ON THE UNIVALENCY OF CERTAIN POWER SERIES

By
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I.

In this paragraph we consider a family of functions

\[ f(z) = a_1z + a_2z^2 + \ldots + a_nz^n + \ldots \]

with the following properties:

1. \( a_1z + a_2z^2 + \ldots + a_nz^n + \ldots \) is convergent for \( |z| < 1 \).
2. \( |a_1| \geq a > 0 \), (\( a \) is fixed, positive),
3. \( \sum_{n=1}^{\infty} \gamma_n |a_n|^2 \leq 1 \), where the \( \gamma_n \)'s are given and positive, (\( n = 1, 2, 3, \ldots \)), and \( \lim_{n\to\infty} \frac{2n-2}{\sqrt[2n]{\gamma_n}} \leq 1 \).

Our object is to give some complements to a theorem of T. Itihara.(1) This theorem can be stated as follows: \( f(z) \) is univalent (schlicht) for \( |z| < \rho_0 \), if the equation \( \frac{a^2}{1-\gamma_1a^2} = \psi(\rho) \) has a root \( \rho_0 \) in the interval \((0, 1)\), where \( \psi(\rho) = \sum_{n=1}^{\infty} \frac{n^2\rho^{2n-2}}{\gamma_n} \).

**Theorem 1.** The derivative \( f'(z) \) does not vanish for \( |z| < 1 \), if the equation \( \frac{a^2}{1-\gamma_1a^2} = \psi(\rho) \) has no root in the interval \((0, 1)\) and for \( |z| < \rho_0 \), if it has a root \( \rho_0 \) there. In the latter case \( \rho_0 \) cannot be replaced by any greater number, for

\[ f_0(z) = -az + \lambda \sum_{n=2}^{\infty} \frac{\rho_0^{n-1}nz^n}{\gamma_n}, \quad \text{where} \quad \lambda = \frac{1-\gamma_1a^2}{a}, \]

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belongs to the family and \( f_0'(\rho_0) = 0 \). And moreover \( f'(z) \) has a zero on the circle \( |z| = \rho_0 \) when and only when

\[
(5) \quad f(z) = e^{i\theta}f_0(e^{-i\varphi}z).
\]

**Proof.** \[ |f'(z)| = \left| \sum_{n=1}^{\infty} na_n z^{n-1} \right| \]

\[ \geq |a_1| - \sum_{n=2}^{\infty} n |a_n| \rho^{n-1} \quad (0 \leq |z| = \rho < 1) \]

\[ = |a_1| - \sum_{n=2}^{\infty} \sqrt{\gamma_n} |a_n| \cdot \frac{n \rho^{n-1}}{\sqrt{\gamma_n}} \]

\[ \geq |a_1| - \sqrt{\sum_{n=2}^{\infty} \gamma_n |a_n|^2} \sqrt{\sum_{n=2}^{\infty} \frac{n^2 \rho^{2n-2}}{\gamma_n}} \]

\[ \geq |a_1| - \sqrt{1 - \gamma_1 a^2} \sqrt{\psi(\rho)} \]

\[ \geq a - \sqrt{1 - \gamma_1 a^2} \sqrt{\psi(\rho)}. \]

Since \( \psi(\rho) \) is an increasing function of \( \rho \) in the interval \((0, 1)\) its value at the origin being zero, either the equation \( \frac{a^2}{1 - \gamma_1 a^2} = \psi(\rho) \) has no root in the interval \((0, 1)\), or it has only one there. In the former case \( f'(z) \) never vanishes for \( |z| < 1 \) and in the latter for \( |z| < \rho_0 \), as \( \psi(\rho) < \psi(\rho_0) = \frac{a^2}{1 - \gamma_1 a^2} \), if \( \rho < \rho_0 \).

Next we suppose that there exists a function \( f(z) \) of the family whose derivative has a zero on the circle \( |z| = \rho_0 \), which may be assumed without loss of generality to be \( z = \rho_0 \). Put \( z = \rho_0 \) in (6). Then, from \( f'(\rho_0) = 0 \), it is easily seen that the signs of inequality must all be removed from the relations (6). Therefore in order that \( f'(\rho_0 e^{i\varphi}) = 0 \), it is necessary that \( f(z) \) should be of the form (5). Conversely \( f(z) = e^{i\theta}f_0(e^{-i\varphi}z) \) belongs to the family, for it is convergent for \( |z| < \frac{1}{\rho_0} \), by \( \lim_{n \to \infty} \frac{2n-2}{\gamma_n} \) \( \leq 1 \) and it is clear that \( \sum_{n=1}^{\infty} \gamma_n |a_n|^2 \leq 1 \) and \( f''(\rho_0 e^{i\varphi}) = 0 \).
From theorem 1 it follows at once:

Theorem 2. In Itihara's theorem $\rho_0$ can not be replaced by any greater number.

