POSTULATES FOR THE SEPARATION OF POINT-PAIRS IN THE FOUNDATIONS OF GEOMETRY

By

Takeshi INAGAKI

Introduction.

E. V. HUNTINGTON and K. E. ROSINGER have given 10 sets of postulates for the separation of point-pairs, and have established many interesting relations among them.(1)

It is the purpose of this paper to generalize these postulates and to replace the sets by a new one.

First in §1, we shall consider the definition for the separation of point-pairs given by K. KUNUGI.(2) When, in the definition, a chain becomes a circle or a straight line, the separation of point-pairs coincides with that of our intuition. And, as the definition is applicable to chains in general, it will be convenient to use it for establishing systematically the postulates for the separation of point-pairs. Now in §2, we shall see that none of the HUNTINGTON-ROSINGER postulates are satisfied by our definition, and, therefore, it will be natural to give, in §3, a new set of more general postulates and to discuss the relations among them. Finally, in §4, we shall discuss their independency.

Explanation of signs used in this paper.

The sign $A = 0$ means that the relation A does not exist. The sign $A \neq 0$ means that the relation A exists. When a chain K contains n distinct points $P_1, P_2, \ldots, P_n$, then

(1) See E. V. HUNTINGTON and K. E. ROSINGER: Postulates for separation of point-pairs, Proceedings of the American Academy of Arts and Sciences, vol. 67 (1932), p. 61–145.

(2) See K. KUNUGI: Axioms for betweenness in the foundations of geometry, Tôhoku Math. Jour., vol. 37 (1933), p. 421.

$$K_{P_{i_1}, P_{i_2}, \ldots, P_{i_k}, [P_{j_1}, P_{j_2}, \ldots, P_{j_l}]}(P_\lambda, P_\mu)$$

means a chain which connects P_λ and P_μ in K, and it contains P_{i_1}, $P_{i_2}, \ldots, P_{i_k}$ containing or not containing some of $P_{j_1}, P_{j_2}, \ldots, P_{j_l}$ and if ν is not any one of $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l, \lambda, \mu$, then this chain does not contain P_ν.

The sign & means "and", the sign V means "or" (in the sense of "at least one"), and the sign $\rightarrow$ means "If......, then".

§ 1. Some theorems.

Difinition.[1] Let A, B, X, Y be four different points in a point set K. Then we say that X and Y are separated in K, by A and B, if the two following conditions are both fulfilled:

(1). Any chain $K(X, Y)$ in K, which connects X and Y, contains at least one of A and B.

(2). There exists a chain $K(X, Y)$ in K, which connects X and Y, and which contains one and only one of A and B.

When two points X and Y are separated in K by A and B, we denote the separation by $AXBY$ (in K).

From the definition follows at once:

Theorem 1. *Let A, B, X, Y be four different points in a chain K. If $AXBY$ (in K), then we have also $AYBX$, $BXAY$, $BYAX$.*

Before going further, we shall give the following axiom:

Axiom 1. *If $K > X, Y, Z$, then, for at least one of chains $K_z(X, Y)$ in K,*

$$K(X, Z) \, V K(Y, Z) \neq 0 \quad \textit{and} \quad K(X, Z) + K(Y, Z) \leq K_Z(X, Y) .$$

From axiom 1, it is deducible that:

Theorem 2. *If a chain K involves n distinct points $P_1, P_2, \ldots, P_n$, then for any one of them, say P_i,*

(1) The definition is due to K. KUNUGI.

$$\sum_j K(P_i, P_j) \neq 0 \qquad i \neq j,\ i, j = 1, 2, 3, \ldots, n.$$

Proof. Let a chain K involve n distinct points $P_1, P_2, \ldots, P_n$, and let P_i be any one of them. Let a chain K' be one of the chains involving P_i, containing the least number of these points. If $K'(P_i, P_j)$ exists for some one point $P_j\,(P_j \neq P_i)$, then the theorem is evident.

