1,p)$-Poincaré inequality, then there exists $q < p$ such that X admits a $(1,q)$-Poincaré inequality. Because X is a complete metric space equipped with a doubling measure, X is proper. Therefore we can use their result.

Definition 2.5. The *p-capacity* of a subset $E \subset X$ is defined by

$$\mathrm{Cap}_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on E.

We say that a property holds p-quasieverywhere (p-q.e.) if the set of points for which the property fails to hold has p-capacity zero. We let

$$N^{1,p}_0(\Omega) = \{u \in N^{1,p}(X) \colon u = 0 \ p\text{-q.e. on } X \setminus \Omega\}.$$

We say that $u \in N^{1,p}_{loc}(\Omega)$ if for every $x \in \Omega$ there is r_x such that $f|_{B(x,r_x)} \in N^{1,p}(B(x,r_x))$. This is clearly equivalent to saying that $f \in N^{1,p}(V)$ for every relatively compact subset V of Ω. We now introduce the notion of p-harmonicity.

Definition 2.6. We call a function $u \in N^{1,p}_{loc}(\Omega)$ a *p-harmonic function* in Ω if u is continuous and

$$\int_U g_u^p \, d\mu \leq \int_U g_{u+\varphi}^p \, d\mu. \tag{2.4}$$

for all relatively compact subsets U of Ω and all functions $\varphi \in N^{1,p}_0(U)$. A function $u \in N^{1,p}_{loc}(\Omega)$ is said to be a *p-superminimizer* in Ω if (2.4) holds for all relatively compact subsets U of Ω and all nonnegative functions $\varphi \in N^{1,p}_0(U)$. We call a function $u \in N^{1,p}_{loc}(\Omega)$ a *p-subminimizer* in Ω if (2.4) holds for all relatively compact subsets U of Ω and all nonpositive functions $\varphi \in N^{1,p}_0(U)$.

Let u and v be p-harmonic functions and let $\alpha, \beta \in \mathbf{R}$. Then $\alpha u+\beta$ is p-harmonic. But in general $u+v$ is not p-harmonic. Kinnunen and Shanmugalingam [10, Theorem 5.2] showed the following local Hölder continuity of p-harmonic functions. Here, we denote by $\operatorname{osc}_E u$ the oscillation $\sup_E u - \inf_E u$.

Theorem 2.7. *Suppose a function u is p-harmonic on $B(x, 2\kappa R)$. Then there are constants $0 < \alpha_0 \leq 1$ and $C \geq 1$ such that*

$$\operatorname*{osc}_{B(x,\kappa r)} u \leq C\left(\frac{r}{R}\right)^{\alpha_0} \operatorname*{osc}_{B(x,\kappa R)} u \quad \text{for } 0 < r \leq R.$$

The constants α_0 and C are independent of u, x, and R.

Next we define p-Dirichlet solutions over Ω. For a function $f \in N^{1,p}(\Omega)$ we denote by $\mathcal{H}_\Omega f$ the Dirichlet solution of f over Ω, i.e., $\mathcal{H}_\Omega f$ is a function on $\overline{\Omega}$ that is p-harmonic in Ω with $f - \mathcal{H}_\Omega f \in N^{1,p}_0(\Omega)$. For $E \subset X$ we denote by $\operatorname{Lip}(E)$ the family of all Lipschitz continuous functions on E. For every $f \in \operatorname{Lip}(\partial\Omega)$ there is a function $Ef \in \operatorname{Lip}(\overline{\Omega})$ such that $f = Ef$ on $\partial\Omega$. Therefore we can define $\mathcal{H}_\Omega f$ by the function $\mathcal{H}_\Omega Ef$; this is independent of the extension Ef. We say that a lower semicontinuous function u on Ω is a *p-superharmonic function* in Ω if

(i) $-\infty < u \leq \infty$;
(ii) u is not identically ∞ in Ω;
(iii) $\mathcal{H}_{\Omega'} v \leq u$ in Ω' for every relatively compact subset Ω' of Ω and all functions $v \in \operatorname{Lip}(\partial\Omega')$ such that $v \leq u$ on $\partial\Omega'$.

If $-u$ is p-superharmonic, then we say that u is *p-subharmonic*.

The following comparison principle is very useful in nonlinear potential theory (see [9, Theorem 7.2]).

Theorem 2.8. *Let u be a p-superharmonic function on Ω and let v be a p-subharmonic function on Ω. If*

$$\limsup_{\Omega \ni x \to \xi} v(x) \leq \liminf_{\Omega \ni x \to \xi} u(x) \tag{2.5}$$

for every $\xi \in \partial\Omega$, and if both sides of (2.5) *are not simultaneously ∞ or $-\infty$, then $v \leq u$ in Ω.*

Definition 2.9. Let f be a function on $\partial\Omega$. Let $\mathcal{U}_f$ be the set of all p-superharmonic functions u on Ω bounded below such that $\liminf_{\Omega \ni x \to \xi} u(x) \geq f(\xi)$ for each $\xi \in \partial\Omega$.

The *upper Perron solution* of f is defined by

$$\overline{\mathcal{P}}_\Omega f(x) = \inf_{u\in\mathcal{U}_f} u(x) \quad \text{for } x \in \Omega.$$

Similarly, we define the *lower Perron solution* of f by

$$\underline{\mathcal{P}}_\Omega f(x) = \sup_{s\in\mathcal{L}_f} s(x) \quad \text{for } x \in \Omega,$$

where $\mathcal{L}_f = -\mathcal{U}_{-f}$ is the set of all p-subharmonic functions s on Ω bounded above such that $\limsup_{\Omega\ni x\to\xi} s(x) \le f(\xi)$ for each $\xi \in \partial\Omega$. If $\overline{\mathcal{P}}_\Omega f = \underline{\mathcal{P}}_\Omega f$, then we write $\mathcal{P}_\Omega f = \overline{\mathcal{P}}_\Omega f$, and we say that f is *resolutive*. We call $\mathcal{P}_\Omega f$ the *Perron solution* of f.

A. Björn, J. Björn and Shanmugalingam [4, Theorem 6.1] showed that if $f \in C(\partial\Omega)$, then f is resolutive. Moreover, if $f \in N^{1,p}(X)$, then f is resolutive and $\mathcal{P}_\Omega f = \mathcal{H}_\Omega f$, by [4, Theorem 5.1]. We define the p-harmonic measure as follows.

Definition 2.10. Let U be an open subset of X and let E be a Borel set in ∂U. The *p-harmonic measure* evaluated at x of E in U is defined by

$$\omega_p(x, E, U) = \overline{\mathcal{P}}_U \chi_E(x) \quad \text{for } x \in U.$$

The p-harmonic measure is not additive because of the non-linear nature of p-harmonic functions. Therefore the p-harmonic measure is not a measure.

Definition 2.11. Let $E \subset U \subset X$. We define the *relative p-capacity* of E in U by

$$\mathrm{Cap}_p(E, U) = \inf_u \int_U g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(U)$ such that $u \ge 1$ on E.

Finally, we observe some properties for $\mathcal{M}$. The following proposition shows an elementary property for $\mathcal{M}$ (see [2, Lemma 2.2]).

Proposition 2.12. *Let $\psi \in \mathcal{M}$. If $c > 1$ and $0 < s \le t \le cs$, then $\psi(s) \le \psi(t) \le c\psi(s)$.*

In Section 1 we have assumed that $\psi \in \mathcal{M}$ is concave. The relevance of concavity of $\psi \in \mathcal{M}$ follows from the following propositions.

Proposition 2.13. *Let φ be a nondecreasing subadditive function on $(0,\infty)$, i.e., if $t_1, t_2 > 0$, then $\varphi(t_1 + t_2) \le \varphi(t_1) + \varphi(t_2)$. Suppose that $\lim_{t\to 0} \varphi(t) = \varphi(0) = 0$. Then there is a function $\psi \in \mathcal{M}$ satisfying*

$$\frac{1}{2}\psi(t) \le \varphi(t) \le \psi(t) \quad \textit{for } t \ge 0.$$

Proposition 2.14. *Let (A, d_A) be a geodesic space and let f be a uniformly continuous function on A. Then*

$$\varphi(t) = \varphi(f, t) = \sup_{\substack{d_A(x,y)\le t \\ x,y\in A}} |f(x) - f(y)| \quad \textit{for } t \ge 0.$$

is a subadditive function on $(0,\infty)$.

See [2, Section 5], [5, Chapter 2 §6], and [11, Section 3] for these accounts.

3. Proof of Theorem 1.2

To prove Theorem 1.2 we recall the following geometric property (see [6, Proposition 4.4]).

Lemma 3.1. *The space X is quasiconvex, i.e., there exists a constant $C_5 \geq 1$ such that every pair of points x, $y \in X$ can be joined by a curve of length at most $C_5 d(x,y)$. Hence if $x \in E \subsetneq X$, then*

$$\operatorname{dist}(x, X \setminus E) \leq \operatorname{dist}(x, \partial E) \leq C_5 \operatorname{dist}(x, X \setminus E).$$

Proof of Theorem 1.2. Since the LHMD(ψ) property implies the GHMD(ψ) property, it is sufficient to show that Condition (ii) implies Condition (iii) and that Condition (i) is equivalent to Condition (ii).

(ii) $\Longrightarrow$ (iii). Suppose (ii) holds. Let $a \in \partial\Omega$ and $r > 0$. Then

$$\psi(r)\chi_{\partial\Omega \setminus B(a,r)}(\xi) \leq \tau_{a,\psi}(\xi) \quad \text{for } \xi \in \partial\Omega.$$

The comparison principle yields

$$\psi(r)\omega_p(x, \partial\Omega \setminus B(a,r), \Omega) \leq \mathcal{P}_\Omega \tau_{a,\psi}(x) \quad \text{for } x \in \Omega.$$

Hence, (ii) implies that

$$\psi(r)\omega_p(x, \partial\Omega \setminus B(a,r), \Omega) \leq C\psi(d(x,a)) \quad \text{for } x \in \Omega.$$

Thus (iii) follows.

(i) $\Longrightarrow$ (ii). Suppose $\|\mathcal{P}_\Omega\|_\psi < \infty$. Since $\tau_{a,\psi} \in \Lambda_\psi(\partial\Omega)$, we have

$$\|\mathcal{P}_\Omega \tau_{a,\psi}\|_{\psi,\Omega} \leq \|\mathcal{P}_\Omega\|_\psi \|\tau_{a,\psi}\|_{\psi,\partial\Omega} < \infty.$$

By definition

$$|\mathcal{P}_\Omega \tau_{a,\psi}(x) - \mathcal{P}_\Omega \tau_{a,\psi}(y)| \leq \|\mathcal{P}_\Omega \tau_{a,\psi}\|_{\psi,\Omega} \psi(d(x,y)) \quad \text{for } x, y \in \Omega.$$

Letting $y \to a$, we see that $\mathcal{P}_\Omega \tau_{a,\psi}(x) \leq \|\mathcal{P}_\Omega \tau_{a,\psi}\|_{\psi,\Omega} \psi(d(x,a))$. Thus (ii) follows with $C = \|\mathcal{P}_\Omega \tau_{a,\psi}\|_{\psi,\Omega}$.

(ii) $\Longrightarrow$ (i). Suppose (ii) holds. Let $f \in \Lambda_\psi(\partial\Omega)$. Since $|\mathcal{P}_\Omega f|$ is bounded by the supremum of $|f|$ over $\partial\Omega$, it is sufficient to show that

$$|\mathcal{P}_\Omega f(x) - \mathcal{P}_\Omega f(y)| \leq C\|f\|_{\psi,\partial\Omega} \psi(d(x,y)) \quad \text{for } x, y \in \Omega. \tag{3.1}$$

Let $x, y \in \Omega$. Without loss of generality, we may assume that $\operatorname{dist}(x, X \setminus \Omega) \geq \operatorname{dist}(y, X \setminus \Omega)$. Let $R = \operatorname{dist}(x, X \setminus \Omega)/2\kappa$. Since $\partial\Omega$ is compact, we can take $x^* \in \partial\Omega$ such that $d(x, x^*) = \operatorname{dist}(x, \partial\Omega)$. Then Lemma 3.1 gives

$$2\kappa R \leq d(x, x^*) \leq 2\kappa C_5 R. \tag{3.2}$$

Let $f_0(\xi) = f(\xi) - f(x^*)$. By definition

$$|f_0(\xi)| \leq \|f\|_{\psi,\partial\Omega} \tau_{x^*,\psi}(\xi) \quad \text{for } \xi \in \partial\Omega.$$

Hence, by the comparison principle and (ii), we obtain

$$|\mathcal{P}_\Omega f_0(z)| \leq C\|f\|_{\psi,\partial\Omega} \psi(d(z, x^*)) \quad \text{for } z \in \Omega. \tag{3.3}$$

Let us consider two cases.

Case 1: $d(x,y) \leq d(x,x^*)/(2\kappa C_5)$. Let $r = d(x,y)$. Then $r \leq R$. Since $\mathcal{P}_\Omega f_0$ is p-harmonic, Theorem 2.7 gives

$$\operatorname*{osc}_{B(x,\kappa r)} \mathcal{P}_\Omega f_0 \leq C \left(\frac{r}{R} \right)^{\alpha_0} \operatorname*{osc}_{B(x,\kappa R)} \mathcal{P}_\Omega f_0.$$

We obtain from (3.2) that

$$d(z,x^*) \le d(x,z) + d(x,x^*) \le (1+2C_5)\kappa R \quad \text{for } z \in B(x,\kappa R).$$

By Proposition 2.12 we have

$$\psi(d(z,x^*)) \le \psi((1+2C_5)\kappa R) \le (1+2C_5)\kappa\psi(R).$$

Thus by (3.3) we obtain

$$\operatorname*{osc}_{B(x,\kappa R)} \mathcal{P}_\Omega f_0 \le 2 \sup_{B(x,\kappa R)} |\mathcal{P}_\Omega f_0| \le C\|f\|_{\psi,\partial\Omega}\psi(R).$$

Hence

$$|\mathcal{P}_\Omega f(x) - \mathcal{P}_\Omega f(y)| = |\mathcal{P}_\Omega f_0(x) - \mathcal{P}_\Omega f_0(y)| \le C\left(\frac{r}{R}\right)^{\alpha_0} \|f\|_{\psi,\partial\Omega}\psi(R). \tag{3.4}$$

Since $\psi \in \mathcal{M}_0$, there is a constant $C > 0$ such that

$$\left(\frac{s}{r}\right)^{\alpha_0} \le C\frac{\psi(s)}{\psi(r)} \quad \text{for } 0 < s < r < 2\kappa \operatorname{diam}(\Omega).$$

Hence by (3.4), we have

$$|\mathcal{P}_\Omega f(x) - \mathcal{P}_\Omega f(y)| \le C\|f\|_{\psi,\partial\Omega}\psi(d(x,y)).$$

Case 2: $d(x,y) \ge d(x,x^*)/(2\kappa C_5)$. We have

$$d(y,x^*) \le d(x,y) + d(x,x^*) \le (1+2\kappa C_5)d(x,y).$$

It follows from Proposition 2.12 and (3.3) that

$$\begin{aligned} |\mathcal{P}_\Omega f(x) - \mathcal{P}_\Omega f(y)| &= |\mathcal{P}_\Omega f_0(x) - \mathcal{P}_\Omega f_0(y)| \le |\mathcal{P}_\Omega f_0(x)| + |\mathcal{P}_\Omega f_0(y)| \\ &\le C\|f\|_{\psi,\partial\Omega}(\psi(d(x,x^*)) + \psi(d(y,x^*))) \\ &\le C\|f\|_{\psi,\partial\Omega}\psi(d(x,y)). \end{aligned}$$

Combining the above two cases, we obtain (3.1). Thus (i) follows. □

4. Equivalence between GHMD(ψ) and LHMD(ψ)

If $\psi = \varphi_\alpha$, then the GHMD(ψ) property and the LHMD(ψ) property are equivalent for Euclidean domains (see [1]) and for a metric measure space (see [3]). If $\psi \ne \varphi_\alpha$, it is not known whether this equivalence holds or not. In this section we show that the equivalence holds under certain additional assumptions.

Let $S(x,r) = \{y \in X : d(x,y) = r\}$ be the sphere with center at x and radius r and let $A(x,r,R)$ be the annulus $B(x,R) \setminus B(x,r)$ with center at x and radii r and R. We say that X is *linearly locally connected* (abbreviated to LLC) if there are constants $C_6 > 1$ and $r_0 > 0$ such that for every $a \in X$ and $0 < r < r_0$ each pair of points x, $y \in S(a,r)$ can be connected by a curve lying in $A(a, r/C_6, C_6 r)$.

Theorem 4.1. *Let Ω be a bounded regular domain. Assume that X is LLC and there is a constant $C > 0$ such that*

$$\frac{\mu(B(a,r))}{\mu(B(a,R))} \le C\left(\frac{r}{R}\right)^p \tag{4.1}$$

whenever $a \in \partial\Omega$ *and* $0 < r \le R < \operatorname{diam}(X)$. *Let* $\psi \in \mathcal{M}_0$. *Suppose that there exist constants* $0 < C < 1$ *and* $r_0 > 0$ *such that*

$$\inf_{0<r<r_0} \frac{\psi(r)}{\psi(Cr)} > 1. \tag{4.2}$$

Then Ω *satisfies the LHMD(*ψ*) property if and only if* Ω *satisfies the GHMD(*ψ*) property.*

Theorem 4.1 is new, even for the classical setting, i.e., for harmonic functions in Euclidean domains.

