GEOMETRY IN A SPACE WITH GENERALIZED METRICS II [1]

By

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Introduction

As a natural generalization of a RIEMANNian space, we have a FINSLER space[2]. In a FINSLER space the distance between two consecutive points is given by $ds = L(x, dx)$ while in a RIEMANNian space it is given by $ds = \sqrt{g_{ij}(x)dx^i dx^j}$, where L is a positively homogeneous function of degree one in the dx^i. In a FINSLER space $g_{ij}(x, dx) = \frac{1}{2}\partial^2 L / \partial dx^i \partial dx^j$ [3] is adopted as the fundamental tensor and the functions $g_{ij}(x, dx)$ are homogeneous of degree zero in the dx^i.

Other generalizations have been made by E. CARTAN[4], L. BERWALD[5], A. KAWAGUCHI[6] and S. HOKARI[7]. The first two studied a theory of a generalized space (CARTAN space[8]) in which geometry is ruled by an $(n-1)$-ple integral. On the other hand the last two established geometry in an n-dimensional metric space with a connection which depends on an m-dimensional $(m < n)$ surface element (x^i, p^i_α) $(i = 1, 2, \ldots, n;\ \alpha = 1, 2, \ldots, m)$.

The author, in his recent work[9], studied an n-dimensional metric space with an euclidean connection having the fundamental tensor g_{ij} depending on a line-element (x, x'), where g_{ij} are homogeneous functions of degree zero in the x'^i and are given *a priori*. The contracted tensor A^i_{0k} (k fixed) was decomposed into $l^i A_k$ by Postulate 5 [10].

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1) The cost of this research has been defrayed from *the Scientific Research Expenditure of the Department of Education.*

2) P. FINSLER [1]. (The numbers in the brackets after the author's name refer to the bibliography at the end of the present paper). See also E. CARTAN [1] and L. BERWALD [1].

3) See E. CARTAN [1], p. 11.

4) E. CARTAN [2].

5) L. BERWALD [2], [3], [4].

6) A. KAWAGUCHI [1], [2].

7) A. KAWAGUCHI and S. HOKARI [1], [2].

8) See BERWALD [2].

9) T. OHKUBO [1].

10) See T. OHKUBO [1], p. 50.

But, in general, it is not natural and, in this paper, it is propcsed to take in place of that postulate the following: "The two vectors σ^i and $K_k{}^i\omega^k$ are identical".

In Chapter I an euclidean connection is determined in the space. The method employed is similar to that of L. BERWALD. But it is to be noticed that the vectors θ_i and σ^i play an important part in this theory (§ 6). In Chapter II the theory of curves is explained and in Chapter III the torsion and curvature tensors are calculated. The last chapter is devoted to the establishment of theory of hypersurfaces, which is almost analogous to that in a FINSLER space.

The author wishes to express his best thanks to Prof. A. KAWAGUCHI for his kind advices.

Chapter I. A general metric space with an euclidean connection.

§ 1. *The space with an euclidean connection.* First it is proposed to explain the notion of a space of line-elements with an euclidean connection. A line-element means the figure formed by a point $(x^1, x^2, \ldots, x^n)$ and a direction issuing from this point; this direction can be defined by n homogeneous parameters $x'^1, x'^2, \ldots, x'^n$ which have vector charactor and only their mutual ratios are essential. The point (x) is called the element. A space of line-elements with an euclidean connection is denoted by $K_n^{(1)}$ and defined analytically by the following two postulates for the metrics and for the parallel displacement of a vector. It is desirable that the metrics is euclidean in the neighbourhood of each line-element:

(I) The square of the distance from any point (x) to its consecutive point $(x+dx)$ with respect to the line-element in (x) is represented by a positive definite quadratic differential form $g_{ij}dx^i dx^j$, the coefficients $g_{ij}=g_{ji}$ depending on only the line-element.

Postulate I means that the g_{ij} are the functions of x^i and x'^i and positively homogeneous of degree zero in the x'^i. Moreover it is assumed that they are analytic in a certain simple-connected region of the x^i and for all the values of the x'^i with exception of the value-system $(0, 0, \ldots, 0)$.

The figure which determines the metrics is therefore two points infinitely near to each other (x), $(x+dx)$ and the line-element (x, x'). The figure is said to be an infinitesimal vector (dx) in the line-element (x, x'). One thus gets the fragment of an euclidean space which is

associated to each line-element and he can also arrive at finite vectors, tensors and any other geometric objects. Hereafter the components of a geometric object are always considered as functions of a line-element, that is, x^i and x'^i, which are homogeneous of degree zero in the x'^i.

For a finite vector Postulate I shows that the square of the length of a vector (X) in a line-element (x, x') is given by

$$(1) \qquad g_{ij}(x, x')X^i X^j .$$

Hereafter the expression $g_{ij}x'^i x'^j$ will be denoted by g and the inverse system of g_{ij} by g^{ij} and raising and lowering of a suffix can be defined by means of g^{ij} and g_{ij}.

The unit vector along a line-element (x, x') has the contravariant and covariant components

$$(2) \qquad l^i = x'^i g^{-\frac{1}{2}}, \qquad l_i = g_{ij}l^j$$

respectively.

To obtain the euclidean connection, it is necessary to define the parallel displacement of a vector from a line-element to its consecutive one.

(II) Let (X) (whose components are necessarily homogeneous functions of degree zero in the x'^i) be any vector in a line-element (x, x') and $(x+dx, x+dx')$ a consecutive line-element of (x, x'). Then the vector $(X+dX)$ in the line-element $(x+dx, x'+dx')$ is said to be parallel to the vector (X) in the line-element (x, x'), if

$$(3) \qquad DX^i = dX^i + C^i_{jk}X^j dx'^k + \Gamma^i_{jk}X^j dx^k = dX^i + \omega^i_j X^j = 0$$

holds good, where the C^i_{jk}, Γ^i_{jk} are given functions of x^i and x'^i, and

$$(4) \qquad \omega^i_j = C^i_{jk}dx'^k + \Gamma^i_{jk}dx^k .$$

As the vector $(X+dX)$ should depend on only two line-elements (x, x') and $(x+dx, x'+dx')$ it must be unaltered when $\rho(x, x')x'^i$ takes the place of the x'^i where the function $\rho(x, x')$ is positively homogeneous of degree zero in the x'^i. This leads to the following:

(a) The functions $\Gamma^i_{jk}(x, x')$ and C^i_{jk} are homogeneous of degree zero and -1 in the x'^i respectively;

$$\text{(b)} \qquad C^i_{jk}x'^k = 0 .$$

The functions C^i_{jk} and Γ^i_{jk} should not be absolutely arbitrary. It is desirable that parallel displacement is euclidean:

(III) The length of a vector in a line-element does not change when it is moved by parallel displacement to a consecutive line-element.

