ON THE GEOMETRY OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS OF THIRD ORDER

By

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The geometry of paths was studied at first in Princeton, and extended to the general case

$$\frac{d^2x^i}{dt^2}+H^i\left(x^j, \frac{dx^j}{dt}\right)=0 \qquad (i, j, \ldots = 1, 2, \ldots, n)$$

by J. DOUGLAS[1] in 1928. Thereafter, these studies gave rise to research about the system of ordinary differential equations of order m

$$\frac{d^mx^i}{dt^m}+H^i\left(t, x^j, \frac{dx^j}{dt}, \ldots, \frac{d^{m-1}x^j}{dt^{m-1}}\right)=0,$$

and its differential geometry was constructed[2]. Moreover J. DOUGLAS has treated of the system of partial differential equations

$$(1) \qquad \frac{\partial^2x^i}{\partial u^\alpha \partial u^\beta}+H^i_{\alpha\beta}\left(x^j, \frac{\partial x^j}{\partial u^\gamma}\right)=0,$$

$$i, j, \ldots = 1, 2, \ldots, n; \quad \alpha, \beta, \ldots = 1, 2, \ldots, K$$

under the peculiar transformation-group of parameters u^α[3]. The geometry of the system of differential equations (1) under the general transformation-group of parameters u^α was well established by E. BORTOLOTTI[4].

(1) J. DOUGLAS, The general geometry of paths, Annals of Mathematics, (2), 29 (1928), 143–168.

(2) D. D. KOSAMBI, Path-spaces of higher order, Quarterly Journal of Mathematics, Oxford Ser., 7 (1936), 97–104; A. KAWAGUCHI and H. HOMBU, Die Geometrie des Systems der partiellen Differentialgleichungen, this Journal, ser. I, 6 (1937), 21–62; S. HOKARI, Zur neuen Behandlung der Geometrie des Systems der gewöhnlichen Differentialgleichungen höherer Ordnung, this Journal, ser. I, 8 (1940), 47–62.

(3) J. DOUGLAS, Systems of K-dimensional manifolds in an N-dimensional space, Mathematische Annalen, 105 (1931), 707–733.

(4) E. BORTOLOTTI, Trasporti non lineari: geometria di un sistema di equazioni alle derivate parziali del 2° ordine, Rendiconti di Lincei, (7), 23 (1936), 16–21, 104–110, 175–180.

The geometry of the system of partial differential equations of m-th order

$$\frac{\partial^m x^i}{\partial u^{\alpha_1}\dots\partial u^{\alpha_m}} + H^i_{\alpha_1\dots\alpha_m}\left(u^\beta, x^j, \frac{\partial x^j}{\partial u^{\beta_1}}, \dots, \frac{\partial^{m-1}x^j}{\partial u^{\beta_1}\dots\partial u^{\beta_{m-1}}}\right) = 0,$$

a generalization of (1), was treated by A. KAWAGUCHI and H. HOMBU, but its studies under the general transformation-group of parameters u^α are not yet completed. When $m = 3$ and $H^i_{\alpha_1\alpha_2\alpha_3}$ is of second order with respect to $\dfrac{\partial^2 x^j}{\partial u^{\beta_1}\partial u^{\beta_2}}$, the connection has already been determined by T. OHKUBO[1] by the method of elimination. Lately D. D. KOSAMBI has written to Prof. A. KAWAGUCHI on the suitability of his method of variation for the study of the system of partial differential equations of second order.

In this paper, the present author wishes to establish the differential geometry of the system of partial differential equations of third order by the method of variation. The coefficients of connection can not be perfectly determined by the given quantities $H^i_{\alpha_1\alpha_2\alpha_3}$ in general. But, when $H^i_{\alpha_1\alpha_2\alpha_3}$ is linear with respect to $\dfrac{\partial^2 x^j}{\partial u^{\beta_1}\partial u^{\beta_2}}$, one can perfectly determine the coefficients of connection by the given $H^i_{\alpha_1\alpha_2\alpha_3}$.

1. *Preparations.* $x^i\,(i = 1, 2, \dots, n)$ are the coordinates of a point in an n-dimensional manifold X_n, and $u^\alpha\,(\alpha = 1, 2, \dots, K;\ K \leqq n)$ the mutually independent parameters. With these parameters, a K-dimensional surface is given by $x^i = x^i(u^\alpha)$. At every point on this K-dimensional surface, a surface element of second order can be determined by

$$(2) \qquad x^i = x^i(u^\alpha), \quad p^i_{\alpha_1} = \frac{\partial x^i}{\partial u^{\alpha_1}}, \quad p^i_{\alpha_1\alpha_2} = \frac{\partial^2 x^i}{\partial u^{\alpha_1}\partial u^{\alpha_2}}.$$

Now adding to every point in X_n an arbitrary system of values (2), one has the manifold $F^{(2)}_n$. This manifold is of $\binom{n+2}{2}+n-1$ dimensions, which is treated in this paper.