The proof is decomposed mainly into two steps. First, we show that the GHMD(ψ) property implies that the uniform perfectness of the boundary (Lemma 4.3). Second, with the aid of the uniform perfectness and a chain property, we will complete proof of Theorem 4.1. See [3, Lemmas 5.1 and 5.2] for Hölder continuity.

Definition 4.2. Let E be a subset of X. We say that E is *uniformly perfect* if there are constants $0 < C_7 < 1$ and $r_0 > 0$ such that $A(x, C_7 r, r) \cap E \neq \emptyset$ for every $x \in E$ and all $0 < r < r_0$.

Lemma 4.3. *Let* Ω *be a bounded regular domain. Assume that* X *is LLC and* μ *satisfies* (4.1). *Let* $\psi \in \mathcal{M}_0$. *Suppose that* ψ *satisfies* (4.2). *If* Ω *satisfies the GHMD(*ψ*) property, then* $\partial\Omega$ *is uniformly perfect.*

For the proof we state the following lemma, which is proved in the same way as [3, Lemma 5.3].

Lemma 4.4. *Assume that* μ *satisfies* (4.1). *If* $0 < 2r \le R < \operatorname{diam}(\Omega)/2$, *then*

$$\frac{\operatorname{Cap}_p(\overline{B(a,r)}, B(a,R))}{\mu(B(a,R))} \le C\left(\log\frac{R}{r}\right)^{1-p} R^{-p}.$$

Proof of Lemma 4.3. Let $a \in \partial\Omega$ and $0 < \rho_1 < \rho_2 < \operatorname{diam}(\Omega)/2$. Suppose $A(a, \rho_1, \rho_2)$ does not intersect $\partial\Omega$. Then it is sufficient to show that the ratio ρ_1/ρ_2 is bounded below by a positive constant C depending only on Ω and ψ.

Without loss of generality, we may assume that $\rho_1 \le \rho_2/(2C_6^2)$. By the LLC property we see that $A(a, C_6\rho_1, \rho_2/C_6) \subset \Omega$. For simplicity, we let $r = C_6\rho_1$ and $R = \rho_2/C_6$. Then

$$A(a, r, R) \subset \Omega. \tag{4.3}$$

Letting ρ_2 be larger if necessary, we may assume that $S(a, C_6 R)$ has a point $b \in \partial\Omega$. Let $K = \overline{B(a,r)} \setminus \Omega$. Observe from (4.3) that $K = B(a,R) \setminus \Omega$. By Lemma 4.4,

$$\frac{\operatorname{Cap}_p(K, \Omega \cup K)}{\mu(B(a,R))} \le \frac{\operatorname{Cap}_p(\overline{B(a,r)}, B(a,R))}{\mu(B(a,R))} \le C\left(\log\frac{R}{r}\right)^{1-p} R^{-p}. \tag{4.4}$$

Let u_K be the p-capacitary potential for the condenser $(K, \Omega \cup K)$, i.e., u_K is p-harmonic on Ω, $u_K = 1$ p-q.e. on K, $u_K = 0$ p-q.e. on $X \setminus (\Omega \cup K)$ and

$$\operatorname{Cap}_p(K, \Omega \cup K) = \int_X g_{u_K}^p \, d\mu.$$

We prove that $u_K \le 1/3$ p-q.e. on $B(b, \beta R)$ for some $0 < \beta < 1$. Since $r \le R/2$ and $A(a, r, R) \cap \partial\Omega = \emptyset$, it follows from the comparison principle and the GHMD(ψ)

property that

$$u_K(x) \le C_2 \frac{\psi(d(x,b))}{\psi(R/2)} \quad \text{for } x \in \Omega \cap B(b, R/2). \tag{4.5}$$

Since ψ satisfies (4.2), there is a constant $0 < C_8 < 1$ such that

$$S = \inf_{0<r<\operatorname{diam}(\Omega)/2} \frac{\psi(r)}{\psi(C_8 r)} > 1.$$

Therefore, we have

$$\frac{\psi(C_8^{j-1}R/2)}{\psi(C_8^j R/2)} \ge S$$

for every positive integer j. Now multiplying the above inequalities over $j = 1, 2, \ldots, N$, we get

$$\frac{\psi(R/2)}{\psi(C_8^N R/2)} \ge S^N.$$

We can find a positive integer N such that

$$\frac{C_2}{S^N} \le \frac{1}{3}.$$

Let $\beta = C_8^N/2$. By the monotonicity of ψ, if $x \in B(b, \beta R)$, then

$$\psi(d(x,b)) \le \psi(\beta R) \le \psi(R/2)/(3C_2).$$

Hence, by (4.5) we obtain

$$u_K(x) \le \frac{1}{3} \quad \text{for } x \in \Omega \cap B(b, \beta R)$$

Since $u_K = 0$ p-q.e. on $B(b, R/2) \setminus \Omega$, we have $u_K \le 1/3$ p-q.e. on $B(b, \beta R)$.

Next we prove that $u_K \ge 2/3$ p-q.e. on $B(a, \beta R)$. It follows from (4.3) and the comparison principle that

$$u_K(x) = 1 - \omega_p(x, \partial\Omega \setminus B(a, R), \Omega) \quad \text{for } x \in \Omega.$$

By the GHMD(ψ), we have

$$\omega_p(x, \partial\Omega \setminus B(a, R), \Omega) \le C_2 \frac{\psi(d(x,a))}{\psi(R)} \quad \text{for } x \in \Omega \cap B(a, R)$$

Hence (4.2) implies

$$u_K(x) \ge \frac{2}{3} \quad \text{for } x \in \Omega \cap B(a, \beta R)$$

Since $u_K = 1$ p-q.e. on $B(a, \beta R) \setminus \Omega \subset B(a, R) \setminus \Omega$, we obtain $u_K \ge 2/3$ p-q.e. on $B(a, \beta R)$.

Let $v = \max\{u_K, 1/3\} - 1/3 \ge 0$. Then

$$\frac{\mu(\{x \in B(a, 2C_6 R) \colon v(x) = 0\})}{\mu(B(a, 2C_6 R))} \ge \frac{\mu(B(b, \beta R))}{\mu(B(a, 2C_6 R))} \ge \gamma$$

where $\gamma > 0$ depends only on β. Hence the p-Sobolev inequality (2.3) and the doubling property of μ imply

$$\left(\int_{B(a, 2C_6 R)} v^p \, d\mu \right)^{1/p} \le CR \left(\int_{B(a, 2\kappa C_6)} g_v^p \, d\mu \right)^{1/p}.$$

By the doubling property of μ we have

$$\int_{B(a,2C_6R)} v^p\, d\mu \geq \int_{B(a,\beta R)} (1/3)^p\, d\mu \geq C\mu(B(a,R)).$$

Hence, we obtain

$$\begin{aligned}\mathrm{Cap}_p(K,\Omega\cup K) &= \int_X g_{u_K}^p\, d\mu \geq \int_{B(a,2\kappa C_6R)} g_v^p\, d\mu \\ &\geq CR^{-p}\int_{B(a,2C_6R)} v^p\, d\mu \geq CR^{-p}\mu(B(a,R)).\end{aligned}$$

This, together with (4.4), implies that r/R is bounded below and therefore so is ρ_1/ρ_2. Thus the lemma is proved. □

To prove Theorem 4.1 we state two lemmas in [3].

Lemma 4.5. *Let $0 < R < \mathrm{diam}(\Omega)/6\kappa$ and let u be a p-subminimizer on $B(z,2\kappa R)$. Suppose $0 \leq u \leq 1$ on $B(z,2\kappa R)$ and*

$$\frac{\mu(\{x \in B(z,R)\colon u(x) > 1-s\})}{\mu(B(z,R))} \leq \gamma < 1$$

for some $0 < s < 1$. Then there exists a constant $t > 0$ such that

$$u \geq 1-t \quad \text{on } B(z.R/2).$$

The constant t is independent of u, z, and R.

Lemma 4.6. *Let $0 < R < \mathrm{diam}(\Omega)/6\kappa$. Let $B(z_1,R/2)\cap B(z_2,R/2) \neq \emptyset$. Suppose u is a p-subminimizer on $B(z_2,2\kappa R)$ with $0 \leq u \leq 1$ in $B(z_2,2\kappa R)$. If $u \leq 1-\varepsilon_1$ on $B(z_1,R/2)$ for some $\varepsilon_1 > 0$, then there is a positive constant $\varepsilon_2 = \varepsilon_2(\varepsilon_1) < 1$ such that $u \leq 1-\varepsilon_2$ on $B(z_2,R/2)$.*

Proof of Theorem 4.1. It is sufficient to show that if Ω satisfies the GHMD(ψ) property, then Ω satisfies the LHMD(ψ) property. Since Ω is uniformly perfect by Lemma 4.3, there are constants $0 < C_7 < 1$ and $r_0 > 0$ such that $A(x,C_7r,r)\cap\partial\Omega \neq \emptyset$ for every $x \in \partial\Omega$ and all $0 < r < r_0$. Let $a \in \partial\Omega$ and $0 < r < r_0$. Then we can find ρ such that $S(a,\rho)\cap\partial\Omega \neq \emptyset$ and $C_7r \leq \rho < r$.

Let c be a small positive number to be determined later. By the LLC property and the doubling property of μ we can find finitely many points $z_1,\dots,z_N \in A(a,\rho/C_6,C_6\rho)$ such that the union $\cup_{j=1}^N B(z_j,cr)$ is a covering of $S(a,\rho)$ that forms a chain, that is, for every $k,l \in \{1,\dots,N\}$ there is a subcollection of balls $B_{j_1},\dots,B_{j_m}$ such that $B_k = B_{j_1}$, $B_l = B_{j_m}$ and $B_{j_i}\cap B_{j_{i+1}} \neq \emptyset$ for $i \in \{1,\dots,m-1\}$. Observe that

$$\bigcup_{j=1}^N B(z_j,4\kappa cr) \subset A(a,\frac{\rho}{C_6}-4\kappa cr, C_6\rho+4\kappa cr) \subset A(a,(\frac{C_7}{C_6}-4\kappa c)r,(C_6+4\kappa c)r). \tag{4.6}$$

Let $c > 0$ be small enough so that $4\kappa c \leq C_7/(2C_6)$. Let $\eta = C_7/(2C_6)$. Consider

$$u = \begin{cases}\omega_p(\partial\Omega\cap B(a,\eta r),\Omega) & \text{on } \Omega,\\ 0 & \text{on } X\setminus\Omega.\end{cases}$$

Then $0 \leq u \leq 1$ on X and u is a p-subminimizer in $X\setminus\overline{B(a,\eta r)} \supset \cup_{j=1}^N B(z_j,4\kappa cr)$. Fix $z^* \in \partial\Omega\cap S(a,\rho)$. Without loss of generality, we may assume that $z^* \in B(z_1,cr)$.

Since

$$B(z^*, (4\kappa - 1)cr) \subset B(z_1, 4\kappa cr) \subset X \setminus \overline{B(a, \eta r)},$$

it follows from the comparison principle that

$$u(x) \le \omega_p(x, \partial\Omega \setminus B(z^*, (4\kappa - 1)cr), \Omega) \quad \text{for } x \in \Omega.$$

Since Ω satisfies the GHMD(ψ) property and ψ satisfies (4.2), we obtain

$$u(x) \le \frac{1}{2} \quad \text{for } x \in B(z^*, \beta r) \cap \Omega$$

for some $\beta > 0$ independent of a and r. Since $u = 0$ on $X \setminus \Omega$, we have $u \le 1/2$ on $B(z^*, \beta r)$. Hence Lemma 4.5 with $R = 2cr$ yields that $u \le 1 - \varepsilon_1$ on $B(z_1, cr)$ for some $\varepsilon_1 > 0$ independent of a and r. Since $\cup_{j=1}^N B(z_j, cr)$ is a chain, we find some ball, say $B(z_2, cr)$, intersecting $B(z_1, cr)$. Then by Lemma 4.6 we have $u \le 1 - \varepsilon_2$ on $B(z_2, cr)$ for some $\varepsilon_2 > 0$. We may repeat this argument finitely many times until, by the finiteness of the cover and its chain property, we eventually obtain $u \le 1 - \varepsilon_0$ on $\cup_{j=1}^N B(z_j, cr)$ for some $\varepsilon_0 > 0$ that is independent of a and r. In particular, $u \le 1 - \varepsilon_0$ on $S(a, \rho)$. Since

$$\omega_p(\partial\Omega \cap B(a, \eta r), \Omega) = 1 - \omega_p(\partial\Omega \setminus B(a, \eta r), \Omega) \quad \text{on } \Omega,$$

it follows that $\omega_p(\partial\Omega \setminus B(a, \eta r), \Omega) \ge \varepsilon_0$ on $\Omega \cap S(a, \rho)$. By the comparison principle we have

$$\frac{1}{\varepsilon_0}\omega_p(\partial\Omega \setminus B(a, \eta r), \Omega) \ge \omega_p(\Omega \cap \partial B(a, \rho), \Omega \cap B(a, \rho)) \quad \text{on } \Omega \cap B(a, \rho).$$

Hence the GHMD(ψ) property and Proposition 2.12 yield

$$\begin{aligned} \omega_p(x, \Omega \cap \partial B(a, r), \Omega \cap B(a, r)) &\le \omega_p(x, \Omega \cap \partial B(a, \rho), \Omega \cap B(a, \rho)) \\ &\le \frac{C_2}{\varepsilon_0}\frac{\psi(d(x, a))}{\psi(\eta r)} \le \frac{C_2}{\varepsilon_0 \eta}\frac{\psi(d(x, a))}{\psi(r)} \end{aligned}$$

for all $x \in \Omega \cap B(a, \rho)$. Because $\rho \ge C_7 r$, we obtain $d(x, a) \ge C_7 r$ for all $x \in \Omega \cap B(a, r) \setminus B(a, \rho)$. Proposition 2.12 yields

$$\omega_p(x, \Omega \cap \partial B(a, r), \Omega \cap B(a, r)) \le 1 \le \frac{\psi(d(x, a))}{\psi(C_7 r)} \le \frac{1}{C_7}\frac{\psi(d(x, a))}{\psi(r)}$$

for all $x \in \Omega \cap B(a, r) \setminus B(a, \rho)$. Thus Ω satisfies the LHMD(ψ) property. □

Remark 4.7. We say that X is Ahlfors Q-regular if there exists a positive constant C such that

$$C^{-1} r^Q \le \mu(B(x, r)) \le C r^Q \quad \text{for every } B(x, r).$$

If X is Ahlfors Q-regular with $Q \ge p$, then μ satisfies (4.1). Moreover if X supports a $(1, p)$-Poincaré inequality and X is Ahlfors Q-regular with $Q \ge p$, then X is LLC (see [6, Proposition 4.5]). Therefore, if X is Ahlfors Q-regular with $Q \ge p$ and $\psi \in \mathcal{M}_0$ satisfies (4.2), then Ω satisfies the LHMD(ψ) property if and only if Ω satisfies the GHMD(ψ) property.

Remark 4.8. Let $\psi = \psi_{\alpha\beta}$. If $\alpha > 0$, then $\psi_{\alpha\beta}$ satisfies (4.2). Therefore if X is Ahlfors Q-regular with $Q \ge p$, then the LHMD($\psi_{\alpha\beta}$) property and the GHMD($\psi_{\alpha\beta}$) property are equivalent. On the other hand, $\psi_{0\beta}$ does not satisfy (4.2), and we do not know whether the equivalence holds or not.