This condition gives the following relations:

$$dg_{ij} = g_{ik}\omega^k_j + g_{jk}\omega^k_i = \omega_{ji} + \omega_{ij}\,, \tag{5}$$

$$g_{ij,k} = \Gamma_{ijk} + \Gamma_{jik}\,, \qquad g_{ij;k} = C_{ijk} + C_{jik}\,, \tag{6}$$

where

$$\Gamma_{ijk} = g_{jl}\Gamma^l_{ik}\,, \qquad C_{ijk} = g_{jl}C^l_{ik}\,, \tag{7}$$

and the partial differentiation with respect to x^i is denoted by $,i$ and that with respect to x'^i by $;i$.

§ 2. *Determination of the functions* C^i_{jk}. In order to determine the coefficients C^i_{jk}, Γ^i_{jk}, which depend on g_{ij} and its derivatives only and define the euclidean connection, additional postulates must be introduced.

(IV) Let (X) and (Y) be two vectors in the same line-element, and $(\bar{D}X)$ and $(\bar{D}Y)$ their covariant differentials when their contravariant components X^i and Y^i remain unaltered and when their common line-element is subject to the same infinitesimal rotation about the center. Then the law of symmetry

$$\vec{X}\,\bar{D}\,\vec{Y} = \vec{Y}\,\bar{D}\,\vec{X}$$

holds good.

By the definition of the $\bar{D}\vec{X}$ one has $\bar{D}X^i = X^j C^i_{jk}\,dx'^k$, hence the above law of symmetry is written as follows:

$$X^i Y^k C_{kih}\,dx'^h = Y^i X^k C_{kih}\,dx'^h\,.$$

As the relation holds for any vectors and for any differentials of the x'^i, it follows that

$$C_{ikh} = C_{kih}\,. \tag{8}$$

From (6) and (8) one obtains

$$C_{ijk} = \frac{1}{2}g_{ij;k}\,. \tag{9}$$

The functions C_{ijk} thus obtained are homogeneous of degree -1 in the x'^i and symmetric with respect to the first two indices. When the functions C_{ijk} are multiplied by g^{jl} and summed up from 1 to n with regard to j, one gets

$$C^l_{ik} = \frac{1}{2} g^{jl} g_{ij;k} \,. \tag{10}$$

The functions C^l_{ik} are homogeneous of degree -1 in the x'^i.

§ 3. *The tensor* A_{ijk}. Now a general method will be shown which permits deducing a new tensor from any given tensor.

The simplest tensors in the space, which we have encountered till now, are the unit vector l^i and the fundamental tensor g_{ij}. Let $f(x, x')$ be a homogeneous function of any degree p in the x'^i, then the expressions

$$f_{//i} = g^{\frac{1}{2}} f_{;i} \tag{11}$$

are the homogeneous functions of the same degree p.

Let a notation be introduced for the contracted product of any geometric object and the unit vector l^i, that is,

$$T_{ij} l^j = T_{i0} \,.$$

On account of the homogeneity of $f(x, x')$ one has

$$f_{//0} = pf \,. \tag{12}$$

Next, consider any tensor (contravariant, covariant or mixed), for example, a covariant tensor with two indices $T_{ij}(x, x')$ whose components are necessarily homogeneous of degree zero in the x'^i. The expressions $T_{ij//k}$ are the components of the tensor with one more covariant index. Put

$$A_{ijk} = \frac{1}{2} g_{ij//k} = g^{\frac{1}{2}} C_{ijk} \,, \tag{13}$$

which are the components of a tensor. From (13) it follows:

The RIEMANN*ian spaces are characterized by* $A_{ijk} = 0$.

On account of the homogeneity of the g_{ij} it follows:

$$A_{ij0} = 0 \,. \tag{14}$$

The tensor A_{ijk} is symmetric for the first two indices. If they are symmetric for all indices, the considered space is FINSLERian. The following relations are easily verified:

$$(15)\qquad \begin{cases} g_{//i} = 2g(l_i + A_{00i}), \quad l^i_{//j} = \delta^i_j - l^i(l_j + A_{00j}), \\ l_{i//j} = 2A_{i0j} + g_{ij} - l_i l_j - l_i A_{00j}. \end{cases}$$

§ 4. *The angular metrics at a point.* The angle between a line-element and its consecutive line-element with the same center is obtained as follows. Let the angle be $d\varphi$, then one has

$$(16)\qquad \cos d\varphi = \frac{l_j(l^j + \bar{\delta} l^j)}{\sqrt{(l^i + \bar{\delta} l^i)(l_i + \bar{\delta} l_i)}} = \frac{1}{\sqrt{1 + \bar{\delta} l^i \bar{\delta} l_i}},$$

because by the definition the relation $l_i l^i = 1$ is obtained from which another relation $l_i \bar{\delta} l^i = l^i \bar{\delta} l_i = 0$ is gotten by covariant differentiation, where the notation $\bar{\delta}$ shows the covariant differentiation when the center remains unaltered. After a short calculation from (16) one gets (neglecting the higher order of the infinitesimal $d\varphi$)

$$(17)\qquad \begin{aligned} d\varphi^2 &= g_{ij} \bar{\delta} l^i \bar{\delta} l^j \\ &= g^{-1}\{g_{ij} dx'^i dx'^j + 2A_{0ij} dx'^i dx'^j + A^l_{0i} A_{0lj} dx'^i dx'^j \\ &\qquad - 2l_i dx'^i A_{00j} dx'^j - (l_i dx'^i)^2 - (A_{00i} dx'^i)^2\}. \end{aligned}$$

It is clear that the last member remains unaltered when x'^i are substituted by $\rho(x, x')x'^i$, where $\rho(x, x')$ is a homogeneous function of degree zero in the x'^i.

§ 5. *Covariant derivatives.* By definition the covariant differentials of any vector $v^i(x, x')$ and the unit vector l^i are written as follows:

$$(18)\qquad Dv^i = dv^i + C^i_{jk} v^j dx'^k + \Gamma^i_{jk} v^j dx^k,$$

$$(19)\qquad \omega^i = Dl^i = g^{-\frac{1}{2}}(\delta^i_k + A^i_{0k}) dx'^k + x'^i d(g^{-\frac{1}{2}}) + \Gamma^i_{0k} dx^k.$$

Now put

$$(20)\qquad \delta^i_k + A^i_{0k} = H^i_{\cdot k}$$

and assume that the determinant $|H^i_{\cdot k}|$ does not vanish identically, then one obtains the inverse system $K_i^{\cdot k}$, that is,

(21) $$H^i_{\cdot k} K_i^{\cdot j} = \delta^j_k .$$

When (19) is multiplied by $g^{\frac{1}{2}} K_i^{\cdot l}$, one has

(22) $$dx'^l = g^{\frac{1}{2}} K_i^{\cdot l} \omega^i - g^{\frac{1}{2}} K_i^{\cdot l} \Gamma^i_{0k} dx^k - g^{\frac{1}{2}} x'^l d(g^{-\frac{1}{2}}) ,$$

because $x'^i K_i^{\cdot l} = x'^l$.

