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CONNECTIONS IN THE MANIFOLD ADMITTING CONTACT TRANSFORMATIONS

By

Tôyomon HOSOKAWA

The theory of connections in the manifold admitting the generalized transformations has been developed by the present author. As an application of the theory, it is proposed now to consider some linear displacements in the general manifold preserving a contact transformation.

Consider an $n$-dimensional manifold $X_n$ with coordinates $x^\nu$ ($\nu = a_1, a_2, \ldots, a_n$), and a covariant vector field of components $p_\lambda$ osculating at each point of $X_n$. The new manifold obtained in this manner is called the general manifold $T_n$. However in this general manifold $T_n$ there is no a priori basis for the comparison of the covariant vectors at different points. Hence we shall define the relation between an osculating covariant vector $p_\lambda$ at a given point $P(x_0^\lambda)$ and $p_\lambda + dp_\lambda$ at any nearly point $P'(x_0^\lambda + dx^\lambda)$, by the following equations:

\begin{equation}
(1) \quad dp_\lambda = \omega_{\lambda\mu}dx^\mu \quad \lambda = a_1, a_2, \ldots, a_n,
\end{equation}

where parameters $\omega_{\lambda\mu}$ are arbitrary functions of $x^\nu$ as well as $p_\lambda$. Consequently we see that if at any point $P(x_0^\lambda)$ of $X_n$ we let osculate a covariant vector $p_\lambda$, then we get an osculating covariant vector at every point of $X_n$, so that our manifold $T_n$ is completely determined. The connection so defined is a generalization of that developed some-

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what by several authors as may be seen. The curves defined by equations (1) are called the base paths.

Let us now consider the transformations of the form

\begin{equation}
\tag{2}
'x^\nu = 'x^\nu(x^\lambda; p_\lambda), \quad 'p_\lambda = 'p_\lambda(x^\nu; p_\nu)
\end{equation}

in the $2n$ variables $x^\nu$ and $p_\lambda$, such that the following equations hold good

\begin{equation}
\tag{3}
d'x^\nu p_\nu = \left( \frac{\partial'x^\nu}{\partial x^\lambda} dx^\lambda + \frac{\partial'x^\nu}{\partial p_\lambda} dp_\lambda \right) p_\nu = dx^\nu p_\nu
\end{equation}

for arbitrary values of the differentials $dx^\lambda$ and $dp_\lambda$, followingly for arbitrary functions $\omega_{\lambda\mu}$.

A transformation (2) satisfying this condition is a contact transformation. From (3) are derived the equations

\begin{equation}
\tag{4}
'p_\nu \frac{\partial'x^\nu}{\partial x^\lambda} = p_\lambda, \quad 'p_\nu \frac{\partial'x^\nu}{\partial p_\lambda} = 0.
\end{equation}

Then one may see that a necessary and sufficient condition that a set of functions $'x^\nu(x; p)$ may determine a contact transformation (2) for which the $'p_\lambda(x; p)$ are uniquely determined is that the functions $'x^\nu(x; p)$ be homogeneous of degree zero in $p$'s, that the Jacobian of the $'x^\nu(x; p)$ with respect to the $x$'s be of rank $n$ and that the identities

\[
\frac{\partial'x^\nu}{\partial x^\lambda} \frac{\partial'x^\mu}{\partial p_\lambda} - \frac{\partial'x^\nu}{\partial p_\lambda} \frac{\partial'x^\mu}{\partial x^\lambda} = 0
\]

be satisfied. Also every contact transformation admits a unique inverse contact transformation:

\begin{equation}
\tag{5}
x^\lambda = x^\lambda('x^\nu; 'p_\nu), \quad p_\lambda = p_\lambda('x^\nu; 'p_\nu).
\end{equation}


(3) L. P. EISENHART: loc. cit., p. 249.
By differentiation of the first set of (2) and (5), we have

\[ d'x^\nu = \left( \frac{\partial'x^\nu}{\partial x^\lambda} + \frac{\partial'x^\nu}{\partial p_\tau} \omega_{\tau\lambda} \right) dx^\lambda, \]
\[ dx^\lambda = \left( \frac{\partial x^\lambda}{\partial'x^\mu} + \frac{\partial x^\lambda}{\partial'p_\tau} \overline{\omega}_{\tau\mu} \right) d'x^\mu, \]

where

\[ d'p_\nu = \overline{\omega}_{\nu\mu} d'x^\mu. \]