Some special cases of theorem 2 will be mentioned:

Theorem 3. $\gamma_n = 1 \ (n = 1, 2, \ldots)$. Then

$$\rho_0 = \left[1 - (1 - a^2)^{\frac{1}{3}} \left\{ \left( 1 + \sqrt{1 + \frac{1 - a^2}{27}} \right)^{\frac{1}{3}} + \left( 1 - \sqrt{1 + \frac{1 - a^2}{27}} \right)^{\frac{1}{3}} \right\} \right]^{\frac{1}{3}}$$

and

$$f_0(z) = \frac{z}{a} \left\{ \frac{1 - a^2}{(1 - \rho_0 z)^2} - 1 \right\}.$$

Theorem 4. $\gamma_n = n \ (n = 1, 2, \ldots)$. Then

$$\rho_0 = (1 - \sqrt{1 - a^2})^{\frac{1}{2}} \quad \text{and} \quad f_0(z) = -\frac{z(a^2 - \rho_0 z)}{a(1 - \rho_0 z)}.$$

Theorem 5. $\gamma_n = n^2 \ (n = 1, 2, \ldots)$. Then

$$\rho_0 = a \quad \text{and} \quad f_0(z) = \int_0^z \frac{z - a}{1 - az} \, dz.$$

II.

Recently J. Dieudonné has proved the following

Theorem 6. Let $w = f(z) = z + \ldots$ be convergent for $|z| < 1$ and $|f(z)| < M$ for $|z| < 1$. Then

1. $f(z)$ is univalent for $|z| < M - \sqrt{M^2 - 1}$ (1),

2. $f(z)$ maps $|z| < M - \sqrt{M^2 - 1}$ on a star-domain with respect to $w = 0$ (2).


(2) J. DIEUDONNÉ: l. c.
the inverse function $z = w + \ldots$ is regular for $|w| < M(M - \sqrt{M^2 - 1})^2$.

(4) the limits in (1), (2) and (3) are attained by the function

$$f(z) = \frac{Mz(1-Mz)}{M-z}.$$ 

Replacing the hypothesis $|f(z)| < M$ by $|f'(z)| < M$, we obtain

**Theorem 7.** Let $w = f(z) = z + \ldots$ be convergent for $|z| < 1$ and $|f'(z)| < M$ for $|z| < 1$. Then

(1) $f(z)$ is univalent for $|z| < \frac{1}{M}$.

(2) $f(z)$ maps $|z| < M - \sqrt{M^2 - 1}$ on a convex domain with respect to $w = 0$.

(3) the inverse function $z = w + \ldots$ is regular for $|w| < M(1 + (M^2 - 1) \log \left(1 - \frac{1}{M^2}\right))$.

(4) the limits in (1), (2) and (3) are attained by the function

$$f(z) = M \int_0^z \frac{1-Mz}{M-z} dz = M\left(Mz + (M^2-1) \log \left(1 - \frac{z}{M}\right)\right).$$

Proof. (1) Consider $\varphi(z) = \frac{f(z)}{M} = \frac{1}{M}z + \frac{a_2}{M}z^2 + \ldots + \frac{a_n}{M}z^n + \ldots$.

Then $\varphi(z)$ satisfies the conditions of theorem 5, for $\varphi(z)$ is regular and $|\varphi'(z)| < 1$ for $|z| < 1$ and Gutzmer's inequality gives $\sum_{n=1}^{\infty} n^2 \left|\frac{a_n}{M}\right|^2 \leq 1$, $(a_1 = 1)$. Hence $\varphi(z)$ is univalent for $|z| < \frac{1}{M}$.

(2) The proof is entirely similar to Dieudonné's for (2) in theorem 6 (a).

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(2) J. Dieudonné: L. c.
(3) By (1) $f(z)$ is univalent for $|z| < \frac{1}{M}$ and

$$|f(z)| \geq M \left(1 + (M^2 - 1) \log \left(1 - \frac{1}{M^2}\right)\right),$$

if $|z| = \frac{1}{M}$. Hence the inverse function $z = w + \ldots$ is regular for $|w| < M \left(1 + (M^2 - 1) \log \left(1 - \frac{1}{M^2}\right)\right)$. Lastly, $f(z) = M(Mz + (M^2 - 1) \log \left(1 - \frac{z}{M}\right))$ has a vanishing derivative at $z = \frac{1}{M}$ and $f(\frac{1}{M}) = M \left(1 + (M^2 - 1) \log \left(1 - \frac{1}{M^2}\right)\right)$.

Sapporo
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(1) E. Landau: l. c.