When (22) is substituted into (18) on account of the relation (b), one obtains

(18)′ $$Dv^i = dv^i + \Gamma^{*i}_{jk} v^j dx^k + T^i_{jk} v^j \omega^k ,$$

where

(23) $$\Gamma^{*i}_{jk} = \Gamma^i_{jk} - T^i_{jr} \Gamma^r_{0k} , \quad T^i_{jk} = A^i_{jl} K_k^{\cdot l} .$$

As A^i_{jk}, $K_k^{\cdot l}$, N^i and ω^k are the components of tensors or vectors, the expressions Γ^{*i}_{jk} are transformed in like manner the coefficients of the affine connection under the transformation of coördinates. Making use of (22) one can write (18) in another form:

(18)″ $$Dv^i = \nabla_h v^i dx^h + \nabla'_h v^i \omega^h ,$$

where

(24) $$\nabla_h v^i = v^i_{\cdot h} - g^{\frac{1}{2}} K_r^{\cdot k} \Gamma^r_{0h} v^i_{\cdot k} + \Gamma^{*i}_{jh} v^j ,$$

(25) $$\nabla'_h v^i = g^{\frac{1}{2}} K_h^{\cdot k} v^i_{\cdot k} + T^i_{jh} v^j ,$$

because $v^i_{\cdot k} x'^k = 0$. We call $\nabla_h v^i$ and $\nabla'_h v^i$ the covariant derivatives of the first and second kinds of v^i respectively. In particular we have for the vector l^i and the tensor g_{ij}:

(26) $$\nabla_h l^i = l^i_{\cdot h} + l^i (l_k + A_{00k}) K_r^{\cdot k} \Gamma^r_{0h} = 0 ,$$

(27) $$\nabla_h g_{ij} = 0 .$$

§ 6. *The vectors θ_i and σ^i.* In this section will be introduced a very important vector θ_i.

Theorem. *Let $g^{-\frac{1}{2}} F_i(x, x')$ be the components of a covariant vector in the space $K_n^{(1)}$ which are the homogeneous functions of degree zero in the x'^i, then the* PFAFFian *forms*

$$\theta_j = \frac{1}{2}(F_{i;j}+F_{j;i})dl^i + \frac{1}{2}\Big\{-g^{\frac{1}{2}}F_{i;j;l}\Gamma^{*l}_{00} + F_{i;j,l}l^l - g^{-\frac{1}{2}}F_{i,j} + g^{-\frac{1}{2}}F_{j,i}\Big\}dx^i \tag{28}$$

are also the components of a vector.

Proof. Under the transformation of the coördinates $x^\lambda = x^\lambda(x^i)$ one gets the relations:

$$(F_{i;j}+F_{j;i})dl^i = (F_{\lambda;\mu}+F_{\mu;\lambda})dl^\lambda Q^\mu_j - (F_{\lambda;\mu}+F_{\mu;\lambda})Q^\lambda_{il}l^l Q^\mu_j dx^i, \tag{29}$$

$$g^{\frac{1}{2}}F_{i;j;l}\Gamma^{*l}_{00}dx^i = -g^{\frac{1}{2}}F_{\lambda;\mu;\nu}\Gamma^{*\nu}_{00}dx^\lambda Q^\mu_j - g^{\frac{1}{2}}F_{\lambda;\mu;\nu}dx^\lambda Q^\nu_{kl}l^k l^l Q^\mu_j, \tag{30}$$

$$F_{i;j,l}l^l dx^i = F_{\lambda;\mu,\nu}l^\nu dx^\lambda Q^\mu_j + g^{\frac{1}{2}}F_{\lambda;\mu;\nu}dx^\lambda Q^\nu_{kl}l^k l^l Q^\mu_j + F_{\lambda;\mu}Q^\lambda_{il}l^l dx^i Q^\mu_j + F_{\lambda;\mu}dx^\lambda Q^\mu_{jk}l^k, \tag{31}$$

$$g^{-\frac{1}{2}}(-F_{i,j}+F_{j,i})dx^i = g^{-\frac{1}{2}}(-F_{\lambda,\mu}+F_{\mu,\lambda})dx^\lambda Q^\mu_j - F_{\lambda;\mu}dx^\lambda Q^\mu_{jk}l^k + F_{\mu;\lambda}Q^\mu_j Q^\lambda_{ik}l^k dx^i, \tag{32}$$

where

$$Q^\lambda_i = \frac{\partial x^\lambda}{\partial x^i}, \qquad Q^\lambda_{ij} = \frac{\partial^2 x^\lambda}{\partial x^i \partial x^j}.$$

When (29), (30), (31) and (32) are substituted into (28), one obtains

$$\theta_i(x^k, x'^k, dx^k, dx'^k) = \theta_\lambda(x^\mu, x'^\mu, dx^\mu, dx'^\mu)Q^\lambda_i,$$

as $g(x, x')$ is the scalar. Q. E. D.

Now put $g_i(x, x') = g_{ij}x'^j$, $2G_{ij} = g_{i;j}+g_{j;i}$ and take g_i instead of the functions F_i in the above theorem. If it is assumed that the determinant $|G_{ij}|$ does not vanish identically, it is possible to define the inverse system E^{ij}, that is, $G_{ij}E^{ik} = \delta_j^k$. As G_{ij} and E^{ik} are the components of the tensors, there follows

Theorem. *The expressions*

$$\sigma^i(x, x', dx, dx') = dl^i + \frac{1}{2}E^{ij}\Big\{-g^{\frac{1}{2}}g_{k;j;l}\Gamma^{*l}_{00} + g_{k;j,l}l^l - g^{-\frac{1}{2}}g_{k,j} + g^{-\frac{1}{2}}g_{j,k}\Big\}dx^k \tag{33}$$

are the components of a vector.

Finally from (18)′ one gets

$$K_k^i \omega^k = dl^i + \Gamma_{0k}^{*i} dx^k . \tag{34}$$

§ 7. *Determination of* Γ_{jk}^i. In order to determine the coefficients Γ_{jk}^i let the last two postulates be introduced:

(V)[1] The two vectors σ^i and $K_k^i \omega^k$ are identical.

