ON PRIMARY LATTICES

By

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In my previous short note concerning the lattice of all subgroups of a finite abelian group, I have introduced new sorts of lattices[1]. I have called namely a modular lattice, whose quotients are always chains or sublattices with no proper neutral element, *primary* (primär) and the direct union of a finite number of primary lattices *semi-primary* (halb-primär). They enable us not solely to illustrate the intermediate relations between all subgroups of an ordinary abelian group, but further to characterize the lattice of submodules in a module with finite bases, where the ring of operators is primary and uniserial in Köthe's sense[2]. We obtain in this way an extension of the well known fact, that an indecomposable complemented modular lattice of finite dimension $n \geqq 4$ characterizes completely the lattice of all linear subspaces of an n-dimensional projective space, in a direction different from that of J. v. Neumann's continuous geometry. As we assume no preliminary knowledge about modular lattices, we shall deal with them in part I. All properties of primary or semi-primary lattices, which are either characteristic for them or indispensable for the later developement, shall be treated in part 2. As we deal with only the finite-dimensional case, their topological aspect was not considered. The remaining parts concern chiefly with the generalization of von Staudt's algebra of throws, to attain the main theorem, that every primary lattice with $m_h \geqq 4$ is isomorphic with a lattice of submodules in a module of the above mentioned sort[3].

Finally I must tender my hea[illegible]hanks to Mr. Nakayama, who has been kind enough to give me [illegible] useful remarks.

(1) Über modulare Verbände, welche die Untergruppen einer endlichen abelschen Gruppe bilden. I. Proc. of the Imp. Acad. Vol. XIX. No. 9, 1943.

(2) G. Köthe, Verallgemeinerte Abelsche Gruppen mit hyperkomplexem Operatorenring. Math. Zeitschrift. 39, 1935.

(3) For the meaning of the condition $m_h \geqq 4$ refer the part 2 of this paper.

PART I.

A set is called a *partially-ordered system*, when within elements a relation $x \geqq y$ is determined, satisfying the following axioms: (A) $x \geqq x$ for each element x, (B) $x \geqq y$ and $y \geqq x$ yield $y = x$, (C) if $x \geqq y$, $y \geqq z$, then $x \geqq z$. If $x \neq y$ and $x \geqq y$, then the relation is denoted by $x > y$. By a least (greatest) element of a subset X of a partially ordered system is meant an element $a \in X$ such that for all $x \in X$ the relation $a \leqq x$ $(x \leqq a)$ holds. If there exists the least element in the set of all the elements u with $u \geqq x$, $u \geqq y$, it is uniquely determined and is called the *join* $x \cup y$ of x and y. If further there exists the greatest element in the set of all the elements v with $v \leqq x$, $v \leqq y$, it is also uniquely determined and is called the *meet* $x \frown y$ of x and y. By a *lattice* is meant a partially ordered system, in which any two elements have always their join and meet. We can then verify easily the following identities $x \cup x = x \frown x = x$, $x \frown y = y \frown x$, $x \cup y = y \cup x$, $x \frown (y \frown z) = (x \frown y) \frown z$, $x \cup (y \cup z) = (x \cup y) \cup z$, $x \frown (x \cup y) = x$, $x \cup (x \frown y) = x$. Conversely, a set, in which two kinds of operations $x \cup y$, $x \frown y$ are so determined, that the above identities hold, becomes partially ordered, if we define $x \geqq y$ to mean $x \frown y = y$. If the greatest element (least element) of a lattice L exists, then it is denoted with $I(O)$. The lattice, whose elements are in a finite number, shall be called *finite*. A finite lattice has evidently I and O. If the modular identity $x \cup (y \frown z) = (x \cup y) \frown z$ holds for $x \leqq z$, then the lattice is called *modular*. The *quotient* $\frac{b}{a}$ or $[a, b]$ is the sublattice of all the elements x between a and b, such that $a \leqq x \leqq b$. In a modular lattice the quotient $\frac{u \cup v}{u}$ is lattice-isomorphic with the quotient $\frac{v}{u \frown v}$, if we make correspo[illegible]he element $y = v \frown x$ to each x with $u \leqq x \leqq u \cup v$. By the *dim*[illegible] d of an element x in a modular lattice L is meant the lar[illegible]umber d, such that $O = x_0 < x_1 < x_2 \cdots < x_d = x$, x_i being elements from L. A modular lattice, whose greatest element I is of finite dimension, is called *of finite dimension*. It holds then the Jordan-Dede[illegible]nd chain condition with the relation $\dim(u \cup v) + \dim(u \frown v) = \dim u + \dim v$. The elements $x_1, x_2, \cdots x_\lambda$ in

L are said to be *independent*, if $(x_1 \cup x_2 \cup \cdots \cup x_{i-1} \cup x_{i+1} \cup \cdots \cup x_\lambda) \cap x_i = 0$, $i = 1, 2, \cdots \lambda$, hold.

Lemma 1. *If* $(x \cup y) \cap z \leqq x$ *in a modular lattice, then* $(x \cup z) \cap y = x \cap y$.

Pr. $x = \{(x \cup y) \cap z\} \cup x = (x \cup y) \cup (z \cup x) = \{(x \cup z) \cap y\} \cup x$, $x \cap y = [\{(x \cup z) \cap y\} \cup x] \cap y = \{(x \cup z) \cap y\} \cup (x \cap y) = (x \cup z) \cap y$.

Lemma 2. *If* $(x_1 \cup x_2 \cup \cdots \cup x_{i-1}) \cap x_i = 0$, $i = 2, 3 \cdots \lambda$, *in a modular lattice, then* $x_1, x_2, \cdots x_\lambda$ *are independent.*

Pr. By induction on λ. Put $X_i' = x_1 \cup x_2 \cup \cdots x_{i-1} \cup x_{i+1} \cdots \cup x_{\lambda-1}$, $i = 1, 2, \cdots \lambda-1$, $X_i = x_\lambda \cup X_i'$, $X_\lambda = x_1 \cup x_2 \cup \cdots \cup x_{\lambda-1}$, then $X_i' \cap x_i = 0$ by the induction-hypothesis and $X_i \cap x_i = (X_i' \cup x_\lambda) \cap x_i = X_i' \cap x_i = 0$, $i = 1, 2, \cdots \lambda-1$, $X_\lambda \cap x_\lambda = 0$ by lemma I, since $(X_i' \cup x_i) \cap x_\lambda = 0$.

Corollary. *If* $x_1, x_2, \cdots x_\lambda$ *are independent in a modular lattice, then, putting* $X_i = x_1 \cup x_2 \cup \cdots \cup x_{i-1} \cup x_{i+1} \cdots \cup x_\lambda$, *holds* $X_1 \cap X_2 \cap \cdots \cap X_\lambda = 0$.

Pr. It suffices to prove $X_1 \cap X_2 \cap \cdots \cap X_i = x_{i+1} \cup x_{i+2} \cup \cdots \cup x_\lambda$; by induction on i. The case $i = 1$ is evident. From $X_1 \cap X_2 \cap \cdots \cap X_s = x_{s+1} \cup x_{s+2} \cup \cdots \cup x_\lambda$ follows $X_1 \cap X_2 \cap \cdots \cap X_s \cap X_{s+1} = (x_{s+1} \cup x_{s+2} \cup \cdots \cup x_\lambda) \cap X_{s+1} = (x_{s+2} \cup \cdots \cup x_\lambda) \cup (x_{s+1} \cap X_{s+1}) = x_{s+2} \cup \cdots \cup x_\lambda$. q. e. d.

Any three elements x, y, z of a modular lattice are said to be *distributive*, if the relation $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$ holds.

Theorem 1. *The distributivity relation is symmetric and dual, i. e. if* x, y, z *are distributive, then we have*

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z) \cdots\cdots\cdots\cdots\cdots (1)$$

$$y \cup (z \cap x) = (y \cup z) \cap (y \cup x) \cdots\cdots\cdots\cdots\cdots (2)$$

$$z \cup (x \cap y) = (z \cup x) \cap (z \cup y) \cdots\cdots\cdots\cdots\cdots (3)$$

and the three corresponding dual relations (4), (5), (6).

Pr. From modular identity and (I) we have $(y \cap z) \cup (y \cap x) = \{x \cup (y \cap z)\} \cap y = (x \cup y) \cap (x \cup z) \cap y = y \cap (z \cup x)$, that is (5), and $(z \cup x) \cap (z \cup y) = z \cup \{y \cap (z \cup x)\} = z \cup (y \cap z) \cup (y \cap x) = z \cup (x \cap y)$, that is (3). Similarly we obtain (4), (2) from (3) and (6) from (2).

Corollary. *If* $x \geqq y$, *then* x, y, z *are distributive for any* z.

If an element a of a modular lattice L is distributive with any two elements in L, then a is called *neutral*. The elements I and O in L is evidently neutral, precisely *improper neutral*. Other neutral elements are said to be *proper neutral*.

Theorem 2. *Every element of a chain is neutral. Generally, if there exists only one element with a given dimension in a modular lattice of finite dimension, then it is neutral.*

Pr. Suppose x is the unique element of dimension λ and y an arbitrary element. If $\mathrm{Dim}\, y > \lambda$ then the quoteint $\frac{y}{O}$ contains at least an element of dimension λ, i.e. the element x. If $\mathrm{Dim}\, y < \lambda$, then the quotient $\frac{I}{y}$ contains at least an element of dimension $\lambda - \mathrm{Dim}\, y$. Since its dimension equals to λ in L, it is nothing else than the element x. Hence x is neutral by the corollary to theorem I.

Suppose that a lattice L have the elements I and O. If for an element a in L an element a' exists, such that $a \smile a' = I$, $a \frown a' = O$, then a is called *complemented* and a' its *complement*.

Theorem 3. *If an element a of a modular lattice L is complemented and neutral, then the complement a' is also neutral and for every element x of L holds the identity $x = (a \frown x) \smile (a' \frown x)$. Conversely, if an element a is complemented and the above identity holds for every element x, then a is neutral.*

Pr. The neutrality of the element a yields

$$(a \frown x) \smile (a' \frown x) = \{a \smile (a' \frown x)\} \frown \{x \smile (a' \frown x)\} = \{a \smile (a' \frown x)\} \frown x$$
$$= (a \smile a') \frown (a \smile x) \frown x = x.$$

Conversely $x = (a \frown x) \smile (a' \frown x)$ and $y = (a \frown y) \smile (a' \frown y)$ yield $a \frown (x \smile y) = \{(a \frown x) \smile (a \frown y) \smile (a' \frown x) \smile (a' \frown y)\} \frown a = (a \frown x) \smile (a \frown y) \smile [\{(a' \frown x) \smile (a' \frown y)\} \frown a] = (a \frown x) \smile (a \frown y)$, whence the neutrality ef the element a follows. But a is the complement of a' and the identity is symmetric with respect to a, a'. Therefore a' is also neutral.

Theorem 4. *If a is a complemented neutral element of a modular lattice L and a' its complement, then every element of L is uniquely representable as the join of two elements from the quotients $\frac{a}{O}$ and $\frac{a'}{O}$*

respectively. Conversely. if every element of a modular lattice L is representable as the join of elements from two sublattices L_1, L_2, which have only O as its common element, then we have $I = e_1 \cup e_2$, $O = e_1 \cap e_2$, where e_1, e_2 are the greatest elements of L_1, L_2 respectively, and moreover every element x from L is uniquely representable as $x = (e_1 \cap x) \cup (e_2 \cap x)$, e_1, e_2 being neutral.

Pr. In order to prove the converse statement, let f_1 be the greatest element in L_1, then $f_1 = f_1 \cap (e_1 \cup e_2) = e_1 \cup (e_2 \cap f_1) = e_1$. Similarly the greatest element in L_2 is identical with e_2. From $x = x_1 \cup x_2$ with $x_i \in L_i$ follows $x \cap e_1 = (x_1 \cup x_2) \cap e_1 = x_1 \cup (x_2 \cap e_1) = x_1$ and $x \cap e_2 = x_2$.

Theorem 5. *If in a modular lattice L the relation $I = e_1 \cup e_2 \cup \cdots \cup e_\lambda$ holds with independent neutral elements e_i, $i = 1, 2, \cdots \lambda$, then every element x of L is reprensentable as $x = (x \cap e_1) \cup (x \cap e_2) \cup \cdots \cup (x \cap e_\lambda)$.*

Pr. By induction on λ.

Corollary and definition. *In the above theorem, the quotient $\frac{e_i}{O}$ is a sublattice of L (also an ideal in L) and shall be denoted by L_i. Every element x of L is uniquely representable as the join of elements $x_i = x \cap e_i$ from L_i. Then the lattice L is said to be a direct union of lattices L_i, $i = 1, 2, \cdots \lambda$, x_i being the L_i-component of x.*

Pr. If $x = x_1 \cup x_2 \cup \cdots \cup x_\lambda$, $x_i \in L_i$, then $x \cap e_i = x_i \cup \{(x_1 \cup x_2 \cup \cdots \cup x_{i-1} \cup x_{i+1} \cup \cdots \cup x_\lambda) \cap e_i\} = x_i$, since $(x_1 \cup x_2 \cup \cdots \cup x_{i-1} \cup x_{i+1} \cup \cdots \cup x_\lambda) \cap e_i \leqq (e_1 \cup e_2 \cup \cdots \cup e_{i-1} \cup e_{i+1} \cup \cdots \cup e_\lambda) \cap e_i = 0$.

Theorem 6. *If L is the direct union of L_i and x_i, y_i the L_i-components of x, y respectively, then L_i-components of $x \cup y$ and $x \cap y$ are $x_i \cup y_i$ and $x_i \cap y_i$ respectively.*

Pr. $$(x \cup y) \cap e_i = (x \cap e_i) \cup (y \cap e_i) = x_i \cup y_i .$$

$$(x \cap y) \cap e_i = (x \cap e_i) \cap (y \cap e_i) = x_i \cap y_i .$$

A lattice, in which every element is complemented and neutral, is called *Boolean algebra.*

Theorem 7. *The set of all complemented neutral elements in a modular lattice L is a sublattice of L and is a Boolean algebra, which is called the center of L.*

Pr. Suppose a_1, a_2 are complemented and neutral in L. Then we have

$$(a_1 \cup a_2 \cup x) \cap (a_1 \cup a_2 \cup y) = a_1 \cup \{(a_2 \cup x) \cap (a_2 \cup y)\}$$
$$= a_1 \cup \{a_2 \cup (x \cap y)\} = (a_1 \cup a_2) \cup (x \cap y).$$

$$(a_1 \cup a_2) \cup (a_1' \cap a_2') = a_1 \cup \{(a_2 \cup a_1') \cap (a_2 \cup a_2')\} = a_1 \cup a_2 \cup a_1' = I,$$

$$(a_1 \cup a_2) \cap (a_1' \cap a_2') = \{(a_1 \cap a_1') \cup (a_2 \cap a_1')\} \cap a_2' = a_2 \cap a_1' \cap a_2' = O.$$

Therefore $a_1 \cup a_2$ is complemented neutral and similarly for $a_1 \cap a_2$, if dually considered. q. e. d.

If the center of L consists of only I and O, then L cannot be a direct union of two sublattices and is called *indecomposable.*

Theorem 8. *If a modular lattice L is the direct union of sublattices L_i, $i = 1, 2, \cdots \lambda$, then the center C of L is the direct union of the centers C_i of L_i.*

Pr. For $a \in C_i$ we have $a = a \cap e_i$ and for arbitrary x, y in L $a \cap (x \cup y) = a \cap e_i \cap (x \cup y) = a \cap (x_i \cup y_i) = (a \cap x_i) \cap (a \cap y_i) = (a \cap x) \cup (a \cap y)$. From $a \cup a' = e_i$, $a \cap a' = O$ follows $a \cup (a' \cup e_1 \cup \cdots \cup e_{i-1} \cup e_{i+1} \cup \cdots \cup e_\lambda) = I$ and $a \cap (a' \cup e_1 \cup \cdots \cup e_{i-1} \cup e_{i+1} \cup \cdots \cup e_\lambda) = a \cap e_i \cap (a' \cup e_1 \cup \cdots e_{i-1} \cup e_{i+1} \cdots \cup e_\lambda) = a \cap [a' \cup \{(e_1 \cup \cdots \cup e_{i-1} \cup e_{i+1} \cdots \cup e_\lambda) \cap e_i\}] = a \cap a' = O$. Hence a belongs to C and $C_i \subseteq C$. Further, if x is an element in C, then $(x \cap e_i) \cup (x' \cap e_i) = (x \cup x') \cap e_i = e_i$, $(x \cap e_i) \cap (x' \cap e_i) = O$ and therefore $x \cap e_i \in C_i$. Now it follows from $x = (x \cup e_1) \cup \cdots \cup (x \cap e_\lambda)$, that x belongs to the direct union of C_i. q.e.d.

Theorem 9. *If L is the direct union of sublattices $L^{(i)}$, $i = 1, 2, \cdots m$, each $L^{(i)}$ being the direct union of $L_j^{(i)}$, $j = 1, 2, \cdots \lambda_i$, then L is the direct union of all $L_j^{(i)}$, $i = 1, 2, \cdots m$; $j = 1, 2, \cdots \lambda_i$.*

Pr. Let $I = e^{(1)} \cup e^{(2)} \cup \cdots \cup e^{(m)}$, $e^{(i)} \in L^{(i)}$, and $e^{(i)} = e_1^{(i)} \cup e_2^{(i)} \cup \cdots \cup e_{\lambda_i}^{(i)}$, $e_j^{(i)} \in L_j^{(i)}$. It suffices to prove that $e_j^{(i)}$ are independent. This can be easily proved by lemma 1.

Theorem 10. *If the dimension of the center C of a modular lattice L is finite and equals to n, then C consists of 2^n elements and L is the direct union of n indecomposable sublattices.*

Pr. By induction on n. In case $n = 1$ evident. If $n > 1$, let $I = e_1 \cup e_2$, $e_1 \cap e_2 = O$, e_i being proper neutral. Then $L = L_1 + L_2$, $C = C_1 + C_2$, where the dimensions n_1, n_2 of C_1, C_2 are smaller than n

with $n = n_1 + n_2$, since $\frac{I}{e_1} \simeq \frac{e_1 \cup e_2}{e_1} \simeq \frac{e_2}{O}$. Then C_i consists of 2^{n_i} elements by induction-hypothesis. Therefore C consists of $2^{n_1} \cdot 2^{n_2} = 2^n$ elements, and is the direct union of n indecomposable sublattices, since C_i is the direct union of n_i indecomposable sublattices. q.e.d.

An element x of a lattice L is said to be *reducible*, if $x = y \cup z$, $x > y$, $x > z$. If this is not the case, then we shall say x *irreducible*. For example every element from a chain is irreducible. Every element from a lattice of finite dimension can be represented as the join of irreducible elements. The independent irreducible elements x_i with $x = x_1 \cup x_2 \cup \cdots \cup x_\nu$ are called a *basis* of the element x. If the center C of a modular lattice consists of 2^n elements, then $I = e_1 \cup e_2 \cup \cdots \cup e_n$ uniquely with irreducible elements e_i from C and every element from C is the join of some e_i. Hence it follows

Theorem 11. *Any modular lattice of finite dimension is uniquely representable as the direct join of indecomposable sublattices.*

Theorem 12. *Every element of the center of a modular lattice has an unique complement. Conversely an element of a complemented modular lattice belongs to the center, if its complement is unique.*

Pr. Suppose x', x'' are complements of an element x in the center. Then $I = x \cup x'$, $x'' = (x'' \cap x) \cup (x'' \cap x') = x'' \cap x'$, whence $x'' \leqq x'$. Similarly we obtain $x' \leqq x''$ and therefore $x' = x''$. Next let a' be the unique complement of an element a, and u be an arbitrary element, which is independent with a. Then u, a, $(u \cup a)'$ are independent and consequently $u \cup (u \cup a)' = a'$, whence $u \leqq a'$. Since $(a \cap x)' \cap x$ is independent with a, where x is arbitrary, so we have $(a \cap x)' \cap x \leqq a'$. Now we infer

$$x = (a \cap x) \cup \{(a \cap x)' \cap x\} \leqq (a \cap x) \cup (a' \cap x) \leqq x$$

Hence a is neutral by theorem 3. q. e. d.

The converse statement in the above theorem does not hold for arbitrary modular lattices. For example, we consider the lattice of seven elements O, u_1, u_2, v_1, v_2, v_3, I, where $u_1 \cup u_2 = v_1$, $u_1 \cap u_2 = O$, $v_1 \cup v_2 = v_1 \cup v_3 = v_2 \cup v_3 = I$, $v_1 \cap v_2 = v_1 \cap v_3 = v_2 \cap v_3 = u_2$, $u_1 \cup v_2 = u_1 \cup v_3 = I$, $u_1 \cap v_2 = u_1 \cap v_3 = O$. This is modular and the element v_2 has its unique complement, but however not neutral, since $(v_2 \cap v_3) \cup (u_1 \cap v_3) = u_2 \neq v_3$.

Theorem 13. *C being the center of a modular lattice L, the set W of all elements $(b \frown c) \smile a$, where $c \in C$ and $a \leqq b$, is a sublattice of the center of the quotient $\frac{b}{a}$.*

Pr. First we prove, that W is a sublattice of L.

$$\{(b \frown c_1) \smile a\} \smile \{(b \frown c_2) \smile a\} = (b \frown c_1) \smile (b \frown c_2) \smile a = \{b \frown (c_1 \smile c_2)\} \smile a,$$

$$\{(b \frown c_1) \smile a\} \frown \{(b \frown c_2) \smile a\} = b \frown (c_1 \smile a) \frown (c_2 \smile a)$$
$$= b \frown \{(c_1 \frown c_2) \smile a\} = \{b \frown (c_1 \frown c_2)\} \smile a.$$

Since we have further

$$\{(b \frown c) \smile a\} \smile \{(b \frown c') \smile a\} = b, \quad \{(b \frown c) \smile a\} \frown \{(b \frown c') \smile a\} = a,$$

every element $(b \frown c) \smile a$ is complemented in $\frac{b}{a}$. From $a \leqq x \leqq b$, $x = (c \frown x) \smile (c' \frown x)$ follows

$$[x \frown \{(b \frown c) \smile a\}] \smile [x \frown \{(b \frown c') \smile a\}] = \{(x \frown c) \smile a\} \smile \{(x \frown c') \smile a\} = x.$$

Hence $(b \frown c) \smile a$ is neutral in the quotient $\frac{b}{a}$ by theorem 3.

Theorem 14. *If a modular lattice L is a direct union of L_i, $i = 1, 2, \cdots n$, then a quotiont $\frac{b}{a}$ of L is isomorphic with a direct union of quotients in L_i.*

Pr. Suppose $I = e_1 \smile e_2 \smile \cdots \smile e_n$ and $L_i \simeq \frac{e_i}{O}$. We infer the independence of $(b \frown e_i) \smile a$ in $\frac{b}{a}$ from the independence of e_i in L, as follows

$$\{(b \frown e_i) \smile a\} \frown [\{(b \frown e_1) \smile a\} \smile \cdots \{(b \frown e_{i-1}) \smile a\} \smile \{(b \frown e_{i+1}) \smile a\} \cdots]$$
$$= [b \frown \{e_i \frown (e_1 \smile \cdots e_{i-1} \smile e_{i+1} \cdots)\}] \frown a = a.$$

Now we have, for $a \leqq x \leqq b$, $x = a \smile \sum (e_i \frown x) = \sum [\{(b \frown e_i) \smile a\} \frown x]$ and the quotient $\frac{b}{a}$ becomes the direct union of $\frac{(b \frown e_i) \smile a}{a}$, $i = 1, 2, \cdots n$, which are isomorphic with the quotients $\frac{b \frown e_i}{a \frown e_i}$ respectively[1].

(1) The sign $\sum A_i$ means the join of all elements A_i, $i = 1, 2, \cdots$.

PART 2

Definition. A modular lattice, whose quotients are always chains or sublattices with no proper neutral element, is called *primary*. For instance, chains, indecomposable complemented modular lattices of finite dimension (projective geometries) and continuous geometries are all primary. Since the inversion of the order within elements has no influence upon the neutrality of an element, so we have

Theorem 15. *A lattice, which is dually isomorphic with a given primary lattice, is primary. Every quotient of a primary lattice is also primary.*

Definition. A lattice, which is a direct union of a finite number of primary lattices, is called *semi-primary*. In particular a complemented modular lattice of finite dimension is semi-primary.

Theorem 16. *A lattice, which is dually isomorphic with a given semi-primary lattice, is semi-primary. Any arbitrary quotient of a semi-primary lattice is also semi-primary.*

Pr. Let $\bar{L}$ be dually isomorphic with a given semi-primary lattice L and denote the corresponding element in $\bar{L}$ of an element x in L with $\bar{x}$. If $I = e_1 \cup e_2 \cup \cdots \cup e_n$, where e_i irreducible elements from the center of L, then, putting $E_i = e_1 \cup e_2 \cup \cdots \cup e_{i-1} \cup e_{i+1} \cup \cdots \cup e_n$, we have $\bar{I} = \bar{E}_i \cap (\bar{E}_1 \cup \cdots \bar{E}_{i-1} \cup \bar{E}_{i+1} \cup \cdots \cup \bar{E}_n)$ and $\bar{E}_1 \cup \bar{E}_2 \cup \cdots \cup \bar{E}_n = \bar{O}$, since $I = E_i \cup (E_1 \cap \cdots E_{i-1} \cap E_{i+1} \cdots \cap E_n)$ and $E_1 \cap E_2 \cap \cdots \cap E_n = O$ hold. Hence $\bar{E}_i$, $i = 1, 2, \cdots n$, are independent and neutral. The quotient $\frac{\bar{E}_i}{\bar{I}}$ is now dually isomorphic with $\frac{I}{E_i} \simeq \frac{e_i}{O}$, whence it follows that $\bar{L}$ is a direct union of primary lattices $\frac{\bar{E}_i}{\bar{I}}$. The proof of the second part of this theorem is immediate in virtue of theorem 14. q. e. d.

As regards theorem 13 we can sharpen this as follows.

Theorem 17. *The center W of the quotient $\frac{b}{a}$ of a semi-primary lattice L is the set of all elements $(b \cap c) \cup a$ with elements c from the center C of L. In particular, if c irreducible in C, then $(b \cap c) \cup a$ is irreducible in W.*

Pr. It suffices to prove, that every element d in W is representable as $(b \frown c) \smile a$ with $c \in C$. Let $I = e_1 \smile e_2 \smile \cdots \smile e_n$ with irreducible elements e_i in C, then we have $d = \sum \{a \smile (e_i \frown d)\}$. Since $a \smile (e_i \frown d) = \{(b \frown e_i) \smile a\} \frown d$ belongs to W, and since $\frac{b}{a}$ is a direct union of primary lattices $\frac{(b \frown e_i) \smile a}{a}$ by theorem 14, so the element $a \smile (e_i \frown d)$ is neutral in $\frac{(b \frown e_i) \smile a}{a}$ and henceforth $a \smile (e_i \frown d) = a$ or $a \smile (e_i \frown d) = (b \frown e_i) \smile a$. It follows then, that d is the join of some $a \smile (e_i \frown b)$, whence $d = a \smile (E \frown b)$, $E \in C$. That $a \smile (e_i \frown b)$ is irreducible in W follows from the fact, that the quotient $\frac{a \smile (e_i \frown b)}{a}$ is primary, if e_i is irreducible in C.

Definition. If a quotient $\frac{a}{O}$ is a chain, then it is called a *chain in L* with a as its *summit*. If moreover it is not contained in another chain in L, then it is called a *maximal chain in L*.

Theorem 18. *In a primary lattice L of finite dimension*[1], *which is not a chain, there are at least three elements, which cover the element O, and the element I covers at least three elements. Further any element d, which does not belong to a chain in L, covers at least three elements.*

Pr. Suppose there exists only one atom in L[2]. Then it would become proper neutral by theorem 2, contradictory to the assumption. Next suppose there are two atoms a, b, then the quotient $\frac{a \smile b}{O}$ must have another atom c, which is different from both a, b, since otherwise the quotient would not be primary. The dual consideration asserts that I covers at least three elements. The last part of the theorem is now obvious, since $\frac{d}{O}$ is primary and not a chain.

Theorem 19. *A semi-primary lattice, whose elements are all neutral, is a direct join of chains.*

(1) Hereafter by primary or semi-primary lattices are meant always only those of finite dimension.

(2) An element, which covers O, is called an *atom*.

Pr. If a primary lattice is not a chain, then by the preceding theorem there exists at least three atoms, which are not neutral.

Theorem 20. *If a modular lattice L is a direct union of L_i, $i = 1, 2, \cdots n$, then any chain in L is a chain in some L_i.*

Pr. It suffices to prove only in the case $n = 2$. Let $I = e_1 \cup e_2$, $L_1 = \frac{e_1}{O}$, $L_2 = \frac{e_2}{O}$, then the relation $a = (a \frown e_1) \cup (a \frown e_2)$ holds for any chain $\frac{a}{O}$, whence $a \frown e_1 = a$ or $a \frown e_2 = a$. The chain is therefore contained in L_1 or L_2.

Theorem 21. *An element a in a semi-primary lattice L is irreducible, if and only if it is the summit of a chain in L.*

Pr. That a is irreducible, if $\frac{a}{O}$ is a chain, is evident. Suppose that a is irreducible, then a is contained in some primary sublattices L_i of L, for the relation $a = \sum (a \frown e_i)$ yields $a \leqq e_i$ for some i. If $\frac{a}{O}$ were not a chain, then it would cover at least two elements b, c, such that $a = b \cup c$, contrary to the assumption. q.e.d.

Since any element of a lattice of finite dimension is representable as the join of irreducible elements, so we have

Theorem 22. *Any element of a semi-primary lattice L can be represented as the join of irreducible elements, which are summits of chains in L.*

Theorem 23. *Any element of a complemented modular lattice L of finite dimension can be represented as the join of independent atoms in L, that is, any element from L has its basis.*

Pr. We will prove by induction on the dimension of the given element a. If, $\dim a > 1$, a covers an element b, then $a = b \cup (b' \frown a)$, $b \frown (b' \frown a) = O$. In virtue of the dimension-relation the element $b' \frown a$ must be an atom. But b is a join of independent atoms by induction-hypothesis, and these together with $b' \frown a$ make a basis of a.

Lemma 3. *If in a complemented modular lattice L $a > b$ and if the atoms a_i, $i = 1, 2, \cdots m$, b_j, $j = 1, 2, \cdots n$ are bases of a, b respectively, then we have a new basis of a, such that it consists of whole b_j, $j = 1, 2, \cdots n$, and a part of a_i.*

Pr. Let a_{i_1} be an atom in $\frac{a}{O}$, which is not contained in b and put $b^{(1)} = b \cup a_{i_1}$. Putting $b^{(\nu+1)} = b^{(\nu)} \cup a_{i_{\nu+1}}$ successively, such that the atom $a_{i_{\nu+1}}$ is not contained in $\frac{b^{(\nu)}}{O}$, we have $\dim b^{(\nu+1)} = \dim b^{(\nu)} + 1 \leq \dim a$, whence we conclude $b^{(m-n)} = a$.