It is my intention to handle here such a system as obeys the following postulates A_1, A_2, A_3, and the special transformation-group under which the postulates hold:

(1) T. OHKUBO, Die Geometrie der Differentialgleichungen dritter Ordnung, Japanese Journal of Mathematics, 14 (1938), 59 66.

A_1 The system of equations is transformed according to the tensor law,

A_2 the equations of variation of the given system are also tensorial when the variation itself is a vector,

A_3 there exists at least one operator, which is vectorial in character and corresponds to total differentiation with respect to one of the independent parameters.

We shall name the quantity in $F_n^{(2)}$ which is transformed according to the tensor law under the transformation-group of coordinates and parameters

$$(3) \qquad \bar{x}^i = \bar{x}^i(x^1, x^2, \ldots, x^n), \quad \bar{u}^\alpha = \bar{u}^\alpha(u^{\dot{1}}, u^{\dot{2}}, \ldots, u^{\dot{K}}),$$

the intrinsic quantity, according to E. BORTOLOTTI.

We can speak of x-transformations or u-transformations alone, and of x-tensors or u-tensors accordingly. Tensor will mean, unless otherwise mentioned, a geometrical object which has the proper law of transformation for both sorts of indices.

Now let us consider a system of partial differential equations of third order

$$(4) \qquad \frac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta \partial u^\gamma} + H^i_{\alpha\beta\gamma}(u^\nu, x^j, p^j_{\nu_1}, p^j_{\nu_1\nu_2}) = 0,$$

$$i, j, \ldots = 1, 2, \ldots, n; \quad \alpha, \beta, \ldots = \dot{1}, \dot{2}, \ldots, \dot{K},$$

and assume that $H^i_{\alpha\beta\gamma}$, the function of surface elements of second order, is continuous, differentiable and symmetric with respect to α, β, γ.

Under the transformation-group (3), the functions $H^i_{\alpha\beta\gamma}$ must have the transformation law

$$(5) \quad -\bar{H}^i_{\alpha\beta\gamma} = -H^j_{\rho\sigma\tau}\frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial u^\rho}{\partial \bar{u}^\alpha}\frac{\partial u^\sigma}{\partial \bar{u}^\beta}\frac{\partial u^\tau}{\partial \bar{u}^\gamma} + p^j_\rho \frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial^3 u^\rho}{\partial \bar{u}^\alpha \partial \bar{u}^\beta \partial \bar{u}^\gamma}$$

$$+ p^j_\rho p^k_\sigma p^l_\tau \frac{\partial u^\rho}{\partial \bar{u}^\alpha}\frac{\partial u^\sigma}{\partial \bar{u}^\beta}\frac{\partial u^\tau}{\partial \bar{u}^\gamma}\frac{\partial^3 \bar{x}^i}{\partial x^j \partial x^k \partial x^l}$$

$$+ 3p^j_{\rho\sigma}\frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial u^\rho}{\partial \bar{u}^{(\alpha}}\frac{\partial^2 u^\sigma}{\partial \bar{u}^\beta \partial \bar{u}^{\gamma)}} + 3p^j_\rho p^k_\sigma \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k}\frac{\partial u^\sigma}{\partial \bar{u}^{(\alpha}}\frac{\partial^2 u^\rho}{\partial \bar{u}^\beta \partial \bar{u}^{\gamma)}}$$

$$+ 3p^k_\sigma p^j_{\tau\rho}\frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k}\frac{\partial u^\tau}{\partial \bar{u}^{(\alpha}}\frac{\partial u^\sigma}{\partial \bar{u}^\beta}\frac{\partial u^\rho}{\partial \bar{u}^{\gamma)}}.$$