(VI) The coefficients Γ_{jk}^{*i} are symmetric with respect to j and k, namely:

$$\Gamma_{jk}^{*i} = \Gamma_{kj}^{*i} . \tag{35}$$

Postulate (V) gives the relations

$$\frac{1}{2} E^{ij} \left\{ -g^{\frac{1}{2}} g_{k;j;l} \Gamma_{00}^{*l} + g_{k;j,0} - g^{-\frac{1}{2}} g_{k,j} + g^{-\frac{1}{2}} g_{j,k} \right\} = \Gamma_{0k}^{*i} . \tag{36}$$

When (36) are multiplied by l^k, one gets

$$(2 G_{il} + g^{\frac{1}{2}} g_{0;i;l}) \Gamma_{00}^{*l} = g_{0;i,0} - g^{-\frac{1}{2}} g_{0,i} + g^{-\frac{1}{2}} g_{i,0} .$$

Put $2G_{il} + g^{\frac{1}{2}} g_{0;i;l} = M_{il}$ and assume that the determinant $|M_{il}|$ does not vanish identically, so that N^{jk} can be defined as the inverse system of M_{il}. Therefore the above equations are solved for Γ_{00}^{*l} as follows:

$$\Gamma_{00}^{*i} = N^{ij} \left\{ g_{0;j,0} - g^{-\frac{1}{2}} {}_{0,j} + g^{-\frac{1}{2}} g_{j,0} \right\} . \tag{37}$$

If (37) are substituted into (36), Γ_{0k}^{*i} is gotten from the given functions g_{ij} and their derivatives:

$$\Gamma_{0k}^{*i} = \frac{1}{2} E^{ij} \left\{ -g_{k;j;l} N^{rl} \left[g^{\frac{1}{2}} g_{0;r,0} - g_{0,r} + g_{r,0} \right] \right. \tag{38}$$

$$\left. + g_{k;j,0} - g^{-\frac{1}{2}} g_{k,j} + g^{-\frac{1}{2}} g_{j,k} \right\} .$$

On the other hand from (23) one gets

$$\Gamma_{0k}^{*i} = \Gamma_{0k}^i - T_{0r}^i \Gamma_{0k}^r = (\delta_r^i - T_{0r}^i) \Gamma_{0k}^r . \tag{39}$$

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1) In a FINSLER space where the arc length of a curve $x^i = x^i(t)$ is given by the integral $\int F\left(x, \frac{dx}{dt}\right) dt$, the coefficients of the euclidean connection Γ_{jk}^i and C_{jk}^i are determined perfectly by the postulates I, II, III, IV, VI only and the postulate V $\sigma^i = K_k^{\ i} \omega^k$ is satisfied identically (See E. CARTAN [1]).

And from (20) by multiplication of $K_j^{\cdot k}$ one has the relations

$$K_j^{\cdot i} + A^i_{0k} K_j^{\cdot k} = \delta^i_j$$

from which it follows

$$\delta^i_j - A^i_{0k} K_j^{\cdot k} = K_j^{\cdot i} . \tag{40}$$

From (39), (40) and (20) is gotten

$$\Gamma^i_{0k} = H^i_{\cdot r} \Gamma^{*r}_{0k} . \tag{41}$$

On account of (6) and (23) one gets

$$\Gamma^*_{ijk} + \Gamma^*_{jik} = g_{ij,k} - 2 T_{ijl} \Gamma^l_{0k} \tag{42}$$

and from (33)

$$\Gamma^*_{ijk} = \Gamma^*_{kji} . \tag{43}$$

From (42) and (43) it follows by the same calculation as in a RIEMANNian space ($A_{ijk} = 0$)

$$\Gamma^*_{ijk} = \gamma_{ijk} + T_{ikl} \Gamma^l_{0j} - T_{jkl} \Gamma^l_{0i} - T_{jil} \Gamma^l_{0k} , \tag{44}$$

where γ_{ijk} are the CHRISTOFFEL's symbols of the first kind with respect to g_{ij}. From (44) and (23) one gets

$$\Gamma_{ijk} = \gamma_{ijk} + T_{ikl} \Gamma^l_{0j} - T_{jkl} \Gamma^l_{0i} . \tag{45}$$

When (39) are substituted into (44) and (45) on account of (21), one obtains

$$\Gamma^*_{ijk} = \gamma_{ijk} + A_{ikr} \Gamma^{*r}_{0j} - A_{jkr} \Gamma^{*r}_{0i} - A_{jir} \Gamma^{*r}_{0k} , \tag{46}$$

$$\Gamma_{ijk} = \gamma_{ijk} + A_{ikr} \Gamma^{*r}_{0j} - A_{jkr} \Gamma^{*r}_{0i} . \tag{47}$$

If the suffix j in Γ^*_{ijk} and Γ_{ijk} is raised up by multiplication of g^{jh}, one obtains

$$\Gamma^{*h}_{ik} = \gamma^h_{ik} + A_{ikr} \Gamma^{*r}_{0j} g^{jh} - A^h_{kr} \Gamma^{*r}_{0i} - A^h_{ir} \Gamma^{*r}_{0k} , \tag{48}$$

$$\Gamma^h_{ik} = \gamma^h_{ik} + A_{ikr} \Gamma^{*r}_{0j} g^{jh} - A^h_{kr} \Gamma^{*r}_{0i} , \tag{49}$$

where γ^h_{ik} are CHRISTOFFEL's symbols of the second kind with respect to g_{ij}.

Finally if (36) are substituted into (48) and (49), Γ^{*h}_{ik} and Γ^h_{ik} are obtained as functions of g_{ij} and their derivatives up to the second.

Thus the coefficients Γ^i_{jk} and C^i_{jk} of the connection in $K_n^{(1)}$ can be completely determined.

Chapter II. Theory of curves.

§ 1. *The geodesic line and the extremal.* In this chapter the unit tangential vector l^i to a curve is taken as line-element and the arc length s as parameter. So we have

$$l^i = \frac{dx^i}{ds} = \dot{x}^i, \tag{1}$$

$$g(x, \dot{x}) = 1. \tag{2}$$

If a contravariant vector l^i is transported in its own direction parallely, there is obtained the line which is called a geodesic line. From Chap. I, (18)′ follows its equations

$$\frac{Dl^i}{ds} = H^i_{\cdot k}\left(\frac{d^2x^k}{ds^2} + \Gamma^{*k}_{jh}\frac{dx^j}{ds}\frac{dx^h}{ds}\right) = 0,$$

which can be written in the form

$$\frac{d^2x^i}{ds^2} + \Gamma^{*i}_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{3}$$

because the determinant of the tensor $H^i_{\cdot k}$ does not vanish.

Next calculate the differential equations of the extremal in the theory of variation $\delta s = 0$. The EULER's equations of the base function $g^{\frac{1}{2}}$ are

$$\frac{d}{ds}(g^{\frac{1}{2}})_{;i} - (g^{\frac{1}{2}})_{,i} = 0, \tag{4}$$

that is,

$$g^{\frac{1}{2}}(g^{\frac{1}{2}})_{;i;j}\frac{d^2x^j}{ds^2} + g^{\frac{1}{2}}(g^{\frac{1}{2}})_{;i,j}\frac{dx^j}{ds} - g^{\frac{1}{2}}(g^{\frac{1}{2}})_{,i} = 0. \tag{5}$$

On the other hand differentiating (2) with respect to s and then multipling with $(g^{\frac{1}{2}})_{;i}$ one gets

$$(g^{\frac{1}{2}})_{;i}(g^{\frac{1}{2}})_{;j}\frac{d^2x^j}{ds^2} + (g^{\frac{1}{2}})_{;i}(g^{\frac{1}{2}})_{,j}\frac{dx^j}{ds} = 0. \tag{6}$$

When (6) is added to (5) side by side, the equations of the extremal are obtained :

$$g_{;i;j}\frac{d^2x^j}{ds^2}+g_{;i,j}\frac{dx^j}{ds}-g_{,i}=0\ ,$$

that is,

$$(2G_{ij}+g^{\frac{1}{2}}g_{0;i;j})\frac{d^2x^j}{ds^2}+gg_{0;i,0}-g^{\frac{1}{2}}g_{0,i}+g^{\frac{1}{2}}g_{i,0}=0\ ,$$

and according to Chapt. I, (37) their contravariant forms are the same as (3). Therefore in our space $K_n^{(1)}$ the geodesic line is identical with the extremal in the theory of variation $\delta s=0$ as in a FINSLER space.

