ON THE UNIVALENCY OF CERTAIN ANALYTIC FUNCTIONS

By

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The object of this paper is to give some generalizations on and complements to Mr. J. DIEUDONNÉ's results concerning the univalency (*Schlichtheit*) of a bounded function(1). First, assuming that $\varphi(z) = c_0 + \ldots.,\ c_0 \neq 0$, is regular and $\varphi(z) \subset D$ in the unit circle(2), where D is a given connected domain, we obtain a general theorem on the univalency of the function $f(z) \equiv z\varphi(z)$, with the aid of Prof. KAKEYA's principle(3). Next we state some applications of this theorem, taking some special domains as D. Finally we investigate the univalency of the function $f(z) = c_0 z + c_1 z^2 + \ldots.,\ c_0,\ c_1$ given, $c_0 \neq 0$, which is regular and bounded in the unit circle: $|f(z)| < 1$.

§ I.

Let $\varphi(z) = c_0 + \ldots.,\ c_0 \neq 0$, be regular and $\varphi(z) \subset D$ in the unit circle, where D is a given connected domain. We will here enunciate a theorem on the univalency of the function $f(z) \equiv z\varphi(z)$.

First we consider a function

$$\psi(z) = \alpha + \beta z + \ldots., \tag{1}$$

which is regular and $\psi(z) \subset D$ in the unit circle. Let us denote by $g(z)$ a function, which maps D conformally on the unit circle. If we

(1) J. DIEUDONNÉ: Ann. d. l'Éc. Norm. Sup., **48**, 1931. Cf. p. 349–351.

(2) The notation $\varphi(z) \subset D$ means that the set of values taken by $\varphi(z)$ in the unit circle belongs to the domain D.

(3) S. KAKEYA: Sci. Rep. of Tokyo Bunrika Daigaku, Section A, I, No. 14, 1931, p. 159–165.

take a branch of $g(z)$, we obtain a function $G(z)$ defined by

$$G(z) \equiv g\{\psi(z)\}, \tag{2}$$

which is uniform, regular in the unit circle and has the following properties:

(i) $|G(z)|<1$ for $|z|<1$, (ii) $G(0)=g(a)$, $G'(0)=\beta g'(a)$.

Hence, applying a theorem of E. LANDAU(4), we have

$$|g(a)|^2+|\beta|\,|g'(a)|\leqq 1$$

or

$$|\beta|\leqq\frac{1-|g(a)|^2}{|g'(a)|}\equiv\Omega(a,D), \tag{3}$$

since $g'(a)\neq 0$. Here we must remark that the positive quantity $\Omega(a, D)$ depends only on a and D, and does not depend either on the selection of a mapping function $g(z)$ or on the selection of a branch of $g(z)$(5).

Next consider the function

$$\psi(\zeta)=\varphi\left(\frac{z_0-\zeta}{1-\bar{z}_0\zeta}\right)=\varphi(z_0)-(1-|z_0|^2)\varphi'(z_0)\zeta+\ldots\ldots, \quad (|z_0|<1). \tag{4}$$

Since $\psi(\zeta)$ is regular and $\psi(\zeta)\subset D$ for $|\zeta|<1$, if we put

$$a=\varphi(z_0), \quad \beta=-(1-|z_0|^2)\varphi'(z_0),$$

we obtain, from (3),

$$|\varphi'(z_0)(1-|z_0|^2)|\leqq\Omega(\varphi(z_0), D).$$

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(4) S. KAKEYA: l. c. p. 161.

(5) Let $g_1(z)$ and $g_2(z)$ be two mapping functions. Then there exists a relation $g_2(z)=L(g_1(z))$, where $L(z)$ is a function of the form $e^{i\theta}\frac{z-\gamma}{\bar{\gamma}z-1}$, $|\gamma|<1$, and so $\frac{1-|g_1(\alpha)|^2}{|g_1'(\alpha)|}=\frac{1-|g_2(\alpha)|^2}{|g_2'(\alpha)|}$. By an entirely similar discussion, we see that $\Omega(\alpha, D)$ does not also depend on the selection of a branch of $g(z)$. Cf. BIEBERBACH: Lehrbuch der Funktionentheorie, II, 2. Aufl., p. 45–46.