For $f(u^\beta, x^j, p^j_{\beta_1}, p^j_{\beta_1\beta_2})$, a quantity in $F^{(2)}_n$, one introduces the non-tensorial operator of differentiation with respect to u^α

$$(6)\qquad \partial_\alpha f \equiv \frac{\partial f}{\partial u^\alpha} + p^i_\alpha \frac{\partial f}{\partial x^i} + p^i_{\alpha\beta_1}\frac{\partial f}{\partial p^i_{\beta_1}} - H^i_{\alpha\beta_1\beta_2}\frac{\partial f}{\partial p^i_{\beta_1\beta_2}},$$

where $\partial_\alpha f$ is the differential of f along an integral surface of (4). Then we may assume the covariant differential of $T^{i\nu}_{j\sigma}$, an intrinsic tensor in $F^{(2)}_n$, along an integral surface of (4), to be of the form

$$(7)\qquad D_\alpha T^{i\nu}_{j\sigma} = \partial_\alpha T^{i\nu}_{j\sigma} + \gamma^i_{\alpha k} T^{k\nu}_{j\sigma} - \gamma^k_{\alpha j} T^{i\nu}_{k\sigma} + \Gamma^\nu_{\alpha\rho} T^{i\rho}_{j\sigma} - \Gamma^\rho_{\alpha\sigma} T^{i\nu}_{j\rho}.$$

The laws of transformation for the two sets of coefficients under the transformation-group (3) must be as follows:

$$(8)\qquad \begin{aligned} &\text{a)}\quad \bar\gamma^i_{\alpha s}\frac{\partial \bar x^s}{\partial x^r} + p^s_\rho \frac{\partial^2 \bar x^i}{\partial x^r \partial x^s}\frac{\partial u^\rho}{\partial \bar u^\alpha} = \gamma^s_{\rho r}\frac{\partial \bar x^i}{\partial x^s}\frac{\partial u^\rho}{\partial \bar u^\alpha},\\ &\text{b)}\quad \Gamma^\rho_{\alpha\beta}\frac{\partial u^\nu}{\partial \bar u^\rho} = \Gamma^\nu_{\sigma\tau}\frac{\partial u^\sigma}{\partial \bar u^\alpha}\frac{\partial u^\tau}{\partial \bar u^\beta} + \frac{\partial^2 u^\nu}{\partial \bar u^\alpha \partial \bar u^\beta}. \end{aligned}$$

From the last relation, we can assume that $\Gamma^\nu_{\alpha\beta}$ is symmetric with respect to both lower indices. Our purpose is to determine $\gamma^i_{\alpha j}$, $\Gamma^\nu_{\alpha\beta}$ each as the function of $H^i_{\alpha\beta\gamma}$.

2. *The first variation.* By the transformation of coordinates $\bar x^i = \bar x^i(x^1, x^2, \ldots, x^n)$, the surface elements of third order are transformed as follows:

$$(9)\qquad \begin{aligned} &\bar x^i = \bar x^i(x^j), \qquad \bar p^i_\alpha = \frac{\partial \bar x^i}{\partial x^j} p^j_\alpha,\\ &\bar p^i_{\alpha\beta} = \frac{\partial \bar x^i}{\partial x^j} p^j_{\alpha\beta} + \frac{\partial^2 \bar x^i}{\partial x^j \partial x^k} p^j_\alpha p^k_\beta,\\ &\bar p^i_{\alpha\beta\gamma} = \frac{\partial \bar x^i}{\partial x^j} p^j_{\alpha\beta\gamma} + \frac{\partial^2 \bar x^i}{\partial x^j \partial x^k} p^j_{\alpha\beta} p^k_\gamma + \frac{\partial^2 \bar x^i}{\partial x^j \partial x^k} p^j_{\alpha\gamma} p^k_\beta\\ &\qquad\qquad + \frac{\partial^2 \bar x^i}{\partial x^j \partial x^k} p^j_{\beta\gamma} p^k_\alpha + \frac{\partial^3 \bar x^i}{\partial x^j \partial x^k \partial x^l} p^j_\alpha p^k_\beta p^l_\gamma. \end{aligned}$$

Now we consider the infinitesimal transformation of coordinates

$$\bar{x}^i = x^i + \varepsilon\lambda^i , \tag{10}$$

where the parameters u^α must remain unaltered. Here λ's, the function of u^β, x^j, $p^j_{\beta_1}$, $p^j_{\beta_1\beta_2}$, represent the components of an intrinsic contravariant vector, and ε is infinitesimal.

(9) and (10) give us following relations:

$$\begin{aligned} &p^i_\alpha = p^i_\alpha + \varepsilon\partial_\alpha\lambda^i , \qquad \bar{p}^i_{\alpha\beta} = p^i_{\alpha\beta} + \varepsilon\partial_\beta\partial_\alpha\lambda^i , \\ &\bar{p}^i_{\alpha\beta\gamma} = p^i_{\alpha\beta\gamma} + \varepsilon\partial_\gamma\partial_\beta\partial_\alpha\lambda^i . \end{aligned} \tag{11}$$

Let us put (10) and (11) into

$$\frac{\partial^3\bar{x}^i}{\partial u^\alpha\partial u^\beta\partial u^\gamma} + \bar{H}^i_{\alpha\beta\gamma}(u^\nu, \bar{x}^j, \bar{p}^j_{\nu_1}, \bar{p}^j_{\nu_1\nu_2}) = 0 ,$$

which are the equations of paths relating to $\bar{x}^i$.