§ 2. *FRENET's formulae.* In this section the space of three dimensions only is considered. As a curve is regarded as the locus of their tangential line-elements, the theory of curves is the same as in the euclidean geometry. The unit tangential vector is l^i and its covariant derivative with respect to s is

$$\frac{Dl^i}{ds}=H^i_{\cdot h}\left(\frac{dl^h}{ds}+\Gamma^{*h}_{0k}\frac{dx^k}{ds}\right)$$

and we put

$$\frac{Dl^i}{ds}=\kappa n^i\ , \tag{7}$$

where n^i is a unit vector along $\frac{Dl^i}{ds}$ and κ is called the curvature. It is seen that the vector n^i is perpendicular to the vector l^i, because when the relation $l^il_i=1$ is differentiated covariantly, $l^i\frac{Dl_i}{ds}+l_i\frac{Dl^i}{ds}=0$ is obtained, namely $l_i\frac{Dl^i}{ds}=0$.

Now let a unit vector b^i be introduced which is perpendicular to each of the vectors l^i and n^i and has the same line-element as l^i and n^i. Then the covariant derivative $\frac{Dn^i}{ds}$ is written in the form

$$\frac{Dn^i}{ds}=\xi l^i+\eta n^i+\zeta b^i\ , \tag{8}$$

but η must be zero because the vector $\frac{Dn^i}{ds}$ is perpendicular to the vector n^i. When the relation $n^i l_i = 0$ is differentiated covariantly, one has $l_i \frac{Dn^i}{ds} = -n^i \frac{Dl_i}{ds}$. From the last equation and (7) one gets

$$(9) \qquad l_i \frac{Dn^i}{ds} = -\kappa .$$

When (8) are multiplied by l_i and summed up for i and substituted by (9), $\xi = -\kappa$ is obtained. So that (8) becomes

$$(10) \qquad \frac{Dn^i}{ds} = -\kappa l^i + \zeta b^i .$$

Analogously one obtains the result:

$$(11) \qquad \frac{Db^i}{ds} = -\zeta n^i .$$

Equations (7), (10) and (11) give the FRENET's formulae.

Chapter III. The torsion and curvature tensors.

§ 1. *The torsion tensor.* The covariant differential (18)′ in Chap. I of any vector v^i is written in the form

$$(1) \qquad Dv^i = dv^i + \omega^{*i}_j v^j ,$$

where

$$(2) \qquad \omega^{*i}_j = \Gamma^{*i}_{jh} dx^h + T^i_{jh} \omega^h = \omega^i_j .$$

If Δv^i is the covariant differential of v^i which corresponds to the increment δ, the torsion of the space is defined dy

$$(3) \qquad (\Delta D - D\Delta) x^i = \Omega^i ,$$

where $Dx^i = dx^i$, $\Delta x^i = \delta x^i$ and d and δ are exchangable.

From (3) one obtains

$$(4) \qquad \Omega^i = [dx^k \omega^{*i}_k] = T^i_{kh} [dx^k \omega^h] .$$

Therefore the tensor T^i_{kh} is the torsion tensor of the space.

§ 2. *The curvature tensors.* The curvature of the space is defined by

$$(\Delta D - D\Delta)v^i = \Omega^i_k v^k . \tag{5}$$

Then one has

$$\Omega^i_k = [\omega^{*h}_k \omega^{*i}_h] - (\omega^{*i}_k)' \tag{6}$$

or

$$\Omega_{ij} = [\omega^{*m}_j \omega^*_{im}] - (\omega^*_{ij})' , \tag{6'}$$

where

$$\Omega_{ij} = \Omega_i^l g_{jl} .$$

These forms are decomposed in the following wise:

$$\Omega_{ij} = \frac{1}{2} R_{ijhk}[dx^h dx^k] + P_{ijhk}[dx^h \omega^k] + \frac{1}{2} S_{ijhk}[\omega^h \omega^k] . \tag{7}$$

But as the relation $l^i \omega_i = 0$ holds good this decomposition is not unique. If the conditions

$$P_{ijh0} = 0 , \qquad S_{ijh0} = 0 \tag{8}$$

are added, the decomposition is unique. Thus there are obtained the curvature tensors R_{ijhk}, P_{ijhk}, S_{ijhk} of the space. From (2) the first term of the second member of (6)′ is written in the form:

$$\begin{aligned}[\omega^{*m}_j \omega^*_{im}] = {} & \frac{1}{2}(\Gamma^{*m}_{jh}\Gamma^*_{imk} - \Gamma^{*m}_{jk}\Gamma^*_{imh})[dx^h dx^k] + T^m_{jk}\Gamma^*_{imh}[\omega^k dx^h] \\ & + \Gamma^{*m}_{jh} T_{imk}[dx^h \omega^k] + \frac{1}{2}(T^m_{jh} T_{imk} - T^m_{jk} T_{imh})[\omega^h \omega^k] .\end{aligned} \tag{9}$$

On the other hand, as $\omega_{ij} = \Gamma_{ijk} dx^k + C_{ijk} dx'^k$ from Chap. I, (18), the second term of the second member of (6)′ is written in the form

$$\begin{aligned}(\omega_{ij})' = \frac{1}{2}\Big\{ & \Gamma_{ijk,h} - \Gamma_{ijh,k} - g^{\frac{1}{2}}\Gamma_{ijk;s} K_r^{\cdot s}\Gamma^r_{0h} + g^{\frac{1}{2}}\Gamma_{ijh;s} K_r^{\cdot s}\Gamma^r_{0k} \\ & - g^{\frac{1}{2}} C_{ijr,h} K_s^{\cdot r}\Gamma^s_{0k} + g^{\frac{1}{2}} C_{ijr,k} K_s^{\cdot r}\Gamma^s_{0h}\Big\}[dx^h dx^k] \\ & + g^{\frac{1}{2}}\Gamma_{ijh;s} K_k^{\cdot s}[\omega^k dx^h] + \frac{1}{2} g^{\frac{1}{2}}(C_{ijr,h} K_k^{\cdot r} - C_{ijr,k} K_h^{\cdot r})[\omega^h \omega^k] .\end{aligned} \tag{10}$$