Any set of \( n \) quantities \( v^\nu(x; p) \), which are transformed by the transformation (2) into \( n \) new quantities \( 'v'(x'; p) \) in such a way that

\[ 'v^\nu = u_\lambda^\nu v^\lambda, \]

will be called a contravariant vector; a covariant vector is a set of \( n \) quantities \( w_\lambda \) which are transformed by (2)

\[ 'w_\mu = v^\lambda_\mu w_\lambda, \]

where

\[ u_\lambda^\nu = \frac{\partial'x^\nu}{\partial x^\lambda} + \frac{\partial'x^\nu}{\partial p_\alpha} \omega_{\alpha\lambda}, \quad v^\lambda_\mu = \frac{\partial x^\lambda}{\partial'x^\mu} + \frac{\partial x^\lambda}{\partial'p_\sigma} \overline{\omega}_{\sigma\mu}. \]

Let it now be assumed that the following relations are satisfied:

\[ \frac{\partial x^\nu}{\partial'p_\sigma} \omega_{\sigma x} \omega_{x\nu} - \frac{\partial p_\lambda}{\partial'p_\sigma} \omega_{\sigma x} + \frac{\partial x^\nu}{\partial x^\mu} \omega_{\lambda\nu} = \frac{\partial p_\lambda}{\partial x^\mu}, \]

and

\[ \frac{\partial x^\nu}{\partial p_\alpha} \omega_{\sigma x} \omega_{x\nu} - \frac{\partial p_\lambda}{\partial p_\alpha} \omega_{\sigma x} + \frac{\partial x^\mu}{\partial x^\nu} \omega_{x\nu} = \frac{\partial p_\lambda}{\partial x^\mu}. \]

But from (9) is obtained

\[ u_\lambda^\nu v^\mu_\lambda = \frac{\partial'x^\nu}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial'x^\mu} + \frac{\partial'x^\nu}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial'p_\sigma} \omega_{\sigma x} + \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial x^\mu}{\partial'p_\alpha} \omega_{\sigma x} \omega_{x\nu}, \]

and on the other hand

\[ \frac{\partial'x^\nu}{\partial x^\mu} \frac{\partial x^\lambda}{\partial'p_\mu} + \frac{\partial'x^\nu}{\partial p_\alpha} \frac{\partial x^\mu}{\partial'p_\mu} = 0. \]
holds good. Therefore from (10), is obtained

(12) \[ u_{\lambda}^{\nu} v_{\mu}^{\lambda} = \delta_{\mu}^{\nu} , \]

and in like manner from (11)

(13) \[ v_{\lambda}^{\nu} u_{\mu}^{\lambda} = \delta_{\mu}^{\nu} , \]

where the \( \delta 's \) are KRONECKER's deltas.

By means of these new definitions it is to be seen that the \( p_{\lambda} \) is a covariant vector, because from (4) we get

\[ 'p_{\nu} \frac{\partial x^{\nu}}{\partial x} + 'p_{\nu} \frac{\partial x^{\nu}}{\partial p_{\sigma}} \omega_{\sigma\lambda} = p_{\lambda} \]

i.e.

\[ u_{\lambda}^{\nu} p_{\nu} = p_{\lambda} , \]

which becomes by (2)

\[ 'p_{\mu} = v_{\mu}^{\lambda} p_{\lambda} . \]

The equations (6) show that the differential \( dx^{\lambda} \) is a contravariant vector.

A tensor of the higher order is defined by the following equations:

\[ 'v_{\lambda_{1} \ldots \lambda_{t}}^{\alpha_{1} \ldots \alpha_{t}} = v_{\beta_{1} \ldots \beta_{s}}^{\gamma_{1} \ldots \gamma_{t}} u_{\gamma_{1} \ldots \gamma_{t}}^{\alpha_{1} \ldots \alpha_{t}} \ldots u_{\alpha_{t}}^{\beta_{1}} v_{\alpha_{1}}^{\beta_{1}} \ldots v_{\alpha_{t}}^{\beta_{s}} . \]

When a quantity is invariant by the transformation (2), it is called a scalar. Then from (13) it can be shown that \( v^{\nu} w_{\nu} \) is a scalar.

Now let "metrics" be introduced in our manifold. The metrics must be an invariance by means of the transformation (2). We consider one parameter continuous group \( G_{1} \) of the contact transformations. An infinitesimal transformation of the group \( G_{1} \) is defined by equations of the form

(14) \[ x^{\lambda} = x^{\lambda} + \frac{\partial C}{\partial p_{\lambda}} \delta t , \quad 'p_{\lambda} = p_{\lambda} - \frac{\partial C}{\partial x^{\lambda}} \delta t , \]
where

\begin{equation}
C = p_\lambda \frac{\partial C}{\partial p_\lambda}.
\end{equation}

The function \(C\) is called the characteristic function of the contact transformation, and is an invariant function by means of the contact transformation (2).