5. Proof of Theorem 1.3 and Corollary 1.4

In this section we give the proof of Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. Let $a \in \partial\Omega$ and $u = \mathcal{P}_\Omega \tau_{a,\psi}$. We will show (ii) in Theorem 1.2 holds, i.e., $u(x) \le C\psi(d(x,a))$. For $\rho > 0$ we define a function $f(\rho)$ by

$$f(\rho) = \sup_{\Omega \cap S(a,\rho)} u(x).$$

It is sufficient to show that

$$f(\rho) \le C\psi(\rho) \tag{5.1}$$

for small $\rho > 0$. Let $0 < \rho < r < \operatorname{diam}(\Omega)$. By definition of $\tau_{a,\psi}$ we see that $u \le \psi(r) + f(r)\chi_{\Omega\cap\partial B(a,r)}$ on $\partial(\Omega \cap B(a,r))$. The comparison principle yields

$$u(x) \le \psi(r) + f(r)\omega_p(x, \Omega \cap \partial B(a,r), \Omega \cap B(a,r))$$

for all $x \in \Omega \cap B(a,r)$. Hence, the LHMD(ψ_1) property implies

$$f(\rho) \le \psi(r) + C_1 f(r)\frac{\psi_1(\rho)}{\psi_1(r)} = \psi(r) + C_1 f(r)\frac{\psi(\rho)}{\psi(r)}\frac{\psi_2(\rho)}{\psi_2(r)}. \tag{5.2}$$

Without loss of generality, we assume that $r_0 < \operatorname{diam}(\Omega)$. We can find a positive integer N such that $C_3^N \le 1/(2C_1)$. By (1.3) we have

$$M = \sup_{0<\rho<r\le r_0} \left\{ \frac{\psi(r)}{\psi(\rho)} : \frac{\psi_2(\rho)}{\psi_2(r)} = C_3^N \right\} < \infty. \tag{5.3}$$

We can find the number $0 < r_0' < r_0$ such that

$$\frac{\psi_2(r_0')}{\psi_2(r_0)} = C_3^N.$$

Let $0 < r < r_0'$. Then by (5.3) we can find a sequence $\{\rho_j\}_{j=1}^n$ such that $r = \rho_0 < \rho_1 < \ldots < \rho_{n-1} < r_0' \le \rho_n < r_0$,

$$\frac{\psi_2(\rho_j)}{\psi_2(\rho_{j+1})} = C_3^N \le \frac{1}{2C_1} \quad \text{for } j = 0, 1, \ldots, n-1,$$

and

$$\frac{\psi(\rho_{j+1})}{\psi(\rho_j)} \le M \quad \text{for } j = 0, 1, \ldots, n-1.$$

Hence, by (5.2) we obtain

$$f(\rho_j) \le \psi(\rho_{j+1}) + \frac{1}{2} f(\rho_{j+1})\frac{\psi(\rho_j)}{\psi(\rho_{j+1})} \quad \text{for } j = 0, 1, \ldots, n-1.$$

These inequalities imply that

$$\begin{aligned}
f(r) = f(\rho_0) &\le \psi(\rho_1) + \psi(\rho_0) \sum_{j=1}^{n-1} \frac{1}{2^j} \frac{\psi(\rho_{j+1})}{\psi(\rho_j)} + \frac{1}{2^n} f(\rho_n) \frac{\psi(\rho_0)}{\psi(\rho_n)} \\
&\le M\psi(\rho_0) + M\psi(\rho_0) \sum_{j=1}^{n-1} \frac{1}{2^j} + f(\rho_n) \frac{\psi(\rho_0)}{\psi(\rho_n)} \\
&\le M\psi(\rho_0) + M\psi(\rho_0) + f(\rho_n) \frac{\psi(\rho_0)}{\psi(\rho_n)} \\
&\le (2M + \frac{\psi(\mathrm{diam}(\Omega))}{\psi(r_0')})\psi(r),
\end{aligned}$$

where $f \le \psi(\mathrm{diam}(\Omega))$ and $r_0' \le \rho_n$ are used in the last inequality. Thus (5.1) follows, and so (ii) in Theorem 1.2 holds. Hence $\|\mathcal{P}_\Omega\|_\psi < \infty$ by Theorem 1.2. □

Proof of Corollary 1.4. Let us prove (1.3) with

$$C_3 = \sup_{0<r\le r_0} \frac{\psi_2(C_4 r)}{\psi_2(r)} < 1.$$

Fix $0 < r \le r_0$. Then

$$\frac{\psi_2(C_4 r)}{\psi_2(r)} \le C_3.$$

By the monotonicity of ψ_2 we can find a number ρ such that $C_4 r \le \rho < r$ and

$$\frac{\psi_2(\rho)}{\psi_2(r)} = C_3.$$

Proposition 2.12 yields that

$$\frac{\psi(r)}{\psi(\rho)} \le \frac{\psi(r)}{\psi(C_4 r)} \le \frac{1}{C_4}.$$

Hence we have

$$\sup_{0<\rho<r\le r_0} \left\{ \frac{\psi(r)}{\psi(\rho)} : \frac{\psi_2(\rho)}{\psi_2(r)} = C_3 \right\} \le \frac{1}{C_4} < \infty.$$

Next we prove that $\lim_{r\to 0} \psi_2(r) = 0$. By the monotonicity of ψ_2 the limit of $\psi_2(r)$ exists, as $r \to 0$. If $\lim_{r\to 0} \psi_2(r) \ne 0$, then we would have

$$\lim_{r\to 0} \frac{\psi_2(r)}{\psi_2(C_4 r)} = 1.$$

This would contradict (1.4). Hence $\lim_{r\to 0} \psi_2(r) = 0$. Since the assumptions of Theorem 1.3 are satisfied, it follows that $\|\mathcal{P}_\Omega\|_\psi < \infty$. □

6. Proof of Corollaries 1.5, 1.6 and 1.7

In this section we prove Corollaries 1.5, 1.6, and 1.7.

Proof of Corollary 1.5. We divide the proof into the following two cases.

Case 1: (i) or (ii) holds. Let $\psi = \psi_{\alpha\beta}$, $\psi_1 = \psi_{\alpha'\beta'}$, and $\psi_2 = \psi_1/\psi$. Let r_0 be a small positive number. Then

$$\psi_2(r) = r^{\alpha'-\alpha}(-\log r)^{-\beta'+\beta} \quad \text{for } 0 < r \le r_0.$$

Hence ψ_2 is increasing on $(0, r_0)$, and for some constant $C_4 \in (0,1)$

$$\inf_{0<r\le r_0} \frac{\psi_2(r)}{\psi_2(C_4 r)} > 1.$$

Since the assumptions of Corollary 1.4 are satisfied, we have $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$.

Case 2: (iii) holds. Let $\psi = \psi_{0\beta}$, $\psi_1 = \psi_{0\beta'}$, and $\psi_2 = \psi_1/\psi$. Let r_0 be a small positive number. Then

$$\psi_2(r) = (\log r)^{-\beta'+\beta} \quad \text{for } 0 < r \le r_0.$$

Hence $\lim_{r\to 0} \psi_2(r) = 0$ and ψ_2 is increasing on $(0, r_0)$. Fix a constant $0 < \eta < 1$ and $0 < r \le r_0$. Let $\lambda = \eta^{1/(\beta-\beta')}$ and $\rho = r^\lambda$. Then we have

$$\frac{\psi_2(\rho)}{\psi_2(r)} = \lambda^{-\beta'+\beta} = \eta,$$

and

$$\frac{\psi(r)}{\psi(\rho)} = \lambda^\beta.$$

Hence

$$\sup_{0<\rho<r\le r_0} \left\{ \frac{\psi(r)}{\psi(\rho)} : \frac{\psi_2(\rho)}{\psi_2(r)} = \eta \right\} = \lambda^\beta < \infty.$$

Thus it follows from Theorem 1.3 that $\|\mathcal{P}_\Omega\|_{\psi_{0\beta}} < \infty$. □

To prove Corollaries 1.6 and 1.7 we observe the following lemma (see [3, Lemma 6.1]).

Lemma 6.1. *A domain Ω satisfies the LHMD(φ_{α_2}) property for some $\alpha_2 > 0$ if and only if $X \setminus \Omega$ is uniformly p-fat.*

Proof of Corollary 1.6. First suppose that $X \setminus \Omega$ is uniformly p-fat. It follows from Lemma 6.1 that there is a constant $\alpha_2 > 0$ such that Ω satisfies the LHMD(φ_{α_2}) property. Let $\alpha_1 = \min\{\alpha_0, \alpha_2\}$. Then Ω satisfies the LHMD(φ_{α_1}) property. Corollary 1.5 yields that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for $0 < \alpha < \alpha_1$ and $\beta \in \mathbf{R}$.

Conversely, suppose that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for some $0 < \alpha < \alpha_0$ and $\beta \in \mathbf{R}$. Assume that X is Ahlfors Q-regular with $Q \ge p$. By Theorem 1.2 Ω satisfies the GHMD($\psi_{\alpha\beta}$) property. It follows from Remark 4.8 that Ω satisfies the LHMD($\psi_{\alpha\beta}$) property. Let $0 < \alpha' < \alpha$. By Corollary 1.5 we obtain that $\|\mathcal{P}_\Omega\|_{\varphi_{\alpha'}} < \infty$. Theorem 1.2 and Theorem 4.1 imply that Ω satisfies the LHMD($\varphi_{\alpha'}$) property. Lemma 6.1 yields that $X \setminus \Omega$ is uniformly p-fat. □

Proof of Corollary 1.7. Suppose that $X \setminus \Omega$ is uniformly p-fat. It follows from Lemma 6.1 that there is a constant $\alpha_2 > 0$ such that Ω satisfies the LHMD(φ_{α_2}) property. Let $\alpha_1 = \min\{\alpha_0, \alpha_2\}$. Then Ω satisfies the LHMD(φ_{α_1}) property. Corollary 1.5 yields that $\|\mathcal{P}_\Omega\|_{\psi_{0\beta}} < \infty$ for every $\beta \in \mathbf{R}$. □

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Logarithmic Hölder Estimates of p-Harmonic Extension Operators in a Metric Measure Space

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Abstract. Let $1 < p < \infty$ and let X be a metric measure space with a doubling measure and a $(1, p)$-Poincaré inequality. Let Ω be a bounded domain in X. For a function f on $\partial\Omega$ we denote by $\mathcal{P}_\Omega f$ the p-harmonic extension of f over Ω. It is well known that if Ω is p-regular and $f \in C(\partial\Omega)$, then $\mathcal{P}_\Omega f$ is continuous in $\overline{\Omega}$. We characterize the family of domains such that logarithmic Hölder continuity of boundary functions f ensures logarithmic Hölder continuity of $\mathcal{P}_\Omega f$.

1. Introduction

Let $X = (X, d, \mu)$ be a complete connected metric measure space endowed with a metric d and a positive complete Borel measure μ such that $0 < \mu(U) < \infty$ for all nonempty bounded open sets U.

By the symbol C we denote an absolute positive constant whose value is unimportant and may change from line to line. Let $B(x, r) = \{y \in X : d(x, y) < r\}$ denote the open ball centered at x with radius r. We assume that μ is doubling, i.e., there is a constant $C \geq 1$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for every $x \in X$ and $r > 0$. Let $1 < p < \infty$. We assume that X admits a $(1, p)$-Poincaré inequality (see [5]).

We denote by Cap_p the p-capacity defined on X (see [2]). Let $\Omega \subset X$ be a bounded domain with $\mathrm{Cap}_p(X \setminus \Omega) > 0$. For a function f on $\partial\Omega$ we denote by $\mathcal{P}_\Omega f$ the p-Perron solution of f over Ω (see [3]). A point $\xi \in \partial\Omega$ is said to be a *p-regular point* (with respect to the p-Dirichlet problem) if

$$\lim_{\Omega \ni x \to \xi} \mathcal{P}_\Omega f(x) = f(\xi)$$

for every $f \in C(\partial\Omega)$. If every boundary point is a p-regular point, then Ω is called *p-regular*. It is well known that if Ω is p-regular and $f \in C(\partial\Omega)$, then $\mathcal{P}_\Omega f$ is p-harmonic in Ω and continuous in $\overline{\Omega}$. It is natural to raise the following question:

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This is the final form of the paper.

Question 1.1. Does improved continuity of a boundary function f guarantee improved continuity of $\mathcal{P}_\Omega f$?

Aikawa and Shanmugalingam [2] studied this question in the context of Hölder continuity. Aikawa [1] investigated this question in the context of general modulus of continuity for the classical setting, i.e., for harmonic functions in a Euclidean domain. The purpose of this paper is to study this question in the context of logarithmic Hölder continuity in a metric measure space.

We consider the function $\psi_{\alpha\beta}$ defined by

$$\psi_{\alpha\beta}(t) = \begin{cases} t^\alpha(-\log t)^{-\beta} & \text{for } 0 < t < t_0, \\ t_0^\alpha(-\log t_0)^{-\beta} & \text{for } t \geq t_0. \end{cases}$$

where either $0 < \alpha < 1$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$; and t_0 is so small that $\psi_{\alpha\beta}$ is concave. Let E be a subset of X and let f be a function on X. We say that f is $\psi_{\alpha\beta}$-Hölder continuous if $|f(x) - f(y)| \leq C\psi_{\alpha\beta}\big(d(x, y)\big)$ for $x, y \in E$. If f is $\psi_{\alpha 0}$-Hölder continuous, then f is α-Hölder continuous in the classical sense. If f is $\psi_{0\beta}$-Hölder continuous, then f is considered to be logarithmic Hölder continuous. In general, $\psi_{\alpha\beta}$-Hölder continuity is a mixture of Hölder continuity and logarithmic Hölder continuity.

Let E be a subset in X. We consider the family $\Lambda_{\psi_{\alpha\beta}}(E)$ of all bounded continuous functions f on E with norm

$$\|f\|_{\psi_{\alpha\beta},E} = \sup_{x\in E}|f(x)| + \sup_{\substack{x,y\in E\\ x\neq y}} \frac{|f(x) - f(y)|}{\psi_{\alpha\beta}\big(d(x, y)\big)} < \infty.$$

We define the operator norm

$$\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} = \sup_{\substack{f\in\Lambda_{\psi_{\alpha\beta}}(\partial\Omega)\\ \|f\|_{\psi_{\alpha\beta},\partial\Omega}\neq 0}} \frac{\|\mathcal{P}_\Omega f\|_{\psi_{\alpha\beta},\Omega}}{\|f\|_{\psi_{\alpha\beta},\partial\Omega}}.$$

Observe that logarithmic Hölder-continuity of a boundary function f ensures logarithmic Hölder-continuity of $\mathcal{P}_\Omega f$ if and only if $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$. Therefore we characterize the family of domains Ω for which $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$.

In this paper, we state the results obtained in [6]. In Section 2, we give the characterizations of the family of domains Ω for which $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$. In Section 3, we characterize the family of domains such that improved continuity of a boundary function f ensures improved continuity of $\mathcal{P}_\Omega f$ in the context of general modulus of continuity. See [6] for their proofs.

2. Results

In this section, we give some characterizations of the family of domains Ω for which $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$.

The interior regularity of p-harmonic functions is known.

Theorem A ([8, Theorem 5.2])**.** *There exists $\alpha_0 \in (0, 1]$ depending only on X and p such that if u is a p-harmonic function in a domain Ω in X, then u is α_0-Hölder continuous on every compact subset of Ω.*

Aikawa [1] estimated Dirichlet solutions by the Poisson integral representation of harmonic functions on balls. Since we are dealing with (nonlinear) p-harmonic functions, we do not have the Poisson integral representation. We instead use the local Hölder continuity of p-harmonic functions, so that we restrict ourselves to $\alpha < \alpha_0$.

Let U be an open set in X and let E be a Borel set in ∂U. We denote by $\omega_p(x, E, U)$ the p-harmonic measure evaluated at x of E in U (see [2]). Note that the p-harmonic *measure* is not a measure, i.e., the p-harmonic measure is not additive. We define two decay properties for p-harmonic measures. We say that Ω enjoys the *Local Harmonic Measure Decay property* with $\psi_{\alpha\beta}$ (abbreviated to the LHMD($\psi_{\alpha\beta}$) property) if there are positive constants C and r_0 depending only on Ω and $\psi_{\alpha\beta}$ such that

$$\omega_p\big(x, \Omega \cap \partial B(a,r), \Omega \cap B(a,r)\big) \le C \frac{\psi_{\alpha\beta}\big(d(x,a)\big)}{\psi_{\alpha\beta}(r)} \quad \text{for } x \in \Omega \cap B(a,r), \tag{2.1}$$

whenever a $\in \partial\Omega$ and $0 < r < r_0$. We say that Ω enjoys the *Global Harmonic Measure Decay property* with $\psi_{\alpha\beta}$ (abbreviated to the GHMD($\psi_{\alpha\beta}$) property) if there are positive constants C and r_0 depending only on Ω and $\psi_{\alpha\beta}$ such that

$$\omega_p(x, \partial\Omega \setminus B(a,r), \Omega) \le C \frac{\psi_{\alpha\beta}\big(d(x,a)\big)}{\psi_{\alpha\beta}(r)} \quad \text{for } x \in \Omega \cap B(a,r), \tag{2.2}$$

whenever a $\in \partial\Omega$ and $0 < r < r_0$. By the comparison principle (see [7, Theorem 7.2]) it is easy to see that (2.1) implies (2.2).

Without loss of generality, we may assume that Ω is a bounded p-regular domain (see [2, Proposition 2.1]). For $a \in \partial\Omega$ we define a test function $\tau_{a,\psi_{\alpha\beta}}$ on $\partial\Omega$ by

$$\tau_{a,\psi_{\alpha\beta}}(\xi) = \psi_{\alpha\beta}\big(d(a,\xi)\big) \quad \text{for } \xi \in \partial\Omega.$$

Then we have the following theorem.

Theorem 2.1. *Let Ω be a bounded p-regular domain. Suppose that α and β satisfy either $0 < \alpha < \alpha_0$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$, where α_0 is as in Theorem* A. *Consider the following four conditions*:

(i) $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$.

(ii) *There is a constant C such that*

$$\mathcal{P}_\Omega \tau_{a,\psi_{\alpha\beta}}(x) \le C\psi_{\alpha\beta}\big(d(x,a)\big) \quad \textit{for } x \in \Omega,$$

whenever $a \in \partial\Omega$.

(iii) *Ω has the* GHMD($\psi_{\alpha\beta}$) *property.*

(iv) *Ω has the* LHMD($\psi_{\alpha\beta}$) *property.*

Then we have

$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)} \impliedby \text{(iv)}.$$

Moreover, if $\alpha > 0$ and if X is Ahlfors Q-regular, i.e.,

$$C^{-1} r^Q \le \mu(B(x,r)) \le C r^Q \quad \textit{for every } B(x,r),$$

then (iii) $\Leftrightarrow$ (iv).

See [6, Theorems 1.2 and 4.1] for the proof of this theorem. Aikawa and Shanmugalingam [2] showed the case $\beta = 0$ of Theorem 2.1.

The implication (iv) $\Rightarrow$ (i) with the same exponent α and β does not necessarily hold in above theorem (see [2, Remark 2.4]). However, we obtain the following theorem.

Theorem 2.2 ([6, Theorem 1.3]). *Let Ω be a bounded p-regular domain. Consider the following conditions*:

(i) $0 < \alpha < \alpha' < \alpha_0$ *and* $\beta, \beta' \in \mathbb{R}$.
(ii) $0 = \alpha < \alpha' < \alpha_0$ *and* $\beta > 0$, $\beta' \in \mathbb{R}$.
(iii) $\alpha = \alpha' = 0$ *and* $0 < \beta < \beta'$.