Consequently, if $\varphi(z_0) \neq 0$, we have

$$\left| z_0 \frac{\varphi'(z_0)}{\varphi(z_0)} \right| \leqq \frac{|z_0|}{1-|z_0|^2} \frac{\Omega(\varphi(z_0), D)}{|\varphi(z_0)|} . \tag{5}$$

On the other hand, it is sufficient that

$$\left| z \frac{\varphi'(z)}{\varphi(z)} \right| < 1 \quad \text{for} \quad |z| < R , \tag{6}$$

to show that $f(z) \equiv z\varphi(z)$ is univalent and starshaped for $|z| < R \leqq 1$[6].

Thus we obtain a general theorem:

Theorem 1. *Let* $\varphi(z) = c_0 + \ldots$, $c_0 \neq 0$, *be regular and* $\varphi(z) < D$ *in the unit circle, where* D *is a given connected domain. Let* $g(z)$ *be an arbitrary branch of a function mapping* D *conformally on the unit circle, and put* $\Omega(a, D) = \dfrac{1-|g(a)|^2}{|g'(a)|}$, *where* a *is a point in* D. *Then* $f(z) \equiv z\varphi(z)$ *is univalent and starshaped with respect to the origin for* $|z| < R$, *provided that*

$$\frac{|z|}{1-|z|^2} \frac{\Omega(\varphi(z), D)}{|\varphi(z)|} < 1 \quad \text{for} \quad |z| < R . \tag{7}$$

§ II.

We consider a circular domain D: $|z| < M$. Since $g(z) = \dfrac{z}{M}$ is a mapping function, it is easy to see that

$$\Omega(a, D) = \frac{M^2 - |a|^2}{M} \quad \text{and} \quad \left| \frac{z\varphi'(z)}{\varphi(z)} \right| \leqq \frac{r}{1-r^2} \frac{1 - \left| \frac{\varphi(z)}{M} \right|^2}{\left| \frac{\varphi(z)}{M} \right|}$$

$$\text{for} \quad |z| = r\text{[7]} , \tag{8}$$

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(6) The following theorem is known: Suppose that $f(z) = c_0 z + \ldots$, $c_0 \neq 0$, is regular in the unit circle. Then $f(z)$ is univalent and starshaped with respect to the origin for $|z| < R$, provided that

$$\Re\left[z \frac{f'(z)}{f(z)} \right] > 0 \quad \text{for} \quad |z| < R \leqq 1 .$$

Cf. A. KOBORI: Memoirs of the College of Science, Kyoto Imperial University, s. A, 1932, p. 279–291.

(7) Hereafter we often denote $|z|$ by r.

if $\varphi(z)$ does not vanish at a point z within the unit circle. Thus

Theorem 2. *Let* $\varphi(z) = c_0 + \ldots$, $c_0 \neq 0$, *be regular and bounded in the unit circle:* $|\varphi(z)| < M$. *Then* $f(z) \equiv z\varphi(z)$ *is univalent and starshaped with respect to the origin for* $|z| < R$, *if*

$$\left|\frac{\varphi(z)}{M}\right| > |z| \quad \textit{for} \quad |z| < R.$$

Particularly, if we put $c_0 = 1$ in theorem 2, then we easily get

$$\frac{1-Mr}{M-r} \leqq \left|\frac{\varphi(z)}{M}\right| \leqq \frac{1+Mr}{M+r} \quad \text{for} \quad |z| = r \tag{9}$$

and so it is sufficient that r should be less than $R = M - \sqrt{M^2-1}$ in order that $\left|\frac{\varphi(z)}{M}\right| > |z|$. Thus we obtain DIEUDONNÉ's theorem:

Theorem 3. *Suppose that* $f(z) = z + \ldots.$ *is regular and bounded in the unit circle:* $|f(z)| < M$. *Then* $f(z)$ *is univalent and starshaped with respect to the origin for* $|z| < M - \sqrt{M^2-1}$. *This limit can be attained by the function* $f_0(z) = \dfrac{Mz(1-Mz)}{M-z}$.