Theorem 24. *Any modular lattice, whose elements are all representable as the join of a finite number of atoms, is complemented.*

Pr. Let $I = c_1 \cup c_2 \cup \cdots \cup c_m$ with independent atoms c_i and let a be an arbitrary element with $a = a_1 \cup a_2 \cup \cdots \cup a_n$, a_i being independent atoms. Then by the preceding lemma we have $I = a \cup (c_{i_1} \cup c_{i_2} \cup \cdots c_{i_{m-n}})$, where $a \cap (c_{i_1} \cup c_{i_2} \cup \cdots c_{i_{m-n}}) = O$. q. e. d.

In theorem 22 we have shown, that every element of a semi-primary lattice L can be represented as the join of irreducible elements. We shall call the set of all elements in L, which are representable as the join of irreducible elements of dimension not greater than ν, the *ν-th derivative of L* and denote it with $L^{(\nu)}$. Evidently $L^{(0)}$ has only the element O. If $L^{(h-1)} < L^{(h)} = L$, then we shall say, that L has a *height h*. The greatest element in $L^{(\nu)}$ is obviously unique and shall be denoted with $e^{(\nu)}$.

Theorem 25. *The ν-th derivative $L^{(\nu)}$ of a semi-primary lattice L is an ideal in L and the dimension of any arbitrary chain in $\frac{e^{(\nu)}}{O}$ is not greater than ν.*(*)

Pr. By induction on ν. First we prove the case $\nu = 1$. For this purpose we have only to prove, that the meet of an element in $L^{(1)}$ with any element in L belongs to $L^{(1)}$. But this verification can be achieved, if we show, that no irreducible element b of dimension 2 can be contained in the quotient $\frac{e^{(1)}}{O}$. Let $e^{(1)} = a_1 \cup a_2 \cup \cdots \cup a_l$ with independent atoms a_i, and, suppose b covers the atom a_s. Then putting $A_s = a_1 \cup \cdots \cup a_{s-1} \cup a_{s+1} \cdots \cup a_l$, we have

(*) By an *ideal in L* is meant a subset S of L, where the join of any two elements from S and the meet of any element from S with an arbitrary element from L belong always to S. That $L^{(\nu)}$ is an ideal in L, means therefore, $L^{(\nu)}$ is identical with the quotient $\frac{e^{(\nu)}}{O}$.

$e^{(1)} = A_s \cup b$, $\dfrac{e^{(1)}}{A_s} \simeq \dfrac{b}{O}$ in contradiction with the modularity of L. If b covers none of the atoms a_i, but covers the atom c, then there exists an A_s, such that $A_s \cap b = O$, since otherwise $c \leqq A_1 \cap A_2 \cap \cdots \cap A_l = O$ would follow. We then have $\dfrac{e^{(1)}}{A_s} \simeq \dfrac{b}{O}$, which is however not the case. In case $\nu > 1$, suppose x is an irreducible element in L with $\dim x = \nu + 1$. Then we have $\dfrac{x \cup e^{(\nu-1)}}{e^{(\nu-1)}} \simeq \dfrac{x}{x \cap e^{(\nu-1)}}$, where $\dfrac{x}{x \cap e^{(\nu-1)}}$ is a chain of dimension 2 by induction-hypothesis. But the first derivative of the quotient $\dfrac{I}{e^{(\nu-1)}}$ does not contain the chain $\dfrac{x \cup e^{(\nu-1)}}{e^{(\nu-1)}}$ of dimension 2. Hence x is not contained in $\dfrac{e^{(\nu)}}{O}$. q.e.d.

The quotients $\dfrac{e^{(\nu)}}{e^{(\nu-1)}}$ of L are complemented modular lattices by theorem 24. If in particular L is primary, then they are moreover indecomposable. We shall call the system $(a^{(1)}, a^{(2)}, \cdots a^{(h)})$, where $a^{(i)} = (a \cap e^{(i)}) \cup e^{(i-1)} = e^{(i)} \cap (a \cup e^{(i-1)})$, h being the height of L, *system of representatives* for an element a, and denote the dimension of the element $a^{(i)}$ in the quotient $\dfrac{e^{(i)}}{e^{(i-1)}}$ with $\gamma_i(a)$.

Theorem 26. $\sum_{i=1}^{h} \gamma_i(a) = \dim a$.

Pr. By adding the both sides of the following relations

$$\dim (a \cap e^{(i)}) = \gamma_i(a) + \dim (a \cap e^{(i-1)}), \quad i = 1, 2, \cdots h. \qquad \text{q. e. d.}$$

By the *invariant* of an element a of a semi-primary lattice L is meant the system $(\gamma_1(a), \cdots \gamma_h(a))$. The invariant of the element I is called the *invariant* of L and denoted with $(\gamma_1, \gamma_2, \cdots \gamma_h)$.

Theorem 27. *If $a \geqq b$ and $a^{(i)} = b^{(i)}$, $i = 1, 2, \cdots h$, then $a = b$.*

Pr. The assumption $a^{(h)} = b^{(h)}$ yields $a \cup e^{(h-1)} = b \cup e^{(h-1)}$. Now we will show, that $a \cup e^{(i-1)} = b \cup e^{(i-1)}$ follows from $a \cup e^{(i)} = b \cup e^{(i)}$. Since $\dfrac{a \cup e^{(i-1)}}{a^{(i)}} = \dfrac{a \cup e^{(i-1)}}{(a \cup e^{(i-1)}) \cap e^{(i)}} \simeq \dfrac{a \cup e^{(i)}}{e^{(i)}} = \dfrac{b \cup e^{(i)}}{e^{(i)}} \simeq \dfrac{b \cup e^{(i-1)}}{b^{(i)}}$ and $a \cup e^{(i-1)} \geqq b \cup e^{(i-1)}$, we have $a \cup e^{(i-1)} = b \cup e^{(i-1)}$ and finally $a = b$.

Definition. A homomorphism of a lattice L into a lattice W

is called *proper*, if it is not an isomorphism and moreover the image W has at least two elements. A lattice L is called *simple*, if there exists no proper homomorphism of L.

Theorem 28. *Every homomorphism of a complemented modular lattice L of finite dimension is equivalent with the endomorphism $x \to x \cup a$, where a is neutral, and the homomorphic image of L is then isomorphic with the quotient $\frac{I}{a}$.*

Pr. Suppose L is homomorphic with a lattice W. The set of antecedents in L of the least element in W is evidently an ideal in L. Let a be the greatest element in this ideal. Then we can prove that two elements x and y in L have the same image $\bar{x}$ in W, if and only if $x \cup a = y \cup a$. $x \to \bar{x}$, $y \to \bar{x}$ yield namely $(x \cap y)' \cap (x \cup y) \to \bar{O}$, whence $(x \cap y)' \cap (x \cup y) \leqq a$. In virtue of the relation $x \cup y = (x \cap y) \cup \{(x \cap y)' \cap (x \cup y)\}$ we conclude $x \cup y \leqq (x \cap y) \cup a$ and therefore $x \cup a = y \cup a$. Since for any arbitrary two elements x and y $(x \cup a) \cap (y \cup a)$ and $x \cap y$ have the same image in W, we obtain $(x \cup a) \cap (y \cup a) = (x \cap y) \cup a$. W is therefore isomorphic with the quotient $\frac{I}{a}$, if we make correspond to $\bar{x}$ the element $x \cup a$.

Corollary. *An indecomposable complemented modular lattice of finite dimension is simple.*

This is however not the case for a chain. We can establish homomorphisms of a chain of dimension n into a chain of dimension $r \leqq n$ in $\frac{(n-1)!}{(r-1)!\,(n-r)!}$ different ways.

Theorem 29. *A primary lattice L is simple, if it is not a chain.*

Pr. By induction on the height h of L. In case $h = 1$ it is evident by the preceding corollary. In case $h > 1$, $\overline{e^{(i)}}$, $i = 1, 2, \cdots h$, can not be all different for a homomorphism of L, unless it is an isomorphism. Indeed, if $a \to \bar{a}$, $b \to \bar{a}$, $a \neq b$, then either $a \cup b > a$ or $a \cup b > b$. Supposing $a \cup b > a$, $(a \cup b)^{(i)} > a^{(i)}$ holds for some i by theorem 27. But $\overline{(a \cup b)^{(i)}} = \overline{a^{(i)}}$ belongs to the quotient $\frac{\overline{e^{(i)}}}{\overline{e^{(i-1)}}}$ and we have $\overline{e^{(i)}} = \overline{e^{(i-1)}}$, since the quotient $\frac{e^{(i)}}{e^{(i-1)}}$ is simple. Now we have only to treat the following two cases

$$\text{(i)} \qquad \bar{O} = \overline{e^{(1)}} = \cdots = \overline{e^{(h-1)}} < \overline{e^{(h)}} = \bar{I}.$$

$$\text{(ii)} \qquad \bar{O} < \overline{e^{(1)}} < \cdots < \overline{e^{(h-1)}} = \overline{e^{(h)}} = \bar{I}.$$

For other cases do not occur by the induction-hypothesís, if we consider the quotient $\frac{e^{(h-1)}}{O}$, which is not a chain by theorem 18.

In case (i), x, y being arbitrary from L, we have

$$\overline{(e^{(h-1)} \smile x) \frown (e^{(h-1)} \smile y)} = (\bar{o} \smile \bar{x}) \frown (\bar{o} \smile \bar{y}) = \bar{o} \smile (\bar{x} \frown \bar{y}) = \overline{e^{(h-1)} \smile (x \frown y)},$$

where $e^{(h-1)} \smile x$, $e^{(h-1)} \smile y$ and $e^{(h-1)} \smile (x \frown y)$ are elements from the quotient $\frac{I}{e^{(h-1)}}$. Since this quotient is simple by the corollary to theorem 28 the present homomorphism of L induces an isomorphism of this quotient. Hence $(e^{(h-1)} \smile x) \frown (e^{(h-1)} \smile y) = e^{(h-1)} \smile (x \frown y)$ holds and $e^{(h-1)}$ is neutral, contrary to the hypothesis, that L is primary. In case (ii), the homomorphism induces an isomorphism of the quotient $\frac{e^{(h-1)}}{O}$, because $\overline{e^{(i)}}$, $i = 1, 2, \cdots h-1$, are all different. Now, x, y being arbitrary from L, we have

$$\overline{(e^{(h-1)} \frown x) \smile (e^{(h-1)} \frown y)} = (\bar{I} \frown \bar{x}) \smile (\bar{I} \frown \bar{y}) = \bar{I} \frown (\bar{x} \smile \bar{y}) = \overline{e^{(h-1)} \frown (x \smile y)}$$

and consequently $(e^{(h-1)} \frown x) \smile (e^{(h-1)} \frown y) = e^{(h-1)} \frown (x \smile y)$ contrary to the assumptíon. q. e. d,

We will show here an example of a modular lattice, which has no proper neutral element, and which however not simple. Consider the modular lattice L, which consists of ten elements O, $E_i^{(1)}$, $i = 1, 2, 3$; $E_j^{(2)}$, $j = 1, 2$; $E_k^{(3)}$, $k = 1, 2, 3$ and I such that $E_1^{(1)} \smile E_2^{(1)} = E_2^{(1)} \smile E_3^{(1)} = E_3^{(1)} \smile E_1^{(1)} = E_1^{(2)}$, $E_1^{(1)} \frown E_2^{(1)} = E_2^{(1)} \frown E_3^{(1)} = E_3^{(1)} \frown E_1^{(1)} = O$, $E_1^{(2)} \smile E_2^{(2)} = E_1^{(3)}$, $E_1^{(2)} \frown E_2^{(2)} = E_3^{(1)}$, $E_1^{(3)} \smile E_2^{(3)} = E_2^{(3)} \smile E_3^{(3)} = E_3^{(3)} \smile E_1^{(3)} = I$, $E_1^{(3)} \frown E_2^{(3)} = E_2^{(3)} \frown E_3^{(3)} = E_3^{(3)} \frown E_1^{(3)} = E_2^{(2)}$, and the modular lattice L', which consists of eight elements $\bar{O}$, $F_i^{(1)}$, $i = 1, 2, 3$; $F_j^{(2)}$, $j = 1, 2, 3$ and $\bar{I}$ such that $F_1^{(1)} \smile F_2^{(1)} = F_2^{(1)} \smile F_3^{(1)} = F_3^{(1)} \smile F_1^{(1)} = F_1^{(2)}$, $F_1^{(1)} \frown F_2^{(1)} = F_2^{(1)} \frown F_3^{(1)} = F_3^{(1)} \frown F_1^{(1)} = \bar{O}$, $F_1^{(2)} \smile F_2^{(2)} = F_2^{(2)} \smile F_3^{(2)} = F_3^{(2)} \smile F_1^{(2)} = \bar{I}$, $F_1^{(2)} \frown F_2^{(2)} = F_2^{(2)} \frown F_3^{(2)} = F_3^{(2)} \frown F_1^{(2)} = F_3^{(1)}$. L is then proper homomorphic with L' by the following correspondence:

$$E_i^{(1)} \to F_i^{(1)},\ i = 1, 2, 3,\ E_1^{(2)} \to F_1^{(2)},\ E_k^{(3)} \to F_k^{(2)},\ k = 1, 2, 3.$$

Definition. If an element a in a semi-primary lattice L has a basis a_i, $i = 1, 2, \cdots r$, then we call the number r of the components a_i *rang* of the element a. If further ν_i is the dimension of a_i, where $\nu_1 \geqq \nu_2 \geqq \cdots \geqq \nu_r$, then we call the system $(\nu_1, \nu_2, \cdots \nu_r)$ *type* of the element a. By basis, rang or type of L is to be meant the basis, the rang or the type of the element I of L respectively.

Lemma 4. *If in a semi-primary lattice r irreducible elements a_i, $i = 1, 2, \cdots r$, are independent, among which $a_1, a_2, \cdots a_{r_\nu}$ are of dimensions greater then $\nu-1$, then $a_i \cup e^{(\nu-1)}$, $i = 1, 2, \cdots r_\nu$, are independent over $e^{(\nu-1)}$.*

Pr. Suppose $a_i \cup e^{(\nu-1)}$ are not independent over $e^{(\nu-1)}$, then $(a_1 \cup a_2 \cup \cdots \cup a_s \cup e^{(\nu-1)}) \cap (a_{s+1} \cup e^{(\nu-1)}) > e^{(\nu-1)}$ for some $s < r_\nu$ and hence $a_{s+1} \cap e^{(\nu)} \leqq a_1 \cup a_2 \cup \cdots \cup a_s \cup e^{(\nu-1)}$. Putting $A_s = a_1 \cup a_2 \cup \cdots \cup a_s$, we have $A_s \cup (a_{s+1} \cap e^{(\nu)}) \leqq A_s \cup e^{(\nu-1)}$ and

$$\frac{A_s \cup (a_{s+1} \cap e^{(\nu)})}{A_s} \simeq \frac{a_{s+1} \cap e^{(\nu)}}{O}, \quad \frac{A_s \cup e^{(\nu-1)}}{A_s} \simeq \frac{e^{(\nu-1)}}{A_s \cap e^{(\nu-1)}}.$$

The quotient $\dfrac{A_s \cup e^{(\nu-1)}}{A_s}$ would then contain a chain of dimension ν, which contradicts with theorem 25.

Lemma 5. *If a_i, $i = 1, 2, \cdots r_\nu$ are of dimension greater than $\nu-1$ among the components of a basis of L and $\lambda \geqq \nu$, then $a_i^r = (a_i \cap e^{(\lambda)}) \cup e^{(\nu-1)}$, $i = 1, 2, \cdots r_\nu$ make a basis of the quotient $\dfrac{e^{(\lambda)}}{e^{(\nu-1)}}$.*

Pr. Since $a_i \cup e^{(\nu-1)}$ are independent over $e^{(\nu-1)}$ by the preceding lemma, so $(a_i \cap e^{(\nu)}) \cup e^{(\nu-1)}$ are independent atoms in the quotient $\dfrac{e^{(\nu)}}{e^{(\nu-1)}}$, whence we have $\gamma_\nu = \dim \dfrac{e^{(\nu)}}{e^{(\nu-1)}} \geqq r_\nu$. On the other hand it holds $\dim I = \sum_{\nu=1}^{h} r_\nu = \sum_{\nu=1}^{h} \gamma_\nu$, whence $\gamma_\nu = r_\nu$ follows and $(a_i \cap e^{(\nu)}) \cup e^{(\nu-1)}$, $i = 1, 2, \cdots r_\nu$, become a basis of $\dfrac{e^{(\nu)}}{e^{(\nu-1)}}$. Next, in order to prove the case $\lambda > \nu$, we can assume $\nu = 1$ without loss of generality, since $a_i \cup e^{(\nu-1)}$, $i = 1, 2, \cdots r_\nu$, are a basis of $\dfrac{I}{e^{(\nu-1)}}$. Putting $a_i \cap e^{(\lambda)} = a_i^{(\lambda)}$, $i = 1, 2, \cdots r$, we now have $\dim (a_1^{(\lambda)} \cup a_2^{(\lambda)} \cup \cdots \cup a_r^{(\lambda)}) = \sum_{s=1}^{\lambda} r_s$ in virtue of the independence of $a_i^{(\lambda)}$. The relation $\dim e^{(\lambda)} = \sum_{\nu=1}^{\lambda} \gamma_\nu$

$= \sum_{i=1}^{\lambda} r_i$ yields then $e^{(\lambda)} \doteq a_1^{(\lambda)} \cup a_2^{(\lambda)} \cup \cdots \cup a_r^{(\lambda)}$.

Corollary. *If a_i, $i = 1, 2, \cdots m$, are independent and irreducible in a semi-primary lattice L, then $a_i \frown e^{(\nu)}$, $i = 1, 2, \cdots m$, are independent and $(a_1 \cup a_2 \cup \cdots \cup a_m) \frown e^{(\nu)} = (a_1 \frown e^{(\nu)}) \cup (a_2 \frown e^{(\nu)}) \cup \cdots \cup (a_m \frown e^{(\nu)})$.*

Pr. Putting $(a_1 \cup a_2 \cup \cdots \cup a_m) \frown e^{(\nu)} = \bar{e}^{(\nu)}$, it becomes the greatest element of the $\bar{\nu}$-th derivative of the quotient $\frac{a_1 \cup a_2 \cdots \cup a_m}{O}$ and $a_i \frown e^{(\nu)} \doteq a_i \frown \bar{e}^{(\nu)}$. Hence $a_i \frown e^{(\nu)}$ are a basis of the quotient $\frac{\bar{e}^{(\nu)}}{O}$.

Theorem 30. *An element a of a semi-primary lattice L is supposed to have bases. The number r_ν of the basis-components with dimensions greater than $\nu - 1$ and the type of a are then uniquely determined irrespetive of the choice of bases.*

Pr. In the proof of lemma 4 we have showed $r_\nu = \gamma_\nu$, where γ_ν is clearly irrespective of the choice of bases.

Lemma 6. *If a quotient $\frac{g}{k}$ is a chain, then there exists a chain, $\frac{l}{O}$, such that $g = l \cup k$.*

Pr. Let $g = a_1 \cup a_2 \cup \cdots \cup a_s$, where a_i, $i = 1, 2, \cdots s$, are irreducible, and let $\frac{a_\rho \cup k}{O}$ be with the highest dimension among the quotients $\frac{a_i \cup k}{O}$. Then we have $a_i \cup k \leqq a_\rho \cup k$, $i = 1, 2, \cdots s$, since otherwise the sublattice $\frac{a_i \cup a_\rho \cup k}{k}$ of the quotient $\frac{g}{k}$ would not be a chain. Therefore we have $g = a_\rho \cup k$.

Theorem 31. *A primary lattice L is suppsed to have a basis, whose components are all of dimension greater than ν. Then every maximal chain in L has a dimension greater than ν.*

Pr. Let r be the rang of L. The case $r = 1$ being trivial, we first consider the case $r = 2$ and make its proof by induction on ν. Given a chain $\frac{b}{O}$ in L with dimension $\lambda < \nu + 1$, let a_1, a_2 be a basis of $e^{(\lambda+1)}$ and let $\frac{d}{O}$ be a maximal chain in $\frac{e^{(\lambda)}}{O}$, which is in-

dependent with $\frac{b}{O}$. By induction-hypothesis $\dim d = \lambda$. Then $e^{(\lambda)} = b \cup d$, and $\frac{e^{(\lambda)}}{b} \simeq \frac{d}{O}$ is a chain of dimension λ, which is contained in the quotient $\frac{e^{(\lambda+1)}}{b}$.

This quotient is however not a chain, since the quotient $\frac{e^{(\lambda+1)}}{e^{(\lambda)}}$ is not a chain. Hence there exists an atom $\frac{c}{b}$ in $\frac{e^{(\lambda+1)}}{b}$, which is independent with $e^{(\lambda)}$ over b. We have now by lemma 6 $c = b \cup f$, where f is a chain, which is not contained in $\frac{e^{(\lambda)}}{O}$. Henceforth we have $\dim c = \dim f = \lambda+1$ and $c = f$, $b < f$. Therefore $\frac{b}{O}$ is not maximal. In case $r > 2$ we prove by induction on r. Given a chain $\frac{b}{O}$ in L with dimension λ, let a_i, $i = 1, 2, \cdots r$, be a basis of $e^{(\lambda+1)}$ and put $A = a_1 \cup a_2 \cup \cdots a_{r-1}$, where we can assume $b \cap a_r = O$ without loss of generality, since otherwise we can replace a_r by a new component a'_r, such that $a'_r \cup a_1 = a_r \cup a_1$, $a'_r \cap b = O$. Since the relation $\frac{a_r \cup A}{A} = \frac{(b \cup a_r) \cup A}{A} \simeq \frac{b \cup a_r}{(b \cup a_r) \cap A}$ holds, and $\dim (b \cup a_r) = 2\lambda+1$, we have, putting $(b \cup a_r) \cap A = d$, $\frac{d}{O} = \frac{d}{d \cap a_r} \simeq \frac{a_r \cup d}{a_r} = \frac{a_r \cup b}{a_r} \simeq \frac{b}{O}$. The quotient $\frac{d}{O}$ is therefore a chain in $\frac{A}{O}$ with dimension λ. Now by induction-hypothesis there exists a chain $\frac{c}{O}$ of dimension $\lambda+1$ in $\frac{A}{O}$, which contains d and is independent with a_r. The quotient $\frac{c \cup a_r}{O}$ is primary of type $(\lambda+1, \lambda+1)$ and moreover contains the chain $\frac{b}{O}$ of dimension λ. Therefore $\frac{b}{O}$ is not maximal.

Theorem 32. *Every element of a primary lattice L has a basis.*

Pr. Since every quotient $\frac{a}{O}$ is primary, we need only prove, that the element I has a basis. Suppose L is not a chain, we choose an irreducible element a_1 of highest dimension in L, and next an irreducible element a_2 of highest dimension, which is independent with a_1. Suppose we have thus successively ν irreducible elements a_i, $i = 1, 2, \cdots \nu$, such that a_i is independent with $A_{i-1} = a_1 \cup a_2 \cup \cdots \cup a_{i-1}$ and moreover of highest dimension. We will prove, that, whenever $A_\nu < I$, we can find an irreducible element $a_{\nu+1}$, which is independent with A_ν. Since $A_\nu < I$, there exists certainly a chain, which is not contained in $\frac{A_\nu}{O}$. We choose among those a chain $\frac{b_{\nu+1}}{O}$, such that $A_\nu \frown b_{\nu+1} = d_{\nu+1}$ is of least dimension. Suppose $d_{\nu+1} > O$ and $c_{\nu+1}$ cover $d_{\nu+1}$ in $\frac{b_{\nu+1}}{O}$ and let $\frac{e_{\nu+1}}{O}$ be a maximal chain in $\frac{A_\nu}{O}$, which contains $d_{\nu+1}$. If $e_{\nu+1} > d_{\nu+1}$, then $\frac{e_{\nu+1} \cup c_{\nu+1}}{O}$ is not a chain and therefore contains an atom h, which does not belong to $\frac{d_{\nu+1}}{O}$, such that $h \cup e_{\nu+1} = c_{\nu+1} \cup e_{\nu+1}$. Then we have $h \frown A_\nu = O$, which contradicts with the assumption. Now suppose $e_{\nu+1} = d_{\nu+1}$, then, putting $\mu = \dim e_{\nu+1} = \dim d_{\nu+1}$, holds the inequality $\mu < \dim b_{\nu+1} \leqq \dim a_1$. By the way of selection of the elements a_i, $i = 1, 2, \cdots \nu$, we have evidently $\dim a_\nu \leqq \dim a_{\nu-1} \leqq \cdots \leqq \dim a_1$. The preceding theorem then asserts that there exists certainly a place, where $\dim a_p > \mu$ and $\dim a_{p+1} \leqq \mu$ hold. If $(a_1 \cup a_2 \cup \cdots \cup a_p) \frown e_{\nu+1} = O$, then $(a_1 \cup a_2 \cup \cdots \cup a_p) \frown b_{\nu+1} = O$, contrary to the assumption, that a_{p+1} is independent with $a_1 \cup a_2 \cup \cdots \cup a_p$ and moreover of highest dimension. Now let $k_{\nu+1} = (a_1 \cup a_2 \cup \cdots \cup a_p) \frown e_{\nu+1} > O$. Then there exists a chain $\frac{f_{\nu+1}}{O}$ of dimension $\mu+1$ in $\frac{a_1 \cup a_2 \cup \cdots \cup a_p}{O}$, which contains $k_{\nu+1}$. The quotient $\frac{f_{\nu+1} \cup c_{\nu+1}}{O}$, being not a chain, contains an atom θ, which is contained neither in $\frac{f_{\nu+1}}{O}$ nor in $\frac{c_{\nu+1}}{O}$. Since $f_{\nu+1} \frown c_{\nu+1} = k_{\nu+1}$, the

quotient $\frac{f_{\nu+1} \cup c_{\nu+1}}{k_{\nu+1}}$ has a basis, whose components are both of dimension $= \mu+1-\dim k_{\nu+1}$. Let $\frac{\phi}{k_{\nu+1}}$ be a maximal chain in this quotient with $\dim \phi = \mu+1$, such that ϕ contains $\theta \cup k_{\nu+1}$. Then $\phi = l \cup k_{\nu+1}$, where l is a chain and $\dim l \leqq \mu+1$. If $\dim l = \mu+1$, then $l \geqq k_{\nu+1}$, whence $l \geqq \theta$. The chain $\frac{k_{\nu+1}}{O}$ then contains the atom θ and henceforth $\frac{f_{\nu+1}}{O}$ contains θ contrary to the hypothesis, that θ is not contained in $\frac{f_{\nu+1}}{O}$. Therefore $\dim l \leqq \mu$ and l is not contained in $\frac{A_\nu}{O}$, since otherwise we would have $c_{\nu+1} \leqq A_\nu$ from the relation $\phi \cup f_{\nu+1} = f_{\nu+1} \cup c_{\nu+1} = l \cup f_{\nu+1}$. On the other hand we have $\dim (l \frown A_\nu) = \dim l + \dim A_\nu - \dim (l \smile A_\nu)$. $l \smile A_\nu = l \smile f_{\nu+1} \smile A_\nu = c_{\nu+1} \smile A_\nu$, $\dim (c_{\nu+1} \smile A_\nu) = \dim c_{\nu+1} + \dim A_\nu - \dim (c_{\nu+1} \frown A_\nu)$, $c_{\nu+1} \frown A_\nu = b_{\nu+1} \frown A_\nu = d_{\nu+1}$, whence $\dim (l \frown A_\nu) = \dim l - \dim c_{\nu+1} + \mu = \dim l - 1 \leqq \mu - 1$, yielding a contradiction $\dim (l \frown A_\nu) < \dim (b_{\nu+1} \frown A_\nu)$. q. e. d.

Theorem 33. *Every element of a semi-primary lattice has a basis, its type being uniquely determined irrespective of the choice of the bases.*

We now fix the order of types of elements in a semi-primary lattice lexicographically; namely $(\nu_1, \nu_2, \cdots) > (\nu'_1, \nu'_2, \cdots)$, if $\nu_1 = \nu'_1 \cdots \nu_s = \nu'_s$ and $\nu_{s+1} > \nu'_{s+1}$.

Theorem 34. *If $a \geqq b$, then holds the relations following:* (1) *rang of $a \geqq$ rang of b,* (2) *type of $a \geqq$ type of b, and* (3) *$m_\nu(a) \geqq m_\nu(b)$, where $m_\nu(a)$ denotes the number of basis-components with dimensions greater than $\nu-1$ for the element a.*

Pr. If we denote with $a^{(i)}$, $i = 1, 2, \cdots h$ the representatives of the element a, we have $a^{(i)} \geqq b^{(i)}$ and consequently $\gamma_i(a) \geqq \gamma_i(b)$. It follows then $m_\nu(a) \geqq m_\nu(b)$ by virtue of the relation $r_\nu = \gamma_\nu$. q.e.d.

The summit of a maximal chain in L is not always a basis-component of L. For example, consider the lattice of all subgroups in an abelian group, which is generated by an element a_1 of order 8 and an element a_2 of oder 2, where $a_2 \neq a_1^4$. Then the cyclic subgroup $\{a_1^2 a_2\}$ is of order 4. This subgroup being the summit

of the maximal chain, which contains $\{a_1^2 a_2\}$, $\{a_1^4\}$ and $\{e\}$, can not become a basis-component of the group. Under what conditions the summit of a maximal chain becomes a basis-component, shall be answered in the following.