Assume that either (i), (ii), *or* (iii) *holds. If Ω has the* LHMD($\psi_{\alpha'\beta'}$) *property, then* $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$.

Proof. Let $a \in \partial\Omega$ and $u = \mathcal{P}_\Omega \tau_{a,\psi_{\alpha\beta}}$. We will show (ii) in Theorem 2.1 holds, i.e., $u(x) \le C\psi_{\alpha\beta}\big(d(x,a)\big)$. For $\rho > 0$ we define a function $f(\rho)$ by

$$f(\rho) = \sup_{\Omega\cap S(a,\rho)} u(x).$$

It is sufficient to show that

$$f(\rho) \le C\psi_{\alpha\beta}(\rho) \tag{2.3}$$

for small $\rho > 0$.

Let $0 < \rho < r < \operatorname{diam}(\Omega)$. By definition of $\tau_{a,\psi_{\alpha\beta}}$ and the comparison principle yields

$$u(x) \le \psi_{\alpha\beta}(r) + f(r)\omega_p(x, \Omega\cap\partial B(a,r), \Omega\cap B(a,r))$$

for all $x \in \Omega\cap B(a,r)$. Let $\psi(r) = \psi_{\alpha'\beta'}(r)/\psi_{\alpha\beta}(r)$. Since Ω has the LHMD($\psi_{\alpha'\beta'}$) property,

$$f(\rho) \le \psi_{\alpha\beta}(r) + C_1 f(r)\frac{\psi_{\alpha'\beta'}(\rho)}{\psi_{\alpha'\beta'}(r)} \le \psi_{\alpha\beta}(r) + C_1 f(r)\frac{\psi_{\alpha\beta}(\rho)}{\psi_{\alpha\beta}(r)}\frac{\psi(\rho)}{\psi(r)} \tag{2.4}$$

for some C_1. Let r_0 be a small positive number. Since

$$\psi(r) = \psi_{\alpha''\beta''} \quad \text{for } 0 < r < r_0$$

where $\alpha'' = \alpha' - \alpha$ and $\beta'' = \beta' - \beta$, we have

$$M = \sup_{0<\rho<r\le r_0}\left\{\frac{\psi_{\alpha\beta}(r)}{\psi_{\alpha\beta}(\rho)} : \frac{\psi(\rho)}{\psi(r)} = \frac{1}{2C_1}\right\} < \infty. \tag{2.5}$$

Let $0 < r < r_0$. Then we can find a sequence $\{\rho_j\}_{j=0}^n$ such that $r = \rho_0 < \rho_1 < \cdots < \rho_{n-1} < r_0 \le \rho_n < \operatorname{diam}(\Omega)$,

$$\frac{\psi(\rho_j)}{\psi(\rho_{j+1})} = \frac{1}{2C_1} \quad \text{for } j = 0, 1, \dots, n-1,$$

and

$$\frac{\psi_{\alpha\beta}(\rho_{j+1})}{\psi_{\alpha\beta}(\rho_j)} \le M \quad \text{for } j = 0, 1, \dots, n-1.$$

Hence, by (2.4) we obtain

$$f(\rho_j) \le \psi_{\alpha\beta}(\rho_{j+1}) + \frac{1}{2}f(\rho_{j+1})\frac{\psi_{\alpha\beta}(\rho_j)}{\psi_{\alpha\beta}(\rho_{j+1})} \quad \text{for } j = 0, 1, \dots, n-1.$$

These inequalities imply that

$$
\begin{aligned}
f(r) = f(\rho_0) &\le \psi_{\alpha\beta}(\rho_1) + \psi_{\alpha\beta}(\rho_0) \sum_{j=1}^{n-1} \frac{1}{2^j} \frac{\psi_{\alpha\beta}(\rho_{j+1})}{\psi_{\alpha\beta}(\rho_j)} + \frac{1}{2^n} f(\rho_n) \frac{\psi_{\alpha\beta}(\rho_0)}{\psi_{\alpha\beta}(\rho_n)} \\
&\le M\psi_{\alpha\beta}(\rho_0) + M\psi_{\alpha\beta}(\rho_0) \sum_{j=1}^{n-1} \frac{1}{2^j} + f(\rho_n) \frac{\psi_{\alpha\beta}(\rho_0)}{\psi_{\alpha\beta}(\rho_n)} \\
&\le M\psi_{\alpha\beta}(\rho_0) + M\psi_{\alpha\beta}(\rho_0) + f(\rho_n) \frac{\psi_{\alpha\beta}(\rho_0)}{\psi_{\alpha\beta}(\rho_n)} \\
&\le (2M + \frac{\psi_{\alpha\beta}(\operatorname{diam}(\Omega))}{\psi_{\alpha\beta}(r_0)}) \psi_{\alpha\beta}(r),
\end{aligned}
$$

where $f \le \psi_{\alpha\beta}\big(\operatorname{diam}(\Omega)\big)$ and $r_0 \le \rho_n$ are used in the last inequality. Thus (2.3) follows, and so (ii) in Theorem 2.1 holds. Hence $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ by Theorem 2.1. □

We give more geometrical characterizations of domains Ω for which $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$. We say that $E \subset X$ is *uniformly p-fat* or satisfies the *p-capacity density condition* if there are constants $C > 0$ and $r_0 > 0$ such that

$$
\frac{\operatorname{Cap}_p\big(E \cap B(a,r), B(a,2r)\big)}{\operatorname{Cap}_p\big(B(a,r), B(a,2r)\big)} \ge C, \tag{2.6}
$$

whenever $a \in E$ and $0 < r < r_0$.

For $\alpha > 0$ we obtain the following theorem.

Theorem 2.3 ([6, Theorem 1.4])**.** *Let Ω be a bounded p-regular domain. If $X \setminus \Omega$ is uniformly p-fat, then there is a constant $0 < \alpha_1 \le \alpha_0$ such that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for $0 < \alpha < \alpha_1$ and $\beta \in \mathbb{R}$. Conversely, if $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for some $0 < \alpha < \alpha_0$ and $\beta \in \mathbb{R}$, then $X \setminus \Omega$ is uniformly p-fat, provided that there is a constant $Q \ge p$ such that X is Ahlfors Q-regular.*

To prove Theorem 2.3 we observe the following lemma (see [2, Lemma 6.1]).

Lemma 2.4. *A domain Ω has the* LHMD(φ_{α_2}) *property for some $\alpha_2 > 0$ if and only if $X \setminus \Omega$ is uniformly p-fat.*

Proof of Theorem 2.3. First suppose that $X \setminus \Omega$ is uniformly p-fat. It follows from Lemma 2.4 that there is a constant $\alpha_2 > 0$ such that Ω has the LHMD(φ_{α_2}) property. Let $\alpha_1 = \min\{\alpha_0, \alpha_2\}$. Then Ω has the LHMD(φ_{α_1}) property. Theorem 2.2 yields that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for $0 < \alpha < \alpha_1$ and $\beta \in \mathbb{R}$.

Conversely, suppose that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for some $0 < \alpha < \alpha_0$ and $\beta \in \mathbb{R}$. Assume that X is Ahlfors Q-regular with $Q \ge p$. By Theorem 2.1 Ω has the LHMD($\psi_{\alpha\beta}$) property. Let $0 < \alpha' < \alpha$. By Theorem 2.2 we obtain that $\|\mathcal{P}_\Omega\|_{\varphi_{\alpha'}} < \infty$. Theorem 2.1 implies that Ω has the LHMD($\varphi_{\alpha'}$) property. Lemma 2.4 yields that $X \setminus \Omega$ is uniformly p-fat. □

Aikawa and Shanmugalingam [2] showed the case $\beta = 0$ of Theorem 2.3. Moreover, for $\alpha = 0$ we obtain the following theorem.

Theorem 2.5 ([6, Theorem 1.5])**.** *If $X \setminus \Omega$ is uniformly p-fat, then $\|\mathcal{P}_\Omega\|_{\psi_{0\beta}} < \infty$ for every $\beta > 0$.*

Proof. Suppose that $X \setminus \Omega$ is uniformly p-fat. It follows from Lemma 2.4 that there is a constant $\alpha_2 > 0$ such that Ω has the LHMD(φ_{α_2}) property. Let $\alpha_1 = \min\{\alpha_0, \alpha_2\}$. Then Ω has the LHMD(φ_{α_1}) property. Theorem 2.2 yields that $\|\mathcal{P}_\Omega\|_{\psi_{0\beta}} < \infty$ for every $\beta > 0$. □

3. Modulus of continuity

In this section, we consider general modulus of continuity of p-Perron solutions $\mathcal{P}_\Omega f$. See [6] for the proof of theorems and the corollary in this section. Note that the case of $p = 2$ and Euclidean domains was studied by Aikawa [1].

Let $\mathcal{M}$ be the family of all positive nondecreasing concave functions ψ on $(0, \infty)$ with $\psi(0) = \lim_{t\to 0} \psi(t) = 0$. In particular, if either $0 < \alpha < 1$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$, then $\psi_{\alpha\beta} \in \mathcal{M}$. For $\psi \in \mathcal{M}$, we say that f is *ψ-Hölder continuous* if $|f(x) - f(y)| \le C\psi\big(d(x, y)\big)$. The modulus of continuity of a uniformly continuous function on any geodesic space is comparable to a certain concave function. See [4, Chapter 2, §6]. Therefore, we have only to check ψ-Hölder continuity for $\psi \in \mathcal{M}$ to study Question 1.1 in the context of modulus of continuity.

Next we define the operator norm for $\psi \in \mathcal{M}$. Let E be a subset in X. We consider the family $\Lambda_\psi(E)$ of all bounded continuous functions f on E with norm

$$\|f\|_{\psi,E} = \sup_{x\in E}|f(x)| + \sup_{\substack{x,y\in E\\ x\ne y}} \frac{|f(x) - f(y)|}{\psi\big(d(x, y)\big)} < \infty.$$

We define the operator norm

$$\|\mathcal{P}_\Omega\|_\psi = \sup_{\substack{f\in\Lambda_\psi(\partial\Omega)\\ \|f\|_{\psi,\partial\Omega}\ne 0}} \frac{\|\mathcal{P}_\Omega f\|_{\psi,\Omega}}{\|f\|_{\psi,\partial\Omega}}.$$

Let ψ, $\varphi \in \mathcal{M}$. We say that $\varphi \precsim \psi$ if there are $r_0 > 0$ and $C > 0$ such that

$$\frac{\varphi(s)}{\varphi(r)} \le C\frac{\psi(s)}{\psi(r)} \quad \text{for } 0 < s < r < r_0.$$

Let $\mathcal{M}_0$ be the family of all $\psi \in \mathcal{M}$ with $t^{\alpha_0} \precsim \psi(t)$, where α_0 is a positive constant such that every p-harmonic function in Ω is locally α_0-Hölder continuous in Ω as explained in Section 2 (see [8]). For example, if either $0 < \alpha < \alpha_0$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$, then $\psi_{\alpha\beta} \in \mathcal{M}_0$. But if $\alpha = \alpha_0$ and $\beta < 0$, then $\psi_{\alpha_0\beta} \notin \mathcal{M}_0$. Hence we see that $\mathcal{M}_0 \subsetneq \mathcal{M}$. We use the locally Hölder continuity of p-harmonic functions as Section 2, so that we restrict ourselves to the case $\psi \in \mathcal{M}_0$.

Let $\psi \in \mathcal{M}$. We say that Ω enjoys the *Local Harmonic Measure Decay property* with ψ (abbreviated to the LHMD(ψ) property) if there are positive constants C and r_0 depending only on Ω and ψ such that

$$\omega_p(x, \Omega \cap \partial B(a, r), \Omega \cap B(a, r)) \le C\frac{\psi\big(d(x, a)\big)}{\psi(r)} \quad \text{for } x \in \Omega \cap B(a, r), \tag{3.1}$$

whenever a $\in \partial\Omega$ and $0 < r < r_0$. We say that Ω enjoys the *Global Harmonic Measure Decay property* with ψ (abbreviated to the GHMD(ψ) property) if there are positive constants C and r_0 depending only on Ω and ψ such that

$$\omega_p(x, \partial\Omega \setminus B(a, r), \Omega) \le C\frac{\psi\big(d(x, a)\big)}{\psi(r)} \quad \text{for } x \in \Omega \cap B(a, r), \tag{3.2}$$

whenever a $\in \partial\Omega$ and $0 < r < r_0$.

For $a \in \partial\Omega$ we define a test function $\tau_{a,\psi}$ on $\partial\Omega$ by

$$\tau_{a,\psi}(\xi) = \psi\big(d(a,\xi)\big) \quad \text{for } \xi \in \partial\Omega.$$

Then we obtain the generalization of Theorem 2.1.

Theorem 3.1 ([6, Theorem 1.2])**.** *Let $\psi \in \mathcal{M}_0$ and let Ω be a bounded p-regular domain. Consider the following conditions*:

(i) $\|\mathcal{P}_\Omega\|_\psi < \infty$.

(ii) *There is a constant C such that*

$$\mathcal{P}_\Omega \tau_{a,\psi} \le C\psi\big(d(x,a)\big) \quad \textit{for } x \in \Omega,$$

whenever $a \in \partial\Omega$.

(iii) *Ω has the* GHMD(ψ) *property.*

(iv) *Ω has the* LHMD(ψ) *property.*

Then we have

$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)} \impliedby \text{(iv)}.$$

Moreover, we have the generalizations of Theorem 2.2.

Theorem 3.2 ([6, Theorem 5.1])**.** *Let $\psi, \psi_1 \in \mathcal{M}_0$. Let $\psi_2 = \psi_1/\psi$. Suppose that $\lim_{r\to 0} \psi_2(r) = 0$ and there are constants $0 < C < 1$ and $r_0 > 0$ such that ψ_2 is increasing on $(0, r_0)$ and*

$$\sup_{0<\rho<r\le r_0} \left\{ \frac{\psi(r)}{\psi(\rho)} : \frac{\psi_2(\rho)}{\psi_2(r)} = C \right\} < \infty. \tag{3.3}$$

If Ω has the LHMD(ψ_1) *property, then $\|\mathcal{P}_\Omega\|_\psi < \infty$.*

Corollary 3.3 ([6, Corollary 5.2])**.** *Let $\psi, \psi_1 \in \mathcal{M}_0$. Let $\psi_2 = \psi_1/\psi$. Suppose that there are constants $0 < C < 1$ and $r_0 > 0$ such that ψ is increasing on $(0, r_0)$ and*

$$\inf_{0<r\le r_0} \frac{\psi_2(r)}{\psi_2(Cr)} > 1. \tag{3.4}$$

If Ω has the LHMD(ψ_1) *property, then $\|\mathcal{P}_\Omega\|_\psi < \infty$.*

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Martin boundary for p-harmonic functions in a cylinder and a cone

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Preprint.

MARTIN BOUNDARY FOR p-HARMONIC FUNCTIONS IN A CYLINDER AND A CONE

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Abstract. Let $1 < p < \infty$. A p-harmonic kernel function is a p-harmonic analogue of Martin kernel functions for harmonic functions. We study p-harmonic kernel functions in a cylinder and a cone in $\mathbb{R}^n$. In case $n = 2$ explicit representations of p-harmonic kernel functions are given.

1. Introduction

Let $\mathbb{R}$ be the set of all real numbers. We denote by $\mathbb{R}^n (n \geq 2)$ the n-dimensional Euclidean space. A point $x \in \mathbb{R}^n$ is denoted by (x', x_n) with $x' = (x_1, \dots, x_{n-1})$. We denote a point $x \in \mathbb{R}^n \setminus \{0\}$ by (r, σ) with $r = |x|$ and $\sigma = x/|x|$. We let ∂E and $\overline{E}$ be the boundary and the closure of a set E in $\mathbb{R}^n$, respectively. We define $\operatorname{dist}(x, E)$ to equal the distance from a point $x \in \mathbb{R}^n$ to a set $E \subset \mathbb{R}^n$. Let $B(x, r)$ and $S(x, r)$ be the open ball and the sphere with center x and radius r, respectively. We use the symbol C to denote an absolute positive constant whose value is unimportant and may change from line to line.

Let $1 < p < \infty$. Let $D \subset \mathbb{R}^n$ be a domain. We say that u is a p-harmonic function in D if $u \in W^{1,p}_{loc}(D)$ is continuous and satisfies the p-Laplace equation $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in D in the weak sense; that is, whenever D' is a relatively compact subdomain of D and $\varphi \in W^{1,p}_0(D')$, we have

$$\int_{D'} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = 0.$$

If $p = 2$, then the p-Laplace equation reduces to the classical linear equation $\Delta u = 0$. The set of all positive p-harmonic functions in D is denoted by $\mathcal{H}_+(D)$.

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Let us recall the classical Martin boundary theory for harmonic functions. Let D be an arbitrary domain with Green function $G(x, y)$. Martin [13] introduced the Martin boundary as the smallest ideal boundary for which $G(x, y)/G(x_0, y)$ has a continuous extension $K(x, y)$. An ideal boundary point y is called minimal if $K(\cdot, y)$ is a minimal harmonic function. The set of all minimal Martin boundary points is called the minimal Martin boundary. Martin proved that every positive harmonic function in D is uniquely represented as the integral of the kernel function $K(x, y)$ over the minimal Martin boundary. The identification of the (minimal) Martin boundary for specific domains is of great interest. There are a number of works on this topic. Hunt-Wheeden [8] gave the first cornerstone. They showed that the Martin boundary of a Lipschitz domain D is homeomorphic to the Euclidean boundary ∂D and every boundary point is minimal. They said that a positive harmonic function u in D is a *kernel function* in D at a boundary point $w \in \partial D$ if u has continuous boundary values 0 on $\partial D \setminus \{w\}$ and $u(x_0) = 1$ ([8, p.507]). They proved that every boundary point has a unique kernel function. This is crucial for the identification of the Martin boundary.