Let K' contain at least two points P_j, P_k which are not P_i. By axiom 1, then, there exists a chain K'' in K', which fulfils the following condition:

$$K'' > P_i, P_j \quad \text{and} \quad K'' \cdot P_k = 0$$

or

$$K'' > P_i, P_k \quad \text{and} \quad K'' \cdot P_j = 0\,.$$

This result is inconsistent with the definition of the chain K'; thus, for some one point of $P_1, P_2, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n$, $K(P_i, P_l)$ exists. Hence

$$\sum_j K(P_i, P_j) \neq 0 \quad i \neq j, \quad i, j = 1, 2, 3, \ldots, n.$$

Q. E. D.

Theorem 3. $ABCD.X \rightarrow ABCX \vee ABXD \vee AXCD \vee XBCD.$

Proof. From $ABCD$, hence $K_{[X]}(B, D)=0$. In the following proof, we shall assume that (α): $K_{[X]}(B, D) = 0$, (β): $K_{A[X]}(B, D) \neq 0$.

Case 1. Suppose $K(D, X) \neq 0$. Let $K(B, X) \neq 0$, then $K(D, X) + K(B, X) = K_X(B, D) \neq 0$. The result is inconsistent with (α). Thus $K(B, X) = 0$. Similarly, let $K_D(B, X) \neq 0$, then $K_{[X]}(B, D) \neq 0$. The result is inconsistent with (α); thus $K_D(B, X) = 0$. Hence $K_{[D]}(B, X) = 0$. Next, from (β) and the supposition $K(D, X) \neq 0$, $K_{A[X]}(B, D) + K(D, X) = K_{A, D}(B, X) \neq 0$. Therefore $ABCX$.

Case 2. Suppose $K(B, X) \neq 0$. Substituting D in place of B in case 1, we get $ADCX$.

Case 3. Suppose $K(B, X) + K(D, X) = 0$, $K(A, X) \neq 0$. By (β), $K_{A[X]}(B, D) \neq 0$. Hence $K_{[D, X]}(B, A) + K(A, X) = K_{A[D]}(B, X) \neq 0$. By (α), $K_{[D]}(B, X) = 0$. Hence $ABCX$.(1)

---

(1) In this case, exchanging B and D, we obtain $AXCD$ also.

Case 4. Suppose $K(A, X) + K(B, X) + K(D, X) = 0$. Then by theorem 2, $K(C, X) \neq 0$. There are three possible cases:

1°) Suppose $K(B, C) \neq 0$. By (α), $K_{[D]}(B, X) = 0$, and $K(B, C) + K(C, X) = K_C(B, X) \neq 0$. Hence $ABCX$.

2°) Suppose $K(C, D) \neq 0$. By (α), $K_{[B]}(D, X) = 0$ and $K(C, D) + K(C, X) = K_C(D, X) \neq 0$. Hence $AXCD$.

3°) Suppose $K(B, C) + K(C, D) = 0$. Then by theorem 2, $K(A, B) \neq 0$, and $K(A, D) \neq 0$. Hence $K(A, B) + K(A, D) = K_A(B, D) \neq 0$.

Next, by the supposition $K(B, C) + K(C, D) = 0$, and axiom 1, we get $K_C(B, D) = 0$. Hence $K_{[C]}(B, D) = 0$. Therefore $ABXD$.

In preceding proofs, we have obtained the three cases, $ABCX$, $ABXD$, $AXCD$. And assuming that $K_{[X]}(B, D) = 0$, $K_{C[X]}(B, D) \neq 0$, and exchanging A and C, we must obtain the following three cases, $ABCX$, $AXCD$, $XBCD$. Thus we have proved the theorem

$$ABCD.X \rightarrow ABCX \vee ABXD \vee AXCD \vee XBCD .$$

Theorem 4. *Suppose G: $ABCD \rightarrow BCDA$ and R: $ABCD \rightarrow DCBA$, then K_1: $ABCD \rightarrow ADCB$, K_2: $ABCD \rightarrow CBAD$.*

Proof. By G, $ABCD \rightarrow BCDA$. By R, $BCDA \rightarrow ADCB$. Hence K_1, $ABCD \rightarrow ADCB$. Next, by R, $ABCD \rightarrow DCBA$. By G, $DCBA \rightarrow CBAD$. Hence K_2, $ABCD \rightarrow CBAD$. Q. E. D.