Next consider a half-plane D: $\Re[z] > 0$. Taking a mapping function $g(z) = \dfrac{1-z}{1+z}$, we have

$$\Omega(a, D) = 2\Re[a] \quad \text{and} \quad \left|z\frac{\varphi'(z)}{\varphi(z)}\right| \leqq \frac{2r}{1-r^2}\frac{\Re[\varphi(z)]}{|\varphi(z)|} \leqq \frac{2r}{1-r^2}$$

$$\text{for} \quad |z| = r. \tag{10}$$

From this we get

Theorem 4. *Let* $\varphi(z)$ *be regular and* $\Re[\varphi(z)] > 0$ *for* $|z| < 1$. *Then* $f(z) \equiv z\varphi(z)$ *is univalent and starshaped for* $|z| < \sqrt{2} - 1$. *This limit can be attained by the function* $f_0(z) = cz\dfrac{1-z}{1+z}$, $(c > 0)$.

§ III.

We now consider a pricked (*punktiert*) circular domain D: $0 < |z| < M$. A function which maps D on the unit circle is

$$g(z) = \frac{1+\log\frac{z}{M}}{1-\log\frac{z}{M}}.$$

By a short calculation, we have

$$\Omega(\alpha, D) = 2\,|\alpha| \log \frac{M}{|\alpha|} \quad \text{and} \quad \left| z\frac{\varphi'(z)}{\varphi(z)} \right| \leqq \frac{2r}{1-r^2} \log \left| \frac{M}{\varphi(z)} \right|$$
$$\text{for} \quad |z| = r. \tag{11}$$

Hence we easily obtain

Theorem 5. *Let $\varphi(z) = c_0 + \ldots.$ be regular for $|z| < 1$ and $0 < |\varphi(z)| < M$ for $0 < |z| < 1$. Then $f(z) \equiv z\varphi(z)$ is univalent and starshaped with respect to the origin for $|z| < r$, if*

$$\left| \frac{\varphi(z)}{M} \right| \geqq e^{-\frac{1-r^2}{2r}} \quad \textit{for} \quad |z| = r.$$

Put $c_0 = 1$ in theorem 5 and consider the function

$$F(z) = \frac{\log\frac{\varphi(z)}{M} - \log\frac{1}{M}}{\log\frac{\varphi(z)}{M} + \log\frac{1}{M}},$$

taking a branch of $\log z$ such that $\log 1 = 0$. Since $F(z)$ is regular and $|F(z)| < 1$ in the unit circle, $F(0)$ being equal to 0, SCHWARZ's lemma gives $|F(z)| \leqq |z|$, whence we have

$$M^{-\frac{1+r}{1-r}} \leqq \left| \frac{\varphi(z)}{M} \right| \leqq M^{-\frac{1-r}{1+r}} \quad \text{for} \quad |z| = r. \tag{12}$$

Thus we can enunciate a theorem analogous to DIEUDONNÉ's.

Theorem 6. *Let $f(z)=z+\dots$ be regular for $|z|<1$ and $0<|f(z)|<M$ for $0<|z|<1$. Then*

(1) *$f(z)$ maps $|z|<R=\log eM-\sqrt{(\log eM)^2-1}$ on a starshaped domain Δ with respect to the origin.*

(2) *If we denote by d the shortest distance of the origin from the boundary of Δ, then $d\geqq\left\{\log eM-\sqrt{(\log eM)^2-1}\right\}M^{-\frac{2}{\log M+\sqrt{\log M(2+\log M)}}}$.*

(3) *These limits (1) and (2) are attained by the function $f_0(z)=zM^{\frac{2e^{i\theta}z}{1+e^{i\theta}z}}$.*

Proof. From (11) and (12) we have

$$\left|z\frac{\varphi'(z)}{\varphi(z)}\right|\leqq\frac{2r}{(1-r)^2}\log M \quad \text{for} \quad |z|=r. \tag{13}$$