In view of this important result, it is natural to extend the notion of kernel functions to p-harmonic functions. Let D be an arbitrary domain with compactification D^*. We write $\partial^* D$ for the ideal boundary $D^* \setminus D$. We say that $u \in \mathcal{H}_+(D)$ is a p-harmonic kernel function in D at $w \in \partial^* D$ with reference point x_0 if u has continuous boundary values 0 on $\partial^* D \setminus \{w\}$ and $u(x_0) = 1$. If each $w \in \partial^* D$ corresponds to a unique p-harmonic kernel function, we say that the p-Martin boundary of D is homeomorphic to $\partial^* D$.

Let Ω be a $C^{2,\alpha}$-domain in $\mathbb{R}^{n-1}$. The domain $\Omega \times \mathbb{R} = \{(x', x_n) : x' \in \Omega, x_n \in \mathbb{R}\}$ is said to be a cylinder generated by Ω. We compactify $\Omega \times \mathbb{R}$ by adding the topological boundary and the ideal boundary $\{+\infty, -\infty\}$, where $\pm\infty$ corresponds to the limit as $x_n \to \pm\infty$, respectively. In this paper we investigate p-harmonic kernel functions in $\Omega \times \mathbb{R}$ at $\pm\infty$ with the aid of a translation operator similar to the stretching operator used by Tolksdorf [15] in his study on p-harmonic functions in a cone. We let

$$\mathcal{H}_+^{+\infty}(\Omega \times \mathbb{R}) = \{u \in \mathcal{H}_+(\Omega \times \mathbb{R}) : u = 0 \text{ on } \partial(\Omega \times \mathbb{R}) \cup \{-\infty\}\},$$

where $u = 0$ at $-\infty$ means $\lim_{x_n \to -\infty} u(x) = 0$. Similarly, we let

$$\mathcal{H}_+^{-\infty}(\Omega \times \mathbb{R}) = \{u \in \mathcal{H}_+(\Omega \times \mathbb{R}) : u = 0 \text{ on } \partial(\Omega \times \mathbb{R}) \cup \{\infty\}\}.$$

By definition $u \in \mathcal{H}_+(\Omega \times \mathbb{R})$ is a p-harmonic kernel function at $+\infty$ (resp. $-\infty$) if and only if $u(x_0) = 1$ and $u \in \mathcal{H}_+^{+\infty}(\Omega \times \mathbb{R})$ (resp. $u \in \mathcal{H}_+^{-\infty}(\Omega \times \mathbb{R})$). The following theorem shows that $+\infty$ and $-\infty$ have a unique p-harmonic kernel function.

Theorem 1.1. *There exist a positive constant λ and a function $f(x')$ of $x' \in \Omega$, depending only on p, n and Ω, such that*

$$\mathcal{H}_+^{+\infty}(\Omega \times \mathbb{R}) = \{C \exp(\lambda x_n) f(x') : C > 0\}, \tag{1.1}$$

$$\mathcal{H}_+^{-\infty}(\Omega \times \mathbb{R}) = \{C \exp(-\lambda x_n) f(x') : C > 0\}. \tag{1.2}$$

Since $\Omega\times\mathbb{R}$ is locally a $C^{2,\alpha}$-domain in $\mathbb{R}^n$, every boundary point in $\partial(\Omega\times\mathbb{R})$ has a unique p-harmonic kernel function, in view of Lewis-Nyström [11]. So, we have the following corollary.

Corollary 1.2. *The p-Martin boundary of $\Omega\times\mathbb{R}$ is homeomorphic to $\partial(\Omega\times\mathbb{R}) \cup \{-\infty, +\infty\}$.*

Remark 1.3. Lewis-Nyström obtained the uniqueness of p-harmonic kernel functions by using their scale invariant Harnack principle for Lipschitz domains and starlike Lipschitz ring domains ([10]) and a very delicate argument. Unlike the linear case, the scale invariant boundary Harnack principle is not enough to deduce the uniqueness of p-harmonic kernel functions. See Remark 2.9 below. This is the reason why the domains in [11] are restricted to C^1 or convex. To avoid such difficulties, we restrict ourselves to $C^{2,\alpha}$-domains in this paper. In this case the scale invariant boundary Harnack principle can be proved rather easily. See Lemma 2.3 below and [2, Theorem 1.2].

In case $n = 2$, we can explicitly calculate λ and f.

Theorem 1.4. *Let $n = 2$ and $\Omega = (0, L)$ with $0 < L < \infty$. Then*

$$\lambda = \frac{p\pi}{2(p-1)L},$$

and $f(x_1)$ has a parametric representation given by

$$\begin{cases} f(s) = \exp\Big(\dfrac{-(p-2)\sin^2 s}{p-1}\Big)\sin s, \\ x_1(s) = \dfrac{1}{\lambda}\Big(\dfrac{p}{2(p-1)}s + \dfrac{p-2}{4(p-1)}\sin 2s\Big). \end{cases} \tag{1.3}$$

Next we consider p-harmonic kernel functions in a cone. Let Σ be a $C^{2,\alpha}$-domain in the unit sphere. The domain $\Gamma = \{(r,\sigma) : 0 < r < \infty, \sigma \in \Sigma\}$ is said to be a cone generated by Σ. We compactify Γ by adding the topological boundary and the ideal boundary $\{\infty\}$, where ∞ is the point at infinity. We study p-harmonic kernel functions in Γ at ∞ and 0 with the aid of the stretching operator used by Tolksdorf [15]. We let

$$\mathcal{H}_+^{\infty}(\Gamma) = \{u \in \mathcal{H}_+(\Gamma) : u = 0 \text{ on } \partial\Gamma\},$$
$$\mathcal{H}_+^{0}(\Gamma) = \{u \in \mathcal{H}_+(\Gamma) : u = 0 \text{ on } (\partial\Gamma \cup \{\infty\}) \setminus \{0\}\},$$

where $u = 0$ on ∞ means $\lim_{|x|\to\infty} u(x) = 0$. By definition $u \in \mathcal{H}_+(\Gamma)$ is a p-harmonic kernel function at ∞ (resp. 0) if and only if $u(x_0) = 1$ and $u \in \mathcal{H}_+^\infty(\Gamma)$ (resp. $u \in \mathcal{H}_+^0(\Gamma)$). The following theorems show that ∞ and 0 have a unique p-harmonic kernel function.

Theorem 1.5. *There exist a positive constant μ and a function $g(\sigma)$ of $\sigma \in \Sigma$, depending only on p, n and Σ, such that*

$$\mathcal{H}_+^\infty(\Gamma) = \{Cr^\mu g(\sigma) : C > 0\}. \tag{1.4}$$

Theorem 1.6. *There exist a positive constant ν and a function $h(\sigma)$ of $\sigma \in \Sigma$, depending only on p, n and Σ, such that*

$$\mathcal{H}_+^0(\Gamma) = \{Cr^{-\nu} h(\sigma) : C > 0\}.$$

Corollary 1.7. *The p-Martin boundary of Γ is homeomorphic to $\partial\Gamma \cup \{\infty\}$.*

Remark 1.8. In case $n = 2$, we can explicitly calculate μ, ν, g and h, although these are involved (Propositions 6.1 and 6.2). If Γ is the upper half space $H = \{(x', x_n) : x_n > 0\}$, then $u(x) = x_n \in \mathcal{H}_+^\infty(H)$ and $\mu = 1$ for any p, n. However, in general, it is difficult to explicitly calculate μ, ν, g and h.

Remark 1.9. Tolksdorf [15] studied functions $u \in \mathcal{H}_+^\infty(\Gamma)$ satisfying the doubling condition:

$$\sup_{\Gamma\cap B(0,2R)} u \le C \sup_{\Gamma\cap B(0,R)} u \quad \text{for } R \ge 1, \tag{1.5}$$

with a constant $C \ge 1$ depending only on u. The set of all $u \in \mathcal{H}_+^\infty(\Gamma)$ satisfying (1.5) is denoted by $\widetilde{\mathcal{H}_+^\infty}(\Gamma)$. By applying the stretching operator, he gave a characterization of $\widetilde{\mathcal{H}_+^\infty}(\Gamma)$ similar to (1.4). Theorem 1.5 implies that the doubling condition (1.5) is superfluous, that is, $\widetilde{\mathcal{H}_+^\infty}(\Gamma) = \mathcal{H}_+^\infty(\Gamma)$.

The plan of this paper is as follows. In the next section we shall state known results of p-harmonic functions. In Sections 3 and 5, we shall prove Theorems 1.1, 1.5 and 1.6 by applying the translation operator and the stretching operator. We will show Theorem 1.4 in Section 4. Finally, we shall explicitly calculate μ, ν, g and h for $n = 2$ in Section 6.

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2. Preliminaries

In this section, let D be a domain in $\mathbb{R}^n$. We state known results for p-harmonic functions such as Hopf's maximum principle and the strong comparison principle (see [15, Section 3]).

Lemma 2.1. *(Hopf's Maximum Principle) Let B be a ball. If $u \in \mathcal{H}_+(B) \cap C^1(\overline{B})$ and $u(x_0) = 0$ for some $x_0 \in \partial B$, then $\nabla u(x_0) \neq 0$.*

Lemma 2.2. *(Strong Comparison Principle) Assume that u is p-subharmonic in D, v is p-superharmonic in D, $v \in C^2(D)$ and $\nabla u \neq 0$ in D. If $u \leq v$ in D and $u \not\equiv v$, then $u < v$ in D.*

For $C^{2,\alpha}$-domains the boundary Harnack principle can be easily established (see [2, Theorem 1.2]).

Lemma 2.3. *(Boundary Harnack Principle) Let D be a bounded $C^{2,\alpha}$-domain. There exist constants $C_1 > 1$, $C_2 > 1$, $r_1 > 0$ with the following property: Let $0 < r < r_1$ and $\xi \in \partial D$. If $u, v \in \mathcal{H}_+(D \cap B(\xi, C_1 r))$ vanishing on $\partial D \cap B(\xi, C_1 r)$, then*

$$\frac{u(x)/u(y)}{v(x)/v(y)} \leq C_2 \quad \text{for } x, y \in D \cap B(\xi, r).$$

By [16, Theorem 1] and [15, Proposition 3.7], we obtain the following $C^{1,\gamma}$-estimate.

Lemma 2.4. *($C^{1,\gamma}$-estimate) Let $B_R = B(x_0, R)$ be a ball with radius $R > 0$. Suppose that $\partial D \cap B_{2R}$ is empty or that $\partial D \cap B_{2R}$ is a $C^{2,\alpha}$-boundary portion of ∂D. If u is a p-harmonic function in $D \cap B(x_0, 2R)$ vanishing on $\partial D \cap B_{2R}$, then there exist constants $C > 0$ and $\gamma \in (0, 1)$ depending only on n, p, R, D and $\|u\|_{L^\infty(D \cap B_{2R})}$ such that*

$$\|u\|_{C^{1,\gamma}(\overline{D \cap B_R})} \leq C.$$

The Schauder theory [6, Theorems 6.13 and Lemma 6.18] implies the following lemma.

Lemma 2.5. *Let $T \subset \partial D$ be a $C^{2,\alpha}$-boundary portion. If u is p-harmonic in D and if $u = 0$ on T, then $u \in C^{2,\alpha}(D \cup T)$ provided $\nabla u \neq 0$ in $D \cup T$.*

We are inspired by the argument in the proof of Hopf's comparison principle [15, Proposition 3.3.1], to give the following lemma.

Lemma 2.6. *Let D be a $C^{2,\alpha}$-domain. Let $\xi \in \partial D$ and let r_0 be a sufficiently small positive constant. Assume that $u, v \in \mathcal{H}_+(D \cap B(\xi, 6r_0))$ vanishing on $\partial D \cap B(\xi, 6r_0)$. Suppose that $v \in C^2(D \cap B(\xi, 6r_0))$ and there exist positive constants m_1, m_2, M_1, M_2 and M_3 such that*

$$m_1 \leq |\nabla v| \leq M_1 \quad \text{in } D \cap B(\xi, 6r_0),$$

$$\sum_{i,j=1}^{n} \left| \frac{\partial^2 v}{\partial x_i \partial x_j} \right| \leq M_2 \quad \text{in } D \cap B(\xi, 6r_0),$$

$$\sup_{D \cap B(\xi, 6r_0)} \leq M_3,$$

$$\inf\{u(x) - v(x) : x \in D \cap B(\xi, 6r_0), \operatorname{dist}(x, \partial D) \geq r_0\} \geq m_2.$$

If $u > v$ in $D \cap B(\xi, 6r_0)$, then there exists a positive constant δ, depending only on $n, p, m_1, m_2, M_1, M_2, M_3$ and r_0, such that $u \geq (1+\delta)v$ in $D \cap B(\xi, r_0)$.

Proof. By [2, Lemma 2.2], the domain D satisfies the ball condition with some $r_1 > 0$; that is, for every $\xi \in \partial D$ there exist $\xi^i \in D$ and $\xi^e \in \mathbb{R}^n \setminus \overline{D}$ such that $B(\xi^i, r_1) \subset D$, $B(\xi^e, r_1) \subset \mathbb{R}^n \setminus \overline{D}$ and $\xi \in S(\xi^i, r_1) \cap S(\xi^e, r_1)$. Let $r_0 < r_1/2$ and let $x_0 \in D \cap B(\xi, r_0)$. Then there is $\eta \in \partial D$ such that $\mathrm{dist}(x_0, \partial D) = |x_0 - \eta|$. By the interior ball condition at η we find $\eta^i \in D$ such that $B(\eta^i, 2r_0) \subset D$ and $\eta \in S(\eta^i, 2r_0)$. Observe that $B(\eta^i, 2r_0) \subset D \cap B(\xi, 6r_0)$ and $x_0 \in B(\eta^i, 2r_0) \setminus B(\eta^i, r_0)$. Without loss of generality, we may assume that $\eta^i = 0$ and $r_0 = 1/2$.

For $b > 1$, we set

$$V(x) = b^{-2}(e^{-b|x|^2} - e^{-b}).$$

By assumption there is a constant $M_2' > 0$ such that

$$\left|\frac{\partial^2(v+V)}{\partial x_i \partial x_j}(x)\right| \le M_2' \quad \text{for } y \in D \cap B(\xi, 6r_0).$$

We claim $v + V$ is p-subharmonic in $B(0,1) \setminus \overline{B(0,1/2)}$ if b is sufficiently large. We will prove the claim later and we finish the proof of Lemma 2.6. By assumption we can choose b such that then $u \ge v + V$ on $\partial B(0,1) \cup \partial B(0,1/2)$. Hence it follows from the comparison principle that $u \ge v + V$ in $B(0,1) \setminus B(0,1/2)$, in particular $u(x_0) \ge v(x_0) + V(x_0)$. On the other hand, we see that $V(x_0) \ge C(1 - |x_0|)$ and $\mathrm{dist}(x, \partial D) = 1 - |x_0| \ge Cv(x_0)$ by the ball condition. Therefore we obtain

$$u(x_0) \ge (1 + C)v(x_0).$$

Since $x_0 \in D \cap B(\xi, r_0)$ is arbitrary, we obtain that

$$u \ge (1 + \delta)v \quad \text{in } D \cap B(\xi, r_0),$$

where δ is a constant depending only on m_1, m_2, M_1, M_2 and M_3.

Finally, we prove that $v + V$ is p-subharmonic in $B(0,1) \setminus \overline{B(0,1/2)}$ if b is sufficiently large. Let $\varepsilon = \min\{1, p-1\}/4$. Assume that

$$b > \max\left\{\frac{4(n+p-2)}{\varepsilon}, \frac{4(1+|p-2|)(1+M_1)nM_2'}{\varepsilon m_1}\right\}.$$

Observe that

$$\Delta_p(v + V) = |\nabla(v + V)|^{p-4}(I_1 + I_2 + I_3 + I_4),$$

where

$$
\begin{aligned}
I_1 &= |\nabla v|^2 \Delta v + (p-2) \sum_{i,j=1}^{n} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}, \\
I_2 &= |\nabla v|^2 \Delta V + (p-2) \sum_{i,j=1}^{n} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial^2 V}{\partial x_i \partial x_j}, \\
I_3 &= (|\nabla V|^2 + 2(\nabla v \cdot \nabla V)) \Delta(v+V), \\
I_4 &= (p-2) \sum_{i,j=1}^{n} \Big(\frac{\partial (v+V)}{\partial x_i} \frac{\partial (v+V)}{\partial x_j} - \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \Big) \frac{\partial^2 (v+V)}{\partial x_i \partial x_j}.
\end{aligned}
$$

Because $v \in \mathcal{H}_+(D \cap B(\xi,3))$, we have $I_1 = 0$ in $D \cap B(\xi,3)$. If $x \in B(0,1) \setminus \overline{B(0,1/2)}$, then

$$
\begin{aligned}
I_2 =& |\nabla v|^2 \Big(4|x|^2 - \frac{2(n-p+2)}{b} \Big) e^{-b|x|^2} + 4(p-2) \sum_{i,j=1}^{n} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} x_i x_j e^{-b|x|^2} \\
\ge& 4|\nabla v|^2 \Big(\min\{1, p-1\}|x|^2 - \frac{2(n+p-2)}{b} \Big) e^{-b|x|^2} \\
\ge& 2 m_1 \varepsilon e^{-b|x|^2},
\end{aligned}
$$

and

$$
\begin{aligned}
|I_3 + I_4| \le& \Big| \Big(\frac{4|x|^2}{b^2} e^{-2b|x|^2} - \frac{4}{b} (\nabla v \cdot x) e^{-b|x|^2} \Big) \Delta(v+V) \Big| \\
&+ \Big| (p-2) \sum_{i,j=1}^{n} \Big(\frac{4}{b^2} x_i x_j e^{-2b|x|^2} - \frac{4}{b} x_i \frac{\partial v}{\partial x_j} e^{-b|x|^2} \Big) \frac{\partial^2 (v+V)}{\partial x_i \partial x_j} \Big| \\
\le& \frac{4(1+|p-2|) n M_2'}{b} \Big(\frac{|x|^2}{b} e^{-b|x|^2} + |x| M_1 \Big) e^{-b|x|^2} \\
\le& m_1 \varepsilon e^{-b|x|^2}.
\end{aligned}
$$

Since $\Delta_p(v+V) \ge 0$ in $B(0,1) \setminus \overline{B(0,1/2)}$, we see that $v+V$ is p-subharmonic in $B(0,1) \setminus \overline{B(0,1/2)}$. □

It is well known that if $\{u_j\}$ is a locally uniformly bounded sequence of p-harmonic functions in D, then there exist a subsequence $\{u_{j_k}\}$ and a p-harmonic function u in D such that $u_{j_k} \to u$ locally uniformly in D. Moreover the following lemma holds.