From (14) are derived

\begin{equation}
\frac{dx^\lambda}{dt} = \frac{\partial C}{\partial p_\lambda}, \quad \frac{dp_\lambda}{dt} = -\frac{\partial C}{\partial x^\lambda},
\end{equation}

and by integration of the above equations we get the finite equations of \(G_1\):

\begin{align*}
'x^\lambda &= 'x^\lambda(x; p, t), \\
p_\lambda &= 'p_\lambda(x; p, t).
\end{align*}

If equations (16) are transformed by means of a contact transformation (2), we obtain

\begin{align*}
\frac{d'x^\lambda}{dt} &= \frac{\partial \bar{C}}{\partial 'p_\lambda}, \\
n\frac{dp_\lambda}{dt} &= -\frac{\partial \bar{C}}{\partial 'x^\lambda},
\end{align*}

where \(\bar{C}\) is the transform of the characteristic function of the group \(G_1^{(2)}\). Accordingly we see that the \(\frac{\partial C}{\partial p_\lambda}\) is a contravariant vector.

In particular we put

\begin{equation}
C = \sqrt{g^{\lambda\mu} p_\lambda p_\mu},
\end{equation}

where the \(g^{\lambda\mu}\)'s are functions of the \(x\)'s as well as \(p\)'s, and are homogeneous of zero-th degree in the \(p\)'s, and the rank of the matrix of the \(g^{\lambda\mu}\)'s is \(n\). But it is evident that the \(g^{\lambda\mu}\)'s are components of a contravariant tensor of the second order. We shall take \(g^{\lambda\mu}\) as the fundamental tensor of the metrics.

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(1) L. P. Eisenhart: loc. cit., p. 252.
If the functions $g^{\lambda\mu}$ be defined by the following equations:

\begin{equation}
(18) \quad g^{\lambda\mu} = \frac{1}{2} \frac{\partial^2 C^2}{\partial p_\lambda \partial p_\mu},
\end{equation}

then by EULER's theorem we get

\[ \frac{\partial g^{\lambda\mu}}{\partial p_\nu} p_\lambda p_\mu = \frac{1}{2} \frac{\partial^2 C^2}{\partial p_\lambda \partial p_\mu \partial p_\nu} p_\lambda p_\mu = 0. \]

Hence from (16), we have

\begin{equation}
(19) \quad \frac{dx_\lambda}{dt} = hg^{\lambda\mu} p_\mu, \quad \frac{dp_\lambda}{dt} = -\frac{h}{2} \frac{\partial g^{\nu\mu}}{\partial x^\lambda} p_\mu p_\nu,
\end{equation}

where $h^{-1} = C$.

From (1) and the first set of (19),

\[ \frac{dp_\lambda}{dt} = hg^{\nu\mu} \omega_{\lambda\mu} p_\nu. \]

If we define arbitrary functions $\omega_{\lambda\mu}$ by the following equations:

\begin{equation}
(20) \quad \omega_{\lambda\sigma} = -\frac{1}{2} g_{\nu 0} \frac{\partial g^{\mu\nu}}{\partial x^\lambda} p_\mu,
\end{equation}

then equations (1) are reduced to the second set of (19).

We shall now define a linear displacement for contravariant and covariant vectors $v^\nu$ and $w_\lambda$:

\begin{equation}
(21) \quad \begin{cases}
\delta v^\nu = dv^\nu + \Gamma^\nu_{\lambda\mu} v^\lambda dx_\mu + \Lambda_\nu^\sigma v^\lambda dp_\sigma, \\
\delta w_\nu = dw_\nu - \Gamma^\lambda_{\nu\mu} w_\lambda dx_\mu - \Lambda_\nu^\sigma w_\lambda dp_\sigma,
\end{cases}
\end{equation}

where $\Gamma^\nu_{\lambda\mu}$ and $\Lambda_\nu^\sigma$ are the functions of $x$'s as well as $p$'s. If the linear displacement is taken along the base paths satisfying (1), we get from the above equations
Connections in the Manifold Admitting Contact Transformations

\begin{equation}
\delta v^\nu = dv^\nu + \Gamma^\nu_{\lambda\mu} v^\lambda dx^\mu,
\end{equation}

\[ \Gamma^\nu_{\lambda\mu} = \Gamma^\nu_{\lambda\mu} + \Lambda^\nu_{\lambda q} \omega_{\sigma\mu} \],

and

\begin{equation}
\delta w_\nu = dw_\nu - \Gamma^\nu_{\nu\mu} w_\mu dx^\mu,
\end{equation}

\[ \Gamma^\nu_{\nu\mu} = \Gamma^\nu_{\nu\mu} + \Lambda^\nu_{\nu q} \omega_{\sigma\mu} \]

where

\begin{equation}
\nabla_{\mu} v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \frac{\partial v^\nu}{\partial p_\sigma} \omega_{\sigma\mu} + \Gamma^\nu_{\lambda\mu} v^\lambda
\end{equation}