When (9) and (10) are substituted into equations (6)′, one gets from (7):

$$(11)\quad R_{ijhk} = \Gamma^{*m}_{jh}\Gamma^{*}_{imk} - \Gamma^{*m}_{jk}\Gamma^{*}_{imh} + \Gamma_{ijk,h} - \Gamma_{ijh,k} - \Gamma_{ijk||s}K_r^{\cdot s}K^r_{0h}$$
$$+ \Gamma_{ijh||s}K_r^{\cdot s}\Gamma^r_{0k} - g^{\frac{1}{2}}C_{ijr,h}K_s^{\cdot r}\Gamma^s_{0k} + g^{\frac{1}{2}}C_{ijr,k}K_s^{\cdot r}\Gamma^s_{0h},$$

$$(12)\quad P_{ijhk} = g^{\frac{1}{2}}C_{ijr,h}K_k^{\cdot r} - \Gamma_{ijh||s}K_k^{\cdot s} + \Gamma^{*m}_{jh}T_{imk} - T^m_{jk}\Gamma^{*}_{imk}$$

and

$$(13)\quad S_{ijhk} = T^m_{jh}T_{imk} - T^m_{jk}T_{imh}.$$

The tensors R_{ijhk} and S_{ijhk} are skew-symmetric with respect to h, k and i, j respectively. For the tensors P_{ijhk} and S_{ijhk} the conditions (8) are satisfied identically. In fact, as $K_s^{\cdot r}l^s = l^r$ it follows that

$$\Gamma_{ijh||s}K_k^{\cdot s}l^k = \Gamma_{ijh;s}x'^s = 0,$$

because Γ_{ijh} are the homogeneous functions of degree zero in the x'^k. Similarly it follows $C_{ijr,h}K_k^{\cdot r}l^k = 0$, $T_{imk}l^k = 0$. Therefore one gets $P_{ijh0} = 0$ and $S_{ijh0} = 0$.

For our connection we have the equations $Dg_{ik} = 0$, therefore the relations

$$(14)\quad (\Delta D - D\Delta)g_{ij} = \Omega_{ij} + \Omega_{ji} = 0$$

hold. As any two of the three exterior products $[dx^h dx^k]$, $[dx^h \omega^k]$, $[\omega^h \omega^k]$ can be made to vanish by a suitable selection of dx^h, δx^h, while the other remains without vanishing, it follows that the first two indices of the three curvature tensors are skew-symmetric by substitution of the last equations in (7); namely

$$(15)\quad R_{ijhk} + R_{jihk} = 0, \quad P_{ijhk} + P_{jihk} = 0, \quad S_{ijhk} + S_{jihk} = 0.$$

From the skew-symmetry of the curvature tensors one gets immediately:

$$R_r^{\cdot r}{}_{hk} = P_r^{\cdot r}{}_{hk} = S_r^{\cdot r}{}_{hk} = 0.$$

§ 3. *BIANCHI's identities.* BIANCHI's identities are given by exterior derivation of Ω^i and Ω^i_j, namely:

$$(16)\quad (\Omega^i)' - [dx^k \Omega^i_k] + [\omega^i_k \Omega^k] = 0,$$

$$(17)\quad (\Omega_{ij})' - [\omega^k_j \Omega_{ik}] + [\omega^k_i \Omega_{jk}] = 0.$$

he gets

$$(8)\quad x^i_\alpha x^\beta_i = \delta^\beta_\alpha\,,\quad g_{\alpha\beta}x^\beta_k = g_{ik}x^i_\alpha\,,\quad g^{\alpha\beta}x^i_\beta = g^{ik}x^\alpha_k\,,\quad g^{\alpha\beta} = g^{ik}x^\alpha_i x^\beta_k\,.$$

Any tensor defined in the normal line-element $(x(u), x'(u))$ on the hypersurface is decomposed in the manner:

$$(9)\qquad T^k_i = T^\beta_\alpha x^\alpha_i x^k_\beta + T_{\alpha 0}x^\alpha_i l^k + T^{0\alpha}l_i x^k_\alpha + T_{00}l_i l^k\,,$$

where $T^\alpha_\beta = T^j_i x^j_\beta x^\alpha_i$, $T_{\alpha 0} = T_{i0}x^i_\alpha$, $T^\alpha_0 = T^i_0 x^\alpha_i$. The T^k_i are called the space components and the T^α_β the hypersurface components of the tensor T^k_i. If $T_{i0} = T^k_0 = 0$, $T_{00} = 0$ and the T^k_i is the hypersurface tensor. An example of a hypersurface tensor is the tensor $g_{ij}-l_i l_j$ and its hypersurface components are $g_{\alpha\beta}$. From (7), (8) and (9) one gets:

$$(10)\qquad g_{\alpha\beta}x^\alpha_i x^\alpha_j = g_{ij}-l_i l_j\,,\quad g^{\alpha\beta}x^i_\alpha x^j_\beta = g^{ij}-l^i l^j\,,$$

$$(11)\qquad x^i_\alpha x^\alpha_j = \delta^i_j - l_j l^i\,.$$

Putting

$$(12)\qquad \begin{aligned} &\Gamma^{*\alpha}_{\beta k} = \Gamma^{*h}_{ik}x^i_\beta x^\alpha_h\,,\quad \Gamma^*_{\alpha 0\beta} = \Gamma^{*h}_{ik}x^i_\alpha x^k_\beta l_h\,,\\ &\Gamma^{*\alpha}_{00} = \Gamma^{*h}_{ik}x^\alpha_h l^i l^k\,,\end{aligned}$$

one obtains the relations:

$$(13)\qquad \begin{cases} \Gamma^{*i}_{00} = \Gamma^{*\alpha}_{00}x^i_\alpha + \Gamma^*_{000}l^i\,,\quad \Gamma^{*\alpha}_{\alpha k} = \Gamma^{*r}_{rk} - \Gamma^{*0}_{0k}\,,\\ \Gamma^{*i}_{\alpha h}\Gamma^{*\alpha}_{jk} = \Gamma^{*i}_{rh}\Gamma^{*r}_{jk} - \Gamma^{*i}_{0h}\Gamma^{*0}_{jk}\,.\end{cases}$$

The second fundamental form of the hypersurface is defined as in the RIEMANNian space by

$$(14)\qquad a_{\alpha\beta}du^\alpha du^\beta = \pi_\beta du^\beta \qquad (a_{\alpha\beta} = a_{\beta\alpha})\,,$$

where

$$(15)\qquad \pi_\rho = l_i Dx^i_\rho = -\omega_i x^i_\rho = -\omega_\rho\,.$$

From (14) and (15) one gets

$$(16)\qquad \begin{aligned} a_{\alpha\beta} &= l_i\left\{\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \Gamma^i_{jk}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta} + C^i_{jk}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x'^k}{\partial u^\beta}\right\}\\ &= \mathfrak{H}^\sigma_\alpha\left(l_i\frac{\partial^2 x^i}{\partial u^\sigma \partial u^\beta} + \Gamma^{*0}_{\sigma\beta}\right)\\ &= \mathfrak{K}^\sigma_{\cdot\alpha}(\qquad ,, \qquad)\,,\end{aligned}$$

where $\mathfrak{H}^{\sigma}_{\cdot\alpha}$ is the inverse system of $\delta^{\sigma}_{\alpha}+T^{0}_{\alpha\delta}g^{\delta\sigma}$ and $\mathfrak{K}^{\sigma}_{\cdot\alpha}=K_{\xi}^{\cdot\delta}g^{\xi\sigma}g_{\delta\alpha}$.