Remark. G and R are those of postulates which were established by E. V. HUNTINGTON and K. E. ROSINGER.

Theorem 5. *If K_1 and K_2 are true, and H: $ABCD \,\&\, ABDC = 0$ is not true in a chain K, then all the twenty-four permutations of four different points A, B, C, D are true.*

Proof. H is not true; therefore $ABCD \,\&\, ABDC$ is true. Hence $ABCD \rightarrow ABDC$ .... (1), therefore $ABCD$ ..... (2), and $ABDC$ .... (3), are both true.

By (3) and K_1, $ACDB$ (4). By (4) and (1), $ACBD$ (5).

By (5) and K_1, $ADBC$ (6). By (6) and (1), $ADCB$.

By (5) and K_2, $BCAD$ (7). By (7) and (1), $BCDA$ (8).
By (7) and K_1, $BDAC$ (9). By (9) and (1), $BDCA$ (10).
By (10) and K_1, $BACD$ (11). By (11) and (1), $BADC$.

Exchanging A and C, B and D respectively we get the theorem.

Theorem 6. *If K_1, K_2 and axiom 1 are true in a chain K, then* $H:\ ABCD \,\&\, ABDC = 0$.

Proof. Suppose that H is not true in K, then, by theorem 5, all the twenty-four permutations of A, B, C, D are true.

Now, by theorem 2,

$$K(A, D)+K(B, D)+K(C, D) \neq 0 .$$

Then it is evident that at least one of $BACD$, $ABCD$, $ACBD$ is not true in K. It is inconsistent with the fact that H is not true in K. Thus we get the theorem.

Theorem 7. *When a chain K contains n distinct points P_1, $P_2, \dots, P_n$, if K is decomposed in pieces involving only two points, then axiom 1 is true in this chain K.*

Proof. Let X, Y, Z be three any different points in $P_1, P_2, \dots P_n$. Hence $K > X, Y, Z$. As K is decomposed in pieces involving only two points, for at least one of chains $K_Z(X, Y)$ in K,

$$K(X, Z) \vee K(Y, Z) \neq 0$$

and

$$K(X, Z)+K(Y, Z) \leq K_Z(X, Y) .$$

Hence axiom 1 is true in K. Q. E. D.

§ 2. Examples which do not fulfil the postulates of E. V. Huntington and K. E. Rosinger.

In translating to our definition, there exist two possible significations of the notation $ABCD$, viz.

(1) B and D are separated in K by A and C.

(2) A and C are separated in K by B and D.

In the following examples, we shall show case (1) on the left-hand side and case (2) on the right-hand side.

1) An example which does not fulfil postulates F and F':

Postulate F: If A, B, C, D are any distinct points, then at least one of the twenty-four permutations $ABCD, ABDC, \dots, DCBA$ will form a true tetrad.

Postulate F′: At least one tetrad is true.

Let
$$K = K(A, B) + K(A, C) + K(A, D) + K(B, C) + K(B, D) + K(C, D).$$

Let
$$K = K(A, B) + K(A, C) + K(A, D) + K(B, C) + K(B, D) + K(C, D).$$

In these cases, there is a chain which contains any two points only of A, B, C, D. Hence postulates F and F' are not true.

2) An example which does not fulfil postulates G, H, R and R'.

Postulate G: $ABCD \rightarrow BCDA$.

Postulate H: $ABCD \,\&\, ABDC = 0$.

Postulate R: $ABCD \rightarrow DCBA$.

Postulate R′: At least one true tetrad is reversible.

Let
$$K = K(A, B) + K(A, C) + K(A, D).$$

G) In this case, $K(A, C) \neq 0$. Hence $BCDA$ is not true; therefore G is not true.

H) In this case, $K(B, C) = 0$, $K_A(B, C) \neq 0$. Hence $ABCD$ and $ABDC$ are true. Therefore H is not true.

R) In this case, $K(A, C) \neq 0$. Hence $DCBA$ is not true. Therefore R is not true.

Let $K = K(A, B) + K(B, C) + K(B, D) + K(C, D)$.