In order that \( \nabla_{\mu} v^\nu \) may be the components of a mixed tensor, \( \Gamma^\nu_{\lambda\mu} \) must satisfy the following equation:

\begin{equation}
\frac{\partial u^\lambda_\nu}{\partial x^\mu} + \frac{\partial u^\lambda_\nu}{\partial p_\sigma} \omega_{\sigma\mu} + \Gamma^\nu_{\lambda\sigma} u^\sigma_\nu = u^\lambda_\nu \Gamma^\nu_{\lambda\mu},
\end{equation}

where \( \Gamma^\nu_{\lambda\nu} \) are functions of \( x \)'s as well as \( p \)'s and \( \Gamma^\nu_{\nu\mu} \) of \( x \)'s as well as \( p \)'s.

In the same manner as that of the general linear displacements, we can calculate the curvature tensor:

\begin{equation}
R_{\nu\mu\lambda\rho} = \frac{\partial \Gamma^\nu_{\lambda\mu}}{\partial x^\rho} - \frac{\partial \Gamma^\nu_{\lambda\rho}}{\partial x^\mu} + \Gamma^\nu_{\mu\alpha} \Gamma^\alpha_{\lambda\rho} - \Gamma^\nu_{\mu\rho} \Gamma^\lambda_{\rho\lambda} + \frac{\partial \Gamma^\nu_{\lambda\rho}}{\partial p_\tau} \omega_{\tau\mu} - \frac{\partial \Gamma^\nu_{\rho\mu}}{\partial p_\tau} \omega_{\tau\nu}.
\end{equation}

When the \( p \)'s are such that \( C \neq 0 \) and \( h = \text{const.} \), we can normalize \( C = 1 \), by replacing \( p_\lambda \) by \( h^{-1} p_\lambda \). Since \( C \) is homogeneous of degree one in the \( p \)'s. Hence from (19) we get

\begin{equation}
\frac{dx^\lambda}{dt} = g^{\lambda\nu} p_\nu, \quad \frac{dp_\lambda}{dt} = -\frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial x^\lambda} p_\mu p_\nu.
\end{equation}

When the rank of the hessian of \( C \) with respect to \( p \)'s is \( n-1 \), the first set of the above equations can be solved with respect to \( p \)'s as

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functions of the $x$'s and $\dot{x}$'s, where $\dot{x}^\lambda = \frac{dx^\lambda}{dt}$. We denote by $\hat{C}$ the function resulting from the substitution in $C$ of these expressions for $p_\lambda$. Then we get

$$\tilde{\theta}_{\lambda \mu} = \frac{1}{2} \frac{\partial^2 \hat{C}}{\partial \dot{x}^\lambda \partial \dot{x}^\mu}, \quad p_\lambda = \tilde{\theta}_{\lambda \mu} \dot{x}^\mu,$$

From the second set of (24), we have

$$\frac{dx^\lambda}{dt^2} + \left\{\begin{array}{l} \lambda \\ \mu \nu \end{array}\right\} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0,$$

where $\left\{\begin{array}{l} \lambda \\ \mu \nu \end{array}\right\}$ are CHRISTOFFEL'S symbol with respect to $\tilde{\theta}_{\mu \lambda}$. Thus the paths defined by (24) are the geodesics of BERWALD-FINSLER'S manifold.\(^{(1)}\) In assumption (20), the parallelism defined by equations (1) is reduced to that by (26). Accordingly from the first set of (25), we have

$$dp_\lambda = -\left\{\begin{array}{l} \nu \\ \mu \lambda \end{array}\right\} \tilde{\theta}_{\nu \sigma} \dot{x}^\sigma dx^\lambda.$$

Hence from (1), we obtain

$$\omega_{\lambda \mu} = -\left\{\begin{array}{l} \nu \\ \lambda \mu \end{array}\right\} \tilde{\theta}_{\nu \sigma} \dot{x}^\sigma.$$

Consequently from the linear displacement (22) and (22') we can reduce the connections which has already been studied by the present author.\(^{(2)}\).