Since the right hand side is an increasing function of r for $0<r<1$, the equation $\frac{2r}{(1-r)^2}\log M=1$ has just one root $R=\log eM-\sqrt{(\log eM)^2-1}$ there. From (12) we have $|f(Re^{i\theta})|\geqq RM^{-\frac{2R}{1-R}}$. Our theorem is completely proved, considering the function

$$f_0(z)=zM^{\frac{2e^{i\theta}z}{1+e^{i\theta}z}},$$

which has

$$f_0(-Re^{-i\theta})=-Re^{-i\theta}M^{-\frac{2R}{1-R}},\ f_0'(-Re^{-i\theta})=\left(1-\frac{2R}{(1-R)^2}\log M\right)M^{-\frac{2R}{1-R}}=0.$$

Lastly we consider a ring-domain D: $m<|z|<M$, which can be mapped on the unit circle by a function

$$g(z)=\left(e^{i\frac{\pi}{2}\frac{\log\frac{z}{\sqrt{mM}}}{\log\sqrt{\frac{M}{m}}}}-1\right):\left(e^{i\frac{\pi}{2}\frac{\log\frac{z}{\sqrt{mM}}}{\log\sqrt{\frac{M}{m}}}}+1\right).$$

In this case our discussion gives

$$\left| z\frac{\varphi'(z)}{\varphi(z)} \right| \leqq \frac{2}{\pi}\frac{r}{1-r^2}\left(\log\frac{M}{m}\right)\cos\frac{\pi\log\frac{|\varphi(z)|}{\sqrt{mM}}}{\log\frac{M}{m}} \quad \text{for} \quad |z| = r\,.$$

Consequently we obtain

Theorem 7. *Let $\varphi(z)$ be regular and $m < |\varphi(z)| < M$ for $|z| < 1$. Then $f(z) \equiv z\varphi(z)$ is univalent and starshaped for $|z| < r$ if*

$$\cos\frac{\pi}{2}\frac{\log\frac{|\varphi(z)|^2}{mM}}{\log\frac{M}{m}} \leqq \frac{\pi}{2}\frac{\frac{1-r^2}{r}}{\log\frac{M}{m}} \quad \textit{for} \quad |z| = r\,.$$

§ IV.

DIEUDONNÉ's theorem (see theorem 3) can be enunciated, with a slight modification, as follows: let $f(z) = c_0 z + \ldots$, c_0 given and $\neq 0$, be regular and $|f(z)| < 1$ for $|z| < 1$. Then $f(z)$ is univalent and starshaped with respect to the origin for

$$|z| < \frac{1}{|c_0|} - \sqrt{\frac{1}{|c_0|^2} - 1}\,.$$

It seems to me very difficult to generalize this theorem for the case where n initial coefficients $c_0, c_1, \ldots, c_{n-1}$ ($c_0 \neq 0$) are given. In this paragraph we will treat the case where only two initial coefficients c_0, c_1 ($c_0 \neq 0$) are given. First suppose that

$$\varphi(z) = c_0 + c_1 z + \ldots \qquad (c_0,\ c_1 \ \text{given}, \quad c_0 \neq 0) \tag{14}$$

is regular and $|\varphi(z)| < 1$ for $|z| < 1$. Then the function

$$g(z) \equiv \frac{\varphi(z) - c_0}{1 - \bar{c}_0\varphi(z)} = \frac{c_1}{1 - |c_0|^2} z + \ldots \tag{15}$$

is regular and $|g(z)|<1$ for $|z|<1$. Hereafter, for brevity, we put

$$\alpha_0=\frac{c_1}{1-|c_0|^2}.^{(8)} \tag{16}$$

From (15) we have

$$\varphi(z)=\frac{c_0+g(z)}{1+\bar{c}_0 g(z)}. \tag{17}$$

Considering $|g(z)|<1$ for $|z|<1$, we have

$$\frac{|c_0|-|g(z)|}{1-|c_0|\,|g(z)|}\leqq|\varphi(z)|\leqq\frac{|c_0|+|g(z)|}{1+|c_0|\,|g(z)|} \tag{18}$$