Expanding $\bar{H}^i_{\alpha\beta\gamma}$ into power series of ε, and neglecting the terms of ε^2 and higher powers, we have the equations of variation

$$\partial_\alpha\partial_\beta\partial_\gamma\lambda^i + \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^k_{\tau\rho}}\partial_\tau\partial_\rho\lambda^k + \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^k_\rho}\partial_\rho\lambda^k + \frac{\partial H^i_{\alpha\beta\gamma}}{\partial x^k}\lambda^k = 0 . \tag{12}$$

Use of the relations (7) enables us to get the following equations

$$\begin{aligned} D_\rho\lambda^k &= \partial_\rho\lambda^k + \gamma^k_{\rho j}\lambda^j , \\ D_\tau D_\rho\lambda^k &= \partial_\tau\partial_\rho\lambda^k + \gamma^k_{\rho j}\partial_\tau\lambda^j + \partial_\tau\gamma^k_{\rho j}\cdot\lambda^j + \gamma^k_{\tau j}D_\rho\lambda^j - \Gamma^\sigma_{\tau\rho}D_\sigma\lambda^k , \\ D_\alpha D_\beta D_\gamma\lambda^k &= \partial_\alpha\partial_\beta\partial_\gamma\lambda^k + \gamma^k_{\gamma j}\partial_\alpha\partial_\beta\lambda^j + \gamma^k_{\beta j}\partial_\alpha D_\gamma\lambda^j - \Gamma^\sigma_{\beta\gamma}\partial_\alpha D_\sigma\lambda^k \\ &\quad + \gamma^k_{\alpha j}D_\beta D_\gamma\lambda^j - \Gamma^\rho_{\alpha\beta}D_\rho D_\gamma\lambda^k - \Gamma^\rho_{\alpha\gamma}D_\beta D_\rho\lambda^k + \partial_\beta\gamma^k_{\gamma j}\cdot\partial_\alpha\lambda^j \\ &\quad + \partial_\alpha\gamma^k_{\gamma j}\cdot\partial_\beta\lambda^j + \partial_\alpha\gamma^k_{\beta j}\cdot D_\gamma\lambda^j - \partial_\alpha\Gamma^\sigma_{\beta\gamma}\cdot D_\sigma\lambda^k + \partial_\alpha\partial_\beta\gamma^k_{\gamma j}\cdot\lambda^j . \end{aligned}$$

Putting these equations into (12), one can derive the invariantive form

$$D_\alpha D_\beta D_\gamma\lambda^i + 2E^{i(\tau\rho)}_{j\alpha\beta\gamma}D_\tau D_\rho\lambda^j + F^{i\rho}_{j\alpha\beta\gamma}D_\rho\lambda^j + G^i_{j\alpha\beta\gamma}\lambda^j = 0 , \tag{13}$$

where

$$E^{i\tau\rho}_{j\alpha\beta\gamma} = \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\tau}} + \Gamma^\rho_{\beta\gamma}\delta^\tau_\alpha\delta^i_j + \Gamma^\rho_{\alpha\gamma}\delta^\tau_\beta\delta^i_j + \Gamma^\tau_{\alpha\beta}\delta^\rho_\gamma\delta^i_j - \gamma^i_{\beta j}\delta^\tau_\alpha\delta^\rho_\gamma - \gamma^i_{\alpha j}\delta^\tau_\beta\delta^\rho_\gamma - \gamma^i_{\gamma j}\delta^\tau_\alpha\delta^\rho_\beta ,$$

$$F^{i\tau}_{j\alpha\beta\gamma} = \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_\tau} - \partial_\alpha\gamma^i_{\beta j}\cdot\delta^\tau_\gamma - \partial_\beta\gamma^i_{\gamma j}\cdot\delta^\tau_\alpha - \partial_\alpha\gamma^i_{\gamma j}\cdot\delta^\tau_\beta + \gamma^i_{\gamma r}\gamma^r_{\beta j}\delta^\tau_\alpha + \gamma^i_{\gamma r}\gamma^r_{\alpha j}\delta^\tau_\beta + \gamma^i_{\beta r}\gamma^r_{\alpha j}\delta^\tau_\gamma$$