G) In this case, $K(B, D) \neq 0$. Hence $BCDA$ is not true; therefore G is not true.

H) In this case,
$$K(A, D) = 0, \ K_B(A, D) \neq 0.$$
Hence $ABCD$ and $ABDC$ are true. Therefore H is not true.

R) In this case, $K(B, D) \neq 0$. Hence $DCBA$ is not true. Therefore R is not true.

R′) In this case, all the true tetrads are $ABCD$, $ABDC$, $ACBD$. However $K(C, A) \neq 0$, $K(D, A) \neq 0$ and $K(A, B) \neq 0$, therefore $DCBA$, $CDBA$, $DBCA$ are not true tetrads. Hence R′ is not true.

R′) In this case, all the true tetrads are $ABCD$, $ABDC$. However, $K(B, D) \neq 0$, $K(B, C) \neq 0$, therefore $DCBA$, $CDBA$ are not true tetrads. Hence R′ is not true.

3) An example which does not fulfil postulate 10.

Postulate 10: $ABCD.X \rightarrow AXCD \ V \ ABCX.$

Let $K = K(A, B) + K(A, C) + K(A, D) + K(C, X)$.

In this case,

$$K_{A[B]}(X, D) + K_{C[B]}(X, D) = 0.$$

Hence $AXCD$ is not true. And then,

$$K_{A[D]}(B, X) + K_{C[D]}(B, X) = 0 .$$

Hence $ABCX$ is not true. Therefore postulate 10 is not true.

Let $K = K(A, B) + K(B, C) + K(C, D) + K(C, X) + K(D, A)$.

In this case,

$$K_B(A, C) \neq 0.$$

Hence $AXCD$ is not true. And then,

$$K_D(A, C) \neq 0 .$$

Hence $ABCX$ is not true. Therefore postulate 10 is not true.

4) An example which does not fulfil postulates 11, 12, and 13.

Postulate 11. $ABXC \,\&\, ABCY \rightarrow ABXY.$

Postulate 12. $ABXC \,\&\, ABCY \rightarrow BXCY.$

Postulate 13. $ABXC \,\&\, ABCY \rightarrow AXCY.$

Let $K = K(A, B) + K(A, X) + K(A, C) + K(X, Y)$.

11) In this case,

$$K_{A[C]}(B, Y) + K_{X[C]}(B, Y) = 0 .$$

Hence $ABXY$ is not true. Therefore postulate 11 is not true.

Let $K = K(B, A) + K(B, C) + K(B, Y) + K(Y, X)$.

11) In this case,

$$K_{B[C]}(A, X) + K_{Y[C]}(A, X) = 0 .$$

Hence $ABXY$ is not true. Therefore postulate 11 is not ture.

12, 13) In these cases, $K(X, Y) \neq 0$. Hence $BXCY$ and $AXCY$ are not true. Therefore postulat s 12 and 13 are not true.

12) In this case, $K(B,C) \neq 0$. Hence $BXCY$ is not true. Therefore postulate 12 is not true.

13) In this case, $K_B(A,C) \neq 0$. Hence $AXCY$ is not true. Therefore postulate 13 is not true.

5) An example which does not fulfil postulates 14, 15, 16, 17, 18 and 19.

Postulate 14: $ABCX \,\&\, ABCY \rightarrow ABXY \vee ABYX.$

Postulate 15: $ABCX \,\&\, ABCY \rightarrow ACXY \vee ACYX.$

Postulate 16: $ABCX \,\&\, ABCY \rightarrow BCXY \vee BCYX.$

Postulate 17: $ABCX \,\&\, ABCY \rightarrow (ABXY \vee ACYX) \,\&\, (ABYX \vee ACXY).$

Postulate 18: $ABCX \,\&\, ABCY \rightarrow (ABXY \vee BCYX) \,\&\, (ABYX \vee BCXY).$

Postulate 19: $ABCX \,\&\, ABCY \rightarrow (ACXY \vee BCYX) \,\&\, (ACYX \vee BCXY).$

Let

$$K = K(A, B) + K(A, X) + K(A, Y) + K(C, B) + K(C, X) + K(C, Y).$$

14) In this case, $K_C(B, Y) \neq 0$, $K_C(B, X) \neq 0$. Hence $ABXY$ and $ABYX$ are not true, therefore postulate 14 is not true.