for $|z|<1$. Since the expression on the left hand side decreases and that on the right increases as $|g(z)|$ increases, using a known inequality

$$|g(z)|\leqq r\frac{|\alpha_0|+r}{1+|\alpha_0|r}, \tag{19}$$

we obtain the following inequality

$$\frac{|c_0|-|\alpha_0|(1-|c_0|)r-r^2}{1+|\alpha_0|(1-|c_0|)r-|c_0|r^2}\leqq|\varphi(z)|\leqq\frac{|c_0|+|\alpha_0|(1+|c_0|)r+r^2}{1+|\alpha_0|(1+|c_0|)r+|c_0|r^2} \tag{20}$$

for $|z|=r$. In order that the left expression of (20) be positive, we have only to take r less than

$$\rho=\frac{1}{2}\left\{-|\alpha_0|(1-|c_0|)+\sqrt{|\alpha_0|^2(1-|c_0|)^2+4|c_0|}\right\}, \tag{21}$$

where ρ is the root between 0 and 1 of the equation

$$|c_0|-|\alpha_0|(1-|c_0|)r-r^2=0. \tag{22}$$

(8) Of course $|\alpha_0|\leqq 1$ and the equality holds if and only if $\varphi(z)=\dfrac{c_0+e^{i\theta}z}{1+\bar{c}_0e^{i\theta}z}$.

Thus we see that $\varphi(z)$ does not vanish for $|z|<\rho$(9). This limit cannot be replaced by any greater number, for the lower bound of $|\varphi(z)|$ of (20) can be attained by the function

$$\varphi_0(z) = \frac{c_0+(c_0\bar{a}_0e^{i\theta}+a_0)z+e^{i\theta}z^2}{1+(\bar{a}_0e^{i\theta}+\bar{c}_0a_0)z+\bar{c}_0e^{i\theta}z^2} \cdot$$

$$(\theta \equiv 2\operatorname{amp} c_1-\operatorname{amp} c_0+\pi \pmod{2\pi}) \tag{23}$$

at a point $z_0=r\,e^{i\varphi_0}$, where $\varphi_0\equiv \operatorname{amp} c_0-\operatorname{amp} c_1+\pi \pmod{2\pi}$, and $\varphi_0(\rho e^{i\varphi_0})=0$. From (17)

$$|\varphi'(z)| = (1-|c_0|^2)\left|\frac{g'(z)}{(1+\bar{c}_0g(z))^2}\right|. \tag{24}$$

From (8)

$$\left|\frac{g'(z)}{z}-\frac{g(z)}{z^2}\right| \leqq \frac{1-\left|\frac{g(z)}{z}\right|^2}{1-r^2}, \tag{25}$$

or

$$|g'(z)| \leqq \frac{(1-|g(z)|)(r^2+|g(z)|)}{r(1-r^2)}. \tag{26}$$

From

$$\left|\frac{z\varphi'(z)}{\varphi(z)}\right| = (1-|c_0|^2)\left|\frac{zg'(z)}{(c_0+g(z))(1+\bar{c}_0g(z))}\right|,$$

using (26), we have

$$\left|\frac{z\varphi'(z)}{\varphi(z)}\right| \leqq \frac{1-|c_0|^2}{1-r^2}\,\frac{(1-|g(z)|)(r^2+|g(z)|)}{(|c_0|-|g(z)|)(1-|c_0|\,|g(z)|)}, \tag{27}$$

provided that $|c_0|>|g(z)|$.

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(9) This result has been obtained by M. KÖSSLER: Cf. Ueber die α-Stellen von beschränkten Potenzreihen. Mémoires de la Société Royale des Sciences de Bohème, Année 1930.