Theorem 35. *The summit of a maximal chain in a semi-primary lattice L can become a basis-component of L, if and only if it is complemented.*

Definition. A chain of highest dimension among those chains, which have a given atom commonly in possesion, is called *absolutely maximal.*

Theorem 36. *An element a in a semi-primary lattice L is complemented, if and only if every absolutely maximal chain in the quotient $\frac{a}{O}$ is always absolutely maximal in L.*

Pr. By induction on the rang s of the quotient $\frac{a}{O}$. In case $s = 1$ the quotient $\frac{a}{O}$ itself is an absolutely maximal chain in L by hypothesis. Let c_i, $i = 1, 2, \cdots r$ be a basis of L with $\dim c_i = \nu_i$, $\nu_1 \geqq \nu_2 \geqq \cdots \geqq \nu_r$, and suppose $\dim a = \alpha$, $\nu_t > \alpha \geqq \nu_{t+1}$, $\nu_0 = \infty$. By theorem 31 we have $(c_1 \cup c_2 \cup \cdots \cup c_t) \frown a = O$ and hence $\frac{a}{O}$ must be a chain of highest dimension, which is independent with $c_1 \cup c_2 \cup \cdots \cup c_t$. Then $\alpha = \nu_{t+1}$ and the element a becomes a component of a basis together with $c_1, c_2, \cdots c_t$. If conversely a is complemented and if $\frac{a}{O}$ were not absolutely maximal, such that there exists a chain $\frac{b}{O}$ with $\dim a < \dim b$, $a \frown b > O$, then we have $b \frown a' = O$ and $\frac{a}{O} = \frac{a \cup a'}{a'} = \frac{b \cup a'}{a'} \simeq \frac{b}{O}$ in contradiction. Next suppose $s > 1$ and a_i, $i = 1, 2, \cdots s$ are a basis of $\frac{a}{O}$ with $\dim a_1 \geqq \dim a_2 \geqq \cdots \geqq \dim a_s$. First we consider the case $\nu_1 = \dim a_1 = \dim a_2 = \cdots = \dim a_s$. Then a_i, $i = 1, 2, \cdots s$, are all able to become components of a basis of L, as in the proof of theorem 32 and therefore complemented. Next consider the case $\nu_1 < \dim a_1 = \dim a_2 = \cdots = \dim a_s$. Then

there exists ν_t, such that $\nu_t > \dim a_1 \geqq \nu_{t+1}$. Putting $C_t = c_1 \cup c_2 \cup \cdots \cup c_t$, we have $a \frown C_t = O$, since otherwise the absolutely maximal chain in $\frac{a}{O}$ which contains an atom in $a \frown C_t$ would not be absolutely maximal in L by theorem 31. Hence we have $a_1 \frown C_t = O$ and necessarily $\dim a_1 = \nu_{t+1}$. Then a_1 can become a basis-component of L together with $c_1, c_2, \cdots c_t$. Since further $(c_1 \cup c_2 \cup \cdots \cup c_t \cup a_1 \cdots \cup a_{i-1}) \frown a_i = O$ and $\dim a_i = \nu_{t+1}$, $i = 1, 2, \cdots s$, so a_i are all basis-components of L and therefore a is complemented. Finally consider the case $\dim a_l > \dim a_{l+1} = \cdots = \dim a_s$. Putting $a_1 \cup a_2 \cup \cdots \cup a_l = A_1$, $a_{l+1} \cup \cdots \cup a_s = A_2$, it follows by the induction-hypothesis, that every absolutely maximal chain in $\frac{A_1}{O}$ or $\frac{A_2}{O}$ is always absolutely maximal in $\frac{a}{O}$ and consequently absolutely maximal in L. A_1 is therefore complemented in L by the induction-hypothesis and L has a basis, which contain all a_i, $i = 1, 2, \cdots l$ as its components. Let it be b_i, $i = 1, 2, \cdots r$ with $\dim b_i = \nu_i$. Since $\nu_1 \geqq \dim a_l > \dim a_s$, there exists ν_t such that $\nu_t > \dim a_s \geqq \nu_{t+1}$. Putting $B_t = b_1 \cup b_2 \cup \cdots \cup b_t$, we have $B_t \frown A_2 = O$ and $\dim a_s = \nu_{t+1}$. The elements $a_{l+1}, \cdots, a_s$ become then basis-components of L together with $b_1, \cdots b_t$. Since $a_1, a_2, \cdots a_l$ are part of $b_1, \cdots b_t$, so a_i are all basis-components of L, whence a is complemented.

The converse statement of the last two cases follows immediately from the fact, that if a is complemented, then every basis-component a_i of a becomes a basis-component of L. In fact the quotient $\frac{a_i}{O}$ will then be absolutely maximal in L according to the converse statement of the first case.

Corollary. *A chain in L can become a basis-component of L if and only if it is absolutely maximal in L.*

Theorem 37. *If a complemented element c in a primary lattice L is of rang r, then we can find $r+1$ complements of c, such that the join of them equals to $e^{(h)}$, where by h is meant the height of the complements.*

Pr. The cases $c = I$ or $c = O$ being trivial, we prove by induction on r. In case $r = 1$, let c' be a complement of c and let

$\frac{d}{O}$ be an absolutely maximal chain of highest dimension in $\frac{c'}{O}$. Further let $\dim c = t$, $\dim d = h$ and d' be a complement of d in $\frac{c'}{O}$. If $t \geqq h$, there exists a maximal chain $\frac{c''}{O}$ in $\frac{c \cup d}{O}$, which does not contain the atom commonly with $\frac{c}{O}$ and $\frac{d}{O}$. Then we have $c \frown c'' = O$, $c \smile c'' = c \smile d$, $\dim c'' = h$, $c \frown (c'' \smile d') = O$, $c \smile (c'' \smile d') = I$. Hence $c'' \smile d'$ is a complement of c and $(c'' \smile d') \smile c' = (c'' \smile d) \smile d' = \{(c \smile d) \frown e^{(h)}\} \smile d' = (c \frown e^{(h)}) \smile c' = e^{(h)}$. If $t < h$, let g be the element of dimension $h-t$ in $\frac{d}{O}$. There exists a maximal chain $\frac{c''}{g}$ of dimension t in $\frac{c \smile d}{g}$, which does not possess the atom commonly with $\frac{c \smile g}{g}$ and $\frac{d}{g}$. We have then $c'' \frown c = O$ and $\frac{c''}{O}$ is a chain of dimension h in $\frac{c \smile d}{O}$. For, if $\frac{c''}{O}$ were not a chain, $(c \smile d) \frown e^{(1)}$ and $c'' \frown e^{(1)}$ would be of rang 2 and $c'' \frown c > O$. Hence we obtain $c'' \smile c = d \smile c'' = c \smile d$ and consequently $c'' \smile d'$ is a complement of c and $c' \smile (c'' \smile d') = I = e^{(h)}$. In case $r > 1$, let c_i, $i = 1, 2, \cdots r$, be a basis of c and let $\frac{b}{O}$ be an absolutely maximal chain of highest dimension in $\frac{c'}{O}$, where c' is a complement of c, and $\dim b = h$. We choose a complement b'_i of c_i in $\frac{c_i \smile b}{O}$, such that $b \smile b'_i = (c_i \smile b) \frown e^{(h)}$. Further let c'_i, f be complements of c_i, b in $\frac{c}{O}$, $\frac{c'}{O}$ respectively. Then we have $(b'_i \smile f) \smile c = (b'_i \smile f) \smile (c_i \smile c'_i) = (b'_i \smile c_i) \smile (f \smile c'_i) = (c_i \smile b) \smile (f \smile c'_i) = c \smile c' = I$, $(b'_i \smile f) \frown c = (b'_i \smile f) \frown c_i = b'_i \frown c_i = O$. Therefore $b'_i \smile f$, $i = 1, 2, \cdots r$, are all complements of c and we have $c' \smile \sum_{i=1}^{r} (b'_i \smile f) = \sum_{i=1}^{r} (b \smile b'_i \smile f) = b \smile f \smile \sum_{i=1}^{r} (c_i \frown e^{(h)}) = \sum_{i=1}^{r} (c_i \smile c') \frown e^{(h)} = e^{(h)}$.

Corollary 1. *Every complemented element in a primary lattice, which is different from O and I, has at least two complements.*

Corollary 2. *An element in a semi-primary lattice, which has*

an unique complement, is neutral and consequently belongs to the center.

Pr. Let c be such an element and $I = e_1 \cup e_2 \cup \cdots \cup e_n$, where e_i indepenpent and irreducible in the center. Since $c \frown e_i$ must have an unique complement in $\frac{e_i}{O}$, we have necessarily $c \frown e_i = e_i$ or O and hence c belongs to the center.

Corollary 3. *If $\frac{c}{O}$ is a chain of highest dimension in a primary lattice L of rang r, then we can choose two of its complements, such that their meet is of rang $r-2$ and complemented, and that their join equals to $e^{(h)}$, where by h is meant the height of the complements.*

Pr. Considering the case $t \geqq h$ in the proof of the above theorem, we have $(c'' \cup d') \frown c' = d'$ and d' is evidently complemented.

Theorem 38. *If $a \cup b$ and $a \frown b$ are both complemented in a semi-primary lattice L, then a and b are complemented in L.*

Pr. Since $a \frown b$ is complemented in the quotients $\frac{a}{O}$ and $\frac{b}{O}$, we have from $a = (a \frown b) \cup \bar{a}$, $b = (a \frown b) \frown \bar{b}$, where $\bar{a} \frown b = O$, $\bar{b} \frown a = O$, the relation $a \cup b = a \cup \bar{b} = b \cup \bar{a}$. Hence a, b are both complemented in $\frac{a \cup b}{O}$ and consequentely also in L.

Lemma 7. *If an element a of a semi-primary lattice can be represented as the join of m irreducible elements, then rang of $a \leqq m$.*

Pr. By induction on m. Suppose $m > 1$ and $a = a_1 \cup a_2 \cup \cdots \cup a_m$, where a_1 is of the highest dimension among the irreducible elements a_i. Since the dimension of any chain in $\frac{a_1}{O}$ is not greater than that of a_1, so a_1 becomes a component of a basis and consequently complemented. Putting $a_2 \cup \cdots \cup a_m = A$, we have $\frac{a}{a_1} = \frac{a_1 \cup A}{a_1} \simeq \frac{A}{A \frown a_1} \simeq a_1'$, where a_1' is a complement of a_1. Since A is the join of $a_i \cup (A \frown a_1)$, $i = 2, 3, \cdots m$, where $\frac{a_i \cup (A \frown a_1)}{A \frown a_1}$ are chains, so we have rang of $a_1' \leqq m-1$ by induction-hypothesis. Hence rang of $a \leqq m$.

Theorem 39. *If $a \geqq b$, then $m_\nu(a) \geqq m_\nu\left(\frac{a}{b}\right)$ and type of $a \geqq$*

type of $\frac{a}{b}$. *In particular, rang of* $a \geqq$ *rang of* $\frac{a}{b}$.

Pr. In case $\nu = 1$, let a_i, $i = 1, 2, \cdots r$, be a basis of a. Then $a = \sum (a_i \cup b)$ and by the above lemma rang of $\frac{a}{b} \leqq$ rang of a. Now, denoting with $e^{(i)}$, $E^{(i)}$ the greatest element of the i-th derivative of the quotient $\frac{a}{O}$, $\frac{a}{b}$ respectively, we have the inequality

$$m_\nu\left(\frac{a}{b}\right) = \text{rang of } \frac{a}{E^{(\nu-1)}} \leqq \text{rang of } \frac{a}{b \cup e^{(\nu-1)}} \leqq \text{rang of } \frac{a}{e^{(\nu-1)}} = m_\nu(a),$$

since $E^{(\nu-1)}$ contains $b \cup e^{(\nu-1)}$.

Lemma 8. *A basis* a_i, $i = 1, 2, \cdots m$, *of a complemented element* a *in a semi-primary lattice* L *and a basis* a'_i, $i = 1, 2, \cdots n$, *of its complement make together a basis of* L.

Pr. From lemma 1 and lemma 2 follows the independency of a_i, a'_j.

Lemma 9. *If an element* a *in a semi-primary lattice* L *is complemented, then* $a \cup e^{(i)}$ *are complemented in* $\frac{I}{e^{(i)}}$.

Pr. This follows directly from Lemma 4 and lemma 8.

Lemma 10. *Given an element* a *with the basis* a_i, $i = 1, 2, \cdots r$ *in a semi-primary lattice* L *and given an another element* b *with* $b \leqq a$, *rang* $b = s < r$, *we can choose* $r-s$ *components* $a_{i_1}, a_{i_2}, \cdots a_{i_{r-s}}$ *among* a_i, *such that* $(a_{i_1} \cup a_{i_2} \cup \cdots \cup a_{i_{r-s}}) \cap b = O$.

Pr. Suppose b_i, $i = 1, 2, \cdots s$, is a basis of b, then $b^{(1)} = b_1^{(1)} \cup b_2^{(1)} \cup \cdots \cup b_s^{(1)}$, where $b_i^{(1)} = b_i \cap e^{(1)}$. Since the first derivative $\frac{a^{(1)}}{O}$ of a is complemented modular, so we have by lemma 3 a new basis of $a^{(1)}$, which consists of $b_i^{(1)}$, $i = 1, 2, \cdots s$ and $a_{i_1}^{(1)}, a_{i_2}^{(1)}, \cdots, a_{i_{r-s}}^{(1)}$ such that $b \cap (a_{i_1} \cup a_{i_2} \cup \cdots \cup a_{i_{r-s}}) = O$.

Theorem 40. *If* $a \geqq b$ *and* b *is complemented in* $\frac{a}{O}$, *then*

$$m_i\left(\frac{a}{b}\right) = m_i(a) - m_i(b), \quad i = 1, 2, \cdots h.$$

If b *is not complemented in* $\frac{a}{O}$, *then at least one of the above*

equalities does not hold.

Pr. It suffices to prove in the case $a = I$. Suppose b has a complement b', then we have by lemma 8 $m_i\left(\frac{I}{b}\right) = m_i(b') = m_i(I) - m_i(b)$. If b is not complemented, then there exists certainly an absolutely maximal chain $\frac{d}{O}$ in $\frac{b}{O}$, which is not absolutely maximal in $\frac{I}{O}$ by theorem 36. So let $\frac{f}{O}$ be a chain in $\frac{I}{O}$, which contains the atom in $\frac{b}{O}$, such that $\dim f > \dim d$. Further we can find a basis of b, to which d belongs as its component. Suppose r, s are the rangs of I and b respectively. In the case $r = s$, we have $b \neq I$, $m_1\left(\frac{I}{b}\right) > O$, and $m_1(I) = m_1(b)$. In case $r > s$, we can find $A = a_{i_1} \cup a_{i_2} \cup \cdots \cup a_{i_{r-s}}$, where a_i are the basis-components of $\frac{I}{O}$, such that $b \cap A = O$ by lemma 10. If $d < f$, we have

$$(b \cup f) \cap (b \cup A) = b \cup \{f \cap (b \cup A)\} = b \cup \{f \cap (d \cup d' \cup A)\}$$
$$= [d \cup \{(d' \cup A) \cap f\}] \cup b = d \cup b = b,$$

where d' is a component of d in $\frac{b}{O}$. $b \cup f$ and $b \cup A$ are therefore independent over b and consequently we have

$$m_1\left(\frac{I}{b}\right) \geqq m_1\left(\frac{b \cup f \cup A}{b}\right) = m_1\left(\frac{b \cup f}{b}\right) + m_1\left(\frac{b \cup A}{b}\right)$$
$$= 1 + m_1(A) > m_1(I) - m_1(b).$$

If $d < f$ does not hold, then $\frac{d \cup f}{O}$ is not a chain and consequently of rang two by lemma 7. We have then a chain $\frac{h}{O}$ such that $d \cup f = f \cup h$, $h \cap f = O$ and $\dim h < \dim d$, whence follows, putting $\dim d = i$, $e^{(i-1)} \cup d < e^{(i-1)} \cup f$. Since d is in $\frac{b}{O}$ complemented, so $d \cup e^{(i-1)}$ is complemented in $\frac{b \cup e^{(i-1)}}{e^{(i-1)}}$ by lemma 4 and

henceforth the chain $\frac{d \cup e^{(i-1)}}{e^{(i-1)}}$ is absolutely maximal in $\frac{b \cup e^{(i-1)}}{e^{(i-1)}}$ but not in $\frac{I}{e^{(i-1)}}$, whence follows

$$m_1\left(\frac{I}{b \cup e^{(i-1)}}\right) > m_1\left(\frac{I}{e^{(i-1)}}\right) - m_1\left(\frac{b \cup e^{(i-1)}}{e^{(i-1)}}\right).$$

Considering the relation $\dim \frac{I}{b \cup e^{(i-1)}} = \dim \frac{I}{e^{(i-1)}} - \dim \frac{b \cup e^{(i-1)}}{e^{(i-1)}}$, it then holds for some j the relation

$$m_j\left(\frac{I}{b \cup e^{(i-1)}}\right) < m_j\left(\frac{I}{e^{(i-1)}}\right) - m_j\left(\frac{b \cup e^{(i-1)}}{e^{(i-1)}}\right) = m_{i+j-1}(I) - m_{i+j-1}(b).$$

Since further, for the $i-1$-th derivative $E^{(i-1)}$ of $\frac{I}{b}$ the relation $b \cup e^{(i-1)} \leqq E^{(i-1)}$ holds, so we have

$$m_{i+j-1}\left(\frac{I}{b}\right) = m_j\left(\frac{I}{E^{(i-1)}}\right) \leqq m_j\left(\frac{I}{b \cup e^{(i-1)}}\right) < m_{i+j-1}(I) - m_{i+j-1}(b).$$

PART 3

Definition. Two elements a and b in a modular lattice are called *quasi-perspective*, if there exists an element c, which is called the *axis*, such that $a \cap c = O$ and $a \cup c = b \cup c$. If furthermore $b \cap c = O$, then a and b are called *perspective with respect to the axis* c and denoted with the symbol $a \sim b$.

Theorem 41. *If a and b are quasi-perspective, then $\frac{a}{O}$ and $\frac{b}{b \cap c}$ are isomorphic by the correspondence $u \to v$ with $O \leqq u \leqq a$, $b \cap c \leqq v = b \cap (u \cup c) \leqq b$, where u and v are also quasi-perspective with respect to the axis c.*

Pr. $v \cup c = (b \cup c) \cap (u \cup c) = (a \cup c) \cap (u \cup c) = u \cup c$, $u \cap c = a \cap c = O$, whence the quasi-perspectivity of u and v with the axis c follows. We see from the relation $(v \cup c) \cap a = (u \cup c) \cap a = u \cup (c \cap a) = u$ that the correspondence $u \to v = b \cap (u \cup c)$ is one-to-one. It remains to prove that this induces moreover an isomorphism. From $u_1 \to v_1 = (u_1 \cup c) \cap b$, $u_2 \to v_2 = (u_2 \cup c) \cap b$, we obtain

$$v_1 \cup v_2 = \{(u_1 \cup c) \cap b\} \cup \{(u_2 \cup c) \cap b\} \leqq (u_1 \cup u_2 \cup c) \cap b.$$

$$u_1 \cup u_2 = \{(v_1 \cup c) \cap a\} \cup \{(v_2 \cup c) \cap a\} \leqq (v_1 \cup v_2 \cup c) \cap a\,,$$
$$(u_1 \cup u_2 \cup c) \cap b \leqq (v_1 \cup v_2 \cup c) \cap b = v_1 \cup v_2\,,$$

which yields $v_1 \cup v_2 = (u_1 \cup u_2 \cup c) \cap b$. Further we have

$$v_1 \cap v_2 = \{(u_1 \cup c) \cap b\} \cap \{(u_2 \cup c) \cap b\} \geqq \{(u_1 \cap u_2) \cup c\} \cap b\,,$$
$$u_1 \cap u_2 = \{(v_1 \cup c) \cap a\} \cap \{(v_2 \cup c) \cap a\} \geqq \{(v_1 \cap v_2) \cup c\} \cap a\,,$$

$(u_1 \cap u_2) \cup c \geqq (v_1 \cap v_2) \cup c$, which yields $v_1 \cap v_2 = \{(u_1 \cap u_2) \cup c\} \cap b$. q.e.d.

We can choose the axis of quasi-perspectivity in such a manner, that it is contained in the quotient $\dfrac{a \cup b}{O}$. Put $c' = c \cap (a \cup b)$. Then we have $a \cup c' = (a \cup c) \cap (a \cup b) = a \cup b$, $b \cup c' = (b \cup c) \cap (a \cup b) = a \cup b$, $a \cap c' = a \cap c = O$, $b \cap c' = b \cap c$, whence the quasi-perspectivity of a and b with respect to the axis c' follows. Since $(u \cup c') \cap b = \{(u \cup c) \cap (a \cup b)\} \cap b = (u \cup c) \cap b$ holds, this change of the axis exerts no influence upon the correspondence.

Theorem 42. *If a and b are perspective with the axis c, then $a \cap e^{(i)}$ and $b \cap e^{(i)}$ are perspective with the axis $c \cap e^{(i)}$ by the correspondence $u \cap e^{(i)} \to v \cap e^{(i)}$, where $O \leqq u \leqq a$, $O \leqq v \leqq b$, and $v = (u \cup c) \cap b$.*

Pr. From $a \cup c = b \cup c$ and $a \cap c = b \cap c = O$ we have by the corollary to lemma 4, the following relations

$$(a \cap e^{(i)}) \cup (c \cap e^{(i)}) = (a \cup c) \cap e^{(i)} = (b \cup c) \cap e^{(i)} = (b \cap e^{(i)}) \cup (c \cap e^{(i)})\,,$$
$$(a \cap e^{(i)}) \cap (c \cap e^{(i)}) = O, \quad (b \cap e^{(i)}) \cap (c \cap e^{(i)}) = O.$$

Further from $v = (u \cup c) \cap b$ we obtain

$$v \cap e^{(i)} = (u \cup c) \cap e^{(i)} \cap b = \{(u \cap e^{(i)}) \cup (c \cap e^{(i)})\} \cap (b \cap e^{(i)})\,.$$

Theorem 43. *Two irreducible elements e_1, e_2 of the same dimension in a primary lattice are perspective.*

Pr. Suppose $e_1 \neq e_2$ and $\dim e_1 = \dim e_2$. The quotient $\dfrac{e_1 \cup e_2}{O}$ being not a chain, there exists certainly an element e_3, such that $e_1 \cup e_2 = e_1 \cup e_3$, $e_1 \cap e_3 = O$. For the case $e_1 \cap e_2 = O$ we may take for e_3 the summit of a maximal chain in $\dfrac{e_1 \cup e_2}{O}$, which contains neither the atom in $\dfrac{e_1}{O}$ nor the atom in $\dfrac{e_2}{O}$. Then $e_1 \cup e_2 = e_1 \cup e_3 = e_2 \cup e_3$,

$e_1 \frown e_3 = e_2 \frown e_3 = O$, proving $e_1 \sim e_2$. For the case $e_1 \frown e_2 > O$, $e_2 \frown e_3 = O$ follows from $e_1 \frown e_3 = O$, whence we have $e_1 \smile e_2 = e_2 \smile e_3$ and consequently $e_1 \sim e_2$.

Theorem 44. *If in a primary lattice two quotients* $\frac{a}{a \frown d}$ *and* $\frac{b}{d}$ *are isomorphic, and if* $(a \frown b) \smile d$ *is complemented in* $\frac{I}{d}$, *then* a *and* b *are quasi-perspective in* $\frac{I}{a \frown d}$.

Pr. Put $a \smile d = a'$, $a' \frown b = (d \smile a) \frown b = d \smile (a \frown b) = d'$ and let c_i, $i = 1, 2, \cdots s$, be a basis of $\frac{d'}{d}$. Since by hypothesis d' is complemented in $\frac{a'}{d}$, we can find a basis of $\frac{a'}{d}$, such that it consists of c_i, $i = 1, 2, \cdots s$ and e_j, $j = 1, 2, \cdots m$. Since d' is also complemented in $\frac{b}{d}$, and since by hypothesis $\frac{b}{d}$ is isomorphic with $\frac{a'}{d}$, we may assume that c_i, $i = 1, 2, \cdots s$, f_j, $j = 1, 2, \cdots m$ is a basis of $\frac{b}{d}$ with $(e_1 \smile \cdots \smile e_m) \frown (f_1 \smile \cdots \smile f_m) = d$, where $\frac{e_i}{d}$ and $\frac{f_i}{d}$ are chains of the same dimension. Now e_i and f_i are perspective in $\frac{I}{d}$ by the preceding theorem and hence there exists g_i, such that $e_i \smile g_i = f_i \smile g_i = e_i \smile f_i$, $e_i \frown g_i = f_i \frown g_i = e_i \frown f_i = d$. Then we have $a \smile \sum g_i = a' \smile \sum g_i = \sum c_i \smile \sum e_i \smile \sum g_i = \sum c_i \smile \sum f_i \smile \sum g_j = b \smile \sum g_i$, $(a \frown \sum g_i) \smile d = a' \frown \sum g_i = d = b \frown \sum g_i$. Hence $a \frown \sum g_i = d \frown a$ and therefore a is quasiperspective to b in $\frac{I}{a \frown d}$ with the axis $\sum g_i$.

Corollary 1. *If* $\frac{a}{O}$ *and* $\frac{b}{O}$ *are isomorphic and if* $a \frown b$ *is complemented, then* a *and* b *are perspective.*

Corollary 2. *If* a *and* b *are perspective, then they are projective. Conversely, if* a *and* b *are projective and further if* $a \frown b$ *is complemented, then they are perspective.*

Pr. From $a \smile c = b \smile c$, $a \frown c = b \frown c = O$ follows immediately

$$\frac{a}{O} = \frac{a}{a \frown c} \simeq \frac{a \smile c}{c} = \frac{b \smile c}{c} \simeq \frac{b}{O}.$$

The converse statement follows from the preceding corollary.

Theorem 45. *A modular lattice L is primary if and only if all atoms in any quotient of L are pairwive perspective.*

Pr. It is evident by theorem 43, that any two atoms in a quotient of a primary lattice are perspective. We will show conversely, that there exists no proper neutral element in L, if L is not a chain, and if L satisfies the above mentioned condition. Let a be an arbitrary element distinct from O and I. For the case, $\frac{a}{O}$ is not a chain, let $\frac{b}{O}$ be simply a chain in L, which is not contained in $\frac{a}{O}$. For the case, $\frac{a}{O}$ is a chain, the chain $\frac{b}{O}$ must be moreover so chosen, that it does not contain the element a. Then we have $b \frown a < a$, $b \frown a < b$ and, denoting the atoms in $\frac{a}{b \frown a}$, $\frac{b}{b \frown a}$ with c, d respectively, we can find f such that $c \smile f = d \smile f = c \smile d$, $c \frown f = d \frown f = b \frown a = c \frown d$, since by hypothesis c and d are perspective in $\frac{a \smile b}{a \frown b}$. Then we have $(d \smile f) \frown a = (c \smile d) \frown a \geqq c$ and $(d \frown a) \smile (f \frown a) = \{(f \frown a) \smile d\} \frown a = \{(b \frown a) \smile d\} \frown a = d \frown a = b \frown a$, whence $(d \frown a) \smile (f \frown a) < (d \smile f) \frown a$, and therefore a is not neutral. q. e. d.

In a primary lattice L let a^* be quasi-perspective to b with axis x and b quasi-perspective to a with axis y, where $O < a^* \leqq a$. Then $\frac{a^*}{O}$ is isomorphic with a quotient $\frac{a}{\kappa}$. We shall call this isomorphism a *projection of a^* into a*, and κ, $a \frown b$, and $(x \smile y) \frown a = C$ respectively *base*, *pole* and *center* of the projection. Here the base $\kappa = \{(x \frown b) \smile y\} \frown a$ is obviously contained in the center C. In the particular case $\kappa = O$ we have automatically $a^* = a$ and then the projection is called a *projective automorphism of* $\frac{a}{O}$,

Theorem 46. *In a projection of a^* into a with base κ, pole P, center C to every element u in $P \frown a^*$ corresponds $u \smile \kappa$ and to every element in $C \frown a^*$ corresponds an element in C.*

Pr. If $u \leqq a^* \frown b$, then $v = (u \smile x) \frown b = u \smile (x \frown b)$, $(v \smile y) \frown a = \{u \smile (x \frown b) \smile y\} \frown a = u \smile [\{(x \frown b) \smile y\} \frown a] = u \smile \kappa$. If $u \leqq (x \smile y) \frown a^*$, then $v = (u \smile x) \frown b \leqq [\{(x \smile y) \frown a^*\} \smile x] \frown b = (x \smile y) \frown (a^* \smile x) \frown b = (x \smile y) \frown b$, and $[\{(x \smile y) \frown b\} \smile y] \frown a = (x \smile y) \frown (b \smile y) \frown a = (x \smile y) \frown a = C$, whence $(v \smile y) \frown a \leqq C$.

Theorem 47. *If in a projection of a^* into a with pole P, $u \to v$ with $v = (u \cup x) \cap b$ and $v \to w$ with $w = (v \cup y) \cap a$, then $u \cap P = v \cap P = w \cap P$.*

Pr. From $(u \cup P) \cap x = O$ follows $v \cap P = (u \cup x) \cap P = u \cap P$ by lemma 1. Further we have $(v \cup P) \cap y \leqq b \cap y = O$, whence $w \cap P = (v \cup y) \cap P = v \cap P$ follows by lemma 1.

Theorem 48. *If $u \cap C = O$, then $v \cap (x \cup y) = b \cap x$, $w \cap C = \kappa$.*

Pr. From the hypothesis $u \cap (x \cup y) = O$, we obtain $v \cap (x \cup y) = (u \cup x) \cap b \cap (x \cup y) = x \cap b$ and $w \cap C = (v \cup y) \cap a \cap (x \cup y) = \kappa$.

Definition. If the axes of a projection of a^* into a are both irreducible, then it is called *normal*.

Theorem 49. *In a normal projection the base κ and the center C are both irreducible.*

Pr. $C = (x \cup y) \cap a$ and $C \cap x = O$ yield

$$\frac{C}{O} \simeq \frac{C \cup x}{x} \leqq \frac{x \cup y}{x} \simeq \frac{y}{x \cap y}.$$

Theorem 50. *If a normal projection of a^* into a has the base κ with dimension ν, then to every irreducible element g of dimension λ in $\frac{a^*}{O}$, which is independent with P, corresponds an irreducible element h of dimension $\lambda + \nu$ in $\frac{a}{O}$, which is independent with P.*

Pr. $h \cap P = O$ follows immediately from $g \cap P = O$ by theorem 47. From $\frac{a}{P} = \frac{a}{a \cap b} \simeq \frac{a \cup b}{b} \leqq \frac{b \cup y}{b} \simeq \frac{y}{O}$ follows, that $a = P \cup l$ with an irreducible element l by lemma 6. Hence we obtain from $v = (g \cup x) \cap b$ and $(v \cup y) \cap a = h$ the relation

$$h \cup P = (v \cup P \cup y) \cap a = (v \cup P \cup y) \cap (P \cup l) = P \cup \{l \cap (v \cup P \cup y)\}$$

Putting $l' = l \cap (v \cup P \cup y)$, we have $\frac{h}{O} \simeq \frac{h \cup P}{P} = \frac{P \cup l'}{P} \simeq \frac{l'}{l' \cap P'}$ proving that $\frac{h}{O}$ is a chain of dimension $\lambda + \nu$, since $\frac{g}{O}$ and $\frac{h}{\kappa}$ are isomorphic in the projection.

Lemma 11. *In a primary lattice L with height h, there exists*

an irreducible element of dimension h, which is independent with two given elements, whose rangs are less than m_h.

Pr. Let a_i, $i = 1, 2, \cdots m_h$, be all the basis-components with dimension h of L, and c_1, c_2 two given elements, whose rangs are less than m_h. The theorem is evident for the case, where among the atoms $a_i^{(1)} = a_i \frown e^{(1)}$ at least such an one exists, that it is contained neither in $\frac{c_1}{O}$ nor in $\frac{c_2}{O}$. If it is not the case, there exists certainly an atom $a_r^{(1)}$ contained in $\frac{c_1}{O}$ but not contained in $\frac{c_1 \frown c_2}{O}$ and also an atom $a_s^{(1)}$ contained in $\frac{c_2}{O}$ but not contained in $\frac{c_1 \frown c_2}{O}$. Then, the quotient $\frac{a_r^{(1)} \smile a_s^{(1)}}{O}$ being not a chain, contains an atom a, which is distinct from both $a_r^{(1)}$ and $a_s^{(1)}$. The absolutely maximal chain, which contains a, is of dimension h by theorem 31 and is independent with both c_1, c_2. q. e. d.

Two projections are said to be *identical*, if their correspondence relations are same in both projections.