15, 16) In these cases, $K(C, Y) \neq 0$, $K(C, X) \neq 0$. Hence $ACXY$, $ACYX$, $BCXY$ and $BCYX$ are not true, therefore postulates 15 and 16 are not true.

Let

$$K = K(A, Y) + K(Y, X) + K(X, A) + K(A, B) + K(B, C).$$

14, 15) In these cases, $K(A, X) \neq 0$, $K(A, Y) \neq 0$. Hence $ABXY$, $ABYX$, $ACXY$ and $ACYX$ are not true, therefore, postulates 14 and 15 are not true.

16) In this case, $K_A(B,X) \neq 0$, $K_A(B,Y) \neq 0$. Hence $BCXY$ and $BCYX$ are not true, therefore postulate 16 is not true.

17, 18, 19) In these cases, postulates 14, 15 and 16 are not true, therefore, it is evident that postulates 17, 18 and 19 are not true.

17, 18, 19) In these cases, postulates 14, 15 and 16 are not true, therefore, it is evident that postulates 17, 18 and 19 are not true.

§ 3. New sets of postulates for the separation of point-pairs.

As shown in §2, the postulates of E. V. Huntington and K. E. Rosinger do not explain our definition for even a chain which is decomposed in pieces involving only two points. They are too strong for our definition for a chain which fulfils axiom 1 only. In this §, let us see how we can establish new postulates for the separation of point-pairs, by using the theorems in §1.

Postulates which we propose to study are:

Postulate D. If $ABCD$, then A, B, C and D are distinct.

Postulate F. If A, B, C and D are any distinct points, then at least one of the twenty-four permutations $ABCD$, $ABDC, \ldots\ldots, DCBA$ will form a true tetrad.

Postulate N. If A, B, C and D are any distinct points, then at least one of the twenty-four permutations $ABCD$, $ABDC, \ldots\ldots, DCBA$ will not form a true tetrad.

Postulate K_1. $ABCD \rightarrow ADCB$[1].

Postulate K_2. $ABCD \rightarrow CBAD$[1].

Postulate C. $ABCD.X \rightarrow ABCX \vee ABXD \vee AXCD \vee XBCD$.

Postulates D and F are identical with those which were considered by E. V. Huntington and K. E. Rosinger. K_1 and K_2 are used in place of G and R respectively. N and C are used in place of H and 10 respectively.

(1) Postulates K_1 and K_2 say that the points A, C and B, D are points of pairs, that is to say, A and C or B and D, respectively, are permutable without changing the tetrad.

As shown in §1, it is evident that postulates D, F, N, K_1, K_2 and C are deduced from postulates D, F, G, H, R and 10 under axiom 1. Conversely, we shall show that postulates D, F, G, H, R and 10 may be deduced by adding some conditions from the new postulates D, F, N, K_1, K_2 and C.

Theorem 8. *Suppose* K_1: $ABCD \to ADCB$, *and* G: $ABCD \to BCDA$ are both true, then R: $ABCD \to DCBA$.

Proof. By K_1, $ABCD \to ADCB$. By G, $ADCB \to DCBA$. Hence $ABCD \to DCBA$. Therefore R is true. Q. E. D.

Theorem 9. *Suppose* K_2: $ABCD \to CBAD$, *and* G: $ABCD \to BCDA$ *are both true, then* R: $ABCD \to DCBA$.

Proof. By G, $ABCD \to BCDA$. By K_2, $BCDA \to DCBA$. Hence $ABCD \to DCBA$. Therefore R is true. Q. E. D.

Theorem 10. *Suppose* K_1: $ABCD \to ADCB$, *and* R: $ABCD \to DCBA$ *are both true, then* G: $ABCD \to BCDA$.

Proof. By K_1, $ABCD \to ADCB$. By R, $ADCB \to BCDA$. Hence $ABCD \to BCDA$. Therefore G is true. Q. E. D.