Since $\frac{1-t}{|c_0|-t}\frac{r^2+t}{1-|c_0|t}$ is an increasing function of t for $0<t<|c_0|$, using the inequality (19), we obtain

$$\left|\frac{z\varphi'(z)}{\varphi(z)}\right| \leqq (1-|c_0|^2)\frac{r(|a_0|+2r+|a_0|r^2)}{(|c_0|-|a_0|(1-|c_0|)r-r^2)(1+|a_0|(1-|c_0|)r-|c_0|r^2)} \tag{28}$$

for $|z|<\rho=\frac{1}{2}\left\{-|a_0|(1-|c_0|)+\sqrt{|a_0|^2(1-|c_0|)^2+4|c_0|}\right\}$. (10)

Since the right hand side of (28) is an increasing function of r in the interval $(0, \rho)$, the equation

$$(1-|c_0|^2)\frac{r(|a_0|+2r+|a_0|r^2)}{(|c_0|-|a_0|(1-|c_0|)r-r^2)(1+|a_0|(1-|c_0|)r-|c_0|r^2)}=1 \tag{29}$$

or

$$c-2a(1-c)r-\left\{3-c^2+a^2(1-c)^2\right\}r^2-2a(1-c)r^3+cr^4=0\,, \tag{30}$$

where $c=|c_0|$, $a=|a_0|=\frac{|c_1|}{1-|c_0|^2}$,

has only one root R between 0 and ρ(11).

Thus we obtain the following

Theorem 8. *Suppose that*

$$f(z)=c_0z+c_1z^2+\dots \quad (c_0,\ c_1\ \textit{given},\ c_0\neq 0)$$

is regular and $|f(z)|<1$ *for* $|z|<1$. *Then*

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(10) By (19), we see that

$$|c_0|>r\frac{|\alpha_0|+r}{1+|\alpha_0|r}\geqq|g(z)| \text{ for } |z|<\rho.$$

(11) The equation (30) has 4 real roots, of which two are positive and the others negative. If we denote a negative root by τ, all roots can be written as follows: $R, \frac{1}{R}, \tau, \frac{1}{\tau}$.

(1) *$f(z)$ is univalent for $|z|<R$ and maps the circle $|z|<R$ on a starshaped domain Δ with respect to the origin, if R is the root between* 0 *and* 1 *of the equation*

$$c-2\alpha(1-c)r-\left\{3-c^2+\alpha^2(1-c)^2\right\}r^2-2\alpha(1-c)r^3+cr^4=0\ ,$$

where

$$c=|c_0|\ ,\qquad \alpha=\frac{|c_1|}{1-|c_0|^2}\ .$$

(2) *If we denote by d the shortest distance of the origin from the boundary of Δ, then*

$$d\geqq R\frac{c-\alpha(1-c)R-R^2}{1+\alpha(1-c)R-cR^2}\ .$$

(3) *These limits* (1) *and* (2) *can be attained by the function*

$$f_0(z)=z\frac{c_0+(c_0\bar{\alpha}_0e^{i\theta}+\alpha_0)z+e^{i\theta}z^2}{1+(\bar{\alpha}_0e^{i\theta}+\bar{c}_0\alpha_0)z+\bar{c}_0e^{i\theta}z^2}\ ,$$

where $\theta\equiv 2$ amp c_1-amp $c_0+\pi$ (mod. 2π), $\alpha_0=\dfrac{c_1}{1-|c_0|^2}$.

Proof. The first assertion is proved above and the second follows directly from (20). Finally, by an elementary calculation, we see that

$$f_0'(Re^{i\varphi_0})=0\qquad\text{and}\qquad |f_0(Re^{i\varphi_0})|=R\frac{c-\alpha(1-c)R-R^2}{1+\alpha(1-c)R-cR^2}\ ,$$

where $\varphi_0\equiv$ amp c_0-amp $c_1+\pi$ (mod. 2π).

Remark. The root R of the equation (30) can be written explicitly:

$$R=\frac{1}{2c}\Big[\alpha(1-c)+\sqrt{(1+c)(1-c)^2\alpha^2+c(3-c)(1+c)}$$
$$-\left\{(2+c)(1-c)^2\alpha^2+c(3+c)(1-c)\right.$$
$$\left.+2\alpha(1-c)\sqrt{(1+c)(1-c)^2\alpha^2+c(3-c)(1+c)}\right\}^{\frac{1}{2}}\Big]\ .$$

When c is fixed, R is a decreasing function of α in the interval (0,1) (see footnote (8)) and especially

$$R=\frac{1}{c}-\sqrt{\frac{1}{c^2}-1}\,, \qquad \text{if} \quad \alpha=1,$$

$$R=\frac{1}{2}\left\{\sqrt{\frac{(3-c)(1+c)}{c}}-\sqrt{\frac{(3+c)(1-c)}{c}}\right\}, \quad \text{if} \quad \alpha=0.$$

§ V.