Theorem 51. *In a primary lattice L with the height h and $m_h \geqq 4$, let $\mathfrak{P}_1$, $\mathfrak{P}_2$ be two normal projections of a^* into a with centers C_i and poles P_i, $i = 1, 2$, where it is assumed that* $\operatorname{rang} a \leqq m_h - 2$, *$P_1 \leqq P_2$, and $C_1 \geqq C_2$ or $C_1 \leqq C_2$. If for a given element t_0, which satisfies the relations $t_0 \frown C_1 = O$, $t_0 \frown C_2 = O$, $a^* = (P_1 \frown a^*) \smile t_0$, holds $\mathfrak{P}_1(t_0) = \mathfrak{P}_2(t_0)$, then $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are identical.*

Pr. Suppose that $\mathfrak{P}_i$ is generated by the quasi-perspectivities $\frac{a^*}{O} \simeq \frac{b_i}{\tau_i}$ with axis x_i and $\frac{b_i}{O} \simeq \frac{a}{\eta_i}$ with axis y_i, where we can assume that $a \smile y_i = b_i \smile y_i = a \smile b_i$, $a^* \smile x_i = b_i \smile x_i = a^* \smile b_i$. If we put $(t_0 \smile x_1) \frown b_1 = v_0^{(1)}$, $(v_0^{(1)} \smile y_1) \frown a = w_0$, $(t_0 \smile x_2) \frown b_2 = v_0^{(2)}$, then we have $(v_0^{(2)} \smile y_2) \frown a = w_0$ by hypothesis. It is required to prove, using these relations, that for any element t the relation $(v_2 \smile y_2) \frown a = w$ follows from $(t \smile x_1) \frown b_1 = v_1$, $(v_1 \smile y_1) \frown a = w$, $(t \smile x_2) \frown b_2 = v_2$. We shall now distinguish two cases: case (i), where $x_2 \frown (a \smile x_1) = O$ and case (ii), where $x_2 \frown (a \smile x_1) > O$.

Since $(t \smile x_2 \smile b_2) \frown x_1 \leqq (a^* \smile x_2) \frown x_1 = O$, we have by lemma 1

$$(1) \qquad v_2 = (t \smile x_1 \smile x_2) \frown b_2 .$$

Since $a \smile x_2 = a \smile b_2 = a \smile y_2 = b_2 \smile y_2$ and $(w \smile y_2 \smile b_2) \frown y_1 \leqq (a \smile x_2) \frown y_1$

$= a \frown y_1 \leqq w$, we have also by lemma 1

(2) $$\bar{v}_2 = (w \smile y_2) \frown b_2 = (w \smile y_2 \smile y_1) \frown b_2 .$$

Then (1) and (2) yield

(3) $$\bar{v}_2 \frown v_2 = (t \smile x_1 \smile x_2) \frown (w \smile y_2 \smile y_1) \frown b_2 .$$

Putting $u = (x_1 \smile x_2) \frown (y_1 \smile y_2)$ and considering $v_1 \leqq w \smile y_1$, $v_1 \leqq t \smile x_1$, we have $u \smile v_1 \leqq (t \smile x_1 \smile x_2) \frown (w \smile y_1 \smile y_2)$. Hence from (3) follows

(4) $$(u \smile v_1) \frown b_2 \leqq v_2 \frown \bar{v}_2 .$$

Now we will show that $(u \smile v_1) \frown b_2 = v_2$. From $v_1 = (t \smile x_1) \frown b_1 = (t \smile x_1 \smile x_2) \frown b_1$, we have

(5) $$u \smile v_1 = (u \smile b_1) \frown (t \smile x_1 \smile x_2).$$

(6) $$\begin{aligned}(t_0 \smile x_1 \smile x_2) \frown (w_0 \smile y_1 \smile y_2) &= (v_0^{(1)} \smile x_1 \smile x_2) \smile (v_0^{(1)} \smile y_1 \smile y_2) \\ &= v_0^{(1)} \smile \{(y_1 \smile y_2) \frown (v_0^{(1)} \smile x_1 \smile x_2)\} = v_0^{(1)} \smile u\end{aligned}$$

where the last inequality follows by lemma 1, since

$$\begin{aligned}v_0^{(1)} \frown (x_1 \smile x_2 \smile y_1 \smile y_2) &= v_0^{(1)} \frown (C_i \smile x_1) = (t_0 \smile x_1) \frown (C_i \smile x_1) \frown b_1 \\ &= b_1 \frown [x_1 \smile \{t_0 \frown (C_i \smile x_1)\}] = b_1 \frown x_1 \leqq x_1 \smile x_2 ,\end{aligned}$$

where $i = 1$ or 2 according as $C_1 \geqq C_2$ or $C_2 \geqq C_1$. Similarly we obtain

(7) $$(t_0 \smile x_1 \smile x_2) \frown (w_0 \smile y_1 \smile y_2) = v_0^{(2)} \smile u .$$

But $(a^* \frown P_1) \smile v_0^{(1)} = (a^* \frown P_1) \smile \{(t_0 \smile x_1) \frown b_1\} = \{(a^* \frown P_1) \smile t_0 \smile x_1\} \frown b_1 = (a^* \smile x_1) \frown b_1 = b_1$ and $(a^* \frown P_1) \smile v_0^{(2)} = b_2$. Hence we have $b_1 \smile u = b_2 \smile u$ in virtue of (6) and (7). Substituting this in (5) we have $u \smile v_1 = (u \smile b_2) \frown (t \smile x_1 \smile x_2)$, whence

$$(u \smile v_1) \frown b_2 = b_2 \frown (t \smile x_1 \smile x_2) = b_2 \frown (t \smile x_2) = v_2 .$$

Then from (4) follows $v_2 \leqq v_2 \frown \bar{v}_2$ and this implies $\bar{v}_2 \geqq v_2$. Hence $(v_2 \smile y_2) \frown a \leqq (\bar{v}_2 \smile y_2) \frown a = [\{(w \smile y_2) \frown b_2\} \smile y_2)] \frown a = (w \smile y_2) \frown (b_2 \smile y_2) \frown a = w$. If we further regard, that $(v_2 \smile y_2) \frown a$ and w are of the same dimension, then we have finally the required result.

Proof of the case (ii). By lemma 11 we can find an irreducible element x_3, such that $x_3 \frown (a \smile x_1) = O$, $x_3 \frown (a \smile x_2) = O$ and $\dim x_3 = \dim y_1$. Since $\dfrac{a}{P_1} = \dfrac{a}{a \frown b_1} = \dfrac{a \smile b_1}{b_1} = \dfrac{b_1 \smile y_1}{b_1} \simeq \dfrac{y_1}{O}$, so the uqo-

tient $\frac{a \cup x_3}{P_1}$ is of type $(\dim y_1, \dim y_1)$ and consequently there exists a complement b_3' of $x_3 \cup P_1$ in $\frac{a \cup x_3}{P_1}$, such that $a \cap b_3' = P_1$, $b_3' \cap (x_3 \cup P_1) = P_1$, and $a \cup b_3' = a \cup x_3 = b_3' \cup x_3$. Putting $b_3 = b_3' \cap (a^* \cup x_3)$, we have $b_3 \cup x_3 = a^* \cup x_3$, $b_3 \cap x_3 = O$. Therefore a^* and b_3 are perspective with respect to the axis x_3. Next, put $v_3 = (t_0 \cup x_3) \cap b_3$, $(w_0 \cup v_3) \cap (C_1 \cup x_3) = y_3$, and we have

$$(8) \qquad v_3 \cup (P_1 \cap a^*) = (a^* \cup x_3) \cap b_3 = b_3$$

Now it holds $t_0 \cup v_1 = t_0 \cup \{b_1 \cap (t_0 \cup x_1)\} = (t_0 \cup b_1) \cap (t_0 \cup x_1) = t_0 \cup x_1$, since $t_0 \cup b_1 = t_0 \cup (a_1^* \cap P_1) \cup b_1 = a^* \cup b_1 = b_1 \cup x_1$. Hence it follows $t_0 \cup w_0 = t_0 \cup \{(v_1 \cup y_1) \cap a\} = (t_0 \cup v_1 \cup y_1) \cap a = (t_0 \cup x_1 \cup y_1) \cap a = t_0 \cup \{(x \cup y_1) \cap a\} = t_0 \cup C_1$. Now from $w_0 \cap C_1 = \kappa_1$, we have $w_0 \cup t_0 = t_0 \cup C = w_0 \cup C_1$ and by virtue of this relation

$$(9) \qquad v_3 \cup y_3 = (w_0 \cup v_3) \cap (C_1 \cup v_3 \cup x_3) = (w_0 \cup v_3) \cap (C_1 \cup t_0 \cup x_3)$$
$$= (w_0 \cup v_3) \cap (C_1 \cup w_0 \cup x_3) = w_0 \cup v_3 = w_0 \cup y_3 .$$

From (8) and (9) we have

$$(10) \qquad b_3 \cup y_3 = (P_1 \cap a^*) \cup v_3 \cup y_3 = (P_1 \cap a^*) \cup w_0 \cup y_3 = a \cup y_3 ,$$

where $(P_1 \cap a^*) \cup w_0 = a$ follows from $(P_1 \cap a^*) \cup t_0 = a^*$ by the isomorphism $\frac{a^*}{O} \simeq \frac{a}{\kappa_1}$. Further we have

$$b_3' \cap y_3 = (w_0 \cup v_3) \cap (C_1 \cup x_3) \cap b_3' = \{v_3 \cup (w_0 \cap b_3')\} \cap (C_1 \cup x_3)$$
$$= \{v_3 \cup (w_0 \cap P_1)\} \cap (C_1 \cup x_3) = v_3 \cap (C_1 \cup x_3) \quad \text{(by theorem 47)}$$
$$= (t_0 \cup x_3) \cap b_3 \cap (C_1 \cup x_3) = x_3 \cap b_3 = O,$$

whence $b_3 \cap y_3 = O$ and so b_3 is quasiperspective to a with the axis y_3 by the relation (10). Since $v_3 \cap y_3 = (t_0 \cup x_3) \cap (C_1 \cup x_3) \cap b_3 = x_3 \cap b_3 = O$ and $\frac{y_3}{O} \simeq \frac{v_3 \cup y_3}{v_3} = \frac{w_0 \cup v_3}{v_3} \simeq \frac{w_0}{w_0 \cap v_3}$, so y_3 is irreducible and we have a normal projection $\mathfrak{P}_3$ of a^* into a by means of x_3, y_3, b_3. Next by virtue of $v_3 \cup x_3 = t_0 \cup x_3$ we have $C_3 = (x_3 \cup y_3) \cap a = (w_0 \cup t_0 \cup x_3) \cap (C_1 \cup x_3) \cap a = (t_0 \cup C_1 \cup x_3) \cap (C_1 \cup x_3) \cap a = C_1 \cap a = C_1$ and $P_3 = a \cap b_3 = P_1 \cap a^* \leq P_1$, whence $(P_3 \cap a^*) \cup t_0 = a^*$. From (9) follows $(v_3 \cup y_3) \cap a = (w_0 \cup v_3) \cap a = w_0 \cup (v_3 \cap a) = w_0 \cup (t_0 \cap P_1) = w_0$ and

therefore $\mathfrak{P}_3(t_0) = \mathfrak{P}_1(t_0) = \mathfrak{P}_2(t_0)$. Now, using the result in the first case, we obtain $\mathfrak{P}_3 \equiv \mathfrak{P}_1$ and $\mathfrak{P}_3 \equiv \mathfrak{P}_2$.

PART 4.

Hereafter L is supposed to be primary and $m_h \geqq 4$ with the height h. a_1, a_2 being two independent irreducible elements in L, the quotient $\frac{a_1 \smile a_2}{O}$ is called a *straight line in L*. In a given straight line l we choose a complemented irreducible element as its *origin* and one of its complements in l shall be fixed as the *infinite point of l*. Every irreducible element in l, which is independent with the infinite point, is called a *finite point*. The dimension of any finite point is evidently not greater than that of the origin. In what follows, we denote with a_1 the origin and with a_2 the infinite point. Now we will establish a composition among finite points of dimension λ by means of a normal projective automorphism in the sublattice $\frac{(a_1 \frown e^{(\lambda)}) \smile a_2}{O}$. If the infinite point contains its center and is contained in its pole, then we say such an automorphism *addition*. If in an addition the image of the point, which is contained in the origin, is the finite point t, we denote this addition with the symbol A_t and the image of any point u with $A_t(u)$.

Theorem 52. *For any finite point t of a straight line there exists the uniquely determined addition* A_t.

Pr. According to the assumption $m_h \geqq 4$, we can find certainly an irreducible element x of dimension λ, which is independent with $a_1 \smile a_2$. We put $a_i^{(\lambda)} = a_i \frown e^{(\lambda)}$, $i = 1, 2$, and choose an irreducible element a_3 of dimension λ in $\frac{a_1 \smile x}{O}$, which is independent with both a_1 and x. Next we put $a_3 \smile a_2 = b$, $a^{(\lambda)} = a_1^{(\lambda)} \smile a_2$, then we have $a^{(\lambda)} \frown x = b \frown x = O$, $a^{(\lambda)} \smile x = b \smile x = a^{(\lambda)} \smile b$. Hence $a^{(\lambda)}$ is perspective to b with the axis x. Putting further $(a_1^{(\lambda)} \smile x) \frown b = v_0$, $(t \smile v_0) \frown (a_2 \smile x) = y$, we have $a_2 \smile y = (a_2 \smile t \smile v_0) \frown (a_2 \smile x) = (a^{(\lambda)} \smile v_0) \frown (a_2 \smile x) = (a_1^{(\lambda)} \smile x \smile a_2) \frown (a_2 \smile x) = a_2 \smile x$, whence $a^{(\lambda)} \smile y = a^{(\lambda)} \smile x = b \smile x = b \smile y$. From $v_0 \frown (t \smile a_2) = v_0 \frown a^{(\lambda)} = O$ follows $a^{(\lambda)} \frown y = a_2 \frown (t \smile v_0) = a_2 \frown t = O$ and $b \frown y = O$ by virtue of the relation $\dim b = \dim a^{(\lambda)}$. Hence

$a^{(\lambda)} \sim b$ with the axis y and y is irreducible by $\frac{y}{O} \simeq \frac{a^{(\lambda)} \cup y}{a^{(\lambda)}} = \frac{a^{(\lambda)} \cup x}{a^{(\lambda)}} \simeq \frac{x}{O}$. Thus we obtain a normal projective automorphism of $\frac{a^{(\lambda)}}{O}$ by means of x, y, b. That this is the required addition A_t, can be seen as follows. $(x \cup y) \frown a^{(\lambda)} = (x \cup t \cup v_0) \frown (a_2 \cup x) \frown a^{(\lambda)} = (x \cup t \cup v_0) \frown a_2 \leqq a_2$ and $C \leqq a_2 = a^{(\lambda)} \frown b = P$. Furthermore $v_0 \cup y = (t \cup v_0) \frown (a_2 \cup x \cup v_0) = t \cup v_0$, whence $(v_0 \cup y) \frown a^{(\lambda)} = t \cup (v_0 \frown a^{(\lambda)}) = t$. By theorem 51 we can conclude further, that such an addition is unique.

Theorem 53. (*commutative law of addition*). *t, s being any two finite points of the same dimension in a straight line, $A_t(s) = A_s(t)$ holds.*

Pr. In the proof of the preceding theorem, put $v_0 = x'$, $a_2 \cup x = b'$, $(x' \cup a_2) \frown (s \cup x) = y'$. Then x' is obviously an irreducible element of dimension $\lambda = \dim t = \dim s$. We have $a_2 \cup v_0 = b$ and $a_2 \cup y' = (a_2 \cup s \cup x) \frown (x' \cup a_2) = (a^{(\lambda)} \cup x) \frown b = b$. Hence $a^{(\lambda)} \cup y' = a_1^{(\lambda)} \cup b = a^{(\lambda)} \cup x' = a_1^{(\lambda)} \cup x$ and $b' \cup y' = x \cup a_2 \cup y' = x \cup b = b' \cup x' = a^{(\lambda)} \cup y'$. Since $a^{(\lambda)} \frown y' = (x' \cup a_2) \frown s = b \frown s = O$, $b' \frown y' = (x' \cup a_2) \frown x = b \frown x = O$, $a^{(\lambda)} \frown x' = a^{(\lambda)} \frown v_0 = a_1^{(\lambda)} \frown b = O$ and $b' \frown x' = (a_2 \cup x) \frown v_0 = O$, so we have $a^{(\lambda)} \sim b'$ with axis x' and $a^{(\lambda)} \sim b'$ with axis y', where y' is irreducible by the relation $\frac{y'}{O} \simeq \frac{a^{(\lambda)} \cup y'}{a^{(\lambda)}} = \frac{a^{(\lambda)} \cup x}{a^{(\lambda)}} \simeq \frac{x}{O}$. Thus we obtain a normal projective automorphism of $\frac{a^{(\lambda)}}{O}$ by means of b', x', y'. It holds furthermore $a^{(\lambda)} \frown b' = a^{(\lambda)} \frown (a_2 \cup x) = a_2$, $x' \cup y' = (x' \cup a_2) \frown (s \cup x \cup x') = b \frown (s \cup x \cup x')$, $(x' \cup y') \frown a^{(\lambda)} = a_2 \frown (s \cup x \cup x') \leqq a_2$ and $(a_1^{(\lambda)} \cup x') \frown b' = (a_1^{(\lambda)} \cup x) \frown (a_2 \cup x) = x$, $(x \cup y') \frown a^{(\lambda)} = (x \cup v_0 \cup a_2) \frown (s \cup x) \frown a^{(\lambda)} = a^{(\lambda)} \cup (s \cup x) = s$. This automorphism is therefore the addition A_s. Since $(t \cup x') \frown b' = (t \cup v_0) \frown (a_2 \cup x) = y$ and $(y \cup y') \frown a^{(\lambda)} = [y \cup \{(v_0 \cup a_2) \frown (s \cup x)\}] \frown a^{(\lambda)} = [y \cup \{b \frown (s \cup x)\}] \frown a^{(\lambda)}$, so we have the required result $A_s(t) = A_t(s)$. q. e. d.

This theorem justifies the adoption of the symbol $t+s$ for the point $A_t(s) = A_s(t)$, if $\dim s = \dim t$.

Theorem 54. (*associative law of addition*). *t, s, u being any three finite points of the same dimension, holds the identity*

$$t+(s+u) = (t+s)+u .$$

Pr. Suppose we have obtained A_t by means of x, y, b just as in the proof of theorem 52. We put further $(s \cup x) \cap b = v_1$, $(a_1^{(\lambda)} \cup v_1) \cap (a_2 \cup x) = z$. Then $a_2 \cup v_1 = (a_2 \cup s \cup x) \cap b = b$,

$$(1) \qquad a^{(\lambda)} \cup z = a_2 \cup a_1^{(\lambda)} \cup z = a_2 \cup \{(a_1^{(\lambda)} \cup v_1) \cap (a^{(\lambda)} \cup x)\}$$

$$= a_2 \cup a_1^{(\lambda)} \cup v_1 = a_1^{(\lambda)} \cup b = a^{(\lambda)} \cup x$$

$$(2) \qquad a^{(\lambda)} \cap z = (a_1^{(\lambda)} \cup v_1) \cap a_2 = a_1^{(\lambda)} \cap a_2 = O$$

by lemma 1, since $v_1 \cap a^{(\lambda)} = (s \cup x) \cap b \cap a^{(\lambda)} = s \cap a_2 = O$. Now z is irreducible and of dimension λ. For in virtue of (1) and (2) $\frac{z}{O} \simeq \frac{a^{(\lambda)} \cup z}{a^{(\lambda)}} = \frac{a^{(\lambda)} \cup x}{a^{(\lambda)}} \simeq \frac{x}{O}$. Further $b \cup z = v_0 \cup a_2 \cup z = v_0 \cup \{(a_1^{(\lambda)} \cup a_2 \cup v_1) \cap (a_2 \cup x)\} = v_0 \cup a_2 \cup x = b \cup x = a^{(\lambda)} \cup x$. This, together with (1), yields $b \cup z = a^{(\lambda)} \cup z$. Hence $a^{(\lambda)} \sim b$ with the axis z and we obtain A_{t+s} by means of z, y, b. For we have $a^{(\lambda)} \cap b = a_2$, $z \cup y \leqq a_2 \cup x$, whence $(z \cup y) \cap a^{(\lambda)} \leqq a_2$ and $(a_1^{(\lambda)} \cup z) \cap b = (a_1^{(\lambda)} \cup v_1) \cap b = v_1$, $(v_1 \cup y) \cap a^{(\lambda)} = A_t(s)$. We see further, that A_s can be obtained by means of z, x, b. In fact $(z \cup x) \cap a^{(\lambda)} \leqq (a_2 \cup x) \cap a^{(\lambda)} \leqq a_2$ and $(a_1^{(\lambda)} \cup z) \cap b = v_1$, $(v_1 \cup x) \cap a^{(\lambda)} = (s \cup x) \cap a^{(\lambda)} = s$. It holds then $(A_s(u) \cup x) \cap b = (u \cup z) \cap b$, whence we have $[\{(A_s(u) \cup x) \cap b\} \cup y] \cap a^{(\lambda)} = [\{(u \cup z) \cap b\} \cup y] \cap a^{(\lambda)}$, proving the identity $A_t(A_s(u)) = A_{t+s}(u)$. q. e. d.

Theorem 55. (*possibility and uniqueness of substraction*). *Given any two finite points s, t of dimension λ, there exists the addition A_u uniquely, such that $A_u(t) = s$, i. e. $u + t = s$.*

Pr. We choose x, b as in the proof of theorem 52 and put $(x \cup t) \cap b = v$, $(v \cup s) \cap (a_2 \cup x) = y$. Then $a_2 \cup v = b$ and $a_2 \cup y = (a_2 \cup v \cup s) \cap (a_2 \cup x) = (a^{(\lambda)} \cup v) \cap (a_2 \cup x) = a_2 \cup x$. Hence $a^{(\lambda)} \cup y = a^{(\lambda)} \cup x$ and $b \cup y = b \cup x$, whence $a^{(\lambda)} \cup y = b \cup y$. Further we have $a^{(\lambda)} \cap y = a_2 \cap (v \cup s) = a_2 \cap s = O$, since $v \cap (a_2 \cup s) = v \cap a^{(\lambda)} = (x \cup t) \cap a_2 = t \cap a_2 = O$, and $b \cap y = (v \cup s) \cap b \cap (a_2 \cup x) = v \cap (a_2 \cup x) = O$. Therefore $a^{(\lambda)}$ is perpective to b with the axis y. Furthermore holds the relation $\frac{y}{O} \simeq \frac{a^{(\lambda)} \cup y}{a^{(\lambda)}} = \frac{a^{(\lambda)} \cup x}{a^{(\lambda)}} \simeq \frac{x}{O}$ and consequently y is irreducible. Since $a^{(\lambda)} \cap b = a_2$ and $(x \cup y) \cap a^{(\lambda)} = (x \cup v \cup s) \cap a_2 \leqq a_2$, we have now an addition by means of x, y, b and, since $v \cup y = (v \cup a_2 \cup x) \cap (v \cup s) = (b \cup x) \cap (v \cup s) = v \cup s$, $(v \cup y) \cap a^{(\lambda)} = (v \cup s) \cap a^{(\lambda)}) = s \cup (v \cap a^{(\lambda)}) = s$ holds, the image of t is indeed the point s. Since from

$A_t(u) = A_t(u')$ follows $u = u'$, so the substraction is unique. q. e. d.

The dimension of the meet of a finite point t and the origin is called *order* of the point t and denoted with $O(t)$. Every finite point, whose order is zero, is called *unit*. In a given straight line $l = \frac{a_1 \cup a_2}{O}$, where a_1 is the origin and a_2 the infinite point, put $a_1 \frown e^{(i)} = a_1^{(i)}$, $a_1^{(i)} \smile a_2 = a^{(i)}$. A normal projection of $a^{(j)}$ into $a^{(i)}$, where $j \leqq i$, is called *multiplication*, if $P \geqq a_2$, $\kappa \leqq C \leqq a_1$ holds, where κ, P, C are base, pole, and center of the projection. We choose one of complemented units in l as a fixed unit point of l and denote it with e_{12}.

Theorem 5.6. *Given a finite point t of dimension i with order ν, there exists uniquely such a multiplication, that the image of the point $e_{12}^{(j)} = e_{12} \frown e^{(j)}$ is the point t, where $j = i - \nu$.*

Pr. We choose an irreducible element x of dimension j in L, which is independent with $a_1 \smile a_2$, and also an irreducible element a_3 of dimension j in $\frac{a_1^{(j)} \smile x}{O}$, which is independent with both a_1 and x. Putting $a^{(j)} = a_1^{(j)} \smile a_2$, $b = a_2 \smile a_3$, we have $a^{(j)} \smile x = b \smile x$, $a^{(j)} \frown x = b \frown x = O$. Hence $a^{(j)}$ is perspective to b with the axis x. Next we put $(e_{12}^{(j)} \smile x) \frown b = v_{12}$, $(t \smile v_{12}) \frown (a_1^{(i)} \smile x) = y$. Then it follows

$$b \frown y = (t \smile v_{12}) \frown (a_1^{(i)} \smile x) \frown b = \{v_{12} \smile (t \frown b)\} \frown (a_1^{(i)} \smile x) = v_{12} \frown (a_1^{(i)} \smile x)$$
$$= (e_{12}^{(j)} \smile x) \frown b \frown (a_1^{(i)} \smile x) = x \frown b = O,$$

$$a_2 \smile v_{12} = (a_2 \smile e_{12}^{(j)} \smile x) \frown b = (a^{(j)} \smile x) \frown b = b,$$

$$v_{12} \smile y = (t \smile v_{12}) \frown (a_1^{(i)} \smile x \smile v_{12}) = (t \smile v_{12}) \frown (a_1^{(i)} \smile e_{12}^{(j)} \smile x) = t \smile v_{12}$$
$$= (t \smile v_{12}) \frown (a_1^{(i)} \smile t \smile x) = t \smile y,$$

whence $b \smile y = a_2 \smile v_{12} \smile y = a_2 \smile t \smile y = a^{(i)} \smile y$. By these relations we see that b is quasiperspective to $a^{(i)}$ and y is irreducible, since $v_{12} \frown y = O$, $v_{12} \smile y = t \smile v_{12}$ and $\frac{y}{O} \simeq \frac{v_{12} \smile y}{v_{12}} = \frac{t \smile v_{12}}{v_{12}} \simeq \frac{t}{O}$. Further $a^{(i)} \frown b = a_2$, $(x \smile y) \frown a^{(i)} \leqq a_1$ and $\kappa = a^{(i)} \frown y = a_1^{(\nu)}$, so we have a multiplication by means of x, y, b. Next it follows

$$(v_{12} \smile y) \frown a^{(i)} = (t \smile v_{12}) \frown a^{(i)} = t \smile (v_{12} \frown a^{(i)}) = t$$

and therefore the image of $e_{12}^{(j)}$ is the point t. q. e. d.

We denote the above multiplication with the symbol M_t and the image of a point u with $M_t(u)$. That such multiplication is unique, is the result of theorem 51, since $e_{12}^{(j)} \frown C = O$, $e_{12}^{(j)} \smile (P \frown a^{(j)}) = e_{12}^{(j)} \smile a_2 = a^{(j)}$ hold. Given a finite point t with $\dim t = i$, $O(t) = \nu$ and an arbitrary finite point s, we define product of t and s as the point $M_t(s^{(i-\nu)})$, where $s^{(i-\nu)} = s \frown e^{(i-\nu)}$, and denote with the symbol ts. The present definition yields then $\dim ts = i$, if $\dim s \geqq i - \nu$, and $\dim ts = \dim s + \nu$, if $\dim s < i - \nu$.

Thorem 57. *If* $\dim t = i$, $O(t) = \nu$, $\dim s = \mu < i - \nu$, *then* $t \cdot s = t^{(\nu+\mu)}s$, *where* $t^{(\nu+\mu)} = t \frown e^{(\nu+\mu)}$.

Pr. We choose x, a_3, y, b just as in theorem 56. Putting $b^{(\mu)} = a_2 \smile a_3^{(\mu)}$, $x^{(\mu)} = x \frown e^{(\mu)}$, $(e_{12}^{(\mu)} \smile x^{(\mu)}) \frown b^{(\mu)} = v^{(\mu)}$, $y^{(\mu)} = (t^{(\mu+\nu)} \smile v^{(\mu)}) \frown (a_1^{(\mu+\nu)} \smile x^{(\mu)})$, we have $M_{t^{(\nu+\mu)}}$ by means of $x^{(\mu)}$, $y^{(\mu)}$, $b^{(\mu)}$. Since $t^{(\nu+\mu)}s = [\{(s \smile x^{(\mu)}) \frown b^{(\mu)}\} \smile y^{(\mu)}] \frown a^{(\mu+\nu)} \leqq [\{(s \smile x) \frown b\} \smile y] \frown a^{(i)} = ts$ and $\dim t^{(\nu+\mu)}s = \dim ts$, so we obtain $t^{(\nu+\mu)}s = ts$.

Theorem 58. *s, t being two finite points with $O(s) = \mu$, $O(t) = \nu$, $\dim ts = j$, then $O(ts) = O(t) + O(s)$, if $j > \mu + \nu$, and $O(ts) = j$, if $j \leqq \mu + \nu$.*

Pr. Suppose M_t is generated by means of x, y, b just as in theorem 56. Then, putting $\dim t = i$, $(s^{(i-\nu)} \smile x) \frown b = v_1$, $(v_1 \smile y) \frown a^{(i)} = ts$ and $a_1^{(i)} \smile y = a_1^{(i)} \smile x$.(1) Hence we obtain

$$(3) \qquad a_1^{(i)} \smile ts = (a_1^{(i)} \smile v_1 \smile y) \frown a^{(i)} = (a_1^{(i)} \smile v_1 \smile x) \frown a^{(i)}$$
$$= (a_1^{(i)} \smile s^{(i-\nu)} \smile x) \frown a^{(i)} = a_1^{(i)} \smile s^{(i-\nu)}.$$

In case $\dim s \geqq i - \nu$, we have $\dim ts = j = i$. Then from (3) it follows

$$\dim (a_1^{(i)} \smile ts) = i + (i - \nu - \mu), \quad \text{if} \quad \nu + \mu < i,$$
$$\dim (a_1^{(i)} \smile ts) = i, \quad \text{if} \quad \nu + \mu \geqq i,$$

whence

$$O(ts) = \nu + \mu, \quad \text{if} \quad \nu + \mu < j$$
$$O(ts) = j, \quad \text{if} \quad \nu + \mu \geqq j$$

In case $\dim s < i - \nu$, we have $j = \dim s + \nu \geqq \mu + \nu$ and from (3) $\dim (a_1^{(i)} \smile ts) = i + \dim s - \mu$, whence follows

(1) This follows from $a_1^{(i)} \smile y \leq a_1^{(i)} \smile x$, $a_1^{(i)} \frown y = a_1^{(\nu)}$ and $\dim y = \dim x + \nu$.

$$O(ts) = (\dim s+\nu+i)-(i+\dim s-\mu) = \mu+\nu .$$

Lemma 12. *The product of a point in the origin with any finite point is in the origin.*

Pr. Suppose $t = a_1^{(i)}$ and M_t is generated by x, y, b. Then $(O\smile x)\frown b = O$, $(O\smile y)\frown a^{(i)} = \kappa = a_1^{(i)} = t$.

Theorem 59. (*associative law of multiplication*). $(ts)u = t(su)$.

Pr. Suppose $\dim t = i$, $O(t) = \nu$, $O(s) = \mu$. If $\mu+\nu \geqq i$, then $ts = a_1^{(i)}$ by theorem 58 and $(ts)u = a_1^{(i)}$ by the preceding lemma. Since $\dim (su) \geqq \mu \geqq i-\nu$, we have $\dim t(su) = i$, $O(su) \geqq \mu$. Hence $\dim t(su) \leqq O(t)+O(su)$ and by theorem 58 $t(su) = a_1^{(i)}$.