Theorem 11. *Suppose* K_2: $ABCD \to CBAD$, *and* R: $ABCD \to DCBA$ *are both true, then* G: $ABCD \to BCDA$.

Proof. By R, $ABCD \to DCBA$. By K_2, $DCBA \to BCDA$. Hence $ABCD \to BCDA$. Therefore G is true. Q. E. D.

Remark. It is clear that either G or R is deducible, by adding R or G, respectively, in either K_1 or K_2.

Theorem 12. *When* G, R *and* N *are true, then* H *is true.*

Proof. By theorems 4 and 5 in §1, it is evident. Q. E. D.

For the purpose of deducing postulate 10 from postulate C, we shall introduce the following axiom on K.

Axiom 2. *A chain* K *is decomposed in pieces involving only two points.*

Theorem 13. *When axiom 2, postulate* C *and either* G *or* R *are true, and* X *is neither* A *nor* C, *then postulate 10 is true.*

Proof. By theorem 7 in §1, axiom 1 is true in this chain. Consequently, we can make use of the results of theorem 3 in §1.

That the case, $XBCD$ and $ABXD$ are true, was shown already in the case 4, 3° in theorem 3 in §1.

In this case, $K(A, X)=K(B, X)=K(D, X)=K(B, C)=K(D, C)=0$,

$$K(A, B) \neq 0\ , \quad K(A, D) \neq 0 \quad \text{and} \quad K(C, X) \neq 0\ .$$

By axiom 2 and the preceding results,

$$K = K(A, B)+K(A, D)+K(A, C)+K(C, X)\ .$$

From the supposition that either G or R is true, $K(A,C)=0$ ... (1).

Hence $$K = K(A, B)+K(A, D)+K(C, X)\ .$$

Therefore, the piece $K(C, X)$ has no commom point with the other pieces $K(A, B)$ and $K(A, D)$. This is inconsistent with the definition of a chain.[1] Hence $K(A, C) \neq 0$. This is inconsistent with (1). Consequently, case 4, 3° in theorem 3 in §1 does not occur. Thus postulate 10 can be deduced from postulate C. Q. E. D.

By theorems 8, 9, 10, 11, 12 and 13, it is clear that the postulates D, F, G, H, R and 10 can be deduced from new postulates D, F, N, K_1, K_2 and C, by applying axiom 2 and either G or R.

The new postulates which we have established are more general than those of E. V. HUNTINGTON and K. E. ROSINGER, and, at the same time, almost all of them (except F) satisfy the definition given in §1 under axiom 1.

Now let us (we propose to) give two sets of postulates for the separation of point-pairs.

(1), for the non reversible case, viz. D, F, N, K_1, K_2, C.

(2), for the reversible case, viz. D, F, N, K_1, R, C.

Remark. It is evident that our new set of postulates for a non reversible case (1) does not coincide with that of E. V. HUNTINGTON and K. E. ROSINGER. We shall show that the new set of postulates

(1) See K. KUNUGI, loc. cit., p. 414.

for a reversible case (2) does not coincide with that of E. V. HUNTINGTON and K. E. ROSINGER.

Let $$K=K(A, B)+K(B, C)+K(C, D)+K(D, A)$$

and let X is either A or C. It is evident that D, F, G, R, N and C are true, but not 10.

Moreover, the preceding example show the consistency of our new sets of postulates for the separation of point-pairs.

§ 4. Independency between K_1, K_2, G, R, N and C.

In this §, we shall consider independency between K_1, K_2, G, R, N and C.

1) G and R are both independent of K_1, K_2, N and C.

Let $$K=K(A, B)+K(B, C)+K(C, D)+K(D, A)+K(A, C)\,.$$

All the true tetrads in K are $ABCD$, $ADCB$, $CBAD$, $CDAB$, therefore K_1, K_2, N and C are true. Therefore G and R are independent of K_1, K_2, N and C.

2) N is independent of K_1, K_2, G, R and C.