Lemma 2.7. *Let D be a $C^{2,\alpha}$-domain. Let $\xi \in \partial D$ and let r_0 be a sufficiently small positive constant. Assume that $\{u_j \in \mathcal{H}_+(D \cap B(\xi, 6r_0)) : u_j = 0$ on $\partial D \cap B(\xi, 6r_0)\}$ is uniformly bounded. Then there exist a subsequence $\{u_{j_k}\}$*

and $u \in \mathcal{H}_+(D \cap B(\xi, 6r_0)) \cap C^1(\overline{D \cap B(\xi, 3r_0)})$ *vanishing on* $\partial D \cap B(\xi, 3r_0)$ *such that*

$$\frac{u_{j_k}}{u} \to 1 \quad \text{uniformly in } D \cap B(\xi, r_0). \tag{2.1}$$

Proof. By Lemma 2.4, there exist a subsequence $\{u_{j_k}\}$ and $u \in \mathcal{H}_+(D \cap B(\xi, 6r_0)) \cap C^1(\overline{D \cap B(\xi, 3r_0)})$ such that

$$u_{j_k} \to u \quad \text{uniformly in } \overline{D \cap B(\xi, 3r_0)},$$
$$\nabla u_{j_k} \to \nabla u \quad \text{uniformly in } \overline{D \cap B(\xi, 3r_0)}.$$

Hence we see that $u = 0$ on $\partial D \cap B(\xi, 3r_0)$. Fix $\varepsilon > 0$ and choose $N > 0$ such that if $k > N$, then

$$|\nabla u - \nabla u_{j_k}| < \varepsilon \quad \text{on } \overline{D \cap B(\xi, 3r_0)}.$$

We prove (2.1). By [2, Lemma 2.2], the domain D satisfies the ball condition with some $r_1 > 0$. Let $r_0 < r_1/2$ and let $x_0 \in D \cap B(\xi, r_0)$. Then there is $\eta \in \partial D$ such that $\mathrm{dist}(x_0, \partial D) = |x_0 - \eta|$. By the interior ball condition at η we find $\eta^i \in D$ such that $B(\eta^i, 2r_0) \subset D$ and $\eta \in S(\eta^i, 2r_0)$. Observe that $B(\eta^i, 2r_0) \subset D \cap B(\xi, 6r_0)$ and $x_0 \in B(\eta^i, 2r_0) \setminus B(\eta^i, r_0)$. Without loss of generality, we may assume that $\eta^i = 0$ and $r_0 = 1$.

For $b > 1$ and $C > 0$, we set

$$V(x) = C(e^{-b|x|^2} - e^{-b}).$$

Since

$$\Delta_p V(x) = (2Cbe^{-b|x|^2})^{p-1}|x|^{p-2}(2b(p-1)|x|^2 - n - p + 2),$$

we can choose $b > 0$ such that V is p-subharmonic in $B(0, 1) \setminus \overline{B(0, 1/2)}$. If $C > 0$ is sufficiently small, then $u \geq V$ on $\partial B(0, 1) \cup \partial B(0, 1/2)$. Hence it follows from the comparison principle that $u \geq V$ in $B(0, 1) \setminus \overline{B(0, 1/2)}$. Then we have $u(x_0) \geq V(x_0)$. Since $V(x_0) \geq C(1 - |x_0|)$ for some constant $C > 0$, we have

$$u(x_0) \geq C(1 - |x_0|) = C|x_0 - \eta|.$$

By the mean value theorem, there exists a constant $0 < c < 1$ such that

$$u(x_0) - u_{j_k}(x_0) = (\nabla u - \nabla u_{j_k})((1 - c)\eta + cx_0) \cdot (x_0 - \eta).$$

Hence if $k \geq N$, then we have

$$\begin{aligned} \left|1 - \frac{u_{j_k}(x_0)}{u(x_0)}\right| &= \left|\frac{u(x_0) - u_{j_k}(x_0)}{u(x_0)}\right| \\ &\leq \frac{|(\nabla u - \nabla u_{j_k})((1 - c)\eta + cx_0)||x_0 - \eta|}{u(x_0)} \\ &\leq \frac{\varepsilon}{C}. \end{aligned}$$

Therefore (2.1) is proved. □

Finally, we observe that the scale invariant boundary Harnack principle implies the uniqueness of kernel functions in the linear case. This observation will be not used in the sequel; it is given only for the emphasis on the difference between the linear case ($p = 2$) and the nonlinear case ($p \neq 2$). Let D be a domain with a boundary point w. By $\mathcal{H}(w)$ we denote the family of all kernel functions at w with reference point x_0. Then the scale invariant boundary Harnack principle implies that there exists a constant $C \geq 1$ such that

$$C^{-1}u(x) \leq v(x) \leq Cu(x) \quad \text{for all } u, v \in \mathcal{H}(w) \text{ and } x \in D. \tag{2.2}$$

Proposition 2.8. *If (2.2) holds, then $\mathcal{H}(w)$ is a singleton.*

Proof. We follow Kemper [9] (see also [1]). Let

$$C_0 = \sup_{u,v\in\mathcal{H}(w),\ x\in D} \frac{u(x)}{v(x)}.$$

Then $1 \leq C_0 < \infty$ by (2.2). It is sufficient to show $C_0 = 1$. Suppose $C_0 > 1$. Take $u, v \in \mathcal{H}(w)$. By the linearity of harmonicity $v_1 = (C_0 v - u)/(C_0 - 1)$ is a positive harmonic function with the same boundary values as u and v such that $v_1(x_0) = (C_0 v(x_0) - u(x_0))/(C_0 - 1) = 1$. Hence $v_1 \in \mathcal{H}(w)$, and so $u \leq C_0 v_1 = C_0(C_0 v - u)/(C_0 - 1)$, which implies

$$\frac{u}{v} \leq \frac{C_0^2}{2C_0 - 1} < C_0 \quad \text{on } D.$$

This contradicts the definition of C_0. □

Remark 2.9. If $p \neq 2$, then the above argument fails, as $v_1 = (C_0 v - u)/(C_0 - 1)$ need not be p-harmonic even if u and v are p-harmonic.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. By $\partial\Omega$ we denote the relative boundary of Ω in $\mathbb{R}^{n-1}$. We observe that $u(x', x_n) = \exp(\lambda x_n) f(x')$ is p-harmonic if and only if λ and $f(x')$ satisfy the equation

$$-\operatorname{div}_{x'}[(\lambda^2 f^2 + |\nabla_{x'} f|^2)^{\frac{p-2}{2}} \nabla_{x'} f] = \lambda^2 (p-1)(\lambda^2 f^2 + |\nabla_{x'} f|^2)^{\frac{p-2}{2}} f \quad \text{on } \Omega, \tag{3.1}$$

where $\nabla_{x'}$ is the gradient in $\mathbb{R}^{n-1}$ and $\operatorname{div}_{x'}$ is the divergence in $\mathbb{R}^{n-1}$.

For $a \in \mathbb{R}$, we define the translation operator $\mathcal{T}_a : \mathcal{H}_+(\Omega \times (-\infty, a)) \to \mathcal{H}_+(\Omega \times (-\infty, 0))$ by

$$\mathcal{T}_a u(x) = C_1(a) u(x + a e_n),$$

where $e_n = (0, \dots, 0, 1)$ and the constant $C_1(a)$ is chosen such that

$$\sup_{\Omega\times(-\infty,0)} \mathcal{T}_a u(x) = 1.$$

Observe that if $u \in \mathcal{H}_+^{+\infty}(\Omega\times\mathbb{R})$, then $\mathcal{T}_a u \in \mathcal{H}_+^{+\infty}(\Omega\times\mathbb{R})$ for all $a \in \mathbb{R}$.

For a domain D and $E \subset \partial D$, we denote by $\omega_p(x, E, D)$ the p-harmonic measure evaluated at x of E in D. See [7, Section 11] for the definition and the property of p-harmonic measure. Let

$$v(x) = \omega_p(x, \Omega\times\{0\}, \Omega\times(-\infty, 0)).$$

Lemma 3.1. *There exists a positive constant ε_0 such that*

$$v(x - ae_n) \le (1 - \varepsilon_0 a)v(x) \quad \textit{for } x \in \Omega\times(-\infty, 0), \tag{3.2}$$

whenever $a \in [0, 1]$.

Proof. The maximum principle implies that

$$\sup_{\Omega\times\{-1\}} v < 1.$$

We set

$$V(x) = 1 + \varepsilon_0 x_n,$$

where

$$\varepsilon_0 = 1 - \sup_{\Omega\times\{-1\}} v.$$

Clearly V is p-harmonic in $\mathbb{R}^n$. Observe that $v \le V$ on $\Omega\times\{-1\}$. Since $v = V = 1$ on $\Omega\times\{0\}$, it follows from the comparison principle that

$$v \le V \quad \text{in } \Omega\times(-1, 0).$$

Hence we obtain that

$$v(x - ae_n) \le V(x - ae_n) = 1 - \varepsilon_0 a = (1 - \varepsilon_0 a)v(x) \quad \text{for } x \in \Omega\times\{0\},$$

whenever $a \in [0, 1]$. Applying the comparison principle, we obtain (3.2). □

By Lemma 2.7 and the comparison principle, there exist a positive increasing sequence $\{a_j\}$ and $v^* \in \mathcal{H}_+(\Omega\times(-\infty, 0)) \cap C^1(\overline{\Omega\times(-\infty, -1/2)})$ such that $a_j + 1 < a_{j+1}$, $v^* = 0$ on $(\partial\Omega\times(-\infty, -1/2)) \cup \{-\infty\}$ and

$$\frac{\mathcal{T}_{-a_j} v}{v^*} \to 1 \quad \text{uniformly in } \Omega\times(-\infty, -1).$$

By Lemma 3.1, we have

$$v^*(x - ae_n) \le (1 - \varepsilon_0 a)v^*(x),$$

whenever $a \in [0, 1]$. It follows that

$$\frac{\partial v^*}{\partial x_n}(x) \ge \varepsilon_0 v^*(x) > 0 \quad \text{for } x \in \Omega\times(-\infty, 0).$$

Applying Hopf's maximum principle (Lemma 2.1), we have

$$\nabla v^* \neq 0 \quad \text{in } \overline{\Omega \times (-\infty, -1/2)}.$$

Therefore Lemma 2.5 implies that $v^* \in C^{2,\alpha}(\overline{\Omega \times (-\infty, -1)})$.

Lemma 3.2. *For $a \geq 0$, we let*

$$\psi_1(a) = \inf_{\Omega\times(-\infty,-1)} \left\{ \frac{v^*(x - ae_n)}{v^*(x)} \right\}.$$

Then

$$v^*(x - ae_n) = \psi_1(a) v^*(x) \quad \text{for } x \in \Omega \times (-\infty, -1),$$

whenever $a \geq 0$.

Proof. Let $a \geq 0$ be fixed. Clearly we have

$$\psi_1(a) v^*(x) \leq v^*(x - ae_n) \quad \text{for } x \in \Omega \times (-\infty, -1).$$

Suppose that there exists a point $x_0 \in \Omega \times (-\infty, -1)$ such that

$$\psi_1(a) v^*(x_0) < v^*(x_0 - ae_n).$$

Since $v^* \in C^{2,\alpha}(\overline{\Omega \times (-\infty, -1)})$, the strong comparison principle (Lemma 2.2) implies that

$$\psi_1(a) v^*(x) < v^*(x - ae_n) \quad \text{for } x \in \Omega \times (-\infty, -1).$$

By Lemma 2.6, there exists a positive constant $\delta > 0$ such that

$$(1 + 2\delta)\psi_1(a) v^*(x) \leq v^*(x - ae_n) \quad \text{for } x \in \Omega \times \{-2\}.$$

Because $(\mathcal{T}_{-a_j} v)/v^* \to 1$ uniformly in $\Omega\times(-\infty, -1)$, there exists $N > 0$ such that if $j > N$, then

$$(1 + \delta)\psi_1(a) v(x) \leq v(x - ae_n) \quad \text{for } x \in \Omega \times \{-a_j - 2\},$$

so that

$$(1 + \delta)\psi_1(a) v(x) \leq v(x - ae_n) \quad \text{for } x \in \Omega \times (-\infty, -a_j - 2),$$

by the comparison principle. Since $a_j + 1 < a_{j+1}$, we have

$$(1 + \delta)\psi_1(a) \mathcal{T}_{a_{j+1}} v(x) \leq \mathcal{T}_{a_{j+1}} v(x - ae_n) \quad \text{for } x \in \Omega \times (-\infty, -1),$$

for $j > N$. By letting $j \to \infty$, we obtain that

$$(1 + \delta)\psi_1(a) v^*(x) \leq v^*(x - ae_n) \quad \text{for } x \in \Omega \times (-\infty, -1).$$

This is a contradiction to the definition of $\psi_1(a)$. Therefore we obtain that

$$v^*(x - ae_n) = \psi_1(a) v^*(x) \quad \text{for } x \in \Omega \times (-\infty, -1),$$

whenever $a \geq 0$. ◻

Observe that $\psi_1(0) = 1$ and $\psi_1(a)$ is a decreasing continuous function of $a \geq 0$. Moreover

$$\psi_1(a + a') = \psi_1(a)\psi_1(a') \quad \text{for } a, a' \geq 0,$$

since

$$\begin{aligned}\psi_1(a + a')v^*(x) &= v^*(x - (a + a')e_n) \\ &= v^*((x - ae_n) - a'e_n) \\ &= \psi_1(a')v^*(x - ae_n) \\ &= \psi_1(a)\psi_1(a')v^*(x) \quad \text{for } x \in \Omega \times (-\infty, -1),\end{aligned}$$

by Lemma 3.2. By an elementary calculation it follows from continuity of ψ_1 that

$$\psi_1(a) = \exp(-\lambda a) \quad \text{for } a \geq 0,$$

with $\lambda = -\log \psi_1(1) > 0$. By Lemma 3.2, we have

$$v^*(x) = \exp(\lambda(x_n - 1))v^*(x', -1) \quad \text{for } x \in \Omega \times (-\infty, -1).$$

Let $f(x') = v^*(x', -1)$ for $x' \in \Omega$. Since $v^* \in C^{2,\alpha}(\overline{\Omega \times (-\infty, -1)})$ and $\nabla v^* \neq 0$ in $\overline{\Omega \times (-\infty, -1/2)}$, we have $f \in C^{2,\alpha}(\overline{\Omega})$ and $\lambda^2 f^2 + |\nabla_{x'} f|^2 > 0$ in $\overline{\Omega}$. Since $v^* \in \mathcal{H}_+(\Omega \times (-\infty, 0))$ and $v^* = 0$ on $(\partial\Omega \times (-\infty, -1/2)) \cup \{-\infty\}$, it follows that λ and f satisfy (3.1) and $f = 0$ on $\partial\Omega$. Therefore we obtain the following lemma.

Lemma 3.3. *There exist a positive constant λ and a function $f \in C^{2,\alpha}(\overline{\Omega})$ such that λ and f satisfy (3.1), $\lambda^2 f^2 + |\nabla_{x'} f|^2 > 0$ in $\overline{\Omega}$ and $f = 0$ on $\partial\Omega$.*

Proof of Theorem 1.1. Since $\mathcal{H}_+^{-\infty}(\Omega \times \mathbb{R}) = \{u(x', -x_n) : u \in \mathcal{H}_+^{+\infty}(\Omega \times \mathbb{R})\}$, it is sufficient to prove (1.1). Let $u_0(x', x_n) = \exp(\lambda x_n) f(x')$, $x' \in \Omega$ and $x_n \in \mathbb{R}$, where λ and f are as in Lemma 3.3. Then we observe $u_0 \in \mathcal{H}_+^{+\infty}(\Omega \times \mathbb{R}) \cap C^{2,\alpha}(\overline{\Omega \times \mathbb{R}})$. Since

$$|\nabla u_0(x)| = \exp(\lambda x_n)(\lambda^2 f^2 + |\nabla_{x'} f|^2)^{1/2} \quad \text{for } x \in \overline{\Omega \times \mathbb{R}},$$

we have $\nabla u \neq 0$ in $\overline{\Omega \times \mathbb{R}}$. We will show that every $u \in \mathcal{H}_+^{+\infty}(\Omega \times \mathbb{R})$ is represented as $u = Cu_0$ with some positive constant C.