Now we have only to treat the case $\mu+\nu < i$. Put $i-\nu = \lambda$, if $\dim s \geqq i-\nu$, and $\dim s = \lambda$, if $\dim s < i-\nu$. Theorem 57 and 58 yield then the results $ts = t^{(\lambda+\nu)}s$, $\dim ts = \lambda+\nu$, $O(ts) = \mu+\nu$. Since further $\dim su \leqq \dim s$, we have $t(su) = t^{(\lambda+\nu)}(su)$ by theorem 57. Therefore we can assume $\dim t = i = \lambda+\nu$ without loss of generality and we see that

$$(4)\quad (ts)u = M_{ts}(u^{(\lambda-\mu)}) , \qquad (5)\quad t(su) = M_t\{M_s(u^{(\lambda-\mu)}\}$$

by virtue of $M_s(u^{(\lambda-\mu)}) = (su)^{(\lambda)}$. We will now prove that the right hand sides of both (4) and (5) are equal. Suppose M_t is generated by x, y, b, where $b = a_2\smile a_3^{(\lambda)}$, $(s^{(\lambda)}\smile x)\frown b = v_1$ and $(v_1\smile y)\frown a^{(i)} = ts$ as in theorem 56. Putting $(e_{12}^{(\lambda)}\smile v_1)\frown(x\smile a_1^{(\lambda)}) = z$, we have $a^{(i)}\frown z = (e_{12}^{(\lambda)}\smile v_1)\frown a_1^{(\lambda)} = e_{12}^{(\lambda)}\frown a_1^{(\lambda)} = O$, where $(a_1^{(\lambda)}\smile e_{12}^{(\lambda)})\frown v_1 \leqq a^{(\lambda)}\frown v_1 = O$. From the relations $z\frown e_{12}^{(\lambda)} = O$, $z\smile e_{12}^{(\lambda)} = (e_{12}^{(\lambda)}\smile v_1)\frown(x\smile a_1^{(\lambda)}\smile e_{12}^{(\lambda)}) = e_{12}^{(\lambda)}\smile v_1$ follows $\dfrac{z}{O} \simeq \dfrac{z\smile e_{12}^{(\lambda)}}{e_{12}^{(\lambda)}} = \dfrac{e_{12}^{(\lambda)}\smile v_1}{e_{12}^{(\lambda)}} \simeq \dfrac{v_1}{O}$ and therefore z is irreducible. Further we have

$$(6)\qquad a_2\smile v_1 = (a_2\smile s^{(\lambda)}\smile x)\frown b = (a^{(\lambda)}\smile x)\frown b = b$$

$$(7)\quad v_1\smile z = (e_{12}^{(\lambda)}\smile v_1)\frown(x\smile v_1\smile a_1^{(\lambda)}) = (e_{12}^{(\lambda)}\smile z)\frown(x\smile s^{(\lambda)}\smile a_1^{(\lambda)})$$
$$= e_{12}^{(\lambda-\mu)}\smile z .$$

From (6) and (7) follows $b\smile z = a_2\smile v_1\smile z = a^{(\lambda-\mu)}\smile z$ and hence $a^{(\lambda-\mu)}$ is quasi-perspective to b with the axis z. Since further $(x\smile z)\frown a^{(\lambda)} = (e_{12}^{(\lambda)}\smile v_1\smile x)\frown a_1^{(\lambda)} \leqq a_1$, so we have a multiplication by means of z, x, b. That this multiplication is indeed $M_{s^{(\lambda)}}$, can be seen from

$$(e_{12}^{(\lambda-\mu)} \cup z) \frown b = (x \cup a_1^{(\lambda)} \cup e_{12}^{(\lambda-\mu)}) \frown v_1 = v_1 ,$$
$$(v_1 \cup x) \frown a^{(\lambda)} = (s^{(\lambda)} \cup x) \frown a^{(\lambda)} = s^{(\lambda)} .$$

Next the quasi-perspectivity of $a^{(\lambda-\mu)}$ and b with axis z and the quasi-perspectivity of b and $a^{(i)}$ with axis y yield a normal projection of $a^{(\lambda-\mu)}$ into $a^{(i)}$. Further $(y \cup z) \frown a^{(i)} \leqq (x \cup a_1^{(i)}) \frown a^{(i)} \leqq a_1$, $(e_{12}^{(\lambda-\mu)} \cup z) \frown b = v_1$, $(v_1 \cup y) \frown a^{(i)} = t \cdot s$. Hence this projection is M_{ts}. Then we have $(u^{(\lambda-\mu)} \cup z) \frown b = v_2$, $(v_2 \cup x) \frown a^{(\lambda)} = M_s(u^{(\lambda-\mu)})$, $\{M_s(u^{(\lambda-\mu)}) \cup x\} \frown b = (v_2 \cup x) \frown b = v_2$, $(v_2 \cup y) \frown a^{(i)} = M_{ts}(u^{(\lambda-\mu)})$. q. e. d.

Theorem 60. *If* $O(t) \leqq O(s)$ *and* $\dim s \leqq \dim t$ *for any two finite points* t, s, *then there exists uniquely a finite point* q, *such that* $s = tq$ *and* $\dim q = \dim s - O(t)$.

Pr. Put $\dim t = i$, $O(t) = \nu$, $a^{(i-\nu)} = a_1^{(i-\nu)} \cup a_2$, $b = a_3 \cup a_2$, $(e_{12}^{(i-\nu)} \cup x) \frown b = v_{12}$, $(v_{12} \cup t) \frown (a_1^{(i)} \cup x) = y$ just as in theorem 56, then we have M_t by means of x, y, b. Now in this projection to s corresponds the element $q = (v \cup x) \frown a^{(i-\nu)}$, where $v = (s \cup y) \frown b$. Since $q \cup x = v \cup x$, $q \frown x = a^{(i-\nu)} \frown x = O$ and $\dfrac{v}{O} \simeq \dfrac{v \cup x}{x} = \dfrac{q \cup x}{x} = \dfrac{q}{O}$, and since $v \cup y = s \cup y$, $v \frown y = b \frown y = O$, $\dfrac{v}{O} \simeq \dfrac{v \cup y}{y} = \dfrac{s \cup y}{y} \simeq \dfrac{s}{a_1^{(\nu)}}$, so q is irreducible and of dimension $\dim s - \nu$. Further, by virtue of $q \frown a_2 = v \frown a_2 = (s \cup y) \frown a_2 = O$, q is finite and $s = tq$.

Theorem 61. *If* $O(s) = \mu$, $O(t) = \nu$, $\mu \geqq \nu$ *and* $\dim s \leqq \dim t + (\mu - \nu)$ *for any two finite points* s, t, *then there exists a finite point* q *such that* $s = qt$ *and* $\dim q = \dim s$.

Pr. First we consider the case $\mu = \nu$ and put $\dim s = i$. Since the case $i = \nu$ is trivial, we assume that $i > \nu$. We determine x, b as in theorem 56 and put $(t^{(i)} \cup x) \frown b = v$, $(s \cup v) \frown (a_1^{(i)} \cup x) = y'$. Then we have $a_1^{(i)} \cup y' = (a_1^{(i)} \cup s \cup v) \frown (a_1^{(i)} \cup x) = (a_1^{(i)} \cup a_2^{(i-\nu)} \cup v) \frown (a_1^{(i)} \cup x) = a_1^{(i)} \cup x$ and $a^{(i)} \frown y' = (s \cup v) \frown a_1^{(i)} = a_1^{(\nu)}$. Hence $\dfrac{y'}{a_1^{(\nu)}} \simeq \dfrac{a^{(i)} \cup y'}{a^{(i)}} = \dfrac{a^{(i)} \cup x}{a^{(i)}} \simeq \dfrac{x}{O}$ and $\dfrac{y'}{a_1^{(\nu)}}$ is a chain of dimension i. By lemma 6 we have then $y' = l \cup a_1^{(\nu)}$, where l is irreducible and of dimension not less than i. Since $l \leqq a_1^{(i)} \cup x$, the dimension of l must be equal to i and consequently $l \frown a_1^{(i)} = O$. The quotient $\dfrac{y'}{O}$ is then of type (i, ν). If we put further $v \frown (a_1^{(i)} \cup x) = v_0$, then $v_0 = (x \cup a_1^{(\nu)}) \frown b$ and

$$\dim v_0 = \dim b + \dim (x \cup a_1^{(\nu)}) - \dim (b \cup x) = \nu .$$

Now v_0 is contained in $\frac{y'}{O}$ and $\frac{y'}{v_0}$ is a chain of dimension i. For

$$\frac{y'}{v_0} = \frac{(s \cup v) \cap (a_1^{(i)} \cup x)}{v \cap (s \cup v) \cap (a_1^{(i)} \cup x)} \simeq \frac{v \cup \{(s \cup v) \cap (a_1^{(i)} \cup x)\}}{v} = \frac{s \cup v}{v} = \frac{s}{O}.$$

Since $m_j\left(\frac{y'}{v_0}\right) = m_j(y') - m_j(v_0)$ holds for every j, so v_0 is complemented in $\frac{y'}{O}$ by theorem 40. The chain $\frac{a_1^{(\nu)}}{O}$ is absolutely maximal in $\frac{y'}{O}$. Therefore a complement y of v_0 in $\frac{y'}{O}$ is irreducible with dimension i and independent with a_1 by the assumption $i > \nu$. Hence y is also a complement of $a_1^{(\nu)}$ in $\frac{y'}{O}$. From $b \cap y' = v \cap (a_1^{(i)} \cup x)$ $= (t^{(i)} \cup x) \cap (a_1^{(i)} \cup x) \cap b = (x \cup a_1^{(\nu)}) \cap b = v_0$ follows $b \cap y = b \cap y' \cap y$ $= v_0 \cap y = O$ and from $a^{(i)} \cup y = a^{(i)} \cup y' = a^{(i)} \cup x$ follows $b \leqq a^{(i)} \cup y$, whence $b \cup y = a^{(i)} \cup y$ and consequently $a^{(i)} \sim b$ with axis y. Since further $a^{(i)} \cap b = a_2$, $(x \cup y) \cap a^{(i)} \leqq (x \cup y') \cap a^{(i)} \leqq a_1^{(i)}$ holds, so we have a multiplication by means of x, y, b. $(t^{(i)} \cup x) \cap b = v$ and

$$(v \cup y) \cap a^{(i)} = (v \cup v_0 \cup y) \cap a^{(i)} = (v \cup y') \cap a^{(i)}$$

$$= (s \cup v) \cap (a_1^{(i)} \cup t^{(i)} \cup x) \cap a^{(i)} = s \cup \{v \cap (a_1^{(i)} \cup t^{(i)})\} = s$$

show that this is the required multiplication.

In case $\mu > \nu$, we choose an arbitrary finite point q' with $\dim q' = i$, $O(q') = \mu - \nu$ and put $s' = q't$. Then $O(s') = O(s)$ and $\dim s' = \dim s$. Hence there exists a finite point q'', such that $s = q''s' = q''(q't)$. Finally, applying the associative law, we obtain $s = (q''q')t$.

Theorem 62. *t, s being two finite points of the same dimension, it holds* $O(t+s) \geqq O(t) = O(s)$, *if* $O(t) = O(s)$, *and* $O(t+s) = \mathrm{Min}\,\{O(t), O(s)\}$, *if* $O(t) \neq O(s)$.

Pr. Put $O(s) = \mu$, $O(t) = \nu$ and assume $\nu \geqq \mu$. In virtue of $A_t(s^{(\mu)}) = A_t(a_1^{(\mu)}) = t^{(\mu)}$ and $A_t(s^{(\mu+1)}) \neq t^{(\mu+1)}$ we have $\dim\{(t+s) \cap t\} = \mu$, whence follows $(t+s) \cap a_1 \geqq s \cap a_1$, that is $O(t+s) \geqq \mu$. In particular $O(t+s) = \mu$, if $\nu > \mu$.

Theorem 63. (*first distributive law*). *If s, u are two finite*

points of the same dimension, then holds $t(s+u) = ts+tu$ *for every finite point* t.

Pr. Suppose $\dim t = i$, $O(t) = \nu$. Then we can assume $\dim s = \dim u = i-\nu$ without loss of generality. Next suppose we obtain M_t by means of x, y, b, such that, putting $\lambda = i-\nu$, $(e^{(\lambda)} \cup x) \frown b = v$, $(v \cup t) \frown (a_1^{(i)} \cup x) = y$. Next we choose an irreducible element x' of dimension λ, which is independent with $a_1 \cup a_2 \cup x$ and an irreducible element a_4 of dimension λ in $\dfrac{a_1^{(\lambda)} \cup x'}{O}$, which is independent with both x' and a_1. Putting $b' = a_4 \cup a_2$, $(a_1^{(\lambda)} \cup x') \frown b' = v'$, $(v' \cup s) \frown (a_2 \cup x') = y'$, we have $a^{(\lambda)} \sim b'$ with axis x' and $b' \sim a^{(\lambda)}$ with axis y'. Since $a^{(\lambda)} \frown b' = a_2$, $(x' \cup y') \frown a^{(\lambda)} \leqq a_2$ and $(v' \cup y') \frown a^{(\lambda)} = s$ hold, so we obtain A_s by means of x', y', b'. Now the perspectivity $a^{(\lambda)} \cup x' \sim b \cup x'$ with axis x and the quasi-perspectivity of $b \cup x'$ and $a^{(i)} \cup x'$ with axis y generates a normal projection $\mathfrak{P}$ of $a^{(\lambda)} \cup x'$ into $a^{(i)} \cup x'$, where the pole is $(a^{(i)} \cup x') \frown (b \cup x') = x' \cup a_2$. Since $v' \frown (x' \cup a_2) = O$, $\mathfrak{P}(v') = q$ is irreducible and of dimension i with $q \frown (x' \cup a_2) = O$, $q \geqq \kappa = a_1^{(\nu)}$ by theorem 50. Putting $b'' = a_2 \cup q$, we have

$$b'' \cup x' = a_2 \cup \mathfrak{P}(v') \cup \mathfrak{P}(x') = a_2 \cup \mathfrak{P}(v' \cup x') = a_2 \cup \mathfrak{P}(a_1^{(\lambda)}) \cup x',$$

where
$$\begin{aligned} \mathfrak{P}(a_1^{(\lambda)}) &= [\{(a_1^{(\lambda)} \cup x) \frown (b \cup x')\} \cup y] \frown (a_1^{(i)} \cup x') \\ &= [\{(a_1^{(\lambda)} \cup x) \frown b\} \cup y] \frown a^{(i)} = a_1^{(i)}. \end{aligned}$$

Therefore we haye $b'' \cup x' = a^{(i)} \cup x'$, $a^{(i)} \frown x' = O$ and $b'' \frown x' = O$, whence $a^{(i)} \sim b''$ with axis x'. Further we have

$$b'' \cup y' = a_2 \cup y' \cup \mathfrak{P}(v') = a_2 \cup x' \cup \mathfrak{P}(v') = b'' \cup x' = a^{(i)} \cup x' = a^{(i)} \cup y',$$

and $b'' \frown y' = a^{(i)} \frown y' = (v' \cup s) \frown a_2 = v' \frown a_2 = O$, whence $a^{(i)} \sim b''$ with axis y'. It holds furthermore $a^{(i)} \frown b'' \geqq a_2$, $(x' \cup y') \frown a^{(i)} \leqq a_2$ and $(a_1^{(i)} \cup x') \frown b'' = (a_1^{(i)} \cup x') \frown (a_2 \cup q) = \mathfrak{P}\{(a_1^{(\lambda)} \cup x') \frown (a_2 \cup v')\} = \mathfrak{P}(v') = q$, $(q \cup y') \frown a^{(i)} = \mathfrak{P}\{(v' \cup y') \frown a^{(\lambda)}\} = \mathfrak{P}(s)$.

Since $\mathfrak{P}$ induces a multiplication M_t in $\dfrac{a^{(i)}}{O}$, so $\mathfrak{P}(s) = ts$. Consequently we have A_{ts} by means of x', y', b'', and

$$\begin{aligned} t(s+u) &= \mathfrak{P}(s+u) = \mathfrak{P}[\{(u \cup x') \frown b'\} \cup y'] \frown \mathfrak{P}a^{(\lambda)} \\ &= [\{(\mathfrak{P}(u) \cup x') \frown b''\} \cup y'] \frown a^{(i)} = ts + \mathfrak{P}(u) = ts + tu. \end{aligned}$$

Theorem 64. (*second distributive law*). *If* u, s *are two finite*

points of the same dimension, then $(s+u)t = st+ut$ *for every finite point* t.

Pr. Putting $\dim s = \dim u = i$, we assume $\lambda = \dim t = i-\nu > O$. First we consider the case $O(s) = O(u) = \nu$, $O(t) = 0$. Then $\mu = O(s+u) \geqq \nu$ by theorem 62. We choose a_3, x, b as usual such that $a^{(\lambda)} \sim b$ with axis x. Putting $(e^{(\lambda)} \smile x) \frown b = p$, $(t \smile x) \frown b = q$ and

$$(8) \quad (s \smile p) \frown (a_1^{(i)} \smile x) = y_s, \qquad (9) \quad (u \smile p) \frown (a_1^{(i)} \smile x) = y_u,$$

we have

$$(10) \quad (y_s \smile q) \frown a^{(i)} = st, \qquad (11) \quad (y_u \smile q) \frown a^{(i)} = ut.$$

If we put $\lambda' = i-\mu$, then $\lambda' \leqq \lambda$ and from the above perspectivity follows $a^{(\lambda')} \sim b^{(\lambda')}$ with axis $x^{(\lambda')}$ and $(e^{(\lambda')} \smile x^{(\lambda')}) \frown b^{(\lambda')} = P^{(\lambda')}$, $(t^{(\lambda')} \smile x^{(\lambda')}) \frown b^{(\lambda')} = q^{(\lambda')}$.

We choose an irreducible element a_4 of dimension i in $\dfrac{a_1^{(i)} \smile x'}{O}$, which is independent with both a_1 and x'. Putting $a_2^{(\lambda)} \smile a_1^{(i)} = a'$, $a_2^{(\lambda)} \smile a_4 = b'$, we have $a' \sim b'$ with axis x'. Further putting $(a_1^{(i)} \smile x') \frown b' = v'$, $(v' \smile s) \frown (a_2 \smile x') = y'$, we have A_s by means of x', y', b'. Hence $[\{(u \smile x') \frown b'\} \smile y'] \frown a^{(i)} = s+u$ and putting $\{(s+u) \smile p^{(\lambda')}\} \frown (a_1^{(i)} \smile x) = y_{s+u}$, we have

$$(12) \qquad (y_{s+u} \smile q^{(\lambda')}) \frown a^{(i)} = (s+u)t.$$

Next it holds for $a_1^{(i)} \smile x = d$ the relations

$$(d \smile x') \smile p = a_1^{(i)} \smile x' \smile x \smile p = a_1^{(i)} \smile e^{(\lambda)} \smile x \smile x'$$
$$= a_1^{(i)} \smile a_2^{(\lambda)} \smile x \smile x' = (a' \smile x') \smile p,$$
$$(d \smile x') \frown p = d \frown p = d \frown (e^{(\lambda)} \smile x) \frown b = x \frown b = O,$$
$$(a' \smile x') \frown p = (a' \smile x') \frown (e^{(\lambda)} \smile x) \frown b = a_2^{(\lambda)} \frown (e^{(\lambda)} \smile x) = O.$$

Hence $a' \smile x' \sim d \smile x'$ with axis p. Next we have

$$(d \smile x') \smile q = (a_1^{(i)} \smile x') \smile (q \smile x) = (a_1^{(i)} \smile x') \smile (t \smile x)$$
$$= a_1^{(i)} \smile x' \smile a_2^{(\lambda)} \smile x = a' \smile x' \smile q$$

and

$$(d \smile x') \frown q = (d \smile x') \frown (t \smile x) \frown b = x \frown b = O,$$
$$)a' \smile x') \frown q = (a' \smile x') \frown (t \smile x) \frown b = a_2^{(\lambda)} \frown (t \smile x) = O.$$

Therefore $a' \cup x' \sim d \cup x'$ with axis q and we obtain a normal projective automorphism T of $a' \cup x'$ by means of p, q, $d \cup x'$, its pole being $(a' \cup x') \frown (d \cup x') = a_1^{(i)} \cup x'$. From (8), (9), (10), (11) and (12) we have $st = T(s)$, $ut = T(u)$, $(s+u)t \leqq T(s+u)$ and $T(x') = x'$, $T(b') \cup x' = T(b' \cup x') = T(a' \cup x')$. Now we proceed as follows

$$T(a_2^{(\lambda)}) = [\{(a_2^{(\lambda)} \cup p) \frown (d \cup x')\} \cup q] \frown (a' \cup x')$$

$$= [\{(a_2^{(\lambda)} \cup p) \frown (d \cup x')\} \cup x \cup t] \frown a_1^{(\lambda)} = (a_1^{(\lambda)} \cup x \cup t) \frown a_2^{(\lambda)} = a_2^{(\lambda)}.$$

Hence $T(b') \cup x' = T(a' \cup x') = T(a_1^{(i)} \cup a_2^{(\lambda)} \cup x') = a_1^{(i)} \cup a_2^{(\lambda)} \cup x' = a' \cup x'$,

$$T(b') \cup T(y') = T(b' \cup y') = T(a' \cup y') = T(a') \cup T(y') = a' \cup T(y') .$$

and $\quad T(b') \frown x' = T(b' \frown x') = O , \quad T(b') \frown T(y') = T(b' \frown y') = O .$

Therefore $a^{(i)} \sim T(b') \cup a_2$ with axis x' and $T(y')$. Since further $T(b') \frown a' = T(a_2^{(\lambda)}) = a_2^{(\lambda)}$, $\{x' \cup T(y')\} \frown a^{(i)}\} \leqq a_2$, and

$$[\{(a_1^{(i)} \cup x') \frown T(b')\} \cup T(y')] \frown a' = T[\overline{\{(a_1^{(i)} \cup x') \frown b'\} \cup y'] \frown a'} = T(s) = st,$$

so we obtain A_{st} by means of x', $T(y')$, $T(b') \cup a_2$. Then

$$st + ut \geqq [\{(ut \cup x') \frown T(b')\} \cup T(y')] \frown a'$$

$$= T[\overline{\{(u \cup x') \frown b'\} \cup y'] \frown a'} = T(s+u).$$

Now we have $(s+u)t \leqq st + ut$, from which we conclude $(s+u)t = st + ut$ considering the dimensions of the both sides.

In case $O(t) = O$ and $O(s) > O(u)$, it follows $O(s+u) = O(u) = O(-u)$.(1) Hence we have $(s+u)t + (-u)t = \{(s+u) + (-u)\}t = st$. From $(-u)t + ut = \{(-u) + u\}t = a_1^{(i)}$ follows then $(s+u)t = st + ut$. In case $O(t) > 0$, it is evidently $O(t + e^{(\lambda)}) = 0$. Hence it follows

$$(s+u)(t+e^{(\lambda)}) = s(t+e^{(\lambda)}) + u(t+e^{(\lambda)}) = st + ut + se^{(\lambda)} + ue^{(\lambda)}$$

$$= st + ut + s + u.$$

On the other hand we have $(s+u)(t+e^{(\lambda)}) = (s+u)t + (s+u)$ by applying the first distributive law. Hence the uniqueness of substraction yields $(s+u)t = st + ut$.

(1) Here $-u$ means a finite point, for which $(-u) + u = a_1^{(i)}$ holds.

PART 5.

As in the preceding parts let L be a primary lattice with height h, $m_h \geqq 4$ and let a_i, 1, 2, $\cdots r$, be its basis with $d_i = \dim a_i$, where $d_1 = d_2 = d_3 = d_4 = h$ holds. In each straight line $\frac{a_1 \cup a_i}{O}$ we choose an irreducible element e_{1i} of dimension d_i as its *unit element*, which is independent with both a_1 and a_i. In the straight line $\frac{a_i \cup a_j}{O}$, where i, j distinct from 1, we fix the element $[a_1 \cup \{(a_i \cup e_{1j}) \cap (a_j \cup e_{1i})\}] \cap (a_i \cup a_j)$ as its *unit element* e_{ij}. Then we will show that (i) e_{ij} is independent with both a_i and a_j. (ii) e_{ij} is irreducible and of dimension $\mathrm{Min}(d_i, d_j)$. In order to prove these, we suppose $d_i \geqq d_j$. Since

$$[a_i \cup \{(a_i \cup e_{1j}) \cap (a_j \cup e_{1i})\}] \cap a_1 = \{(a_i \cup a_1^{(d_i)} \cup a_j) \cup (a_i \cup e_{1j})\} \cap a_1$$
$$= [a_i \cup \{e_{1j} \cap (a_i \cup a_1^{(d_i)} \cup a_j)\}] \cap a_1 = (a_i \cup e_{1j}) \cap a_1 = e_{1j} \cap a_1 = 0\,,$$

so lemma I yields

$$e_{ij} \cap a_i = \{(a_i \cup e_{1j}) \cap (a_j \cup e_{1i})\} \cap a_i = a_i \cap (a_j \cup e_{1i}) = a_i \cap e_{1i} = 0\,.$$

Similarly $e_{ij} \cap a_j = O$. The validity of (ii) can be seen by

$$\frac{e_{ij}}{O} \simeq \frac{a_i \cup e_{ij}}{a_i} = \frac{(a_1 \cup a_i \cup e_{ij}) \cap (a_i \cup a_j)}{a_i} = \frac{a_i \cup a_j}{a_i} \simeq \frac{a_j}{O}\,.$$

Theorem 65. *Among the unit elements e_{ij} of straight lines $\frac{a_i \cup a_j}{O}$ in L hold the relations*

$$[a_k \cup \{(a_i \cup e_{kj}) \cap (a_j \cup e_{ki})\}] \cap (a_i \cup a_j) = e_{ij}^{(d_k)},$$

where k is distinct from i and j.

Pr. By definition the case $k = 1$ is evident. For the case $i = 1$ (or $j = 1$) we have

$$(a_1 \cup e_{kj}) \cap (a_j \cup e_{1k}) = [a_1 \cup \{(a_k \cup e_{1j}) \cap (a_j \cup e_{1k})\}] \cap (a_j \cup e_{1k})$$
$$= \{(a_k \cup e_{1j}) \cap (a_j \cup e_{1k})\} \cup \{a_1 \cap (a_j \cup e_{1k})\} = (a_k \cup e_{1j}) \cap (a_j \cup e_{1k})\,,$$

whence $[a_k \cup \{(a_1 \cup e_{kj}) \cap (a_j \cup e_{1k})\}] \cap (a_1 \cup a_j) = (a_k \cup e_{1j}^{(d_k)}) \cap (a_1 \cup a_j)$
$$= e_{1j}^{(d_k)}.$$

The cases, where all i, j, k are distinct from 1, can be treated as follows

$$a_i \cup e_{kj} = [a_1 \cup \{(a_i \cup a_k \cup e_{1j}) \cap (a_i \cup a_j \cup e_{1k})\}] \cap (a_i \cup a_k \cup a_j)\,;$$

$$a_j \cup e_{ki} = [a_1 \cup \{(a_j \cup a_k \cup e_{1i}) \cap (a_i \cup a_j \cup e_{1k})\}] \cap (a_i \cup a_k \cup a_j)\,.$$

Now the relation $a_1 \cap (a_i \cup a_j \cup e_{1k}) = O$ yields

$$(a_i \cup e_{kj}) \cap (a_j \cup e_{ki}) = [a_1 \cup \{(a_i \cup a_k \cup e_{1j}) \cap (a_j \cup a_k \cup e_{1i}) \cap (a_i \cup a_j \cup e_{1k})\}] \cap (a_i \cup a_k \cup a_j)\,,$$

$$[a_k \cup \{(a_i \cup e_{kj}) \cap (a_j \cup e_{ki})\}] \cap (a_i \cup a_j)$$
$$= [a_1 \cup \{(a_i \cup e_{1j}^{(d_k)}) \cap (a_j \cup e_{1i}^{(d_k)})\}] \cap (a_i \cup a_j) \leqq e_{ij}\,.^{(1)} \quad \text{q. e. d.}$$

For an irreducible element l of dimension λ in the straight line $\dfrac{a_i \cup a_j}{O}$ we choose a basis-component a_k such that $\dim a_k \geqq \lambda$, $k \neq i$, $k \neq j$. Then we can find an irreducible element l' of dimension λ with $a_k \cup l = a_k \cup l'$, $a_k \cap l' = O$, $(a_i \cup a_j) \cap l' = O$. We put

$$\bar{l} = [\{(a_j \cup l') \cap (a_i \cup a_k)\} \cup \{(a_i \cup l') \cap (a_j \cup a_k)\}] \cap (a_i \cup a_j)$$

and will prove that $\bar{l}$ is also an irreducible element of dimension λ, uniquely determined irrespective of the choice of a_k, l'. Suppose $\cap a_j = O$ and take a_i, a_j respectively as the origin and the infinite point of the straight line. Using $a_j \cup l = a_j \cup a_i^{(\lambda)}$, we have

$$(a_j \cup a_i^{(\lambda)}) \cup a_k = (a_j \cup l') \cup a_k\,, \quad (a_j \cup l') \cap a_k = l' \cap a_k = O$$

by lemma 1, since $(a_k \cup l) \cap a_j = l \cap a_j = O$. Therefore $a_j \cup a_i^{(\lambda)} \sim a_j \cup l'$ with axis a_k. Putting $(a_i \cup l') \cap (a_j \cup a_k) = m$, we obtain $m \cap a_j = (a_i \cup l') \cap a_j = a_i \cap a_j = O$ and

$$\frac{m}{O} \simeq \frac{m \cup a_j}{a_j} = \frac{a_j \cup a_k^{(\lambda)}}{a_j} \simeq \frac{a_k^{(\lambda)}}{O}\,.$$

Hence m is irreducible and of dimension λ. Further we have

$$(a_j \cup a_i^{(\lambda)}) \cup m = a_j \cup a_i^{(\lambda)} \cup a_k^{(\lambda)} = a_j \cup a_k^{(\lambda)} \cup l' = (a_j \cup l') \cup m$$

and $m \cap (a_j \cup a_i^{(\lambda)}) = O$. $m \cap (a_j \cup l') = m \cap a_j = O$ by lemma 1, since

$$l' \cap (a_j \cup m) = l' \cap (a_j \cup a_k^{(\lambda)}) = l' \cap (a_k \cup l) \cap (a_j \cup a_k^{(\lambda)})$$
$$= l' \cap [a_k^{(\lambda)} \cap \{a_j \cap (a_k \cup l)\}] = l' \cap a_k^{(\lambda)} = O\,.$$

(1) The calculation of dimension of the left-hand side yields the required result.