$$- \Gamma^\tau_{\beta\gamma}\gamma^i_{\alpha j} - \Gamma^\tau_{\alpha\gamma}\gamma^i_{\beta j} - \Gamma^\tau_{\alpha\beta}\gamma^i_{\gamma j} + \partial_\alpha\Gamma^\tau_{\beta\gamma}\cdot\delta^i_j + \Gamma^\sigma_{\beta\gamma}\Gamma^\tau_{\alpha\sigma}\delta^i_j - 2\gamma^r_{\rho j}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^r_{\rho\tau}} + \Gamma^\tau_{\sigma\rho}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\sigma}} ,$$

$$G^i_{j\alpha\beta\gamma} = \frac{\partial H^i_{\alpha\beta\gamma}}{\partial x^j} - \partial_\alpha\partial_\beta\gamma^i_{\gamma j} + \partial_\beta\gamma^i_{\gamma r}\cdot\gamma^r_{\alpha j} + \partial_\alpha\gamma^i_{\gamma r}\cdot\gamma^r_{\beta j} - \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^r_{\rho\tau}}\partial_\tau\gamma^r_{\rho j} - \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^r_\rho}\gamma^r_{\rho j}$$

$$+ \gamma^i_{\gamma r}\partial_\alpha\gamma^r_{\beta j} + \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^k_{\rho\tau}}\gamma^k_{\rho r}\gamma^r_{\tau j} - \gamma^i_{\gamma r}\gamma^r_{\beta k}\gamma^k_{\alpha j} .$$

In order to determine $\gamma^i_{\alpha j}$, $\Gamma^\nu_{\alpha\beta}$ as intrinsic, we assume that all possible contractions of $E^{i(\tau\rho)}_{j\alpha\beta\gamma}$ vanish. Then we have

$$\begin{aligned}
&2E^{i(\tau\rho)}_{j\alpha\tau\rho} = 2\frac{\partial H^i_{\alpha\tau\rho}}{\partial p^j_{\rho\tau}} + 2(K+2)\Gamma^\rho_{\alpha\rho}\delta^i_j - (K+2)(K+1)\gamma^i_{\alpha j} = 0 ,\\
&2E^{i(\tau\rho)}_{i\alpha\tau\rho} = 2\frac{\partial H^i_{\alpha\tau\rho}}{\partial p^i_{\rho\tau}} + 2n(K+2)\Gamma^\rho_{\alpha\rho} - (K+2)(K+1)\gamma^i_{\alpha i} = 0 ,\\
&2E^{i(\tau\rho)}_{i\tau\beta\gamma} = 2\frac{\partial H^i_{\tau\beta\gamma}}{\partial p^i_{\rho\tau}} + n(K+3)\Gamma^\rho_{\beta\gamma} + n(\Gamma^\tau_{\tau\beta}\delta^\rho_\gamma + \Gamma^\tau_{\tau\gamma}\delta^\rho_\beta)\\
&\qquad\qquad - (K+2)(\delta^\rho_\gamma\gamma^i_{\beta i} + \delta^\rho_\beta\gamma^i_{\gamma i}) = 0 .
\end{aligned} \tag{14}$$

3. *The second variation.* From (14), $\gamma^i_{\alpha j}$ and $\Gamma^\nu_{\alpha\beta}$ can be determined, if we can express $\Gamma^\rho_{\alpha\rho}$ as function of $H^i_{\alpha\beta\gamma}$. For this purpose, we give a vector variation

$$\bar{u}^\alpha = u^\alpha + \varepsilon\mu^\alpha \tag{15}$$

to u^α, where μ^α are functions of u^β and the components of u-contravariant vector.

From (15), the following relations are obtained:

$$p^i_\alpha = \bar{p}^i_\alpha + \varepsilon\bar{p}^i_\beta\partial_\alpha\mu^\beta ,$$

$$p^i_{\alpha\beta} = \bar{p}^i_{\alpha\beta} + \varepsilon(\bar{p}^i_\rho\partial_\beta\partial_\alpha\mu^\rho + p^i_{\alpha\rho}\partial_\beta\mu^\rho + \bar{p}^i_{\beta\rho}\partial_\alpha\mu^\rho) + [\varepsilon^2] ,$$

$$p^i_{\alpha\beta\gamma} = \bar{p}^i_{\alpha\beta\gamma} + \varepsilon(\bar{p}^i_\rho \partial_\gamma \partial_\beta \partial_\alpha \mu^\rho + \bar{p}^i_{\alpha\rho} \partial_\gamma \partial_\beta \mu^\rho + \bar{p}^i_{\beta\rho} \partial_\gamma \partial_\alpha \mu^\rho$$
$$+ \bar{p}^i_{\gamma\rho} \partial_\beta \partial_\alpha \mu^\rho + 3\bar{p}^i_{\rho(\alpha\beta} \partial_{\gamma)} \mu^\rho) + [\varepsilon^2],$$

where $[\varepsilon^2]$ denotes terms of second and higher order of ε.