If one puts

$$\pi_{\beta}^{\alpha}=\gamma^{\alpha}_{\beta\tau}du^{\tau}=x^{\alpha}_{i}Dx^{i}_{\beta}=-x^{i}_{\beta}Dx^{\alpha}_{i}\,, \tag{17}$$

the functions $\gamma^{\alpha}_{\beta\tau}$ are the coefficients of the parallel displacement induced in the hypersurface. Because for a hypersurface vector $(v_i l^i=0)$ one has

$$\delta v_{\alpha}=(Dv_i)x^i_{\alpha}=D(v_i x^i_{\alpha})-v_i Dx^i_{\alpha}=dv_{\alpha}-\pi^{\tau}_{\alpha}v_{\tau}\,,$$

where the parallel displacement in the hypersurface is denoted by the symbol δ.

From (17) one gets

$$\gamma^{\alpha}_{\beta\tau}=x^{\alpha}_{i}\frac{\partial^2 x^i}{\partial u^{\beta}\partial u^{\tau}}+\Gamma^{*\alpha}_{\beta\tau}-T^{\alpha}_{\beta\sigma}g^{\sigma\delta}a_{\delta\tau}\,. \tag{18}$$

§ 2. *The torsion and curvature tensors of the hypersurface.* The decompositions of ω^i and Dx^i_{α} are

$$\omega^i=\omega^{\alpha}x^i_{\alpha}=-\pi^{\alpha}x^i_{\alpha}\,, \tag{19}$$

and

$$Dx^i_{\alpha}=\pi^{\beta}_{\alpha}u^i_{\beta}-\omega_{\alpha}l^i=\pi^{\beta}_{\alpha}x^i_{\beta}+\pi_{\alpha}l^i=dx^i_{\alpha}+\omega^i_k x^k_{\alpha}\,. \tag{20}$$

The above two equations lead to the following:

$$\frac{\partial l^i}{\partial u^{\alpha}}+\Gamma^{*i}_{0k}\frac{\partial x^k}{\partial u^{\alpha}}=-a^{\sigma}_{\alpha}\frac{\partial x^k}{\partial u^{\sigma}}K_{k\cdot}^{\cdot i} \tag{19$'$}$$

and

$$\frac{\partial^2 x^i}{\partial u^{\alpha}\partial u^{\beta}}+\Gamma^{*i}_{\alpha\beta}=\gamma^{\sigma}_{\alpha\beta}x^i_{\sigma}+T^i_{\alpha\sigma}a^{\sigma}_{\beta}+a_{\alpha\beta}l^i\,. \tag{20$'$}$$

As $Dg_{ij}=0$, one gets

$$\delta g_{\alpha\beta}=0 \tag{21}$$

or

$$\frac{\partial g_{\alpha\beta}}{\partial u^{\tau}}=\gamma_{\alpha\beta\tau}+\gamma_{\beta\alpha\tau}\qquad(\gamma_{\alpha\beta\tau}=g_{\sigma\beta}\gamma^{\sigma}_{\alpha\tau})\,. \tag{22}$$

This shows that the induced connection is euclidean. The torsion tensor of the hypersurface is defined by

$$B^\sigma = [du^\alpha \pi^\sigma_\alpha] = \frac{1}{2} B^\sigma_{\alpha\beta} [du^\alpha du^\beta] , \tag{23}$$

where

$$B^\sigma_{\alpha\beta} = \gamma^\sigma_{\alpha\beta} - \gamma^\sigma_{\beta\alpha} = a^\varepsilon_\alpha T^\sigma_{\beta\varepsilon} - a^\varepsilon_\beta T^\sigma_{\alpha\varepsilon} \tag{24}$$

by virtue of (18).

From (22) by suitable exchange of indices and addition are obtained, on account of (24),

$$\begin{aligned} \gamma_{\alpha\beta\gamma} &= \frac{1}{2}\left\{ \frac{\partial g_{\alpha\beta}}{\partial u^\gamma} + \frac{\partial g_{\beta\gamma}}{\partial u^\alpha} - \frac{\partial g_{\gamma\alpha}}{\partial u^\beta} \right\} + \frac{1}{2}\{ B_{\alpha\beta\gamma} + B_{\gamma\alpha\beta} - B_{\beta\gamma\alpha} \} \\ &= \qquad\qquad ,, \qquad\qquad + a^\varepsilon_\alpha T_{\gamma\beta\varepsilon} - a^\varepsilon_\beta T_{\gamma\alpha\varepsilon} . \end{aligned} \tag{25}$$

Moreover from (24) we have the relations:

$$B_{\alpha\beta\gamma} = -B_{\gamma\beta\alpha} , \quad B_{\alpha\beta\gamma} + B_{\beta\gamma\alpha} + B_{\gamma\alpha\beta} = 0 . \tag{26}$$

On the other hand the curvature tensor of the hypersurface is defined by

$$B^\beta_\alpha = [\pi^\tau_\alpha \pi^\beta_\tau] - (\pi^\beta_\alpha)' = \frac{1}{2} B_\alpha{}^\beta{}_{\gamma\delta} [du^\gamma du^\delta] , \tag{27}$$

where

$$B_\alpha{}^\beta{}_{\gamma\delta} = \frac{\partial \gamma^\beta_{\alpha\gamma}}{\partial u^\delta} - \frac{\partial \gamma^\beta_{\alpha\delta}}{\partial u^\gamma} + \gamma^\tau_{\alpha\gamma} \gamma^\beta_{\tau\delta} - \gamma^\tau_{\alpha\delta} \gamma^\beta_{\tau\gamma} . \tag{28}$$

§ 3. *BIANCHI's identities. Conditions of integrability.* BIANCHI's identities are:

$$(B^\alpha)' - [du^\tau B^\alpha_\tau] + [\pi^\alpha_\tau B^\tau] = 0 \tag{29}$$

and

$$(B_{\alpha\beta})' - [\pi^\tau_\beta B_{\alpha\tau}] + [\pi^\tau_\alpha B_{\beta\tau}] = 0 \quad (B_{\alpha\beta} = g_{\gamma\beta} B^\gamma_\alpha) . \tag{30}$$

(29) yields the identities:

$$B_\alpha{}^\sigma{}_{\beta|\gamma} + B^\tau_{\alpha\gamma} B^\sigma_{\tau\beta} - B_\alpha{}^\sigma{}_{\beta\gamma} + \text{cyclic } (\alpha\beta\gamma) = 0 , \tag{31}$$

where $B_\alpha{}^\sigma{}_{\beta|\gamma}$ represents the covariant derivation in the sense of the connection on the hypersurface and $+\text{cyclic}(\alpha\beta\gamma)$ shows the expres-

sions made from the written expression by exchange of the indices α, β and γ cyclically. On the other hand from (30) one gets:

$$B_{\alpha\beta\gamma\delta|\varepsilon}+B_{\alpha\beta\delta\tau}B^{\tau}_{\varepsilon\gamma}+\text{cyclic}\,(\gamma\delta\varepsilon)=0\,. \tag{32}$$