Therefore $a_j \cup l'$ is perspective to $a_j \cup a_i^{(\lambda)}$ with axis m. Since further $(a_j \cup a_i^{(\lambda)}) \cap (a_j \cup l') = a_j$, $(a_k \cup m) \cap (a_j \cup a_i^{(\lambda)}) \leqq a_j$ holds, we have really an addition in $\frac{a_i \cup a_j}{O}$ by means of a_k, m, $a_j \cup l'$. Now we put

$$p = (a_i \cup a_k) \cap (a_j \cup l') = (a_i^{(\lambda)} \cup a_k) \cap (a_j \cup l').$$

Then $$\bar{l} = (p \cup m) \cap (a_i \cup a_j) = (p \cup m) \cap (a_i^{(\lambda)} \cup a_j).$$

Hence $\bar{l}$ is the image of $a_i^{(\lambda)}$ in this addition and therefore $\bar{l}$ is irreducible with dimension λ. Since further

$$(l \cup a_k) \cap (a_j \cup l') = l',$$

$$(l' \cup m) \cap (a_i^{(\lambda)} \cup a_j) = (a_i \cup l') \cap (a_j \cup a_k \cup l') \cap (a_i^{(\lambda)} \cup a_j) = a_i^{(\lambda)},$$

so we have $A_{\bar{l}}(l) = a_i^{(\lambda)}$, that is $l = -\bar{l}$. Therefore $\bar{l}$ is uniquely determined irrespective of the choice of a_k, l by theorem 51.

The element $\bar{l}$ is called *conjugate* to the element l.

Theorem 66. $(e_{ki} \cup e_{kj}) \cap (a_i \cup a_j) = \overline{e_{ij}^{(d_k)}}$, *where* $\overline{e_{ij}}$ *is conjugate to* e_{ij} *and* $\overline{e_{ij}} \cap a_i = O$, $\overline{e_{ij}} \cap a_j = O$.

Pr. Putting $(a_i \cup e_{kj}) \cap (a_j \cup e_{ki}) = e'_{ij}$, we have $a_k \cup e'_{ij} = a_k \cup e_{ij}^{(d_k)}$ by theorem 65, and $e'_{ij} \cap a_k = O$, $e'_{ij} \cap (a_i \cup a_j) = O$. Now it follows

$$\begin{aligned}\overline{e_{ij}^{(d_k)}} &= [\{(a_j \cup e'_{ij}) \cap (a_i \cup a_k)\} \cup \{(a_i \cup e'_{ij}) \cap (a_j \cup a_k)\}] \cap (a_i \cup a_j) \\ &= [\{(a_j \cup e_{ki}^{(d_j)}) \cap (a_i \cup a_k)\} \cup \{(a_i \cup e_{kj}^{(d_i)}) \cap (a_j \cup a_k)\}] \cap (a_i \cup a_j) \\ &= (e_{ki}^{(d_j)} \cup e_{kj}^{(d_i)}) \cap (a_i \cup a_j) = (e_{ki} \cup e_{kj}) \cap (a_i \cup a_j)\end{aligned}$$

Theorem 67. $(\overline{e_{ij}} \cup \overline{e_{jk}}) \cap (a_i \cup a_k) = \overline{e_{ik}^{(d_j)}}$.

Pr. Choose a_ρ, such that $d_\rho = h$, $\rho \neq i$, $\rho \neq j$, $\rho \neq k$. Then we have

$$\overline{e_{ij}} = (e_{\rho i} \cup e_{\rho j}) \cap (a_i \cup a_j) = (e_{\rho i} \cup e_{\rho j} \cup e_{\rho k}) \cap (a_i \cup a_j).$$

$$\overline{e_{jk}} = (e_{\rho i} \cup e_{\rho j} \cup e_{\rho k}) \cap (a_j \cup a_k)$$

$$\begin{aligned}\overline{e_{ij}} \cup \overline{e_{jk}} &= [\{(e_{\rho i} \cup e_{\rho j} \cup e_{\rho k} \cup a_j) \cap (a_i \cup a_j)\} \cup a_k] \cap (e_{\rho i} \cup e_{\rho j} \cup e_{\rho k}) \\ &= (a_i^{(d_j)} \cup a_j \cup a_k) \cap (e_{\rho i} \cup e_{\rho j} \cup e_{\rho k})\end{aligned}$$

$$(\overline{e_{ij}} \cup \overline{e_{jk}}) \cap (a_i \cup a_k) = (e_{\rho i} \cup e_{\rho k}) \cap (a_i^{(d_j)} \cup a_k) = \overline{e_{ik}^{(d_j)}}. \qquad \text{q. e. d.}$$

Given two different straight lines $\frac{a_k \cup a_i}{O}$ and $\frac{a_k \cup a_j}{O}$, $a_k \cup a_i^{(d_j)}$

is perspective to $a_k \cup a_j^{(d_i)}$ with axis $\overline{e_{ij}}$. This perspectivity shall be denoted with the symbol $P\binom{k\ i}{k\ j}$ and the image of a point q with $P\binom{k\ i}{k\ j}q$. For the sake of completeness we shall give the symbol $P\binom{k\ i}{k\ i}$ the meaning of the identical correspondence such that $P\binom{k\ i}{k\ i}q = q$.

Theorem 68.

$$(1)\quad P\binom{k\ i}{k\ j}a_k = a_k\,, \qquad (2)\quad P\binom{k\ i}{k\ j}a_i^{(d_j)} = a_j^{(d_i)}\,,$$

$$(3)\quad P\binom{k\ i}{k\ j}e_{ki}^{(d_j)} = e_{kj}^{(d_i)}\,, \qquad (4)\quad P\binom{k\ i}{k\ j}\overline{e_{ki}^{(d_j)}} = \overline{e_{kj}^{(d_i)}}\,.$$

Pr. Suppose $d_j \geqq d_i$. Then (1) follows from $(a_k \cup \overline{e_{ij}}) \frown (a_k \cup a_j^{(d_i)}) = a_k$, and (2) from $(a_i \cup \overline{e_{ij}}) \frown (a_k \cup a_j^{(d_i)}) = a_j^{(d_i)}$. Theorem 66 yields $e_{ki} \cup \overline{e_{ij}^{(d_k)}} = e_{ki} \cup e_{kj}^{(d_i)}$ and consequently $(e_{ki} \cup \overline{e_{ij}}) \frown (a_k \cup a_j^{(d_i)}) = e_{kj}^{(d_i)}$, which proves (3), and finally by theorem 67 we have $(\overline{e_{ki}} \cup \overline{e_{ij}}) \frown (a_k \cup a_j^{(d_i)}) = \overline{e_{kj}^{(d_i)}}$, which proves (4).

Theorem 69. *If c is irreducible in $\dfrac{a_k \cup a_i}{O}$ and if $P\binom{k\ i}{k\ j}c = d$, then $P\binom{k\ i}{k\ j}\bar{c} = \bar{d}$.*

P.. Choose a_ρ, such that $d_\rho = h$, $\rho \neq k, i, j$ and an irreducible element c' such that $a_\rho \cup c = a_\rho \cup c'$, $c' \frown a_\rho = O$, $c' \frown (a_k \cup a_i) = O$. Then by definition $\bar{c} = [\{(a_k \cup c') \frown (a_\rho \cup a_i)\} \cup \{(a_i \cup c') \frown (a_\rho \cup a_k)\}] \frown (a_i \cup a_k)$. The perspectivity of $a_\rho \cup a_k \cup a_i^{(d_j)}$ and $a_\rho \cup a_k \cup a_j^{(d_i)}$ with axis $\overline{e_{ij}}$ being denoted with the symbol $\mathfrak{P}$, we have

$$P\binom{k\ i}{k\ j}\bar{c} = [\{(a_k \cup \mathfrak{P}(c')) \frown (a_\rho \cup a_j^{(d_i)})\} \cup \{(a_j^{(d_i)} \cup \mathfrak{P}(c')) \frown (a_\rho \cup a_k)\}] \frown (a_j^{(d_i)} \cup a_k)\,.$$

But $a_\rho \cup d = a_\rho \cup \mathfrak{P}(c) = a_\rho \cup \mathfrak{P}(c')$ and $\mathfrak{P}(c') \frown a_\rho = O$, $\mathfrak{P}(c') \frown (a_k \cup a_j^{(d_i)}) = O$. Hence $P\binom{k\ i}{k\ j}\bar{c} = \bar{d}$. q. e. d.

The image of a given element t exists for $P\binom{k\ i}{k\ j}$, when t belongs to $a_k \cup a_i^{(d_j)}$. Every element, whose image exists, is said to be *proper* with respect to the perspectivity. If for every element t, which is proper with respect to $\mathfrak{P}_2$ and $\mathfrak{P}_3$, $\mathfrak{P}_2(t)$ being proper with

respect to $\mathfrak{P}_1$, the relation $\mathfrak{P}_1(\mathfrak{P}_2(t)) = \mathfrak{P}_3(t)$ holds, then we say that $\mathfrak{P}_3$ is *product* of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ and write $\mathfrak{P}_1\mathfrak{P}_2 = \mathfrak{P}_3$: Further an element t is said to be *proper* with respect to the product of perspectivities $\mathfrak{P}_1, \mathfrak{P}_2, \cdots \mathfrak{P}_\lambda$, if $\mathfrak{P}_1(\mathfrak{P}_2 \cdots \mathfrak{P}_\lambda(t))$ exists. Two products are said to be *equivalent*, if for every element, which is proper with respect to both, its image is the same for both products.

Theorem 70. *If i, j, k, l are all different, then*

$$P\begin{pmatrix} k & j \\ k & l \end{pmatrix} P\begin{pmatrix} k & i \\ k & j \end{pmatrix} = P\begin{pmatrix} k & i \\ k & l \end{pmatrix}.$$

Pr. Put $\mathrm{Min}\,(d_i, d_j) = \nu$. Then we have

$$P\begin{pmatrix} k & j \\ k & l \end{pmatrix} P\begin{pmatrix} k & i \\ k & j \end{pmatrix} t = [\{(t \smile \overline{e_{ij}}) \frown (a_k \smile a_j^{(d_i)})\} \smile \overline{e_{jl}}] \frown (a_k \smile a_l^{(d_j)})$$
$$= (t \smile \overline{e_{ij}} \smile \overline{e_{jl}}) \frown (a_k \smile a_l^{(\nu)}).$$

But from theorem 67 follows $\overline{e_{ij}^{(dl)}} \smile \overline{e_{jl}} = \overline{e_{il}^{(dj)}} \smile \overline{e_{jl}}$, which yields

$$P\begin{pmatrix} k & j \\ k & l \end{pmatrix} P\begin{pmatrix} k & i \\ k & j \end{pmatrix} t \geqq (t \smile \overline{e_{il}^{(dj)}}) \frown (a_k \smile a_l^{(\nu)}) = P\begin{pmatrix} k & i \\ k & l \end{pmatrix} t.$$

Theorem 71. *If i, j, k, ρ are all different, then*

$$P\begin{pmatrix} k & j \\ k & \rho \end{pmatrix} P\begin{pmatrix} i & j \\ k & j \end{pmatrix} = P\begin{pmatrix} i & \rho \\ k & \rho \end{pmatrix} P\begin{pmatrix} i & j \\ i & \rho \end{pmatrix}.$$

Pr. $$P\begin{pmatrix} k & j \\ k & \rho \end{pmatrix} P\begin{pmatrix} i & j \\ k & j \end{pmatrix} t = [\{(t \smile \overline{e_{ik}}) \frown (a_j \smile a_k^{(d_i)})\} \smile \overline{e_{j\rho}}] \frown (a_k \smile a_\rho^{(d_j)})$$
$$= (t \smile \overline{e_{ik}} \smile \overline{e_{j\rho}}) \frown (a_k^{(d_i)} \smile a_\rho^{(d_j)}).$$

On the other hand we have

$$P\begin{pmatrix} i & \rho \\ k & \rho \end{pmatrix} P\begin{pmatrix} i & j \\ i & \rho \end{pmatrix} t = [\{(t \smile \overline{e_{j\rho}}) \frown (a_i \smile a_\rho^{(d_j)})\} \smile \overline{e_{ik}}] \frown (a_\rho \smile a_k^{(d_i)})$$
$$= (t \smile \overline{e_{j\rho}} \smile \overline{e_{ik}}) \frown (a_\rho^{(d_j)} \smile a_k^{(d_i)}).$$

Theorem 72. *If i, j, ρ, k are all different, then*

$$P\begin{pmatrix} k & i \\ j & i \end{pmatrix} P\begin{pmatrix} k & j \\ k & i \end{pmatrix} P\begin{pmatrix} i & j \\ k & j \end{pmatrix} = P\begin{pmatrix} \rho & i \\ j & i \end{pmatrix} P\begin{pmatrix} \rho & j \\ \rho & i \end{pmatrix} P\begin{pmatrix} i & j \\ \rho & j \end{pmatrix}.$$

Pr. $P\binom{\rho\ i}{j\ i}P\binom{\rho\ j}{\rho\ i}P\binom{i\ j}{\rho\ j} = P\binom{k\ i}{j\ i}P\binom{\rho\ i}{k\ i}P\binom{\rho\ j}{\rho\ i}P\binom{i\ j}{\rho\ j}$ by theorem 70

$= P\binom{k\ i}{j\ i}P\binom{k\ j}{k\ i}P\binom{\rho\ j}{k\ j}P\binom{i\ j}{\rho\ j}$ by theorem 71

$= P\binom{k\ i}{j\ i}P\binom{k\ j}{k\ i}P\binom{i\ j}{k\ j}$ again by theorem 70.

q. e. d.

Now we define $P\binom{i\ j}{k\ \rho} = P\binom{k\ j}{k\ \rho}P\binom{i\ j}{k\ j}$, if $k \neq j$, $i \neq j$, $k \neq \rho$. If moreover i, j, k, ρ are all different, then this is the perspectivity of $a_i^{(d_k)} \cup a_j^{(d_\rho)}$ and $a_k^{(d_i)} \cup a_\rho^{(d_j)}$ with axis $\overline{e_{j\rho}} \cup \overline{e_{ik}}$ by theorem 71. Further we define $P\binom{i\ j}{j\ i} = P\binom{k\ i}{j\ i}P\binom{k\ j}{k\ i}P\binom{i\ j}{k\ j}$, where $d_k = h$ and $k \neq i, j$.

Theorem 73. $P\binom{k\ \rho}{k\ \sigma}P\binom{i\ j}{k\ \rho} = P\binom{i\ j}{k\ \sigma}$.

Pr. The case $i = k$ or $j = \rho$ is evident. For the case $i \neq k$, $j \neq \rho$ and $j \neq k$

$$P\binom{k\ \rho}{k\ \sigma}P\binom{i\ j}{k\ \rho} = P\binom{k\ \rho}{k\ \sigma}P\binom{k\ j}{k\ \rho}P\binom{i\ j}{k\ j} = P\binom{k\ j}{k\ \sigma}P\binom{i\ j}{k\ j} = P\binom{i\ j}{k\ \sigma}.$$

For the case $i \neq k$, $j \neq \rho$, $j = k$, $i = \rho$ we can assume $\sigma \neq i$ and

$$\begin{aligned} P\binom{j\ i}{j\ \sigma}P\binom{i\ j}{j\ i} &= P\binom{j\ i}{j\ \sigma}P\binom{\varsigma\ i}{j\ i}P\binom{\varsigma\ j}{\varsigma\ i}P\binom{i\ j}{\varsigma\ j} \\ &= P\binom{\varsigma\ \sigma}{j\ \sigma}P\binom{\varsigma\ i}{\varsigma\ \sigma}P\binom{\varsigma\ j}{\varsigma\ i}P\binom{i\ j}{\varsigma\ j} = P\binom{\varsigma\ \sigma}{j\ \sigma}P\binom{\varsigma\ j}{\varsigma\ \sigma}P\binom{i\ j}{\varsigma\ j} \\ &= P\binom{\varsigma\ \sigma}{j\ \sigma}P\binom{i\ \sigma}{\varsigma\ \sigma}P\binom{i\ j}{i\ \sigma} = P\binom{i\ \sigma}{j\ \sigma}P\binom{i\ j}{i\ \sigma} = P\binom{i\ j}{j\ \sigma}, \end{aligned}$$

where $\varsigma \neq i, j, \sigma$. For the case $i \neq k$, $j \neq \rho$, $j = k$, $i \neq \rho$, $i = \sigma$ we have $P\binom{j\ \rho}{j\ i}P\binom{i\ j}{j\ \rho} = P\binom{j\ \rho}{j\ i}P\binom{i\ \rho}{j\ \rho}P\binom{i\ j}{i\ \rho} = P\binom{i\ j}{j\ i}$. Finally in the case $i \neq k$, $j \neq \rho$, $j = k$, $i \neq \rho$, $i \neq \sigma$,

$$\begin{aligned} P\binom{j\ \rho}{j\ \sigma}P\binom{i\ j}{j\ \rho} &= P\binom{j\ \rho}{j\ \sigma}P\binom{i\ \rho}{j\ \rho}P\binom{i\ j}{i\ \rho} = P\binom{i\ \rho}{j\ \sigma}P\binom{i\ j}{i\ \rho} \\ &= P\binom{i\ \sigma}{j\ \sigma}P\binom{i\ \rho}{i\ \sigma}P\binom{i\ j}{i\ \rho} = P\binom{i\ \sigma}{j\ \sigma}P\binom{i\ j}{i\ \sigma} = P\binom{i\ j}{j\ \sigma}. \end{aligned}$$

Theorem 74. $P\binom{i\ j}{k\ \rho}P\binom{i\ \sigma}{i\ j} = P\binom{i\ \sigma}{k\ \rho}$.

Pr. The case $i = k$ or $j = \rho$ is evident. Suppose $i \neq k$, $j \neq \rho$, $j \neq k$. By the preceding theorem we have

$$P\begin{pmatrix} i & j \\ k & \rho \end{pmatrix} P\begin{pmatrix} i & \sigma \\ i & j \end{pmatrix} = P\begin{pmatrix} k & j \\ k & \rho \end{pmatrix} P\begin{pmatrix} i & j \\ k & j \end{pmatrix} P\begin{pmatrix} i & \sigma \\ i & j \end{pmatrix} = P\begin{pmatrix} k & j \\ k & \rho \end{pmatrix} P\begin{pmatrix} i & \sigma \\ k & j \end{pmatrix} = P\begin{pmatrix} i & \sigma \\ k & \rho \end{pmatrix}.$$

If $i \neq k$, $j \neq \rho$ and $j = k$, then we have for the case $i \neq \rho$

$$P\begin{pmatrix} i & j \\ j & \rho \end{pmatrix} P\begin{pmatrix} i & \sigma \\ i & j \end{pmatrix} = P\begin{pmatrix} i & \rho \\ j & \rho \end{pmatrix} P\begin{pmatrix} i & j \\ i & \rho \end{pmatrix} P\begin{pmatrix} i & \sigma \\ i & j \end{pmatrix} = P\begin{pmatrix} i & \rho \\ j & \rho \end{pmatrix} P\begin{pmatrix} i & \sigma \\ i & \rho \end{pmatrix} = P\begin{pmatrix} i & \sigma \\ j & \rho \end{pmatrix},$$

and for the case $i = \rho$

$$\begin{aligned} P\begin{pmatrix} i & j \\ j & i \end{pmatrix} P\begin{pmatrix} i & \sigma \\ i & j \end{pmatrix} &= P\begin{pmatrix} \varsigma & i \\ j & i \end{pmatrix} P\begin{pmatrix} \varsigma & j \\ \varsigma & i \end{pmatrix} P\begin{pmatrix} i & j \\ \varsigma & j \end{pmatrix} P\begin{pmatrix} i & \sigma \\ i & j \end{pmatrix} \\ &= P\begin{pmatrix} \zeta & i \\ j & i \end{pmatrix} P\begin{pmatrix} \zeta & j \\ \zeta & i \end{pmatrix} P\begin{pmatrix} i & \sigma \\ \zeta & j \end{pmatrix} = P\begin{pmatrix} \zeta & i \\ j & i \end{pmatrix} P\begin{pmatrix} i & \sigma \\ \zeta & i \end{pmatrix} = P\begin{pmatrix} i & \sigma \\ j & i \end{pmatrix} \end{aligned}$$

by the preceding theorem. q. e. d.

Theorem 73 and 74 show that through successive perspectivities the final image of a given point depends only on the final straight line, whereupon the process of perspectivities ends, and never depends on the choice of intermediating straight lines.

Theorem 75. *$P\begin{pmatrix} i & j \\ j & i \end{pmatrix} t = t^{-1}$ for every unit t in $\dfrac{a_i \cup a_j}{O}$, where t^{-1} means a point, for which $M_t(t^{-1}) \leqq e_{ij}$ holds.*

Pr. We put $\dim t = \lambda$. From $t \cup a_i \leqq a_i \cup a_j$, $t \cup a_j \leqq a_i \cup a_j$ follows $\dim a_i \geqq \lambda$, $\dim a_j \geqq \lambda$. Suppose $P\begin{pmatrix} i & j \\ j & i \end{pmatrix} = P\begin{pmatrix} k & i \\ j & i \end{pmatrix} P\begin{pmatrix} k & j \\ k & i \end{pmatrix} P\begin{pmatrix} i & j \\ k & j \end{pmatrix}$, where $\dim a_k \geqq \lambda$, and put

$$(t \cup \overline{e_{ki}^{(\lambda)}}) \cap (a_k^{(\lambda)} \cup a_j^{(\lambda)}) = t', \quad (t' \cup \overline{e_{ij}^{(\lambda)}}) \cap (a_i^{(\lambda)} \cup a_k^{(\lambda)}) = t'',$$

$$(t'' \cup \overline{e_{kj}^{(\lambda)}}) \cap (a_i^{(\lambda)} \cup a_j^{(\lambda)}) = t'''$$

Then we haue obviously $t''' = P\begin{pmatrix} i & j \\ j & i \end{pmatrix} t$ and

$$a_i^{(\lambda)} \cup t'' = a_i^{(\lambda)} \cup a_k^{(\lambda)}, \qquad a_k^{(\lambda)} \cup t'' = a_i^{(\lambda)} \cup a_k^{(\lambda)},$$

whence $(a_i^{(\lambda)} \cup a_j) \cup t'' = (a_k^{(\lambda)} \cup a_j) \cup t''$. Further

$$(a_i^{(\lambda)} \cup a_j) \cap t'' = (t' \cup \overline{e_{ij}^{(\lambda)}}) \cap a_i^{(\lambda)} = \overline{e_{ij}^{(\lambda)}} \cap a_i^{(\lambda)} = O,$$

$$(a_k^{(\lambda)} \cup a_j) \cap t'' = (t' \cup \overline{e_{ij}^{(\lambda)}}) \cap a_k^{(\lambda)} = t' \cap a_k^{(\lambda)} = O.$$

Therefore $a_i^{(\lambda)} \cup a_j \sim a_k^{(\lambda)} \cup a_j$ with axis t'' and we have a projective automorphism of $a_k^{(\lambda)} \cup a_j$ by means of t'', $\overline{e_{ki}^{(\lambda)}}$ and $a_k^{(\lambda)} \cup a_j$. If we take a_j as the infinite point and a_i as the orgin of the straight line

$\frac{a_i \cup a_j}{O}$, then this automorphism becomes the multiplication M_{-t}. In fact $(a_i^{(\lambda)} \cup a_j) \cap (a_k^{(\lambda)} \cup a_j) = a_j$, $(t'' \cup e_{ki}^{(\lambda)}) \cap (a_i^{(\lambda)} \cup a_j) \leqq a_i$, and $(\overline{e_{ij}^{(\lambda)}} \cup t'') \cap (a_k^{(\lambda)} \cup a_j) = (t' \cup \overline{e_{ij}^{(\lambda)}}) \cap (a_k^{(\lambda)} \cup a_j) = t'$, $(t' \cup \overline{e_{ki}^{(\lambda)}}) \cap (a_i^{(\lambda)} \cup a_j) = t$. The image of the point $\overline{e_{ij}^{(\lambda)}} = -e_{ij}^{(\lambda)}$ being t, we see that the image of $e_{ij}^{(\lambda)}$ is the point $-t$ by the use of the first distributive law. Now it follows $(t''' \cup t'') \cap (a_k^{(\lambda)} \cup a_j) = \overline{e_{kj}^{(\lambda)}}$ and $(\overline{e_{kj}^{(\lambda)}} \cup \overline{e_{ki}^{(\lambda)}}) \cap (a_i^{(\lambda)} \cup a_j) = \overline{e_{ij}^{(\lambda)}}$ by theorem 67. Hence $(-t) \cdot t''' = -e_{ij}^{(\lambda)}$, whence $t''' = t^{-1}$. q. e. d.

The quotient $\frac{a_i \cup a_j \cup a_k}{O}$, where a_i, a_j, a_k are any three basis-components of L, is called a plane in L. The perspectivity of two planes $\frac{a_i \cup a_j \cup a_k}{O}$ and $\frac{a_i \cup a_j \cup a_{k'}}{O}$ with axis $e_{kk'}$ shall be denoted by $P\begin{pmatrix} i, j, k \\ i, j, k' \end{pmatrix}$. Then we define further

$$P\begin{pmatrix} i, j, k \\ i, j', k' \end{pmatrix} = P\begin{pmatrix} i, j, k' \\ i, j', k' \end{pmatrix} P\begin{pmatrix} i, j, k \\ i, j, k' \end{pmatrix}, \qquad \text{if } j \neq k'.$$

$$P\begin{pmatrix} i, j, k \\ i, k, j \end{pmatrix} = P\begin{pmatrix} i, \rho, j \\ i, k, j \end{pmatrix} P\begin{pmatrix} i, \rho, k \\ i, \rho, j \end{pmatrix} P\begin{pmatrix} i, j, k \\ i, \rho, k \end{pmatrix}, \qquad \text{if } \rho \neq i,\ \rho \neq j,\ \rho \neq k.$$

$$P\begin{pmatrix} i, j, k \\ i', j', k' \end{pmatrix} = P\begin{pmatrix} i, j', k' \\ i', j', k' \end{pmatrix} P\begin{pmatrix} i, j, k \\ i, j', k' \end{pmatrix}, \qquad \text{if } i \neq j',\ i \neq k'$$

$$P\begin{pmatrix} i, j, k \\ j, i, k' \end{pmatrix} = P\begin{pmatrix} i, j, k' \\ j, i, k' \end{pmatrix} P\begin{pmatrix} i, j, k \\ i, j, k' \end{pmatrix} \qquad \text{and so forth.}$$

Theorem 76. *The perspectivity* $P\begin{pmatrix} i, j, k \\ i', j', k' \end{pmatrix}$ *in* $\frac{a_i \cup a_j \cup a_k}{O}$ *induces* $P\begin{pmatrix} j, k \\ j', k' \end{pmatrix}$ *in* $\frac{a_j \cup a_k}{O}$.

Pr. In case $i = i'$, $j = j'$, $k \neq k'$ we have for $t \leqq a_j \cup a_k$

$$P\begin{pmatrix} i, j, k \\ i, j', k' \end{pmatrix} t = (t \cup \overline{e_{kk'}}) \cap (a_i \cup a_j \cup a_{k'}^{(d_k)})$$
$$= (t \cup \overline{e_{kk'}}) \cap (a_j \cup a_{k'}^{(d_k)}) = P\begin{pmatrix} j, k \\ j, k' \end{pmatrix} t.$$

The proofs of the other cases can be readily obtained by virtue of theorem 73 and 74.

PART 6.

As in the preceding parts we consider a primary lattice L with a basis $a_1, a_2, \cdots a_r$, where $r \geqq 4$ and $\dim a_1 = \dim a_2 = \dim a_3$

$= \dim a_4 = h$ holds, h being the height of L, and we denote with A_i the join of all a_ν, $\nu = 1, 2, \cdots i-1, i+1, \cdots r$ except a_i, with A_{ij} the join of all a_ν except a_i and a_j, etc. If l is an irreducible element of dimension λ in L, we put $l_{ij} = (A_{ij} \cup l) \cap (a_i \cup a_j)$, and will prove

Theorem 77. *l_{ij} is irreducible and, if, putting* $\dim (l \cap A_i) = \lambda_i$, $\lambda_j \geqq \lambda_i$, *then* $l_{ij} \cap a_j = O$, $\dim l_{ij} = \lambda - \lambda_i$, $\dim (l_{ij} \cap a_i) = \lambda_j - \lambda_i$.

Pr. Since $\dfrac{l_{ij}}{O} = \dfrac{l_{ij}}{l_{ij} \cap A_{ij}} \simeq \dfrac{A_{ij} \cup l_{ij}}{A} = \dfrac{A_{ij} \cup l}{A_{ij}} \simeq \dfrac{l}{l \cap A_{ij}}$ holds, l_{ij} is irreducible and $\dim l_{ij} = \dim l - \dim (l \cap A_{ij}) = \lambda - \lambda_i$. Since further

$$\frac{l_{ij} \cap a_i}{O} = \frac{(A_{ij} \cup l) \cap a_i}{\{(A_{ij} \cup l) \cap a_i\} \cap A_{ij}} \simeq \frac{\{(A_{ij} \cup l) \cap a_i\} \cup A_{ij}}{A_{ij}}$$

$$= \frac{A_{ij} \cup (l \cap A_j)}{A_{ij}} \simeq \frac{l \cap A_j}{l \cap A_{ij}}$$

we have $\dim (l_{ij} \cap a_i) = \dim (l \cap A_j) - \dim (l \cap A_{ij}) = \lambda_j - \lambda_i$. Similarly we have $\dfrac{l_{ij} \cap a_j}{O} \simeq \dfrac{l \cap A_i}{l \cap A_{ij}}$ whence $l_{ij} \cap a_j = O$ follows in virtue of $\lambda_j \geqq \lambda_i$.

Theorem 78. *If* $\lambda_i \geqq \mathrm{Min}(\lambda_j, \lambda_k)$, *then* $\overline{l_{kj}} \leqq l_{ki} \cup l_{ji}$. *If moreover* $\lambda_i \geqq \lambda_j \geqq \lambda_k$ *holds, then* $\overline{l_{kj}} \cup l_{ji} = l_{ki} \cup l_{ji}$.

Pr. Choose a_ρ, such that $\rho \neq i, j, k$ and $\dim a_\rho = h$ and an irreduclbie element l', such that $a_\rho \cup l_{kj} = a_\rho \cup l'$, $l' \cap a_\rho = O$, $l' \cap (a_k \cup a_j) = O$. Putting $t = (l' \cup a_i) \cap (A_{ijk} \cup l)$, we have

$$A_{ijk} \cup t = (A_{kj} \cup l') \cap (A_{ijk} \cup l) = (A_{kj} \cup l) \cap (A_{ijk} \cup l) = A_{ijk} \cup l,$$

and $\quad t \cup a_i = (l' \cup a_i) \cap (A_{jk} \cup l) = (l' \cup a_i) \cap (A_{jk} \cup l') = a_i \cup l'$.