When ε is infinitesimal, we may replace $\bar{p}^i_\alpha$, $\bar{p}^i_{\alpha\beta}$, $\bar{p}^i_{\alpha\beta\gamma}$ with p^i_α, $p^i_{\alpha\beta}$, $p^i_{\alpha\beta\gamma}$. Expanding $\bar{p}^i_{\alpha\beta\gamma}$ into power series of ε, it follows

$$\bar{p}^i_{\alpha\beta\gamma} = -H^i_{\alpha\beta\gamma} - \varepsilon\mu^\rho \frac{\partial H^i_{\alpha\beta\gamma}}{\partial u^\rho} + \varepsilon p^j_\nu \partial_\rho \mu^\nu \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_\rho}$$
$$+ \varepsilon(p^j_\nu \partial_\tau \partial_\rho \mu^\nu + p^j_{\rho\nu} \partial_\tau \mu^\nu + p^j_{\nu\tau} \partial_\rho \mu^\nu) \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\tau}} + [\varepsilon^2].$$

From these relations, we get the equations of variation

$$\text{(16)} \qquad p^i_\rho \partial_\gamma \partial_\beta \partial_\alpha \mu^\rho + \partial_\tau \partial_\rho \mu^\nu \cdot p^j_\nu \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\tau}} + 3p^i_{\rho(\gamma} \partial_\beta \partial_{\alpha)} \mu^\rho$$
$$+ 3p^i_{\rho(\alpha\beta} \partial_{\gamma)} \mu^\rho + p^j_\nu \partial_\rho \mu^\nu \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_\rho} + p^j_{\rho\nu} \partial_\tau \mu^\nu \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\tau}}$$
$$+ p^j_{\nu\tau} \partial_\rho \mu^\nu \frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\tau}} - \mu^\rho \frac{\partial H^i_{\alpha\beta\gamma}}{\partial u^\rho} = 0.$$

On the other hand

$$D_\alpha \mu^\rho = \partial_\alpha \mu^\rho + \Gamma^\rho_{\alpha\nu} \mu^\nu,$$
$$D_\beta D_\alpha \mu^\rho = \partial_\beta \partial_\alpha \mu^\rho + \Pi^{\rho\tau}_{\alpha\nu\beta} \cdot D_\tau \mu^\nu + \partial_\beta \Gamma^\rho_{\alpha\nu} \cdot \mu^\nu,$$
$$D_\gamma D_\beta D_\alpha \mu^\rho = \partial_\gamma \partial_\beta \partial_\alpha \mu^\rho + \wedge^{\rho\tau\sigma}_{\alpha\nu\beta\gamma} \cdot D_\sigma D_\tau \mu^\nu + \Pi^{\rho\tau}_{\alpha\nu\beta}(\Gamma^\sigma_{\gamma\tau} D_\sigma \mu^\nu - \Gamma^\nu_{\gamma\sigma} D_\tau \mu^\sigma),$$

where

$$\text{(17)} \qquad \begin{aligned} \Pi^{\rho\tau}_{\alpha\nu\beta} &= \Gamma^\rho_{\alpha\nu}\delta^\tau_\beta + \Gamma^\rho_{\beta\nu}\delta^\tau_\alpha - \Gamma^\tau_{\beta\alpha}\delta^\rho_\nu, \\ \wedge^{\rho\tau\sigma}_{\alpha\nu\beta\gamma} &= \Pi^{\rho\tau}_{\alpha\nu\beta}\delta^\sigma_\gamma + \Gamma^\rho_{\gamma\nu}\delta^\sigma_\beta\delta^\tau_\alpha - \Gamma^\sigma_{\gamma\beta}\delta^\tau_\alpha\delta^\rho_\nu - \Gamma^\tau_{\gamma\alpha}\delta^\sigma_\beta\delta^\rho_\nu. \end{aligned}$$