By Lemma 2.7 and the comparison principle, there exist a positive increasing sequence $\{a_j\}$ and $u^* \in \mathcal{H}_+(\Omega \times (-\infty, 0)) \cap C^1(\overline{\Omega \times (-\infty, -1/2)})$ such that $a_j + 1 < a_{j+1}$, and $u^* = 0$ on $(\partial\Omega \times (-\infty, -1/2)) \cup \{-\infty\}$ and

$$\frac{\mathcal{T}_{a_j} u}{u^*} \to 1 \quad \text{uniformly in } \Omega \times (-\infty, -1).$$

Let

$$\overline{C} = \sup_{\Omega \times (-\infty, -1)} \left\{ \frac{u^*}{u_0} \right\} \quad \text{and} \quad \underline{C} = \inf_{\Omega \times (-\infty, -1)} \left\{ \frac{u^*}{u_0} \right\}.$$

For an arbitrary $\varepsilon > 0$, there exists $N > 0$ such that if $j > N$, then

$$(1-\varepsilon)\exp(-\lambda a_j)C_1(a_j)^{-1}\underline{C}u_0 \le u \le (1+\varepsilon)\exp(-\lambda a_j)C_1(a_j)^{-1}\overline{C}u_0 \quad \text{in } \Omega\times(-\infty, a_j-1), \tag{3.3}$$

where

$$C_1(a_j)^{-1} = \sup_{\Omega\times(-\infty,a_j)} u.$$

It follows from the boundary Harnack principle (Lemma 2.3) that $0 < \underline{C} \le \overline{C} < \infty$. We claim $\overline{C} = \underline{C}$. We will prove the claim later and we finish the proof of Theorem 1.1. By (3.3), there exists a positive constant C such that

$$C^{-1} \le \exp(-\lambda a_j)C_1(a_j)^{-1} \le C.$$

Hence there exist a subsequence $\{b_j\}$ of $\{a_j\}$ and a positive constant K^* such that

$$K^* = \lim_{j\to\infty} \exp(-\lambda b_j)C_1(b_j)^{-1}. \tag{3.4}$$

Taking the subsequence $\{b_j\}$ in (3.3) and passing to the limit as $j \to \infty$ and then $\varepsilon \to 0$, we obtain

$$u = K^*\overline{C}u_0 \quad \text{in } \Omega\times\mathbb{R}.$$

Thus Theorem 1.1 is proved.

Finally we prove that $\underline{C} = \overline{C}$. Suppose that $\underline{C} < \overline{C}$. Since $u_0 \in C^{2,\alpha}(\overline{\Omega\times\mathbb{R}})$ and $\nabla u_0 \neq 0$ on $\overline{\Omega\times\mathbb{R}}$, it follows from the strong comparison principle (Lemma 2.2) and Lemma 2.6 that there exists a positive constant $\delta > 0$ such that

$$(\underline{C} + 2\delta)u_0 \le u^* \quad \text{on } \Omega\times\{-2\}.$$

By $(\mathcal{T}_{b_j}u)/u^* \to 1$ uniformly in $\Omega\times(-\infty,-1)$ and (3.4), we obtain that

$$K^*(\underline{C} + \delta)u_0 \le u \quad \text{on } \Omega\times\{b_j - 2\}.$$

The comparison principle implies that

$$K^*(\underline{C} + \delta)u_0(x) \le u(x) \quad \text{for } x \in \Omega\times(-\infty, b_j - 2).$$

Since $b_{j-1} + 1 \le b_j$, by letting $x = y + b_{j-1}e_n$ we obtain

$$K^*\exp(\lambda b_{j-1})C_1(b_{j-1})(\underline{C} + \delta)u_0(y) \le \mathcal{T}_{b_{j-1}}u(y) \quad \text{for } y \in \Omega\times(-\infty,-1).$$

By letting $j \to \infty$, we obtain

$$(\underline{C} + \delta)u_0(y) \le u^*(y) \quad \text{for } y \in \Omega\times(-\infty,-1).$$

This would contradict the definition of $\underline{C}$. Therefore we have $\underline{C} = \overline{C}$. □

4. Proof of Theorem 1.4

In this section, we explicitly calculate λ and f in case $n = 2$. We observe that $u(x_1, x_2) = \exp(\lambda x_2) f(x_1)$ is p-harmonic if and only if λ and $f(x_1)$ satisfy the equation

$$(p-1)\lambda^4 f^3 + (2p-3)\lambda^2 f f'^2 + \lambda^2 f^2 f'' + (p-1) f'^2 f'' = 0. \tag{4.1}$$

Proof of Theorem 1.4. Since $u(x_1, x_2) = \exp(\lambda x_2) f(x_1) \in \mathcal{H}_+^{+\infty}((0, L) \times \mathbb{R})$, it follows that λ and $f(x_1)$ satisfy (4.1). Then we obtain that

$$(p-1)(\lambda^2 f^2 + f'^2)(\lambda^2 f + f'') + (p-2)\lambda^2 (f(\lambda^2 f^2 + f'^2) - f^2(\lambda^2 f + f'')).$$

Multiplying by $2f'^2/(\lambda^2 f^2 + f'^2)^2$, we have

$$\begin{aligned}
&(p-1)\frac{2f'(\lambda^2 f + f'')}{\lambda^2 f^2 + f'^2} + (p-2)\lambda^2 \frac{2ff'(\lambda^2 f^2 + f'^2) - 2f^2 f'(\lambda^2 f + f'')}{(\lambda^2 f^2 + f'^2)^2} \\
=& \frac{d}{dx}\Big((p-1)\log(\lambda^2 f^2 + f'^2) + (p-2)\lambda^2 \frac{\lambda^2 f^2}{\lambda^2 f^2 + f'^2}\Big) \\
=& 0.
\end{aligned}$$

Hence there exists a constant C such that

$$(p-1)\log(\lambda^2 f^2 + f'^2) + (p-2)\lambda^2 \frac{\lambda^2 f^2}{\lambda^2 f^2 + f'^2} = C. \tag{4.2}$$

We introduce the Prüfer substitution [14, pp.239-242]

$$\begin{cases} \lambda f(x_1) = \rho(x_1) \sin s(x_1), \\ f'(x_1) = \rho(x_1) \cos s(x_1), \end{cases} \tag{4.3}$$

where $0 < \rho(x_1) < \infty$ and $0 \le s(x_1) \le \pi$. Since $f \in C^{2,\alpha}([0, L])$, we see that $\rho, s \in C^1([0, L])$. By (4.2), we obtain

$$\rho(x_1) = \exp\Big(\frac{C - \sin^2 s(x_1)}{2(p-1)}\Big). \tag{4.4}$$

On the other hand, (4.3) implies that

$$\lambda \rho \cos s = \lambda \frac{df}{dx_1} = \frac{d\rho}{dx_1} \sin s + \frac{ds}{dx_1} \rho \cos s.$$

By differentiating (4.4), we have

$$\begin{aligned}
\frac{d\rho}{dx_1} &= \exp\Big(\frac{C - \sin^2 s(x_1)}{2(p-1)}\Big) \frac{(p-1)\sin s \cos s}{p-2} \frac{ds}{dx_1} \\
&= \rho \frac{(p-1)\sin s \cos s}{p-2} \frac{ds}{dx_1}.
\end{aligned}$$

Hence we obtain

$$\frac{dx_1}{ds} = \frac{1}{\lambda}\Big(1 - \frac{p-2}{p-1}\sin^2 s\Big).$$

Then there exists a constant C^* such that

$$x_1(s) = C^* + \frac{1}{\lambda}\Big(\frac{p}{2(p-1)}s + \frac{p-2}{4(p-1)}\sin 2s\Big).$$

Since $x_1(s)$ is strictly increasing and $f(0) = f(L) = 0$, we see that $x_1(0) = 0$ and $x_1(\pi) = L$. Hence we have

$$C^* = 0, \quad \lambda = \frac{p\pi}{2(p-1)L}.$$

By letting $C = \log\lambda$ we obtain the parametric representation (1.3). Thus Theorem 1.4 is proved. □

5. Proof of Theorems 1.5 and 1.6

In this section, we show Theorems 1.5 and 1.6. By $\partial\Sigma$ we denote the relative boundary of Σ in the unit sphere. For $0 \le R_1 < R_2 \le \infty$, we define subsets $\Gamma(R_1, R_2)$ and $\Sigma(R_1)$ by

$$\Gamma(R_1, R_2) = \{(r, \sigma) : R_1 < r < R_2, \sigma \in \Sigma\},$$
$$\Sigma(R_1) = \Gamma \cap S(0, R_1).$$

Firstly, we consider $\mathcal{H}_+^\infty(\Gamma)$. We observe that $u(r, \sigma) = r^\mu g(\sigma)$ is p-harmonic if and only if μ and g satisfy the equation

(5.1)
$$-\operatorname{div}_\sigma[(\mu^2 g^2 + |\nabla_\sigma g|^2)^{\frac{p-2}{2}}\nabla_\sigma] = \mu(\mu(p-1)+n-p)(\mu^2 g^2 + |\nabla_\sigma g|^2)^{\frac{p-2}{2}} g \quad \text{on } \Sigma,$$

where ∇_σ is the covariant derivative identified with the tangential gradient and $\operatorname{div}_\sigma$ is the divergence operator acting on vector field on the unit sphere.

For $R > 0$, we define the stretching operator $\mathcal{S}_R^0 : \mathcal{H}_+(\Gamma(0, R)) \to \mathcal{H}_+(\Gamma(0, 1))$ by

$$\mathcal{S}_R^0 u(x) = C_2(R)u(Rx),$$

where the constant $C_2(R)$ is chosen such that

$$\sup_{\Gamma(0,1)} \mathcal{S}_R^0 u(x) = 1.$$

Observe that if $u \in \mathcal{H}_+^\infty(\Gamma)$, then $\mathcal{S}_R^0 u \in \mathcal{H}_+^\infty(\Gamma)$ for all $R > 0$. Let

$$v(x) = \omega_p(x, \Sigma(1), \Gamma(0, 1)),$$

where we recall that $\omega_p(\cdot, \Sigma(1), \Gamma(0, 1))$ is the p-harmonic measure of $\Sigma(1)$ in $\Gamma(0, 1)$.

Lemma 5.1. *There exists a positive constant ε_0 such that*

(5.2) $$v(Rx) \le (1 - \varepsilon_0(1 - R))v(x) \quad \textit{for } x \in \Gamma(0, 1),$$

whenever $R \in [1/2, 1]$.

Proof. The maximum principle implies that

$$\sup_{\Sigma(1/2)} v < 1.$$

For $b > 0$ and $C > 0$, we set

$$V(x) = 1 + C(e^{-b} - e^{-b|x|^2}).$$

Since

$$\Delta_p V(x) = -C(2be^{-b|x|^2})^{p-1}|x|^{p-2}(2b(p-1)|x|^2 - n - p + 2),$$

we can choose $b > 0$ such that V is p-superharmonic in $\Gamma(1/2, 1)$. Observe that if C is sufficiently small, then $v \le V$ on $\Sigma(1/2)$. Since $v = V = 1$ on $\Sigma(1)$, it follows from the comparison principle that

$$v \le V \quad \text{in } \Gamma(1/2, 1).$$

Hence if $\varepsilon_0 > 0$ is sufficiently small, then

$$\begin{aligned} v(Rx) &\le V(Rx) \\ &= 1 + C(e^{-b} - e^{-b|Rx|^2}) \\ &\le 1 - \varepsilon_0(1-R) \\ &= (1 - \varepsilon_0(1-R))v(x) \quad \text{for } x \in \Sigma(1), \end{aligned}$$

whenever $R \in [1/2, 1]$. Applying the comparison principle, we obtain (5.2). □

By Lemma 2.7 and the comparison principle, there exist a positive decreasing sequence $\{R_j\}$ and $v^* \in \mathcal{H}_+(\Gamma(0,1)) \cap C^1(\overline{\Gamma(0,3/4)} \setminus \{0\})$ such that $R_j/2 > R_{j+1}$, $v^* = 0$ on $\partial\Gamma(0,3/4) \setminus \Sigma(3/4)$ and

$$\frac{\mathcal{S}_R^0 v}{v^*} \to 1 \quad \text{uniformly in } \Gamma(0, 1/2).$$

By Lemma 5.1 we have

$$v^*(Rx) \ge (1 - \varepsilon_0(1-R))v^*(x) \quad \text{in } \Gamma(0,1),$$

whenever $R \in [1/2, 1]$. It follows that

$$\frac{\partial v^*}{\partial r}(x) \ge \varepsilon_0 v^*(x) > 0 \quad \text{for } x \in \Gamma(0,1).$$

By Lemma 2.1 we have

$$\nabla v^* \ne 0 \quad \text{in } \overline{\Gamma(0,3/4)} \setminus \{0\}.$$

It follows from Lemma 2.5 that $v^* \in C^{2,\alpha}(\overline{\Gamma(0,1/2)} \setminus \{0\})$.

In a way similar to the proof of Lemma 3.2, we obtain the following lemma.

Lemma 5.2. *For* $0 < R \le 1$, *we let*

$$\psi_2(R) = \inf_{\Gamma(0,1/2)} \left\{ \frac{v^*(Rx)}{v^*(x)} \right\}.$$

Then

$$v^*(Rx) = \psi_2(R) v^*(x) \quad \textit{for } x \in \Gamma(0,1/2),$$

whenever $0 < R \le 1$.

Observe that $\psi_2(1) = 1$ and $\psi_2(R)$ is an increasing continuous function of $0 < R \le 1$. Moreover, it follows from Lemma 5.2 that

$$\psi_2(RR') = \psi_2(R)\psi_2(R') \quad \text{for } 0 < R, R' \le 1.$$

By an elementary calculation, it follows from continuity of ψ_2 that

$$\psi_2(R) = R^\mu \quad \text{for } 0 < R \le 1,$$

with $\mu = -\log \psi_2(1/e) > 0$.

By Lemma 5.2, we have

$$v^*(r,\sigma) = (2r)^\mu v^*(\sigma/2) \quad \text{for } 0 < r < 1/2, \sigma \in \Sigma.$$

Let $g(\sigma) = v^*(\sigma/2)$ for $\sigma \in \Sigma$. Since $v^* \in C^{2,\alpha}(\overline{\Gamma(0,1/2)} \setminus \{0\})$ and $\nabla v^* \ne 0$ in $\overline{\Gamma(0,3/4)} \setminus \{0\}$, we have $g \in C^{2,\alpha}(\overline{\Sigma})$ and $\mu^2 g^2 + |\nabla_\sigma g|^2 > 0$ on $\overline{\Sigma}$. Since $v^* \in \mathcal{H}_+(\Gamma(0,1))$ and $v^* = 0$ on $\partial\Gamma(0,3/4) \setminus \Sigma(3/4)$, it follows that μ and g satisfy (5.1) and $g = 0$ on $\partial\Sigma$. Therefore we obtain the following lemma.

Lemma 5.3. *There exist a positive constant* μ *and a function* $g \in C^{2,\alpha}(\overline{\Sigma})$ *such that* μ *and* g *satisfy* (5.1), $\mu^2 g^2 + |\nabla_\sigma g|^2 > 0$ *in* $\overline{\Sigma}$ *and* $g = 0$ *on* $\partial\Sigma$.

Proof of Theorem 1.5. Let $u_0(r,\sigma) = r^\mu g(\sigma)$, $0 < r < \infty$ and $\sigma \in \Sigma$, where μ and g are as in Lemma 5.3. Then we observe $u_0 \in \mathcal{H}_+^\infty(\Gamma) \cap C^{2,\alpha}(\overline{\Gamma} \setminus \{0\})$. Since

$$|\nabla u_0(x)| = r^{\mu-1}(\mu^2 g^2 + |\nabla_\sigma g|^2)^{1/2} \quad \text{for } x \in \overline{\Gamma} \setminus \{0\},$$

we have $\nabla u_0 \ne 0$ in $\overline{\Gamma} \setminus \{0\}$. We will show that every $u \in \mathcal{H}_+^\infty(\Gamma)$ is represented as $u = Cu_0$ with some positive constant C.