Now it follows

$$\begin{aligned}
\overline{l_{kj}} &= [\{(a_k \cup l') \cap (a_j \cup a_\rho)\} \cup \{(a_j \cup l') \cap (a_k \cup a_\rho)\}] \cap (a_k \cup a_j) \\
&= [\{(A_{j\rho} \cup l') \cap (a_j \cup a_\rho)\} \cup \{(A_{k\rho} \cup l') \cap (a_k \cup a_\rho)\}] \cap (a_k \cup a_j) \\
&= [a_i \cup \{(A_{j\rho} \cup t) \cap (a_j \cup a_\rho)\} \cup \{(A_{k\rho} \cup t) \cap (a_k \cup a_\rho)\}] \cap (a_k \cup a_j) \\
&= [a_i \cup \{(A_{j\rho i} \cup t) \cap (a_j \cup a_\rho \cup a_i)\} \cup \{(A_{ik\rho} \cup t) \cap (a_i \cup a_k \cup a_\rho)\}] \cap (a_k \cup a_j).
\end{aligned}$$

But $A_{j\rho i} \cup A_{i\rho k} \cup t \cup a_k \cup a_j = A_{\rho i} \cup t$ and $(A_{\rho i} \cup t) \cap a_i = A_{\rho i} \cap a_i = O$, where $(A_{\rho i} \cup a_i) \cap t = A_\rho \cap t = a_i \cap (A_{ijk} \cup l) = a_i \cap A_{ijk} = O$, since

$l \frown (a_i \smile A_{ijk}) = l \frown A_{jk} \leqq A_{ijk}$ by hypothesis. Hence we have by lemma 1

$$\bar{l}_{kj} = [\{(A_{j\rho i} \smile t) \frown (a_j \smile a_\rho \smile a_i)\} \smile \{(A_{ik\rho} \smile t) \frown (a_i \smile a_k \smile a_\rho)\}] \frown (a_k \smile a_j)$$

$$\leqq [\{(A_{ji} \smile t) \frown (a_j \smile a_i)\} \smile \{(A_{ik} \smile t) \frown (a_i \smile a_k)\} \smile a_\rho] \frown (a_k \smile a_j)$$

$$= [\{(A_{ji} \smile l) \frown (a_j \smile a_i)\} \smile \{(A_{ik} \smile l) \frown (a_i \smile a_k)\}] \frown (a_k \smile a_j)$$

$$= (l_{ij} \smile l_{ki}) \frown (a_k \smile a_j), \text{ whence}$$

$$(1) \qquad \bar{l}_{kj} \leqq l_{ji} \smile l_{ki}.$$

If moreover $\lambda_i \geqq \lambda_j \geqq \lambda_k$, then $\dim \bar{l}_{kj} = \lambda - \lambda_k$, $\dim l_{ki} = \lambda - \lambda_k$, $\dim l_{ji} = \lambda - \lambda_j$. Since $l_{ij} \frown a_i = O$ and $\bar{l}_{kj} \frown a_j = O$, we have $l_{ij} \frown l_{ki} = O$, $\bar{l}_{kj} \frown l_{ij} = O$ and consequently $\dim (l_{ki} \smile l_{ji}) = \dim (\bar{l}_{kj} \smile l_{ji})$. Hence from (1) we conclude $\bar{l}_{kj} \smile l_{ji} = l_{ki} \smile l_{ji}$. q. e. d.

The perspectivity $a_k \smile a_i^{(dj)} \sim a_k \smile a_j^{(di)}$ with axis e_{ij} shall be denoted with the symbol $\bar{P}\binom{k\ i}{k\ j}$. Then we have.

Theorem 79. *If l is an irreducible element in $\dfrac{a_k \smile a_i^{(dj)}}{O}$, with $l \frown a_k = O$, then $\bar{P}\binom{k\ i}{k\ j} l = P\binom{k\ i}{k\ j} \bar{l}$, $\bar{P}\binom{k\ i}{k\ j} \bar{l} = P\binom{k\ i}{k\ j} l$.*

Pr. Put $g = (l \smile a_j) \frown (a_k \smile e_{ij})$ and $\lambda = \dim l$. Then g is irreducible and of dimension λ with $\lambda \leqq d_i$, d_j. For $g \frown a_j = a_j \frown (a_k e \smile_{ji}) = O$ and $\dfrac{g}{O} \simeq \dfrac{g \smile a_j}{a_j} = \dfrac{l \smile a_j}{a_j} \simeq \dfrac{l}{O}$.

Now we have

$$g_{ij} = (A_{ij} \smile g) \frown (a_i \smile a_j) = (a_k \smile g) \frown (a_i \smile a_j) = (l \smile a_j \smile a_k) \frown e_{ij} = e_{ij}^{(\lambda)},$$

$$g_{ki} = (A_{ki} \smile g) \frown (a_i \smile a_k) = (a_j \smile g) \frown (a_i \smile a_k) = (l \smile a_j) \frown (a_k \smile a_i) = l.$$

These relations yield

$$\bar{P}\binom{ki}{kj} l = (l \smile e_{ij}) \frown (a_k \smile a_j^{(d_i)}) \geqq (g_{ki} \smile g_{ij}) \frown (a_k \smile a_j^{(d_i)})$$

But, since $g \frown A_i = g \frown A_j = (l \smile a_j) \frown a_k = l \frown a_k = O$ by hypothesis, so we obtain $\bar{P}\binom{ki}{kj} l \geqq \bar{g}_{kj}$ by theorem 78. Further we have

$$P\binom{ki}{kj} l = (l \smile \bar{e}_{ij}) \frown (a_k \smile a_j^{(d_i)}) = (g_{ki} \smile \bar{g}_{ij}) \frown (a_k \smile a_j^{(d_i)})$$

$$\leqq (g_{kj} \smile g_{ki}) \frown (a_k \smile a_j^{(d_i)}) = g_{kj},$$

Hence we have $\bar{P}\binom{ki}{kj} l \geqq P\binom{ki}{kj} \bar{l}$ by theorem 69, which proves the first

formula of the theorem. The second one follows from this by observing that $\bar{l} = \bar{\bar{l}}$.

Theorem 80. *If $\lambda_i \geqq \lambda_j \geqq \lambda_k$, and if we put $P(^{ki}_{12})g_{ki} = h_{ki}$, $P(^{ji}_{12})g_{ji} = h_{ji}$, $P(^{kj}_{12})g_{kj} = h_{kj}$, then h_{ki}, h_{ji} and h_{kj} are finite with the relation $h_{ki} = h_{kj}h_{ji}$.*

Pr. We put $P(^{kj}_{13})g_{kj} = l_{kj}$, $P(^{ji}_{32})g_{ji} = l_{ji}$ yielding $P(^{13}_{12})l_{kj} = h_{kj}$, $P(^{32}_{12})l_{ji} = h_{ji}$. By theorem 77 we have $g_{ki} \frown a_i = O$, $g_{ji} \frown a_i = O$, $g_{kj} \frown a_j = O$, whence $l_{kj} \frown a_3 = O$, $l_{ji} \frown a_2 = O$, $h_{ki} \frown a_2 = O$, $h_{kj} \frown a_2 = O$, $h_{ji} \frown a_2 = O$. Putting $\lambda = \dim g$, we have $\dim l_{kj} = \dim l_{ki} = \dim h_{kj} = \dim h_{ki} = \lambda - \lambda_k$, $\dim h_{ji} = \dim l_{ji} = \lambda - \lambda_j$. We put further $a = a_1^{(\lambda-\lambda_k)} \smile a_2$, $b = a_3^{(\lambda-\lambda_j)} \smile a_2$, $a^* = a_1^{(\lambda-\lambda_j)} \smile a_2$. Then $a^* \sim b$ with axis $e_{13}^{(\lambda-\lambda_j)}$. Since $\dim(\bar{l}_{kj} \frown a_1) = \dim(\bar{g}_{kj} \frown a_k) = \lambda_j - \lambda_k$, and $a_1^{(\lambda-\lambda_k)} \smile \bar{l}_{kj} = a_1^{(\lambda-\lambda_k)} \smile a_3^{(\lambda-\lambda_j)} = a_3^{(\lambda-\lambda_j)} \smile \bar{l}_{kj}$, so we have $a \smile \bar{l}_{kj} = b \smile \bar{l}_{kj}$, where $b \frown \bar{l}_{kj} = a_3^{(\lambda-\lambda_j)} \frown \bar{l}_{kj} = O$. Thus we obtain $M_{\bar{h}_{kj}}$ by means of $e_{13}^{(\lambda-\lambda_j)}$, $\bar{l}_{kj}$, b. For $a \frown b = a_2$, $(e_{13}^{(\lambda-\lambda_j)} \smile \bar{l}_{kj}) \frown a \leqq a_1$, and $(e_{12}^{(\lambda-\lambda_j)} \smile e_{13}^{(\lambda-\lambda_j)}) \frown b = \bar{e}_{23}^{(\lambda-\lambda_j)}$, $P(^{13}_{12})\bar{l}_{kj} = \bar{h}_{kj}$. But theorem 79 yields $(\bar{h}_{ji} \smile e_{13}^{(\lambda-\lambda_j)}) \frown b = (\bar{h}_{ji} \smile e_{13}) \smile (a_2 \smile a_3) = \bar{P}(^{12}_{32})\bar{h}_{ji} = P(^{12}_{32})h_{ji} = l_{ji}$. Since moreover from $g_{ji} \smile \bar{g}_{kj} = g_{ji} \smile g_{ki}$ follows $l_{ji} \smile \bar{l}_{kj} = l_{ji} \smile h_{ki}$ by $P(^{kij}_{123})$, so we have $(l_{ji} \smile \bar{l}_{kj}) \frown a = h_{ki}$, that is $h_{ki} = \bar{h}_{kj} \cdot \bar{h}_{ji} = h_{kj} \cdot h_{ji}$. q. e. d.

In the following we use the notation $P(^{ij}_{12})g_{ij} = h_{ij}$, if $\lambda_i \leqq \lambda_j$, and $P(^{ij}_{21})g_{ij} = h_{ji}$, if $\lambda_i \geqq \lambda_j$. If $\lambda_i = \lambda_j$, then h_{ij} is an unit and $h_{ji} = h_{ij}^{-1}$ by theorem 75. Given an arbitrary irreducible element g in L, we can find A_ρ such that $g \frown A_\rho = O$ by the corollary to lemma 2. Then $\lambda = \dim g \leqq \dim a_\rho$ and $h_{\rho i}$, $i = 1, 2, \cdots r$ are all finite points of dimension λ, where $h_{\rho\rho}$ means $e_{12}^{(\lambda)}$. Every system $\{\varepsilon h_{\rho i}\}$, $i = 1, 2, \cdots r$ with arbitrary unit ε is now called a *system of coordinates of the irreducible element g*. If $A_\rho \frown g = O$ and $A_\sigma \frown g = O$ $(\sigma \neq \rho)$, then $h_{\rho\sigma}$ is an unit and we have $h_{\rho i} = h_{\rho\sigma}h_{\sigma i}$, if $i \neq \sigma$, $i \neq \rho$, by the preceding theorem. Furthermore $h_{\rho\rho} = h_{\rho\sigma}h_{\sigma\rho}$, $h_{\rho\sigma} = h_{\rho\sigma}h_{\sigma\sigma}$. Hence two systems of coordinates $\{h_i\}$ and $\{h'_i\}$ of an irreducible element are in a relation $h_i = \varepsilon h'_i$, $i = 1, 2, \cdots r$ with an unit ε. Among the coordinates h_i there exists always at least an unit h_ρ such that $d_\rho \geqq \lambda$ and, since from $g_{\rho i} \smile a_\rho \leqq a_\rho \smile a_i$ the relation $h_{\rho i} \smile a_1 \leqq a_1 \smile a_2^{(d_i)}$ follows, it holds $O(h_i) \geqq \lambda - d_i$. Conversely to every system of finite points h_i, $i = 1, 2, \cdots r$ of dimension λ in $\dfrac{a_1 \smile a_2}{O}$, where h_ρ is an unit with $d_\rho \geqq \lambda$ and $O(h_i) \geqq \lambda - d_i$, there corresponds

an irreducible element in L, such that h_i, $i = 1, 2, \cdots r$, are its coordinates.

In order to prove this assertion, we put $h_\rho^{-1}h_i = h_{\rho i}$, $P(^{12}_{\rho i})h_{\rho i} = g_{\rho i}$, where $h_{\rho i}$ is proper, since $h_{\rho i} \cup a_1^{(d_\rho)} \leqq a_1^{(d_\rho)} \cup a_2^{(d_i)}$, and put

$$g = \prod_{i \neq \rho}(A_{\rho i} \cup g_{\rho i}). \qquad \text{Then} \quad g_{\rho i} \cap A_\rho = g_{\rho i} \cap a_i = O,$$

$$A_\rho \cap g = \prod_{i \neq \rho}\{(A_{\rho i} \cup g_{\rho i}) \cap A_\rho\} = \prod_{i \neq \rho} A_{\rho i} = O,$$

by the corollary to lemma 2, and

$$A_{\rho i} \cup g = (A_{\rho i} \cup g_{\rho i}) \cap \prod_{j \neq i, \rho}(a_j \cup A_{\rho j} \cup g_{\rho i}) = (A_{\rho i} \cup g_{\rho i}) \cap \prod_{j \neq i, \rho}(A_\rho \cup g_{\rho j})$$
$$= (A_{\rho i} \cup g_{\rho i}) \cap (A_\rho \cup a_\rho^{(\lambda)}) = A_{\rho i} \cup g_{\rho i}.$$

Hence it holds $\dfrac{g}{O} \simeq \dfrac{A_{\rho i} \cup g}{A_{\rho i}} = \dfrac{A_{\rho i} \cup g_{\rho i}}{A_{\rho i}} \simeq \dfrac{g_{\rho i}}{O}$. g is therefore an irreducible element of dimension λ with the relation $(A_{\rho i} \cup g) \cap (a_\rho \cup a_i) = g_{\rho i}$, proving that h_i are coordinates of g. We prove further, that for the coordinates $h_i^{(1)}$, $h_i^{(2)}$ of two different elements $g^{(1)}$, $g^{(2)}$ the relations $\varepsilon h_i^{(1)} = h_i^{(2)}$ do not hold with an unit ε. If $\varepsilon h_{\rho i}^{(1)} = h_{\sigma i}^{(2)}$, $i = 1, 2, \cdots r$ with an unit ε, then, $h_{\rho\sigma}^{(1)}$ and $h_{\sigma\rho}^{(2)}$ being units, we have $\lambda_\rho = \lambda_\sigma = O$ for both $g^{(1)}$ and $g^{(2)}$. Hence $h_{\sigma i}^{(2)} = h_{\sigma\rho}^{(2)} h_{\rho i}^{(2)}$ and consequently $h_{\rho i}^{(1)} = h_{\rho i}^{(2)}$, $g_{\rho i}^{(1)} = g_{\rho i}^{(2)}$. From this we obtain $A_{\rho i} \cup g^{(1)} = A_{\rho i} \cup g^{(2)} = A_{\rho i} \cup g_{\rho i}^{(1)}$, whence $g^{(1)} = \prod_{i \neq \rho}(A_{\rho i} \cup g^{(1)}) = g^{(2)}$. q. e. d.

We shall now investigate the relation, which the coordinates of every point on a given straight line satisfy.

Theorem 81. *The coordinates h_i, $i = 1, 2, \cdots r$ of any point P on a stright line $\dfrac{P_1 \cup P_2}{O}$, where P_1, P_2 are independent points with coordinates $h_i^{(1)}$ and $h_i^{(2)}$ respectively, satisfy the relation*

$$h_i = \lambda_1 h_i^{(1)} + \lambda_2 h_i^{(2)}, \quad i = 1, 2, \cdots r$$

with two fixed points λ_1 and λ_2 of dimension dim P, *where either λ_1 or λ_2 is an unit.*

Pr. We can assume that $\dim P_1 = s \geqq \dim P_2 = t$, and find A_{kj} such that $A_{kj} \cap (P_1 \cup P_2) = O$ by lemma 10. For an arbitrary point P in $\dfrac{P_1 \cup P_2}{O}$ it holds either $P \cap A_k = O$ or $P \cap A_j = O$. If we put $P_1 \cup P_2 = l$, $A_k \cap l = w$, then we have

$$\frac{w}{O} \simeq \frac{w \cup A_{kj}}{A_{kj}} = \frac{A_k \frown (l \cup A_{kj})}{A_{kj}} = \frac{(a_j \cup A_{kj}) \frown (l \cup A_{kj})}{A_{kj}}$$
$$= \frac{A_{kj} \cup \{a_j \frown (l \cup A_{kj})\}}{A_{kj}} \simeq \frac{a_j \frown (l \cup A_{hj})}{O}.$$

Hence w is a point and $w \frown A_j = O$, whence $w_{ij} \frown a_i = O$, $\dim w_{ij} = \dim w$ for all $i \neq j$. We first consider the case $P_1 \frown A_k = O$, $P \frown A_k = O$ and put $P(^{ij}_{23})w_{ij} = W_{ij}$, $P(^{kj}_{13})P_{kj} = P^*_{kj}$, $\lambda = \dim P = \dim P_{kj} = \dim P^*_{kj}$, $\nu = \dim (P \frown A_j) = \dim (P_{kj} \frown a_k) = \dim (P^*_{kj} \frown a_1)$. Then $P(^{12}_{12})P^*_{kj} = P(^{kj}_{12})P_{kj} = h_{jk}$, where $h_{kj} \frown a_2 = O$ and, since

$$P_1 \cup w = P_1 \cup \{A_k \frown (P_1 \cup P_2)\} = (P_1 \cup A_k) \frown (P_1 \cup P_2)$$
$$= e^{(s)} \frown (P_1 \cup P_2) = P_1 \cup P_2 = l,$$

it holds $\dim w = \dim P_2 = t = \dim W_{ij}$. We choose an irreducible element W^*_{ij} of dimension λ in $\dfrac{a_2 \cup a_3}{O}$, which contains W_{ij}, if $\lambda > t$, and put $W^*_{ji} = W^{(\lambda)}_{ij}$, if $\lambda \leqq t$. Then, putting $P(^{23}_{21})W^*_{ji} = l^*_{ji}$, $P(^{ij}_{21})W^{(\lambda)}_{ij} = l_{ji}$, we have $l^*_{ji} \geqq l_{ji}$, $l^*_{ji} \frown a_2 = O$.

Next it holds obviously $a^{(\lambda-\nu)}_1 \cup a_2 \sim a^{(\lambda-\nu)}_3 \cup a_2$ with axis $e^{(\lambda-\nu)}_{13}$. Since, in virtue of $P_{kj} \frown a_k = a^{(\nu)}_k$, the relations

$$(a^{(\lambda)}_1 \cup a_2) \cup P^*_{kj} = (a^{(\lambda-\nu)}_3 \cup a_2) \cup P^*_{kj}, \quad P^*_{kj} \frown (a^{(\lambda-\nu)}_3 \cup a_2) = P^*_{kj} \frown a_3 = O$$

hold, so $a^{(\lambda-\nu)}_3 \cup a_2$ is quasi-perspective to $a^{(\lambda)}_1 \cup a_2$ with axis P^*_{kj}. Further $(a^{(\lambda)}_1 \cup a_2) \frown (a^{(\lambda-\nu)}_3 \cup a_2) = a_2$, $(e^{(\lambda-\nu)}_{13} \cup P^*_{kj}) \frown (a^{(\lambda)}_1 \cup a_2) \leqq a_1$

$$(e^{(\lambda-\nu)}_{12} \cup e^{(\lambda-\nu)}_{13}) \frown (a^{(\lambda-\nu)}_3 \cup a_2) = \bar{e}^{(\lambda-\nu)}_{23}, \quad (\bar{e}^{(\lambda-\nu)}_{23} \cup P^*_{kj}) \frown (a^{(\lambda)}_1 \cup a_2) = h_{kj}.$$

Thus we have M_{hkj} by means of $e^{(\lambda-\nu)}_{13}$, P^*_{kj} and $a^{(\lambda-\nu)}_3 \cup a_2$. If we put $M = (W^{*(\lambda-\nu)}_{ji} \cup P^*_{kj}) \frown (a^{(\lambda)}_1 \cup a_2)$, so we have from

$$(\bar{l}^{*(\lambda-\nu)}_{ji} \cup e^{(\lambda-\nu)}_{13}) \frown (a^{(\lambda-\nu)}_3 \cup a_2) = \bar{P}(^{12}_{32})\bar{l}^{*(\lambda-\nu)}_{ji} = W^{*(\lambda-\nu)}_{ji}$$

the relation $M = h_{kj}\bar{l}^*_{ji} = -h_{kj}l^*_{ji}$. It holds furthermore $a^{(\lambda)}_1 \cup a_2 \sim P^*_{kj} \cup a_2$ with axis W^*_{ij}. Since $(a^{(\lambda)}_1 \cup a^{(\lambda)}_3) \frown (a_2 \cup P^*_{kj}) = P^*_{kj}$, $(P^*_{kj} \cup W^*_{ij}) \frown (a^{(\lambda)}_1 \cup a_2) = M$, so we have A_M by means of $a^{(\lambda)}_3$, W^*_{ij} and $a_2 \cup P^*_{kj}$. Now we put $P_{kij} = (A_{kij} \cup P) \frown (a_k \cup a_i \cup a_j)$ and obtain

$$\frac{P_{kij}}{O} \simeq \frac{P_{kij} \cup A_{kij}}{A_{kij}} = \frac{A_{kij} \cup P}{A_{kij}} \simeq \frac{P}{O}.$$

P_{kij} is therefore a point of dimension λ and similarly for

$$\bar{P}_{kij} = (A_{kij} \smile P_1^{(\lambda)}) \frown (a_k \smile a_i \smile a_j)^{(1)}.$$

Then it holds

$$w_{ij}^{(\lambda)} = (A_{ijk} \smile P_1^{(\lambda)} \smile P_2^{(\lambda)}) \frown (a_i \smile a_j)\,.$$

$$\begin{aligned} P_{kij} \smile w_{ij}^{(\lambda)} &= \{A_{kij} \smile P) \frown (a_k \smile a_i \smile a_j)\} \smile \{(A_{kij} \smile P_1^{(\lambda)} \smile P_2^{(\lambda)}) \frown (a_i \smile a_j)\} \\ &= (A_k \smile P) \frown (A_{kij} \smile P_1^{(\lambda)} \smile P_2^{(\lambda)}) \frown (a_k \smile a_i \smile a_j) \\ &= (A_k \smile P_1^{(\lambda)}) \frown (A_{kij} \smile P_1^{(\lambda)} \smile P_2^{(\lambda)}) \frown (a_k \smile a_i \smile a_j) = \bar{P}_{kij} \smile w_{ij}^{(\lambda)}\,. \end{aligned}$$

If we put $P\binom{kij}{123}\bar{P}_{kij} = \bar{Q}_{kij}$ and $P\binom{kij}{123}P_{kij} = Q_{kij}$, then

$$(2) \qquad Q_{kij} \smile W_{ij}^* = \bar{Q}_{kij} \smile W_{ij}^*\,.$$

We have further $P_{kij} \smile a_j^{(\lambda)} = P_{ki} \smile a_j^{(\lambda)}$ from $(P_{kij} \smile a_j) \frown (a_k \smile a_i) = P_{ki}$, whence follows $Q_{kij} \smile a_3^{(\lambda)} = h_{ki} \smile a_3^{(\lambda)}$, and similarly $Q_{kij} \smile a_2 = P_{kj}^* \smile a_2$. Hence we have

$$(h_{ki} \smile a_3^{(\lambda)}) \frown (P_{kj}^* \smile a_2) = (Q_{kij} \smile a_3^{(\lambda)}) \frown (Q_{kij} \smile a_2) = Q_{kij}\,,$$

where $Q_{kij} \frown (a_2 \smile a_3) = O$ follows from $P_{kij} \frown (a_i \smile a_j) = O$, and putting $(Q_{kij} \smile W_{ij}^*) \frown (a_1^{(\lambda)} \smile a_2) = U$, it holds

$$(3) \qquad U = M + h_{ki} = -h_{kj} l_{ji}^* + h_{ki}\,.$$

In virtue of (2) we have similarly $U = -h_{kj}^{(1)(\lambda)} l_j^* + h_{ki}^{(1)(\lambda)}$. This, together with (3), yields $h_{ki} - h_{ki}^{(1)(\lambda)} = (h_{kj} - h_{kj}^{(1)(\lambda)}) l_{ji}^*$. If $\lambda > t$, then $\dim (P \frown P_1) \geqq \lambda - t$, $O(h_{kj} - h_{kj}^{(1)(\lambda)}) \geqq \lambda - t$, and hence

$$(4) \qquad h_{ki} - h_{ki}^{(1)(\lambda)} = (h_{kj} - h_{kj}^{(1)(\lambda)}) l_{ji}\,.$$

If $\lambda \leqq t$, then $l_{ji}^* = l_{ij}$ and therefore (4) holds also in this case.

We next consider the case $P_1 \frown A_k > O$ and $P \frown A_k = O$. Then it holds $P_1 \frown w > O$ and therefore $P_2 \frown A_k = O$, $P_1 \frown P = O$, whence $\dim P \leqq t = \dim P_2$. Further we have

$$\frac{w}{O} \simeq \frac{P_2 \smile w}{P_2} = \frac{(P_2 \smile A_k) \frown (P_1 \smile P_2)}{P_2} \simeq \frac{P_1 \frown (P_2 \smile A_k)}{O} \geqq \frac{P_1 \frown e^{(t)}}{O} = \frac{P_1^{(t)}}{O}\,,$$

which yields $\dim w \geqq t$. Since the point P lies on the straight line $\dfrac{P_1^{(t)} \smile P_2}{O}$, we can apply the above result for the present case, if we only relace the point P_1 with P_2. We have henceforth

(1) Here $\bar{P}_{kij}$ does not signify the conjugate point of P_{kij}.

(5) $$h_{ki}-h_{ki}^{(2)(\lambda)} = (h_{kj}-h_{kj}^{(2)(\lambda)})l_{ji}\,.$$

In similar manner we treat the case $P\frown A_j = O$, $P_1\frown A_j = O$ and the case $P\frown A_j = O$, $P_1\frown A_j > O$, and we have

(6) $$h_{ji}-h_{ji}^{(1)(\lambda)} = (h_{jk}-h_{jk}^{(1)(\lambda)})m_{ki}\,, \quad \text{or}$$

(7) $$h_{ji}-h_{ji}^{(2)(\lambda)} = (h_{jk}-h_{jk}^{(2)(\lambda)})m_{ki}\,,$$

where $A_j\frown l = u$, $P\binom{ik}{21}u_{ik} = m_{ki}$.

Now, assuming $P_1\frown A_k = O$ and $P_2\frown A_k = O$, it follows from (4)

(8) $$h_{ki}^{(2)}-h_{ki}^{(1)(t)} = (h_{kj}^{(2)}-h_{kj}^{(1)(t)})l_{ji}\,,$$

where $h_{kj}^{(2)}-h_{kj}^{(1)(t)}$ is an unit. For

$$\begin{aligned} g_{kj}^{(2)}\frown g_{kj}^{(1)} &= (A_{kj}\smile P_1)\frown(A_{kj}\smile P_2)\frown(a_k\smile a_j) \\ &= [A_{kj}\smile\{P_1\frown(A_{kj}\smile P_2)\}\frown(a_k\smile a_j) = A_{kj}\frown(a_k\smile a_j) = O. \end{aligned}$$

Putting $h_{kj}^{(2)}-h_{kj}^{(1)(t)} = \varepsilon^{-1}$, we have from (4) and (8)

$$h_{ki}-h_{ki}^{(1)(\lambda)} = (h_{kj}-h_{kj}^{(1)(\lambda)})\varepsilon(h_{ki}^{(2)}-h_{ki}^{(1)(t)})$$

for the coordinates of the point P of dimension λ with $P\frown A_k = O$. If we put further $(h_{kj}-h_{kj}^{(1)(\lambda)})\varepsilon = \Theta$, then Θ is a finite point of dimension λ and $h_{ki} = \Theta h_{ki}^{(2)}+(e^{(\lambda)}-\Theta)h_{ki}^{(1)}$. Furthermore it is evident from theorem 62 that either Θ or $e^{(\lambda)}-\Theta$ is an unit.

If $P\frown A_k > O$, then $P\frown A_j = O$, $w\frown A_j = O$ and $l = P_1\smile w$, $\lambda = \dim P \leqq t = \dim w$. Hence we have from (7)

(9) $$h_{ji}-l_{ji}^{(\lambda)} = h_{jk}m_{ki}\,.$$

Since further $u\frown A_k = O$, $P_1\frown A_k = O$, we have $m_{ki}-h_{ki}^{(1)(\mu)} = -h_{kj}^{(1)(\mu)}l_{ji}$ with $\mu = \dim u$ and, substituting this in (9), we obtain

(10) $$\begin{aligned} h_{ji} &= h_{jk}h_{ki}^{(1)}+(e_{12}^{(\lambda)}-h_{jk}h_{kj}^{(1)})l_{ji} \\ &= h_{jk}h_{ki}^{(1)}+(e_{12}^{(\lambda)}-h_{jk}h_{kj}^{(1)})\varepsilon(h_{ki}^{(2)}-h_{ki}^{(1)(t)}).^{(1)} \end{aligned}$$

If we put further, $(e_{12}^{(\lambda)}-h_{jk}h_{kj}^{(1)})\varepsilon = \Theta$, then $h_{ji} = \Theta h_{ki}^{(2)}+(h_{jk}-\Theta)h_{ki}^{(1)}$, where in case $O(h_{jk}-\Theta) > O$ it follows from $(h_{jk}-\Theta)\varepsilon^{-1} = h_{jk}h_{jk}^{(2)}-e_{12}^{(\lambda)}$ that $O(h_{jk}) = O$, whence $O(\Theta) = O$.

We can treat the case $P_1\frown A_j = O$, $P_2\frown A_j = O$ similarly as above.

(1) That $\mu \geqq \lambda$, follows from $P\smile u \geqq (P\smile A_j)\frown(P_1\smile P) = P\smile P_1^{(\lambda)}$.

For the case $P_1 \frown A_k = O$, $P_2 \frown A_k > O$, we have $l = P_1 \smile w$ and from (10) we obtain

$$h_{ji}^{(2)} = h_{jk}^{(2)} h_{ki}^{(1)} + (e_{12}^{(t)} - h_{jk}^{(2)} h_{kj}^{(1)}) l_{ji} . \tag{11}$$

Since $O(h_{jk}^{(2)}) > O$, $e_{12}^{(t)} - h_{jk}^{(2)} h_{kj}^{(1)}$ is an unit and we have from (4) and (11)

$$h_{ki} - h_{ki}^{(1)(\lambda)} = (h_{kj} - h_{kj}^{(1)(\lambda)}) \varepsilon (h_{ji}^{(2)} - h_{jk}^{(2)} h_{ki}^{(1)}) ;$$

where $e_{12}^{(t)} - h_{jk}^{(2)} h_{kj}^{(1)} = \varepsilon^{-1}$ and $P \frown A_k = O$. It follows then

$$h_{ki} = \Theta h_{ji}^{(2)} + (e_{12}^{(\lambda)} - \Theta h_{jk}^{(2)}) h_{ki}^{(1)} .$$

with $\Theta = (h_{kj} - h_{kj}^{(1)}) \varepsilon$. That either Θ or $e_{12}^{(\lambda)} - \Theta h_{jk}^{(2)}$ is an unit, can be easily seen. If $P \frown A_k > O$, then $P \frown A_j = O$ and $\dim P \leqq t$. Consequently $P \leqq P_1^{(t)} \smile P_2 = l'$. Hence we have to consider the new straight line l', where $P_1^{(t)} \frown A_j > O$. $P_2 \frown A_j = O$, $P \frown A_j = O$, and, interchanging k with j and P_2 with $P_1^{(t)}$ in the above result, we obtain

$$h_{ji} = \Theta h_{ki}^{(1)} + (e_{12}^{(\lambda)} - \Theta h_{kj}^{(1)}) h_{ji}^{(2)} ,$$

where $\Theta = (h_{jk} - h_{jk}^{(2)(\lambda)}) \varepsilon$, $\varepsilon^{-1} = e_{12}^{(t)} - h_{kj}^{(1)(t)} h_{jk}^{(2)}$.