By virtue of these relations, we can write down the equations of variation (16) in the following invariantive form

$$\text{(18)} \qquad p^i_\rho D_\gamma D_\beta D_\alpha \mu^\rho + P^{\tau\sigma i}_{\alpha\beta\gamma\nu} \cdot D_\sigma D_\tau \mu^\nu + Q^{i\sigma}_{\alpha\beta\gamma\nu} D_\sigma \mu^\nu - R^i_{\alpha\beta\gamma\nu} \mu^\nu = 0,$$

where

$$P^{\tau\sigma i}_{\alpha\beta\gamma\nu} = 2\left(3p^i_{\nu(\gamma}\delta^{(\sigma}_{\beta}\delta^{\tau)}_{\alpha}+p^j_{\nu}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{(\tau\sigma)}}-p^i_{\rho}\wedge^{\rho(\tau\sigma)}_{\alpha\nu\beta\gamma}\right),$$

$$Q^{i\sigma}_{\alpha\beta\gamma\nu} = p^i_{\rho}\Pi^{\rho\sigma}_{\alpha\tau\beta}\Gamma^{\tau}_{\gamma\nu}-p^i_{\rho}\Pi^{\rho\tau}_{\alpha\nu\beta}\Gamma^{\sigma}_{\gamma\tau}-p^j_{\varkappa}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\tau}}\Pi^{\varkappa\sigma}_{\rho\nu\tau}-3p^i_{\rho(\gamma}\Pi^{\rho\sigma}_{\alpha|\nu|\beta)}$$

$$(19)\qquad +p^j_{\nu}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\sigma}}+p^j_{\rho\nu}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\sigma}}+p^j_{\nu\tau}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\sigma\tau}}-3H^i_{\nu(\alpha\beta}\delta^{\sigma}_{\gamma)},$$

$$R^i_{\alpha\beta\gamma\nu} = p^j_{\varkappa}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\tau}}\partial_{\tau}\Gamma^{\varkappa}_{\rho\nu}+3p^i_{\rho(\gamma}\partial_{\beta}\Gamma^{\rho}_{\alpha)\nu}+p^j_{\varkappa}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho}}\Gamma^{\varkappa}_{\rho\nu}$$

$$+p^j_{\rho\sigma}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\tau}}\Gamma^{\sigma}_{\tau\nu}+p^j_{\sigma\tau}\frac{\partial H^i_{\alpha\beta\gamma}}{\partial p^j_{\rho\tau}}\Gamma^{\sigma}_{\rho\nu}+\frac{\partial H^i_{\alpha\beta\gamma}}{\partial u^{\nu}}.$$

4. *Determination of the coefficients of connection.* In the case of the system of partial differential equations of second order, $\Gamma^{\nu}_{\alpha\beta}$ can be determined from the coefficients of $D_{\tau}\mu^{\nu}$ as D. D. KOSAMBI has shown. Hence in our case also at first we shall consider all possible contractions of $P^{\tau\sigma i}_{\alpha\beta\gamma\nu}$ as equal to zero.

For example, let us consider $P^{\tau\sigma i}_{\alpha\beta\sigma\tau}$, then it follows

$$(20)\qquad p^i_{\rho}(\wedge^{\rho\tau\sigma}_{\alpha\tau\beta\sigma}+\wedge^{\rho\sigma\tau}_{\alpha\tau\beta\sigma}) = A^i_{\alpha\beta},$$

where

$$A^i_{\alpha\beta} = 2\left(3p^i_{\tau(\sigma}\delta^{(\sigma}_{\beta}\delta^{\tau)}_{\alpha}+p^j_{\tau}\frac{\partial H^i_{\alpha\beta\sigma}}{\partial p^j_{(\tau\sigma)}}\right).$$

We multiply the tensor $H^{\rho}_{i} = \dfrac{\partial^2 H^j_{\alpha\beta\gamma}}{\partial p^j_{\alpha\beta}\partial p^i_{\gamma\rho}}$ on both sides of (20), and contract them, then we have

$$(21)\qquad (K+1)\Pi^{\rho\tau}_{\alpha\tau\beta}-\Gamma^{\sigma}_{\sigma\beta}\delta^{\rho}_{\alpha}-\Gamma^{\sigma}_{\sigma\alpha}\delta^{\rho}_{\beta} = B^{\rho}_{\alpha\beta},$$

where

$$B^{\rho}_{\alpha\beta} = \frac{A^i_{\alpha\beta}H^{\rho}_{i}}{p^j_{\sigma}H^{\sigma}_{j}}.$$

From (21) $B^{\rho}_{\alpha\rho} \equiv 0$ is easily concluded and $\Gamma^{\rho}_{\alpha\rho}$ cannot be determined. We cannot determine $\Gamma^{\rho}_{\alpha\rho}$ from any possible contractions of $P^{\tau\sigma i}_{\alpha\beta\gamma\nu}$.