From Chap. III, (6) and the definition of covariant differential one gets

$$[Dx^k_\alpha\omega^i_k]-(Dx^i_\alpha)'=\Omega^i_k x^k_\alpha\,. \tag{33}$$

On account of (20) the first term of the first member of (33) becomes

$$[Dx^k_\alpha\omega^i_k]=[\pi^\beta_\alpha\pi^\tau_\beta]x^i_\tau+[\pi^\beta_\alpha\pi_\beta]l^i-[\pi^\beta_\alpha dx^i_\beta]+[\pi_\alpha\omega^i_k]l^k\,. \tag{34}$$

On the other hand the second term becomes

$$(Dx^i_\alpha)'=[dx^i_\beta\pi^\beta_\alpha]+(\pi^\beta_\alpha)'x^i_\beta+\pi'_\alpha l^i-[\pi^\sigma\pi_\alpha]x^i_\sigma-[\omega^i_k\pi_\alpha]l^k\,. \tag{35}$$

Therefore if (35) is substracted from (34), one obtains, taking heed of (27),

$$\Omega^i_k x^k_\alpha=B^\sigma_\alpha x^i_\sigma+[\pi^\sigma\pi_\alpha]x^i_\sigma+\{[\pi^\sigma_\alpha\pi_\sigma]-\pi'_\alpha\}l^i\,. \tag{36}$$

When (36) is multiplied by x^β_i and l_i and summed up for i, then one gets

$$\Omega^i_k x^k_\alpha x^\beta_i=\Omega^\beta_\alpha=D_\alpha{}^\beta{}_{\gamma\delta}[du^\gamma du^\delta]=B^\beta_\alpha+[\pi^\beta\pi_\alpha] \tag{37}$$

and

$$\Omega^0_k x^k_\alpha=\Omega^0_\alpha=[\pi^\beta_\alpha\pi_\beta]-\pi'_\alpha \tag{38}$$

respectively, where

$$D_\alpha{}^\beta{}_{\gamma\delta}=R_\alpha{}^\beta{}_{\gamma\delta}-P_\alpha{}^\beta{}_{\gamma\sigma}a^\sigma_{\cdot\delta}+P_\alpha{}^\beta{}_{\delta\sigma}a^\sigma_\gamma+S_\alpha{}^\beta{}_{\sigma\varepsilon}a^\sigma_\gamma a^\varepsilon_\delta\,.$$

(37) becomes explicitly

$$\begin{aligned}R_\alpha{}^\beta{}_{\gamma\delta}-P_\alpha{}^\beta{}_{\gamma\sigma}a^\sigma_\delta+P_\alpha{}^\beta{}_{\delta\sigma}a^\sigma_\gamma+S_\alpha{}^\beta{}_{\sigma\varepsilon}a^\sigma_\gamma a^\varepsilon_\delta\\ =B_\alpha{}^\beta{}_{\gamma\delta}+a^\beta_\gamma a_{\alpha\delta}-a^\beta_\delta a_{\alpha\gamma}\,.\end{aligned} \tag{39}$$

Equations (39) are the GAUSS' equations for the hypersurface. From (38) it follows:

$$R_\alpha{}^0{}_{\gamma\delta}-P_\alpha{}^0{}_{\gamma\sigma}a^\sigma_\delta+P_\alpha{}^0{}_{\delta\sigma}a^\sigma_\gamma+S_\alpha{}^0{}_{\sigma\varepsilon}a^\sigma_\gamma a^\varepsilon_\delta=a_{\alpha\gamma|\delta}-a_{\alpha\delta|\gamma}+B^\varepsilon_{\gamma\delta}a_{\alpha\varepsilon}\,. \tag{40}$$

This is the CODAZZI's equations for the hypersurface.

By exterior derivation of (37) and (38) one obtains more identities on account of (30), (27) and (37), (38) themselves:

$$(41) \qquad (\Omega_{\alpha\beta})' = [\pi^{\tau}_{\beta}\,\Omega_{\alpha\tau}] - [\pi^{\tau}_{\alpha}\,\Omega_{\beta\tau}] + [\pi_{\beta}\,\Omega_{\alpha 0}] - [\pi_{\alpha}\,\Omega_{\beta 0}]\,,$$

$$(42) \qquad (\Omega_{\alpha 0})' = [\pi^{\tau}_{\alpha}\,\Omega_{\tau 0}] - [\beta^{\tau}_{\alpha}\,\pi_{\tau}]\,.$$

Bibliography.

L BERWALD, [1], Über FINSLERsche und verwandte Räume, C.R., 2. Congr. Math. des Pays Slaves, Prag, 1935, 1–16.

———, [2], Über die n-dimensionalen CARTANschen Räume und eine Normalform der zweiten Variation eines $(n-1)$-fachen Oberflächenintegrals, Acta Math., 71, 191–248.

———, [3], Über FINSLERsche und CARTANsche Geometrie I. Geometrische Erklärungen der Krümmung und des Hauptskalars eines zweidimensionalen FINSLERschen Raumes, Mathematica, Cluj, 16 (1940).

———, [4], Über FINSLERsche und CARTANsche Geometrie II. Invarianten bei der Variation vielfacher Integrale und Parallelhyperflächen in CARTANschen Räumen, Comp. Math., 7, Fasc. 1 (1939), 141–176.

E. CARTAN, [1], Les espaces de FINSLER, Actualités scientifiques, 79, Paris, Hermann et Cie, 1934.

———, [2], Les espaces métriques fondés sur la notion d'aire, Actualités scientifiques, 72, Paris, Hermann et Cie, 1933.

A. KAWAGUCHI, [1], Ein metrischer Raum, der eine Verallgemeinerung des FINSLERschen Raumes ist, Monatsh. für Math. und Phys., 43 (1936), 289–297.

———, [2], Die Differentialgeometrie höherer Ordnung II. Über die n-dimensionalen metrischen Räume mit vom m-dimensionalen Flächenelement abhängigem Zusammenhang, Journ. Fac. of Science, Hokkaido Imp. Univ., Ser. I, 9 (1940), 153–188,

A. KAWAGUCHI and S. HOKARI, [1], Die Grundlegung der Geometrie der fünfdimensionalen metrischen Räume auf Grund des Begriffs des zwei-dimensionalen Flächeninhalts, Proc. of the Imp. Acad., 16 (1940), 313–319.

———, [2], Die Grundlegung der Geometrie der n-dimensionalen metrischen Räume auf Grund des Begriffs des K-dimensionalen Flächeninhalts, ibid., 320–325.

T. OHKUBO, [1], Geometry in a space with a generalized metrics, Tensor, 3 (1940), 48–55 (Japanese).