Let $$K=K_A(B, D)+K_C(B, D)+K_B(A, C)+K_D(A, C)\,,$$ where

the chains $K_A(B, D)$, $K_C(B, D)$, $K_B(A, C)$, $K_D(A, C)$ are chains which cannot be decomposed in other chains. Since any chain in K contains some three points of A, B, C, D, it is evident that all the twenty-four permutations $ABCD$, $ABDC$, ..., $DCBA$ are all true. Hence K_1, K_2, G, R and C are true. Therefore N is independent of K_1, K_2, G, R and C.

3) C is independent of K_1, K_2, G, R and N.

Let $K=K(A, B)+K(B, C)+K(C, D)+K(D, A)+K_{B,C,D}(A, X)$,

where the chain $K_{B,C,D}(A, X)$ is a chain which cannot be decomposed in other chains.

It is clear that K_1, K_2, G, R and N are true in K, but not C. Therefore C is independent of K_1, K_2, G, R and N.

In order to show that K_1 is independent of K_2, N and C, we shall consider an oriented chain K.

$K(\overrightarrow{X, Y})$ means a chain in K which connects X and Y, from X to Y.

We shall introduce the following definition for the separation of point-pairs.

Definition. Let A, B, X, Y be four different points in an oriented chain K. Then we may say that X and Y are separated in K, by A and B, if the two following conditions are both fulfilled:

$$(1) \qquad K(\overrightarrow{X, Y}) = 0 .$$

$$(2) \qquad K_A(\overrightarrow{X, Y}) + K_B(\overrightarrow{X, Y}) \neq 0 .$$

When two points X and Y are separated in K by A and B, we denote it by $AXBY$ (in K).

From the definition it follows at once that:

If $AXBY$ (in K), then we have also $BXAY$. But we cannot say in general that $AYBX$, $BYAX$.

4) K_1 is independent of K_2, N and C.

Let $\qquad K = K(\overrightarrow{B, A}) + K(\overrightarrow{A, D}) + K(\overrightarrow{B, C}) + K(\overrightarrow{C, D})$.

All the true tetrads in K are $ABCD$, $CBAD$; K_2, N and C are all true, but not K_1. Therefore K_1 is independent of K_2, N and C.

Next, we shall introduce the following definition for the separation of point-pairs.

Definition. Let A, B, X, Y be four different points in an oriented chain K. Then we may say that X and Y are separated in K, by A and B, if the following condition is fulfilled:

One and only one of A and B is between[1] X and Y in K.

When two points X and Y are separated in K, by A and B, and if B is between X and Y in K, we denote it by $AXBY$ (in K).

(1) B is between X and Y, if each chain in K which connects X and Y, from X to Y, according to definite direction in K, contains B.

From the definition it follows at once that:

We cannot say in general that $AXBY$ means $AYBX$. If $AXBY$ is true, then $BXAY$ is not true.

5) K_2 is independent of K_1, N and C.

Let

$$K = K(\overrightarrow{A, D}) + K(\overrightarrow{D, C}) + K(\overrightarrow{C, B}) + K(\overrightarrow{B, C}) + K(\overrightarrow{C, D}) + K(\overrightarrow{D, A}) .$$

All the true tetrads in K are $ABCD$, $ADCB$, $BCDA$, $BADC$. Hence K_1, N and C are true, but not K_2. Therefore K_2 is independent of K_1, N and C.

6) K_1, K_2 and G are all independent of R, N and C.

Let $$K = K(\overrightarrow{C, D}) + K(\overrightarrow{C, B}) + K(\overrightarrow{B, A}) + K(\overrightarrow{B, C}) .$$

All the true tetrads in K are $ABCD$, $DCBA$; R, N and C are true, but not K_1, K_2, G. Therefore K_1, K_2 and G are independent of R, N and C.

7) K_1, K_2 and R are all independent of G, N and C.

Let $$K = K(\overrightarrow{A, B}) + K(\overrightarrow{B, C}) + K(\overrightarrow{C, D}) + K(\overrightarrow{D, A}) .$$

All the true tetrads in K are $ABCD$, $BCDA$, $CDAB$, $DABC$. Thus G, N and C are all true, but not K_1, K_2, R. Therefore K_1, K_2 and R are all independent of G, N and C.