In this paragraph we shall make some remarks on the univalency of certain analytic functions. Suppose that $f(z)=\alpha+\beta z+\dots$ (α, β given and $\beta\neq 0$) is regular and $f(z)\subset D$ in the unit circle, where D is a given *simply* connected domain. It is well known that there exists one and only one function $\varphi(z)$ which maps D in a one-to-one and conformal manner on a circle under the condition: $\varphi(\alpha)=0$, $\varphi'(\alpha)=\frac{1}{\beta}$. We denote the radius of the mapped circle by M. Next consider the function $F(z)=\varphi(f(z))$. Then $F(z)=z+\dots$ is regular and bounded in the unit circle: $|F(z)|<M$. Here we can, without difficulty, assert the following:

1°. The existence of $F(z)$ follows from that of $f(z)$, and conversely. Hence, it is necessary and sufficient that $M\geqq 1$[12] for the existence of $f(z)$.

2°. If $f(z)$ takes the same value at n given points $z_1, z_2, \dots, z_n$ within the unit circle, then $F(z)$ also takes the same value with the same multiplicity, its converse being also true.

Hence, as an immediate result from a theorem due to DIEUDONNÉ[13], we can state the following theorem: $f(z)$ *assumes at most* p *times*

(12) The equality holds if and only if $f(z)=\varphi^{-1}(z)$ where φ^{-1} is an inverse function of φ.

(13) DIEUDONNÉ has proved that $F(z)$ assumes at most p times the same value for $|z|<\rho_p$ and ρ_p can be attained. Cf. l.c. p. 348. Or my paper: Proc. of Phys.-Math. Soc. of Japan, 3rd Ser., Vol. 14, 1932, p. 304-309. Cf. p. 308.

the same value for $|z| < \rho_p$, *where* ρ_p *is the root between* 0 *and* 1 *of the equation* $1 + r^2 + r^4 + \ldots + r^{2p} - (p+1) M r^p = 0$ *and moreover* ρ_p *cannot be replaced by any greater number.*

For example, we consider a half-plane D: $\Re[z] > 0$. Without difficulty, we have

$$\varphi(z) = \frac{2\Re[\alpha]}{\beta} \frac{z-\alpha}{z+\bar{\alpha}} \qquad \text{and} \qquad M = \frac{2\Re[\alpha]}{|\beta|}.$$

Hence, it is necessary and sufficient that $2\Re[\alpha] \geqq |\beta|$ for the existence of $f(z)$. By the theorem above we get at once

Theorem 9.[14] *Let* $f(z) = \alpha + \beta z + \ldots$ (α, β *given and* $\beta \neq 0$) *be regular and* $\Re[f(z)] > 0$ *for* $|z| < 1$. *Then* $f(z)$ *is univalent for*

$$|z| < \rho_1 = \frac{2\Re[\alpha]}{|\beta|} - \sqrt{\left(\frac{2\Re[\alpha]}{|\beta|}\right)^2 - 1}\ ,$$

and the limiting case can be attained by the function

$$f_0(z) = \frac{M\alpha e^{-i\theta} + (\bar{\alpha} - e^{-i\theta}\alpha)z - \bar{\alpha} M z^2}{M e^{-i\theta} - (1 + e^{-i\theta})z + M z^2} \quad \left(\theta = \operatorname{amp} \beta, \quad M = \frac{2\Re[\alpha]}{|\beta|}\right),$$

which has a derivative vanishing at a point on the circle $|z| = \rho_1$.

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(14) Compare with theorem 4 in KOBORI's paper: Memoirs of the College of Science, Kyoto Imperial University, Ser. A. Vol. XIV, 1931, p. 251–255.