Finally for the case $P_1 \frown A_k > O$, $P_2 \frown A_k = O$, we can treat as follows. If $P_2 \frown A_j = O$, then it reduces to the second case and, if $P_2 \frown A_j = O$, then we can treat in the same way as the third case.

Theorem 82. (*Converse of theorem 81*) *If $h_i^{(1)}$, $h_i^{(2)}$ represent the coordinates of two independent points P_1, P_2 respectively, and if λ_1, λ_2 are two finite points of the same dimension s, where at least one is an unit and* $\dim \lambda_1 h_i^{(1)} = \dim \lambda_2 h_i^{(2)} = s$, *then $h_i = \lambda_1 h_i^{(1)} + \lambda_2 h_i^{(2)}$ are coordinates of a point of dimension s on the straight line* $\dfrac{P_1 \smile P_2}{O}$.

Pr. We can assume here that λ_1 is an unit and $t = \dim P_2 \leqq \dim P_1 = s = \dim \lambda_i$. Supposing $(P_1 \smile P_2) \frown A_{jk} = O$, we first consider the case $P_1 \frown A_k = O$, $P_2 \frown A_k = O$. Then we can assume further that $h_i^{(1)} = h_{ki}^{(1)}$ and $h_i^{(2)} = h_{ki}^{(2)}$. Since $h_{kj}^{(1)} \frown h_{kj}^{(2)} = O$, either $\lambda_1 + \lambda_2$ or $\lambda_1 h_{kj}^{(1)} + \lambda_2 h_{kj}^{(2)}$ is an unit. If $\lambda_1 + \lambda_2$ is an unit, we put

$$(\lambda_1 + \lambda_2) h_{kj} = \lambda_1 h_{kj}^{(1)} + \lambda_2 h_{kj}^{(2)} , \quad P(^{12}_{kj}) h_{kj} = P_{kj} ,$$

$$P(^{12}_{kj}) h_{kj}^{(1)} = P_{kj}^{(1)} , \quad (A_{jk} \smile P_{kj}) \frown l = P .$$

We shall now show that P is a point of dimension s on l. Since

$O(\lambda_2) \geqq s-\dim P_2$, we obtain $O(h_{kj}-h_{kj}^{(1)}) \geqq s-\dim P_2$, whence $\dim (P_{kj} \frown P_{kj}^{(1)}) \geqq s-\dim P_2$ follows. Hence we have

$$A_{jk} \smile P_{kj} \leqq A_{jk} \smile P_{kj}^{(1)} \smile P_{kj}^{(2)} = A_{jk} \smile P_1 \smile P_2$$

and then
$$\frac{P}{O} \simeq \frac{A_{jk} \smile P}{A_{jk}} = \frac{A_{jk} \smile P_{kj}}{A_{jk}} \simeq \frac{P_{kj}}{O}$$

and $(P \smile A_{jk}) \frown (a_j \smile a_k) = P_{kj}$. Since the point P lies on $l = P_1 \smile P_2$ and $P_1 \frown A_k = O$, $P_2 \frown A_k = O$, $P \frown A_k = O$, we have for the coordinates of P $h_{ki} = \Theta_{ki}^{(2)} + (e_{12}^{(s)} - \Theta) h_{ki}^{(1)}$, where $\Theta = (h_{kj} - h_{kj}^{(1)})\varepsilon$ and $\varepsilon^{-1} = h_{kj}^{(2)} - h_{kj}^{(1)(t)}$. But $(\lambda_1 + \lambda_2)\Theta = \lambda_2 (h_{kj}^{(2)} - h_{kj}^{(1)(t)})\varepsilon = \lambda_2$ and $(\lambda_1 + \lambda_2)(e_{12}^{(s)} - \Theta) = \lambda_1$. Hence $(\lambda_1 + \lambda_2) h_{ki} = h_i$ are coordinates of P. If $\lambda_1 + \lambda_2$ is not an unit and if $\lambda_1 h_{kj}^{(1)} + \lambda_2 h_{kj}^{(2)}$ is an unit, then both λ_1 and λ_2 are units and $t = s$. Putting $(A_{jk} \smile P_{jk}) \frown l = P$, $\lambda_1 + \lambda_2 = (\lambda_1 h_{kj}^{(1)} + \lambda_2 h_{kj}^{(2)}) h_{jk}$ and $P_{jk}^{(12)} h_{jk} = P_{jk}$, we have

$$\frac{P}{O} \simeq \frac{A_{jk} \smile P_{jk}}{A_{jk}} \simeq \frac{P_{jk}}{O}.$$

Hence P is a point of dimension s with $P \frown A_j = O$ and $(P \smile A_{jk}) \frown (a_j \smile a_k) = P_{jk}$. Its coordinates are now $h_{ji} = \Theta h_{ki}^{(2)} + (h_{jk} - \Theta) h_{ki}^{(1)}$ with $(e_{12}^{(s)} - h_{jk} h_{kj}^{(1)})\varepsilon = \Theta$, $\varepsilon^{-1} = h_{kj}^{(2)} - h_{kj}^{(1)}$. But $(\lambda_1 h_{kj}^{(1)} + \lambda_2 h_{kj}^{(2)}) \Theta = \lambda_2$ and $(\lambda_1 h_{kj}^{(1)} + \lambda_2 h_{kj}^{(2)})(h_{jk} - \Theta) = \lambda_1$ yield $(\lambda_1 h_{kj}^{(1)} + \lambda_2 h_{kj}^{(2)}) h_{ji} = h_i$, proving that these are coordinates of the point P.

For the case $P_1 \frown A_j = O$ and $P_2 \frown A_j = O$ we can treat analogously as above. In case $P_1 \frown A_k = O$, $P_2 \frown A_k > O$, we can assume that $h_i^{(1)} = h_{ki}^{(1)}$ and $h_i^{(2)} = h_{ji}^{(2)}$. Since $O(h_{jk}^{(2)}) > O$, $\lambda_1 + \lambda_2 h_{jk}^{(2)}$ must be an unit. Putting $(\lambda_1 + \lambda_2 h_{jk}^{(2)}) h_{kj} = \lambda_1 h_{kj}^{(1)} + \lambda_2$, $P_{kj}^{(12)} h_{kj} = P_{kj}$ and $(A_{jk} \smile P_{kj}) \frown l = P$, we see that P is a point of dimension s with $P \frown A_k = O$, $(P \smile A_{jk}) \frown (a_j \smile a_k) = P_{jk}$ and its coordinates are $h_{ki} = \Theta h_{ji}^{(2)} + (e_{12}^{(s)} - \Theta h_{jk}^{(2)}) h_{ki}^{(1)}$ with $\Theta = (h_{kj} - h_{kj}^{(1)})\varepsilon$, $e_{12}^{(t)} - h_{jk}^{(2)} h_{kj}^{(1)} = \varepsilon^{-1}$. Since $(\lambda_1 + \lambda_2 h_{jk}^{(2)})\Theta = \lambda_2$ and $(\lambda_1 + \lambda_2 h_{jk}^{(2)})(e_{12}^{(s)} - \Theta h_{jk}^{(2)}) = \lambda_1$, so we have $(\lambda_1 + \lambda_2 h_{jk}^{(2)}) h_{ki} = h_i$. Finally, if $P_1 \frown A_k > O$, $P_2 \frown A_k = O$, we have either the case $P_1 \frown A_j = O$, $P_2 \frown A_j = O$ or the case $P_1 \frown A_j = O$, $P_2 \frown A_j > O$. The latter case can be treated in the same way as above, if we interchange j and k.

Theorem 83. *Let $a^{(1)}, a^{(2)}, \cdots a^{(\nu)}$ be a basis of an element a in L and P be an irreducible element (point) of dimendion λ in $\frac{a}{O}$ with coordinates h_i, then there exists finite points λ_k, $k = 1, 2, \cdots \nu$*

of dimension λ *in* $\frac{a_1 \cup a_2}{O}$, *such that* $h_i = \sum_{k=1}^{\nu} \lambda_k h_i^{(k)}$, $O(\lambda_k) \geqq \lambda - \dim a^{(k)}$ *and at least one of* λ_k *is an unit, where* $h_i^{(k)}$, $i = 1, 2, \cdots r$ *are coordinates of* $a^{(k)}$. *Conversely, if* λ_k, $k = 1, 2, \cdots \nu$ *are all finite points of dimension* λ *with* $O(\lambda_k) \geqq \lambda - \dim (a^{(k)})$, *of which at least one is an unit, then* $h_i = \sum_{k=1}^{\nu} \lambda_k h_i^{(k)}$ *are coordinates of a point of dimension* λ *in* $\frac{a}{O}$.

Pr. By induction on ν. The case $\nu = 2$ is evident from theorems 81 and 82. Suppose, $\nu > 2$ and P does not belong to $\frac{a^{(\nu)}}{O}$. Putting $A^{(\nu)} = a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(\nu-1)}$, $(P \cup a^{(\nu)}) \cap A^{(\nu)} = d$, we have

$$\frac{d}{O} \simeq \frac{d \cup a^{(\nu)}}{a^{(\nu)}} = \frac{P \cup a^{(\nu)}}{a^{(\nu)}} \simeq \frac{P}{P \cap a^{(\nu)}}.$$

Hence $\frac{d}{O}$ is a chain with $O < \dim d \leqq \lambda$ and consequently the point P lies on the straight line $\frac{a^{(\nu)} \cup d}{O}$. By theorem 81 we have $h_i = \lambda_\nu h_i^{(\nu)} + \mu d_i$, where λ_ν and μ are finite points of dimension λ, of which at least one is an unit, and where d_i are coordinates of the point d. But by induction-hypothesis we have $d_i = \sum_{k=1}^{\nu-1} \lambda'_\lambda h_i^{(k)}$, where λ'_k are all finite points with $\dim \lambda'_k = \dim d$ and at least one of λ'_k is an unit. Hence it follows $h_i = \sum_{k=1}^{\nu-1} \mu \lambda'_k h_i^{(k)} + \lambda_\nu h_i^{(\nu)}$, where either λ_ν or at least one of $\mu \lambda'_k$ is an unit.

The converse statement of the theorem can be proved also by induction on ν. If λ_ν is not an unit, then at least one of $\lambda_1, \lambda_2, \cdots \lambda_{\nu-1}$ is an unit and $h'_i = \sum_{k=1}^{\nu-1} \lambda_k h_i^{(k)}$ will be coordinates of a point d with dimension λ in $\frac{A^{(\nu)}}{O}$. Hence $h_i = h'_i + \lambda_\nu h_i^{(\nu)}$ are coordinates of a point with dimension λ in $\frac{d \cup a^{(\nu)}}{O} \leqq \frac{a}{O}$. In case, λ_ν is an unit, let $O(\lambda_P) = v$ be the least one among $O(\lambda_i)$, $i = 1, 2, \cdots \nu - 1$ and put $\lambda_k = \lambda_P \mu_k$, $k = 1, 2, \cdots \nu - 1$ with $\dim \mu_k = \lambda - v$. Then $h''_i = \sum_{k=1}^{\nu-1} \mu_k h_i^{(k)}$ are coordinates of a point c with $\dim c = \lambda - v$ in $\frac{A^{(\nu)}}{O}$, since $\mu_P = e_{12}^{(\lambda - v)}$ is an unit. Therefore $h_i = \lambda_P h''_i + \lambda_\nu h_i^{(\nu)}$ are coordinates of a point with dimension λ in $\frac{c \cup a^{(\nu)}}{O} \leqq \frac{a}{O}$.

PART 7.

Let the height of the given primary lattice L be h and a_i, $i = 1, 2, \cdots r$ be its basis with $r \geqq 4$, $d_1 = d_2 = d_3 = d_4 = h$. Then all finite points of dimension h in the straight line $\frac{a_1 \cup a_2}{O}$ generates a ring R with respect to the addition and the multiplication defined in part 4. The set of all points P in R with $O(P) \geqq i$ is now an ideal J_i in R by theorems 58 and 62. That conversely every left or right ideal in R is such an ideal, can be seen from theorems 60 and 61. Indeed, if u is a point in an ideal J with the least order $O(u) = i$, then we can find by theorems 60 and 61 points q_1 and q_2 in R for any point t with $O(t) \geqq i$, such that $t = q_1 u$ or $t = u q_2$, whence $J_i \leqq J$ and hence $J = J_i$ follows. Now we see that the ideal J_1 is a prime ideal without divisors and consequently the residue class ring R/J_1 is a field. (generally non-commutative). If we select a point τ from each class of R/J_1 as its representant, then we can represent uniquely any finite point t in R as $t = \tau_0 + \tau_1\pi + \tau_2\pi^2 + \cdots + \tau_{h-1}\pi^{h-1}$ with a fixed point π of order I.

Now, r being the rank of the lattice L, we consider a R-module M of rank r, i.e. a module with the operator ring R and with a basis $x_1, x_2, \cdots x_r$. To each irreducible element (point) P of dimension λ in L we make correspond a submodule M_P of rank 1, such that $M_P = (\sum_{i=1}^{r} \pi^{h-\lambda} h_i x_i)$, where h_i are coordinates of P and π is a fixed point in R with $O(\pi) = 1$. Given any submodule $(\sum b_i x_i)$ of rank 1 in M, such that $O(b_i) \geqq h - \dim a_i$, there exists always its corresponding point P in L. In order to prove this, let $\operatorname{Min} O(b_i) = O(b_\nu) = t$, then $\pi^{-t} b_i = h_i$ are coordinates of a point of dimension $h-t$ in L, since $O(h_i) \geqq h - t - d_i$ and h_ν is an unit. That this correspondence is one-to-one can be proved as follows. Let h_i and h_i' be two coordinates of a point of dimension λ with $h_i' = \varepsilon h_i$, ε being an unit. Then $\pi^{h-\lambda} h_i' = (\pi^{h-\lambda} \varepsilon \pi^{-(h-\lambda)}) \pi^{h-\lambda} \varepsilon_i$ and $\pi^{h-\lambda} \varepsilon \pi^{-(h-\lambda)}$ is an unit in R. Hence the submodule $(\sum \pi^{h-\lambda} h_i' x_i)$ is identical with the submodule $(\sum \pi^{h-\lambda} h_i' x_i)$. Conversely if $(\sum b_i x_i)$ and $(\sum b_i' x)$ are the same submodule, then $b_i' = \varepsilon b_i$ with an unit ε and $\operatorname{Min} O(b_i') = \operatorname{Min} O(b_i) = t$, $\pi^{-t} b_i' = (\pi^{-t} \varepsilon \pi^t) \pi^{-t} b_i$, where $\pi^{-t} \varepsilon \pi^t$ is evidently an unit.

Theorem 84. *Let M_1, M_2 be the corresponding submodule for*

the point P_1, P_2 respectively. From $P_1 \leqq P_2$ follows $M_1 \leqq M_2$ and vice versa.

Pr. Suppose $P_2 \frown A_k = O$ and $h_{ki}^{(1)}$, $h_{ki}^{(2)}$, $i = 1, 2, \cdots r$ are coordinates of P_1, P_2 respectively, where $\dim P_1 = t_1$, $\dim P_2 = t_2$. We have $M_1 = \sum b_i^{(1)} x_i$, $M_2 = \sum b_i^{(2)} x_i$ with $b_i^{(1)} = \pi^{h-t_1} h_{ki}^{(1)}$, $b_i^{(2)} = \pi^{h-t_2} h_{ki}^{(2)}$. From $P_1 \leqq P_2$ follows $t_1 \leqq t_2$ and $h_{ki}^{(2)(t_1)} = h_{ki}^{(1)}$, whence $b_i^{(1)} = \pi^{t_2-t_1} b_i^{(2)}$ and consequently $M_1 \leqq M_2$. Conversely, if $M_1 \leqq M_2$, then we have $b_i^{(1)} = \varepsilon \pi^{\mu} b_i^{(2)}$, where ε is an unit. Hence $\pi^{(h-t_1)} h_i^{(1)} = \varepsilon \pi^{\mu+(h-t_2)} h_i^{(2)}$ and consequently $t_2 - t_1 = \mu$, $h_i^{(1)} = \sigma h_i^{(2)(t_1)}$, σ being an unit, which yields $P_1 \leqq P_2$.

Lemma 13. *$h_i^{(\nu)}$, $i = 1, 2, \cdots r$ being coordinates of independent points P_ν, $\nu = 1, 2, \cdots s$, it holds*

$$\operatorname{Min}\{O(\sum_{j=1}^{s} \lambda_j h_1^{(j)}),\ O(\sum_{j=1}^{s} \lambda_j h_2^{(j)}),\ \cdots\} = \operatorname{Min}(O(\lambda_1),\ O(\lambda_2),\ \cdots).$$

Pr. By induction on s. First we treat the case $s = 2$, where we can suppose that $O(\lambda_2) \geqq O(\lambda_1)$ and $h_k^{(1)}$ is an unit. If $O(\lambda_2 h_k^{(2)}) > O(\lambda_1)$, then we have $O(\lambda_1 h_k^{(1)} + \lambda_2 h_k^{(2)}) = O(\lambda_1)$ and $O(\lambda_1 h_i^{(1)} + \lambda_2 h_i^{(2)}) \geqq O(\lambda_1)$; $i = 1, 2, \cdots r$. If $O(\lambda_2 h_k^{(2)}) = O(\lambda_1)$, then $O(\lambda_2) = O(\lambda_1)$ and $h_k^{(2)}$ is an unit. Hence we can assume $h_k^{(1)} \leqq e_{12}$, $h_k^{(2)} \leqq e_{12}$. If $O(\lambda_1 h_j^{(1)} + \lambda_2 h_i^{(2)}) > O(\lambda_1)$ for every j, it would follow $O(h_j^{(2)} - h_j^{(1)}) > O$ for every j, which yields however the dependence of P_1 and P_2, contrary to the hypothesis. If $s > 2$, we can put by induction-hypothesis

$$\operatorname{Min}\{O(\sum_{j=1}^{s-1} \lambda_j h_1^{(j)}),\ \cdots\} = \operatorname{Min}(O(\lambda_1),\ \cdots,\ O(\lambda_{s-1})) = \nu.$$

We conclude therefore that $h_i' = \pi^{-\nu} \lambda_1 h_i^{(1)} + \cdots + \pi^{-\nu} \lambda_{s-1} h_i^{(s-1)}$ are coordinates of a point P' of dimension $\lambda - \nu$, which is independent with P_s. Then it follows $\sum_{i=1}^{s} \lambda_j h_i^{(j)} = \pi^{\nu} h_i' + \lambda_s h_i^{(s)}$ and

$$\begin{aligned}\operatorname{Min}\{&O(\sum_{j=1}^{s} \lambda_j h_1^{(j)}) \cdots O(\sum_{j=1}^{s} \lambda_j h_r^{(i)})\} \\ &= \operatorname{Min}(O(\pi^{\nu}),\ O(\lambda_s)) = \operatorname{Min}(O(\lambda_1),\ O(\lambda_2),\ \cdots O(\lambda_s)).\end{aligned}$$

Theorem 85. *If $a^{(i)}$, $i = 1, 2, \cdots s$ are independent points in L, then the corresponding submodules $M_i = (\sum_{j=1}^{r} A_j^{(i)} x_j)$, $i = 1, 2, \cdots s$ are independent. Moreover to every point in the quotient $\dfrac{a^{(1)} \smile a^{(2)} \smile \cdots a^{(\lambda)}}{O}$ corresponds uniquely a submodule in the module M of rank s, which*

is generated by $\sum_{j=1}^{r} A_j^{(i)} x_j$, $i = 1, 2, \cdots s$. *Conversely to every submodule of rank* 1 *in* M *corresponds a point in* $\dfrac{a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(\lambda)}}{O}$.

Pr. First we consider the case $s = 2$. Since any submodule, which is contained both in M_1 and M_2, corresponds to a point in $\dfrac{a^{(1)} \cap a^{(2)}}{O}$ by theorem 84, so M_1 and M_2 are indepedent. The coordinates of a point of dimension λ in $\dfrac{a^{(1)} \cup a^{(2)}}{O}$ are $\lambda_1 a_j^{(1)} + \lambda_2 a_j^{(2)}$, where $a_j^{(i)}$, $i = 1, 2$ are coordinates of the point $a^{(i)}$ respectively. Then it corresponds the submodule generated by

$$\sum_{j=1}^{r} \pi^{h-\lambda}(\lambda_1 a_j^{(1)} + \lambda_2 a_j^{(2)}) x_j = \pi^{h-\lambda} \lambda_1 \pi^{-(h-n_1)} \sum_{j=1}^{r} A_j^{(1)} x_j + \pi^{h-\lambda} \lambda_2 \pi^{-(h-n_2)} \sum_{j=1}^{r} A_j^{(2)} x_j ,$$

where $n_i = \dim a^{(i)}$, $i = 1, 2$, $\lambda \leqq \mathrm{Max}\,(n_1, n_2)$ end $O(\pi^{h-\lambda}\lambda_1) \geqq h - n_1$, $O(\pi^{h-\lambda}\lambda_2) \geqq h - n_2$. Hence it is contained in the submodule, generated by M_1 and M_2. Conversely, if a submodule $\{\sum_{j=1}^{r} (r_1 A_j^{(1)} + r_2 A_j^{(2)}) x_j\}$ of rank 1 is given, we have by lemma 13

$$\mathrm{Min}\,\{O(r_1 A_1^{(1)} + r_2 A_1^{(2)}), \cdots\} = \mathrm{Min}\,\{O(r_1 \pi^{h-n_1}), O(r_2 \pi^{h-n_2})\} .$$

Denoting this with ν, it corresponds the point, whose coordinates are

$$\pi^{-\nu}(r_1 A_j^{(1)} + r_2 A_i^{(2)}) = \pi^{-\nu} r_1 \pi_1^{h-n_1} a_j^{(1)} + \pi^{-\nu} r_2 \pi^{h-n_2} a_j^{(2)} ,$$

where $\pi^{-\tau} r_1 \pi^{h-n_1}$, $\pi^{-\nu} r_2 \pi^{h-n_2}$ are both of dimension $h - \nu$ and at least one of them is an unit. The point is therefore of dimension $h - \nu$ in $\dfrac{a^{(1)} \cup a^{(2)}}{O}$.

The general case will be proved by inductien on s. Any submodule of rank 1 in the submodule generated by $\sum_{j=1}^{r} A_j^{(1)} x_j, \cdots \sum_{j=1}^{r} A_j^{(\mu-1)} x_j$ shall correspond a point in $\dfrac{a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(\mu-1)}}{O}$ by hypothesis, if $\mu \leqq s$. $\sum A_j^{(1)} x_j, \cdots, \sum A_j^{(s)} x_j$ are then independent in virtue of the relations $(a^{(1)} \cup \cdots \cup a^{(\mu-1)}) \cap a^{(\mu)} = O$, $\mu = 2, \cdots s$. The coordinates of a point of dimension λ in $\dfrac{a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(s)}}{O}$ are $h_i = \sum_{j=1} \lambda_j a_i^{(j)}$ and its corresponding submodule is generated by

$$\sum_{i=1}^{r} \pi^{h-\lambda} h_i x_i = \sum_{i=1}^{r} \pi^{h-\lambda} \sum_{j=1}^{s} \lambda_j a_i^{(j)} x_i = \sum_{j=1}^{s} \pi^{h-\lambda} \lambda_j \pi^{-(h-n_1)} \sum_{i=1}^{r} A_i^{(j)} x_i ,$$

where $n_i = \dim a^{(i)}$, $O(\lambda_j) \geqq \lambda - n_j$. Conversely, if a submodule $(\sum_{i=1}^{r}\sum_{j=1}^{s} r_j A_i^{(j)} x_i)$ of rank 1 is given with $\mathrm{Min}(\sum_{j=1}^{s} r_j A_i^{(j)}) = \nu$, then $\nu = \mathrm{Min}\,(r_j \pi^{h-n_j})$ and the coordinates of the corresponding point are $\sum_{j=1}^{s} \pi^{-\nu} r_j \pi^{h-n_j} a_i^{(j)}$, where $\pi^{-\nu} r_j \pi^{h-n_j}$ are of dimension $h-\nu$ in R and at least one of them is an unit. Hence the point belongs to $\dfrac{a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(s)}}{O}$. q. e. d.

Now we make correspond to an element with a basis $c^{(1)}$, $c^{(2)}$, $\cdots$, $c^{(\lambda)}$ in L, the submodule in M, which is generated by $\sum_{j=1}^{r} C_j^{(1)} x_j$, $\cdots \sum_{j=1}^{r} C_j^{(\lambda)} x_j$. That this correspondenoe is unique, can be seen from the preceding theorem. We see furthermore, that, if $c \leqq b$ in L, then $M_c \leqq M_b$ and vice versa, where by M_b, M_c are meant the corresponding submodules of b, c respectively. Therefore we have

Theorem 86. *A primary lattice of rank $r \geqq 4$ with a basis $a_1, a_2, \cdots a_r$, where $\dim a_1 = \dim a_2 = \dim a_3 = \dim a_4 = h$, h being the height of L, is isomorphic with the lattice of all submodules in a submodule M of a R-module M' with basis $x_1, x_2, \cdots x_r$, where, R being an uniserial complete primary ring, M has a basis $\pi^{\lambda_i} x_i$, $i = 1, 2, \cdots r$ with π in R and $\lambda_i = h - \dim a_i$, $O(\pi) = 1$.*

Theorem 87. *If K is an uniserial complete primary ring, then the lattice of all submodules in any K-module M is primary.*

Pr. Since K is complete primary and uniserial, K has only such ideals $J_i = (\pi^i)$, $i = 1, 2, \cdots l$, where $J_l = (O) < J_{l-1} < \cdots < J_1 < K$ holds. Every submodule A of rank 1 in M is generated by an element $\sum a_i x_i$, where $x_1, x_2, \cdots$ are basis of M and $a_i \in K$. If $\eta \in J_1$, then the submodule $A_\eta = (\sum \eta a_i x_i)$ is properly contained in A, because the ideal (ηa_i) is different from the ideal (a_i) and hence $\varepsilon \eta a_i = a_i$ with $\varepsilon \in K$ does not hold. We have only to show by theorem 45, that, a submodule N being contained in a submodule N^*, two atoms in the quotient $\dfrac{N^*}{N}$ are always perspective. We represent these two atoms as $A \cup N$ and $B \cup N$, where $A = (\sum a_i x_i)$, $B = (\sum b_i x_i)$. We shall now show that $A \cup N$ and $B \cup N$ are perspective with axis $C \cup N$ in $\dfrac{N^*}{N}$, where C is the submodule $\{\sum (a_i + b_i) x_i\}$. Since it holds evidently $A \cup C \cup N = B \cup C \cup N$, it

remains to prove that $(A \cup N) \cap (C \cup N) = N$, $(B \cup N) \cap (C \cup N) = N$. Since $A \cup N$ is an atom in $\frac{N^*}{N}$, A would belong to $C \cup N$, if $(A \cup N) \cap (C \cup N) = N$ does not hold. But then it would follow that a submodule $\{\sum(a_i - \eta a_i - \eta b_i)x_i\}$ with $\eta \in K$ is contained in N. Since B can not be contained in $A \cup N$, so η belongs to the ideal J_1. Then both A_η and B_η are contained in N and consequently also A in N, contrary to the assumption. Similarly we can prove $(B \cup N) \cap (C \cup N) = N$.

Theorem 88. *If R is the ring of all matrices with degree n in an uniserial complete primary ring K, then the lattice L of all submodules in a R-module of rank m is isomorphic with the lattice L^* of all submodules in a K-module of rank mn.*

Pr. Let x_i, $i = 1, 2, \cdots m$ be a basis of the R-module M and consider a K-module M' with basis $e_{1j}x_i$, $i = 1, 2, \cdots m$, $j = 1, 2, \cdots n$, where e_{ij} are matrix units. To every submodule A in M, which is generated by $A^{(\nu)} = \sum_{i,j,k} a_{ijk}^{(\nu)} e_{kj} x_i$, $a_{ijk}^{(\nu)} \in K$, $\nu = 1, 2, \cdots r$, we correspond the submodule A' in M', which is generated by $A_{\lambda\mu} = \sum_{i,j} a_{ij\mu}^{(\lambda)} e_{1j} x_i$, $\lambda = 1, 2, \cdots r$, $\mu = 1, 2, \cdots n$. Then to every submodule in M', which is generated by $B_\nu = \sum_{i,j} b_{ij}^{(\nu)} e_{1j} x_i$, $b_{ij}^{(\nu)} \in K$, $\nu = 1, 2, \cdots s$ corresponds the submodule in M, which is generated by $B'_\nu = \sum_k \sum_{i,j} b_{ij}^{(\nu)} e_{kj} x_i$. Now, if a submodule $C = (\sum_{i,j,k} c_{ijk} e_{kj} x_i)$ of rank 1 in M is contained in the submodule A, it holds

$$\sum_{j,k} c_{ijk} e_{kj} = \sum_\nu \Lambda_\nu \sum_{j,k} a_{ijk}^{(\nu)} e_{kj}, \quad i = 1, 2, \cdots m$$

with $\Lambda_\nu = \sum_{s,t} \lambda_{st}^{(\nu)} e_{st} \in R$, whence $c_{ijk} = \sum_{\nu,t} \lambda_{kt}^{(\nu)} a_{ijt}^{(\nu)}$. Then

$$\sum_{i,j} c_{ijk} e_{1j} x_i = \sum_{\nu,t} \lambda_{kt}^{(\nu)} \sum_{i,j} a_{ijt}^{(\nu)} e_{1j} x_i .$$

Hence the corresponding submodule C' in M', which is generated by $\sum_{i,j} c_{ijk} e_{1j} x_i$, $k = 1, 2, \cdots n$, is contained in A'. Conversely, if a submodule $(\sum d_{ij} e_{1j} x_i)$, $d_{ij} \in K$, is contained in A', then $d_{ij} = \sum_{\mu,\nu} \lambda_{\mu\nu} a_{ij\nu}^{(\mu)}$ and $\sum_k \sum_{i,j} d_{ij} e_{kj} x_i = \sum_\mu (\sum_{k,s} \lambda_{\mu s} e_{ks}) \sum_{i,j,\nu} a_{ij\nu}^{(\mu)} e_{\nu j} x_i$. Hence the corresponding submodule in M is contained in A. The above consideration shows us that the correspondence is one-to-one and isomorphic. q. e. d.

By the preceding theorem we see readily that, if R is a direct

sum of uniserial primary rings R_i, $i = 1, 2, \cdots \lambda$ and if R_i is the ring of all matrices with degree n_i in a complete primary ring K_i, then the lattice L of all submodules in a R-module of rank s is isomorphic with the direct union of lattices L_i, where L_i is the lattice of all submodules in a K_i-module of rank $n_i s$. Therefore L is semi-primary. Conversely, if L is a direct union of primary lattices L_i and $m_{h_i}(L_i) \geqq 4$, then L is isomorphic with a lattice of submodules in a R-module, where h_i being the height of L_i and R is a direct sum of uniserial primary rings. It is to be remarked further that by theorem 88 the lattice of all submodules in a K-module of rank n is isomorphic with the lattice of all left ideals in the ring of all matrices with degree n in K. Then theorem 86 asserts that a primary lattice of rank r with $m_h \geqq 4$ is isomorphic with a lattice of left ideals in the ring of all matrices with degree r in an uniserial complete primary ring.