By Lemma 2.7 and the comparison principle, there exist a positive increasing sequence $\{R_j\}$ and $u^* \in \mathcal{H}_+(\Gamma(0,1)) \cap C^1(\overline{\Gamma(0,3/4)} \setminus \{0\})$ such that $2R_j < R_{j+1}$, and $u^* = 0$ on $\partial\Gamma(0,3/4) \setminus \Sigma(3/4)$ and

$$\frac{\mathcal{S}^0_{R_j} u}{u^*} \to 1 \quad \text{uniformly in } \Gamma(0,1/2).$$

Let

$$\overline{C} = \sup_{\Gamma(0,1/2)} \left\{ \frac{u^*}{u_0} \right\} \quad \text{and} \quad \underline{C} = \inf_{\Gamma(0,1/2)} \left\{ \frac{u^*}{u_0} \right\}.$$

For an arbitrary $\varepsilon > 0$, there exists $N > 0$ such that if $j > N$, then

(5.3) $$(1-\varepsilon)R_j^{-\mu}C_2(R_j)^{-1}\underline{C}u_0 \le u \le (1+\varepsilon)R_j^{-\mu}C_2(R_j)^{-1}\overline{C}u_0 \quad \text{in } \Gamma(0, R_j/2),$$

where

$$C_2(R_j)^{-1} = \sup_{\Gamma(0,R_j)} u.$$

In a way similar to the proof of Theorem 1.1, we obtain $\underline{C} = \overline{C}$ and there exist a subsequence $\{r_j\}$ of $\{R_j\}$ and a positive constant K^* such that

$$K^* = \lim_{j\to\infty} r_j^{-\mu}C_2(r_j)^{-1}.$$

Taking the subsequence $\{r_j\}$ in (5.3) and passing to the limit as $j \to \infty$ and then $\varepsilon \to 0$, we obtain

$$u = K^*\overline{C}u_0 \quad \text{in } \Gamma.$$

Thus Theorem 1.5 is proved. □

Next we consider $\mathcal{H}_+^0(\Gamma)$. We observe that $u(r,\sigma) = r^{-\nu}h(\sigma)$ is p-harmonic if and only if ν and h satisfy the equation

(5.4)
$$-\operatorname{div}_\sigma[(\nu^2h^2 + |\nabla_\sigma h|^2)^{\frac{p-2}{2}}\nabla_\sigma] = \nu(\nu(p-1)-n+p)(\nu^2h^2+|\nabla_\sigma h|^2)^{\frac{p-2}{2}}h \quad \text{on } \Sigma,$$

where ∇_σ is the covariant derivative identified with the tangential gradient and $\operatorname{div}_\sigma$ is the divergence operator acting on vector field on the unit sphere.

For $R > 0$, we define the stretching operator $\mathcal{S}_R^\infty : \mathcal{H}_+(\Gamma(R,\infty)) \to \mathcal{H}_+(\Gamma(1,\infty))$ by

$$\mathcal{S}_R^\infty u(x) = C_3(R)u(Rx),$$

where the constant $C_3(R)$ is chosen such that

$$\sup_{\Gamma(1,\infty)} \mathcal{S}_R^\infty u(x) = 1.$$

Observe that if $u \in \mathcal{H}_+^0(\Gamma)$, then $\mathcal{S}_R^\infty u \in \mathcal{H}_+^0(\Gamma)$ for all $R > 0$. Let

$$v(x) = \omega_p(x, \Sigma(1), \Gamma(1,\infty)),$$

where we recall that $\omega_p(\cdot, \Sigma(1), \Gamma(1,\infty))$ is the p-harmonic measure of $\Sigma(1)$ in $\Gamma(1,\infty)$. In way similar to the proof of Lemma 5.1, we obtain the following lemma.

Lemma 5.4. *There exists a positive constant ε_0 such that*

$$v(Rx) \le (1-\varepsilon_0(R-1))v(x) \quad \text{for } x \in \Gamma(1,\infty),$$

whenever $R \in [1,2]$.

By Lemma 2.7 and the comparison principle, there exist a positive increasing sequence $\{R_j\}$ and $v^* \in \mathcal{H}_+(\Gamma(1,\infty)) \cap C^1(\overline{\Gamma(3/2,\infty)})$ such that $2R_j < R_{j+1}$, $v^* = 0$ on $(\partial\Gamma(3/2,\infty) \setminus \Sigma(3/2)) \cup \{\infty\}$ and

$$\frac{\mathcal{S}_R^\infty v}{v^*} \to 1 \quad \text{uniformly in } \Gamma(2,\infty).$$

By Lemma 5.4 we have

$$v^*(Rx) \le (1 - \varepsilon_0(R-1))v^*(x) \quad \text{in } \Gamma(1,\infty),$$

whenever $R \in [1,2]$. It follows that

$$\frac{\partial v^*}{\partial r}(x) \le -\varepsilon_0 v^*(x) < 0 \quad \text{for } x \in \Gamma(1,\infty).$$

By Lemma 2.1 we have

$$\nabla v^* \ne 0 \quad \text{in } \overline{\Gamma(3/2,\infty)}.$$

It follows from Lemma 2.5 that $v^* \in C^{2,\alpha}(\overline{\Gamma(2,\infty)})$.

In a way similar to the proof of Lemma 3.2, we obtain the following lemma.

Lemma 5.5. *For $R \ge 1$, we let*

$$\psi_3(R) = \inf_{\Gamma(2,\infty)} \left\{ \frac{v^*(Rx)}{v^*(x)} \right\}.$$

Then

$$v^*(Rx) = \psi_3(R) v^*(x) \quad \textit{for } x \in \Gamma(2,\infty),$$

whenever $R \ge 1$.

Observe that $\psi_3(1) = 1$ and $\psi_3(R)$ is a decreasing continuous function of $R \ge 1$. Moreover, it follows from Lemma 5.5 that

$$\psi_3(RR') = \psi_3(R)\psi_3(R') \quad \text{for } R, R' \ge 1.$$

By an elementary calculation, it follows from continuity of ψ_3 that

$$\psi_3(R) = R^{-\nu} \quad \text{for } R \ge 1,$$

with $\nu = -\log \psi_3(e) > 0$. By Lemma 5.5, we have

$$v^*(r,\sigma) = (r/2)^\nu v^*(2\sigma) \quad \text{for } 2 < r < \infty, \sigma \in \Sigma.$$

Let $h(\sigma) = v^*(2\sigma)$ for $\sigma \in \Sigma$. Since $v^* \in C^{2,\alpha}(\overline{\Gamma(2,\infty)})$ and $\nabla v^* \ne 0$ in $\overline{\Gamma(3/2,\infty)}$, we have $h \in C^{2,\alpha}(\overline{\Sigma})$ and $\nu^2 h^2 + |\nabla_\sigma h|^2 > 0$ on $\overline{\Sigma}$. Since $v^* \in \mathcal{H}_+(\Gamma(1,\infty))$ and $v^* = 0$ on $(\partial\Gamma(3/2,\infty) \setminus \Sigma(3/2)) \cup \{\infty\}$, it follows that ν and h satisfy (5.4) and $h = 0$ on $\partial\Sigma$. Therefore we obtain the following lemma.

Lemma 5.6. *There exist a positive constant ν and a function $h \in C^{2,\alpha}(\overline{\Sigma})$ such that ν and h satisfy* (5.4), *$\nu^2 h^2 + |\nabla_\sigma h|^2 > 0$ in $\overline{\Sigma}$ and $h = 0$ on $\partial\Sigma$.*

Proof of Theorem 1.6. Let $u_0(r,\sigma) = r^{-\nu}h(\sigma)$, $0 < r < \infty$ and $\sigma \in \Sigma$, where ν and h are as in Lemma 5.6. Then we observe $u_0 \in \mathcal{H}_+^0(\Gamma) \cap C^{2,\alpha}(\overline{\Gamma} \setminus \{0\})$. Since

$$|\nabla u_0(x)| = r^{-\nu-1}(\nu^2 h^2 + |\nabla_\sigma h|^2)^{1/2} \quad \text{for } x \in \overline{\Gamma} \setminus \{0\},$$

we have $\nabla u_0 \neq 0$ in $\overline{\Gamma}\setminus\{0\}$. We will show that every $u \in \mathcal{H}_+^0(\Gamma)$ is represented as $u = Cu_0$ with some positive constant C.

By Lemma 2.7 and the comparison principle, there exist a positive decreasing sequence $\{R_j\}$ and $u^* \in \mathcal{H}_+(\Gamma(1,\infty)) \cap C^1(\overline{\Gamma(3/2,\infty)})$ such that $R_j/2 > R_{j+1}$, and $u^* = 0$ on $(\partial\Gamma(3/2,\infty) \setminus \Sigma(3/2)) \cup \{\infty\}$ and

$$\frac{\mathcal{S}_{R_j}^{\infty} u}{u^*} \to 1 \quad \text{uniformly in } \Gamma(2,\infty).$$

Let

$$\overline{C} = \sup_{\Gamma(2,\infty)} \left\{ \frac{u^*}{u_0} \right\} \quad \text{and} \quad \underline{C} = \inf_{\Gamma(2,\infty)} \left\{ \frac{u^*}{u_0} \right\}.$$

For an arbitrary $\varepsilon > 0$, there exists $N > 0$ such that if $j > N$, then

$$(1-\varepsilon)R_j^{\nu}C_3(R_j)^{-1}\underline{C}u_0 \le u \le (1+\varepsilon)R_j^{\nu}C_3(R_j)^{-1}\overline{C}u_0 \quad \text{in } \Gamma(2R_j,\infty), \tag{5.5}$$

where

$$C_3(R_j)^{-1} = \sup_{\Gamma(R_j,\infty)} u.$$

In a way similar to the proof of Theorem 1.1, we obtain $\underline{C} = \overline{C}$ and there exist a subsequence $\{r_j\}$ of $\{R_j\}$ and a positive constant K^* such that

$$K^* = \lim_{j\to\infty} r_j^{\nu} C_3(r_j)^{-1}.$$

Taking the subsequence $\{r_j\}$ in (5.5) and passing to the limit as $j \to \infty$ and then $\varepsilon \to 0$, we obtain

$$u = K^*\overline{C}u_0 \quad \text{in } \Gamma.$$

Thus Theorem 1.6 is proved. □

6. Calculations of μ, ν, g and h for $n = 2$

In this section we explicitly calculate μ, ν, g and h for $n = 2$. Our method goes back to Aronsson [3], who studied p-harmonic functions in the whole plane $\mathbb{R}^2 \setminus \{0\}$ of the form $u(r,\sigma) = r^k F(\sigma)$ and gave a representation of F depending on k. Although he assumed $2 < p < \infty$, his technique is appliable for $1 < p \le 2$.

We introduce the spherical coordinates (r,θ) in $\mathbb{R}^2$ which are related to the coordinates $(x_1,x_2) \in \mathbb{R}^2 \setminus \{0\}$ by

$$x_1 = r\sin\theta, \quad x_2 = r\cos\theta,$$

where $0 < r < \infty$, $-\pi \le \theta < \pi$. For $0 < \phi < \pi$, we let

$$\Gamma_\phi = \{(r,\theta) : |\theta| < \phi\}.$$

For simplicity, we let

$$\kappa = \frac{p-2}{p-1}.$$

Firstly we consider μ and g. Since $u(r,\theta) = r^\mu g(\theta) \in \mathcal{H}_+^\infty(\Gamma_\phi)$, we have

$$\int_{-\phi}^{\phi} (\mu^2 g^2 + g'^2)^{\frac{p-2}{2}} g'^2 d\theta = \mu(\mu(p-1) + 2 - p) \int_{-\phi}^{\phi} (\mu^2 g^2 + g'^2)^{\frac{p-2}{2}} g^2 d\theta.$$

Hence we obtain $\mu - \kappa > 0$. Define the function $\theta : [-\pi/2, \pi/2] \to [-\phi, \phi]$ by

$$\theta(s) = s + \frac{1-\mu}{\sqrt{\mu(\mu-\kappa)}} \arctan\Big(\sqrt{\frac{\mu}{\mu-\kappa}} \tan s\Big).$$

Since

$$\frac{d\theta}{ds} = \frac{1-\kappa\cos^2 s}{\mu - \kappa\cos^2 s},$$

we see that $\theta(s)$ is strictly increasing and there exists the inverse function $s(\theta)$. Let

$$u_1(r,\theta) = r^\mu\Big(1 - \frac{\kappa}{\mu}\cos^2 s(\theta)\Big)^{\frac{\mu-1}{2}} \cos s(\theta).$$

Then we observe that $u_1(r,\theta)$ is p-harmonic in Γ_ϕ (see [3]).

Proposition 6.1. *Let $n = 2$ and $\Gamma = \Gamma_\phi$. Then μ and g in Theorem 1.5 is given by*

$$\begin{cases} \mu = \dfrac{2\pi^2 - \kappa(\pi-2\phi)^2 + (\pi-2\phi)\sqrt{4\pi^2(1-\kappa) + \kappa^2(\pi-2\phi)^2}}{8(\pi-\phi)\phi}, \\ g(\theta) = \Big(1 - \dfrac{\kappa}{\mu}\cos^2 s(\theta)\Big)^{(\mu-1)/2} \cos s(\theta), \end{cases}$$

where $s(\theta)$ is given as above.

Proof. It is sufficient to calculate μ. Since $\theta(s)$ is strictly increasing, it follows from $u_1 \in \mathcal{H}_+^\infty(\Gamma_\phi)$ that $\theta(\pi/2) = \phi$. Therefore μ satisfies the equation

$$\frac{\pi}{2} + \frac{1-\mu}{\sqrt{\mu(\mu-\kappa)}} \cdot \frac{\pi}{2} = \phi. \tag{6.1}$$

Squaring and rewriting give

$$4(\pi-\phi)\phi\mu^2 - [2\pi^2 - \kappa(\pi-2\phi)^2]\mu + \pi^2 = 0.$$

The roots of this quadratic equation are

$$\mu_1 = \frac{2\pi^2 - \kappa(\pi-2\phi)^2 + |\pi-2\phi|\sqrt{4\pi^2(1-\kappa) + \kappa^2(\pi-2\phi)^2}}{8(\pi-\phi)\phi}$$

and

$$\mu_2 = \frac{2\pi^2 - \kappa(\pi - 2\phi)^2 - |\pi - 2\phi| \sqrt{4\pi^2(1-\kappa) + \kappa^2(\pi - 2\phi)^2}}{8(\pi - \phi)\phi}.$$

Observe that $\kappa < \mu_2 \leq 1 \leq \mu_1$. It follows from (6.1) that if $0 < \phi \leq \pi/2$, then $\mu \geq 1$; if $\pi/2 \leq \phi < \pi$, then $\mu \leq 1$. Hence we obtain

$$\mu = \frac{2\pi^2 - \kappa(\pi - 2\phi)^2 + (\pi - 2\phi) \sqrt{4\pi^2(1-\kappa) + \kappa^2(\pi - 2\phi)^2}}{8(\pi - \phi)\phi}.$$

□

Next we consider ν and h. Since $u(r,\theta) = r^{-\nu}h(\theta) \in \mathcal{H}_+^0(\Gamma)$, we have

$$\int_{-\phi}^{\phi} (\nu^2 h^2 + h'^2)^{\frac{p-2}{2}} h'^2 d\theta = \nu(\nu(p-1) - 2 + p) \int_{-\phi}^{\phi} (\nu^2 h^2 + h'^2)^{\frac{p-2}{2}} h^2 d\theta.$$

Hence, we have $\nu + \kappa > 0$. Define the function $\theta : [-\pi/2, \pi/2] \to [-\phi, \phi]$ by

$$\theta(t) = t - \frac{1+\nu}{\sqrt{\nu(\nu+\kappa)}} \arctan\Big(\sqrt{\frac{\nu}{\nu+\kappa}} \tan t \Big).$$

We see that $\theta(t)$ is strictly decreasing and there exists the inverse function $t(\theta)$. Let

$$u_2(r,\theta) = r^{\nu}\Big(1 + \frac{\kappa}{\nu}\cos^2 t(\theta)\Big)^{-\frac{\nu+1}{2}} \cos t(\theta).$$

Then we observe that $u_2(r,\theta)$ is p-harmonic in Γ_ϕ (see [3]).

Proposition 6.2. *Let $n = 2$ and $\Gamma = \Gamma_\phi$. Then ν and h in Theorem 1.6 is given by*

$$\begin{cases} \nu = \dfrac{2\pi^2 - \kappa(\pi + 2\phi)^2 + (\pi + 2\phi) \sqrt{4\pi^2(1-\kappa) + \kappa^2(\pi + 2\phi)^2}}{8(\pi + \phi)\phi}, \\ h(\theta) = \Big(1 + \dfrac{\kappa}{\nu}\cos^2 t(\theta)\Big)^{(-\nu-1)/2} \cos t(\theta), \end{cases}$$

where $t(\theta)$ is given as above.

Proof. It is sufficient to calculate ν. Since $\theta(t)$ is strictly decreasing, it follows from $u_2 \in \mathcal{H}_+^0(\Gamma_\phi)$ that $\theta(\pi/2) = -\phi$. Therefore ν satisfies the equation

$$\frac{\pi}{2} - \frac{1+\nu}{\sqrt{\nu(\nu+\kappa)}} \cdot \frac{\pi}{2} = -\phi.$$

Squaring and rewriting give

$$4(\pi + \phi)\phi\nu^2 + [-2\pi^2 + \kappa(\pi + 2\phi)^2]\nu - \pi^2 = 0.$$

The roots of this quadratic equation are

$$\nu_1 = \frac{2\pi^2 - \kappa(\pi + 2\phi)^2 + (\pi + 2\phi) \sqrt{4\pi^2(1-\kappa) + \kappa^2(\pi + 2\phi)^2}}{8(\pi + \phi)\phi} > 0$$

and

$$\nu_2 = \frac{2\pi^2 - \kappa(\pi + 2\phi)^2 - (\pi + 2\phi)\sqrt{4\pi^2(1-\kappa) + \kappa^2(\pi + 2\phi)^2}}{8(\pi + \phi)\phi} < 0.$$

Since $\nu > 0$, we obtain

$$\nu = \frac{2\pi^2 - \kappa(\pi + 2\phi)^2 + (\pi + 2\phi)\sqrt{4\pi^2(1-\kappa) + \kappa^2(\pi + 2\phi)^2}}{8(\pi + \phi)\phi}.$$

□

Remark 6.3. Dobrowolski [5] gave μ but not g. Lundström-Vasilis [12] calculated ν and h for case $p > 2$ in the same way as in the proof of Proposition 6.2. On the other hand, for case $1 < p < 2$, they considered $p/(p-1)$-harmonic stream functions and so they obtained the explicit representation of ν and h. See [4] for details of stream functions.

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