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A REMARK TO A COVARIANT DIFFERENTIATION PROCESS

By

Satoshi MICHIHIRO

In one of his papers CRaIG\(^{(1)}\) has introduced a covariant derivative

\[
T_{\beta\cdots x^{(m-1)}\tau}^{\alpha} - m T_{\beta\cdots x^{(m)}\lambda}^{\alpha} \left\{ \begin{array}{l} \lambda \\ \gamma \end{array} \right\},
\]

from a tensor \( T_{\beta}^{\alpha} \ldots \) whose components depend on \( n \) coordinates \( x \) and their \( m \) derivatives \( x, x', x'', \ldots, x^{(m)} \), that is, are of order \( m \), where

\[
\left\{ \begin{array}{l} \lambda \\ \gamma \end{array} \right\} = x'^{\sigma} \Gamma_{\sigma\tau}^{\lambda} + \frac{1}{2} x''^{\rho} f_{\tau\delta\gamma} f^{\delta\lambda},
\]

which was obtained by TAYLOR\(^{(2)}\), and partial differentiation was denoted by the subscript. Thereafter, extending this process, JOHNSON\(^{(3)}\) has introduced the following covariant derivative

\[
T_{\beta\cdots x^{(m-2)}\tau}^{\alpha} - (m-1) T_{\beta\cdots x^{(m-1)}\lambda}^{\alpha} \left\{ \begin{array}{l} \lambda \\ \gamma \end{array} \right\} - \frac{m(m-1)}{2} T_{\beta\cdots x^{(m)}\lambda}^{\alpha} \left\{ \begin{array}{l} \lambda \\ \gamma \end{array} \right\},
\]

where

\[
\left| \begin{array}{l} \lambda \\ \gamma \end{array} \right| = Q_{2\tau}^{\alpha} - Q_{2\alpha}^{\tau} \left\{ \begin{array}{l} \alpha \\ \gamma \end{array} \right\} + Q^{\alpha} A_{\alpha\tau}^{\lambda}, \quad Q^{\alpha} = x'^{\alpha} + \Gamma_{\beta\tau}^{\alpha} x'^{\beta} x'^{\tau},
\]

\[
A_{\alpha\tau}^{\lambda} = \Gamma_{\alpha\tau}^{\lambda} - \frac{1}{2} f^{\lambda\beta\alpha} \left( f_{\tau\delta\gamma} \left\{ \begin{array}{l} \tau \\ \alpha \end{array} \right\} + f_{\beta\delta\gamma} \left\{ \begin{array}{l} \tau \\ \gamma \end{array} \right\} - f_{\alpha\tau\delta} \left\{ \begin{array}{l} \tau \\ \beta \end{array} \right\} \right).
\]

To obtain this result, she eliminated \( \partial x^{(m)}/\partial y^{k} \) making use of a very complicated calculation. Accordingly it seems almost impossible to extend her method further.

In the present paper it is proposed to introduce a general covariant differentiation process which involves (1) and (2) as special cases.

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The writer expresses his heartfelt thanks to Prof. A. KAWAGUCHI for his kind advices.

In a KAWAGUCHI space\(^{(1)}\) of order \(m\) and dimension \(n\) a covariant differentiation of a contravariant vector \(X^i\) is described in the form

\[
\frac{\delta X^i}{dt} = \frac{dX^i}{dt} + \Gamma^i_j X^j .
\]

The notations

\[
X^{(0)i} = X^i , \quad X^{(s)i} = \frac{d^s X^i}{dt^s} \quad (s = 1, 2, \ldots)
\]

will be adopted for simplicity.

Let \(X^i\) be any contravariant vector, then its \(r\)-th covariant derivative is expressed by a linear combination of \(X^i\) and its ordinary derivatives with respect to \(t\)

\[
\frac{\delta^r X^i}{dt^r} = \sum_{s=0}^{r} \binom{r}{s} (\frac{r}{s}) \Gamma^i_j X^{(r-s)j} ,
\]

where

\[
\Gamma^{(0)}_j = \delta^i_j , \quad \Gamma^{(1)}_j = \Gamma^i_j , \quad \Gamma^{(s)}_j = \frac{d\Gamma^i_j}{dt} + \Gamma^k_j \Gamma^i_k \quad (s = 2, 3, \ldots, r).
\]

This can be proved by mathematical induction. That is, if (3) is correct for \(r\),

\[
\frac{\delta^{r+1} X^i}{dt^{r+1}} = \sum_{s=0}^{r} \binom{r}{s} \left( \frac{r}{s} \right) \Gamma^{(a)}_j X^{(r+1-s)j} + \frac{d\Gamma^i_j}{dt} X^{(r-s)j} + \Gamma^k_j \Gamma^i_k X^{(r-s)j}
\]

\[
= \sum_{s=0}^{r} \binom{r}{s} \left( \frac{r}{s} \right) \Gamma^{(a)}_j X^{(r+1-s)j} + \Gamma^{(s+1)}_j X^{(r-s)j}
\]

\[
= \sum_{s=0}^{r+1} \binom{r+1}{s} \Gamma^{(s)}_j X^{(r+1-s)j} .
\]

And (3) is primitive for \(r = 1\). Therefore (3) is true for all \(r\).