Next, we shall consider all possible contractions of $Q^{i\sigma}_{\alpha\beta\gamma\nu}$ as equal to zero. Now we have

$$Q^{i\nu}_{\alpha\beta\gamma\nu} = Y^{i\nu}_{\alpha\beta\gamma\nu} - p^{j}_{\varkappa}\frac{\partial H^{i}_{\alpha\beta\gamma}}{\partial p^{j}_{\rho\tau}}\Pi^{\varkappa\nu}_{\rho\nu\tau} - 3p^{i}_{\rho(\gamma}\Pi^{\rho\nu}_{\alpha|\nu|\beta)} = 0\,,$$

where

$$Y^{i\sigma}_{\alpha\beta\gamma\nu} = p^{j}_{\nu}\frac{\partial H^{i}_{\alpha\beta\gamma}}{\partial p^{j}_{\sigma}} + p^{j}_{\rho\nu}\frac{\partial H^{i}_{\alpha\beta\gamma}}{\partial p^{j}_{\rho\sigma}} + p^{j}_{\nu\tau}\frac{\partial H^{i}_{\alpha\beta\gamma}}{\partial p^{j}_{\sigma\tau}} - 3H^{i}_{\nu(\alpha\beta}\delta^{\sigma}_{\gamma)}\,,$$

accordingly

$$p^{j}_{\sigma}\frac{\partial H^{i}_{\alpha\beta\gamma}}{\partial p^{j}_{\rho\tau}}\Gamma^{\sigma}_{\rho\tau} + 3p^{i}_{\sigma(\gamma}\delta^{\rho}_{\alpha}\delta^{\tau}_{\beta)}\Gamma^{\sigma}_{\rho\tau} - Y^{i\nu}_{\alpha\beta\gamma\nu} = 0\,. \tag{22}$$

We shall now go on to a special case when $H^{i}_{\alpha\beta\gamma}$ is linear with respect to $p^{j}_{\alpha\beta}$. Differentiating (22) by $p^{j}_{\lambda\mu}$, we have, by use of $\dfrac{\partial^2 H^{i}_{\alpha\beta\gamma}}{\partial p^{j}_{\rho\tau}\partial p^{k}_{\lambda\mu}} = 0$,

$$\delta^{i}_{j}(\delta^{\lambda\mu}_{(\sigma\gamma)}\delta^{\rho\tau}_{\alpha\beta} + \delta^{\lambda\mu}_{(\sigma\beta)}\delta^{\rho\tau}_{\gamma\alpha} + \delta^{\lambda\mu}_{(\sigma\alpha)}\delta^{\rho\tau}_{\beta\gamma})\Gamma^{\sigma}_{\rho\tau} = Y^{i\nu\lambda\mu}_{\alpha\beta\gamma\nu j}\,,$$

where

$$Y^{i\nu\lambda\mu}_{\alpha\beta\gamma\nu j} = \frac{\partial Y^{i\nu}_{\alpha\beta\gamma\nu}}{\partial p^{j}_{\lambda\mu}}\,.$$

From this, it follows that

$$\delta^{\lambda\mu}_{(\sigma\gamma)}\Gamma^{\sigma}_{\alpha\beta} + \delta^{\lambda\mu}_{(\sigma\beta)}\Gamma^{\sigma}_{\gamma\alpha} + \delta^{\lambda\mu}_{(\sigma\alpha)}\Gamma^{\sigma}_{\beta\gamma} = \frac{1}{n}Y^{i\nu\lambda\mu}_{\alpha\beta\gamma\nu i}\,.$$

Putting $\lambda = \alpha$, $\mu = \beta$, and contracting these indices, one gets

$$\Gamma^{\sigma}_{\gamma\sigma} = \frac{1}{n(K+2)}Y^{i\nu\alpha\beta}_{\alpha\beta\gamma\nu i}\,, \tag{23}$$

from which we can determine $\Gamma^{\sigma}_{\gamma\sigma}$ as the function of $H^{i}_{\alpha\beta\gamma}$; and accordingly $\gamma^{i}_{\alpha j}$, $\Gamma^{\nu}_{\alpha\beta}$ from (14). Therefore it is stated:

Theorem. *When $H^i_{\alpha\beta\gamma}$ in the system of partial differential equations (4) is linear with respect to $p^j_{\alpha\beta}$, then the intrinsic parameters of connection $\gamma^i_{\alpha j}$ and $\Gamma^\nu_{\alpha\beta}$ can be perfectly determined from $H^i_{\alpha\beta\gamma}$.*

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