Inversely:

Let \(X^i\) be any contravariant vector, then its \(r\)-th ordinary derivative with respect to \(t\) is expressed by a linear combination of \(X^i\) and its covariant derivatives

(4) \[ X^{(r)i} = \sum_{s=0}^{r} \binom{r}{s} \Pi_{j}^{i} \frac{\delta^{r-s}X^{j}}{dt^{r-s}} , \]

where

\[ \Pi_{j}^{i} = \frac{d\Pi_{j}^{i}}{dt} + \Pi_{k}^{i} \Pi_{j}^{k} \epsilon \quad (s=2, 3, \ldots, r) . \]

In one of his papers Prof. A. KAWAGUCHI has introduced the covariant derivation(1)

(5) \[ D_{ij}(T)X^{j}p = \sum_{\alpha=p}^{p} \binom{\alpha}{\rho} T_{i<\alpha)j} X^{(\alpha-r:)j} \]

(\[ p=1, 2, \ldots, p \])

along a curve, \( T_{i} \) being any covariant vector of order \( p \) and \( X^{i} \) an arbitrary contravariant vector of any order. Use of (4) leads the right member of (6) to

\[ D_{ij}(T)X^{j}p = \sum_{\alpha=p}^{p} \binom{\alpha}{\rho} T_{i\langle\alpha)j} X^{(\alpha-r:)j} \]

Therefore it is obtained covariant tensors

(7) \[ \sum_{\alpha=1}^{p} \binom{\alpha}{\lambda} T_{i\langle\alpha)k} \Pi_{j}^{k} \]

Let any tensor \( T_{i\cdots:} \) be considered, then

(8) \[ \sum_{\alpha=1}^{p} \binom{\alpha}{\lambda} T_{i\cdots\langle\alpha)k} \Pi_{j}^{k} \]

are tensors whose indices are one more than those of \( T_{i\cdots:} \), where \( \Pi_{j}^{k} \) (\( s=0, 1, \ldots, p \)) are defined by (5).

For \( p=m, \lambda=m-1 \), one obtains

(9) \[ T_{i\cdots\langle(0)k} \Pi_{j}^{k} + m T_{i\cdots\langle(0)k} \Pi_{j}^{k} \]

and, for $p = m$, $i = m - 2$, one obtains

\[ T^{\ldots(m-2)k}_{i\cdots} \Pi_{j}^{k} + (m-1)T^{\ldots(m-1)k}_{i\cdots} \Pi_{j}^{k} + \frac{m(m-1)}{2} T^{l}_{i\cdots(m)j} \Pi_{j}^{k} \]

from (8).

CRAIG has used \{ \text{index} \} in place of our $\Gamma^{i}_{j}$, and (9) reduces to his covariant differentiation (1). (10) is different partly from JOHNSON’S. But one can reconcile the present process to hers, as follows. Making use of

\[ \frac{\delta dx^{i}}{dt} = dx^{i} + \left\{ \text{index} \right\} dx^{j}, \]

one has

\[ df_{ij} = \left( f_{ij(0)k} - f_{ij(1)l} \left\{ \text{index} \right\} \right) dx^{k} + f_{ij(1)l} \frac{\delta dx^{l}}{dt}, \]

and in the same way

\[ df_{jk} = \left( f_{jk(0)i} - f_{jk(1)l} \left\{ \text{index} \right\} \right) dx^{i} + f_{jk(1)l} \frac{\delta dx^{l}}{dt}, \]

\[ df_{ki} = \left( f_{ki(0)j} - f_{ki(1)l} \left\{ \text{index} \right\} \right) dx^{j} + f_{ki(1)l} \frac{\delta dx^{l}}{dt}. \]

From these it is found that $\Lambda_{jk}^{i}$ are transformed just as well as the parameters of an affine connection under a coordinate transformation. Therefore one can introduce the following base connection

\[ \delta Q^{i} = dQ^{i} + \Lambda_{jk}^{i} Q^{j} dx^{k} = dx^{i} + \left\{ \text{index} \right\} dx^{j} + Q_{l}^{i} \frac{\delta dx^{k}}{dt}. \]

From (11), (12) and

\[ T^{\ldots(m-2)j}_{i\cdots} dx^{j} + (m-1)T^{\ldots(m-1)j}_{i\cdots} dx^{j} + \frac{m(m-1)}{2} T^{l}_{i\cdots(m)j} dx^{j}, \]

which is a tensor obtained immediately by using the vector $dx^{i}$ instead of $X^{i}$ in (6), her covariant derivative (2) is obtained.

August, 1940.