ON PRIMARY LATTICES

By

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In my previous short note concerning the lattice of all subgroups of a finite abelian group, I have introduced new sorts of lattices. I have called namely a modular lattice, whose quotients are always chains or sublattices with no proper neutral element, primary (primär) and the direct union of a finite number of primary lattices semi-primary (halb-primär). They enable us not solely to illustrate the intermediate relations between all subgroups of an ordinary abelian group, but further to characterize the lattice of submodules in a module with finite bases, where the ring of operators is primary and uniserial in Köthe’s sense. We obtain in this way an extension of the well known fact, that an indecomposable complemented modular lattice of finite dimension $n \geq 4$ characterizes completely the lattice of all linear subspaces of an $n$-dimensional projective space, in a direction different from that of J. v. Neumann’s continuous geometry. As we assume no preliminary knowledge about modular lattices, we shall deal with them in part I. All properties of primary or semi-primary lattices, which are either characteristic for them or indispensable for the later development, shall be treated in part 2. As we deal with only the finite-dimensional case, their topological aspect was not considered. The remaining parts concern chiefly with the generalization of von Staudt’s algebra of throws, to attain the main theorem, that every primary lattice with $m_h \geq 4$ is isomorphic with a lattice of submodules in a module of the above mentioned sort.

Finally I must tender my hearty thanks to Mr. Nakayama, who has been kind enough to give me some useful remarks.


(3) For the meaning of the condition $m_h \geq 4$ refer the part 2 of this paper.
PART I.

A set is called a \textit{partially-ordered system}, when within elements a relation \( x \geqq y \) is determined, satisfying the following axioms:

(A) \( x \geqq x \) for each element \( x \), (B) \( x \geqq y \) and \( y \geqq x \) yield \( y = x \), (C) if \( x \geqq y \), \( y \geqq z \), then \( x \geqq z \). If \( x \neq y \) and \( x \geqq y \), then the relation is denoted by \( x > y \). By a least (greatest) element of a subset \( X \) of a partially ordered system is meant an element \( a \in X \) such that for all \( x \in X \) the relation \( a \leqq x \) (\( x \leqq a \)) holds. If there exists the least element in the set of all the elements \( u \) with \( u \geqq x \), \( u \geqq y \), it is uniquely determined and is called the join \( x \vee y \) of \( x \) and \( y \). If further there exists the greatest element in the set of all the elements \( v \) with \( v \leqq x \), \( v \leqq y \), it is also uniquely determined and is called the meet \( x \wedge y \) of \( x \) and \( y \). By a \textit{lattice} is meant a partially ordered system, in which any two elements have always their join and meet. We can then verify easily the following identities

\[
x \vee x = x \wedge x = x, \quad x \wedge y = y \vee x, \quad x \vee y = y \wedge x, \quad x \vee (y \wedge z) = (x \vee y) \wedge z, \quad x \wedge (y \vee z) = (x \wedge y) \vee z.
\]

Conversely, a set, in which two kinds of operations \( x \vee y \), \( x \wedge y \) are so determined, that the above identities hold, becomes partially ordered, if we define \( x \geqq y \) to mean \( x \wedge y = y \). If the greatest element (least element) of a lattice \( L \) exists, then it is denoted with \( I(O) \). The lattice, whose elements are in a finite number, shall be called \textit{finite}. A finite lattice has evidently \( I \) and \( O \). If the modular identity \( x \vee (y \wedge z) = (x \vee y) \wedge z \) holds for \( x \leqq z \), then the lattice is called \textit{modular}. The quotient \( \frac{b}{a} \) or \([a, b]\) is the sublattice of all the elements \( x \) between \( a \) and \( b \), such that \( a \leqq x \leqq b \). In a modular lattice the quotient \( \frac{u \vee v}{u} \) is lattice-isomorphic with the quotient \( \frac{v}{u \wedge v} \), if we make correspondence the element \( y = v \wedge x \) to each \( x \) with \( u \leqq x \leqq u \wedge v \). By the \textit{dimension} \( d \) of an element \( x \) in a modular lattice \( L \) is meant the largest number \( d \), such that \( O = x_0 < x_1 < x_2 \cdots < x_d = x \), \( x_i \) being elements from \( L \). A modular lattice, whose greatest element \( I \) is of finite dimension, is called \textit{of finite dimension}. It holds then the 'Jordan-Dede'\' chain condition with the relation

\[
\dim (u \vee v) + \dim (u \wedge v) = \dim u + \dim v.
\]

The elements \( x_1, x_2, \cdots x_d \) in.
$L$ are said to be *independent*, if $(x_1 \cup x_2 \cup \cdots \cup x_{i-1} \cup x_{i+1} \cdots \cup x_\lambda) \cap x_i = 0$, $i = 1, 2, \cdots \lambda$, hold.

**Lemma 1.** If $(x \cap y) \cap z \leq x$ in a modular lattice, then $(x \cap y) \cap y = x \cap y$.

**Pr.** $x = \{(x \cap y) \cap z\} \cup x = \{(x \cap y) \cap (z \cap x)\} \cup x$, $x \cap y = [(x \cap y) \cap y] \cup y = \{(x \cap z) \cap y\} \cup (x \cap y) = (x \cap z) \cap y$.

**Lemma 2.** If $(x_1 \cup x_2 \cup \cdots \cup x_{i-1}) \cap x_i = 0$, $i = 2, 3 \cdots \lambda$, in a modular lattice, then $x_1, x_2, \cdots x_\lambda$ are independent.

**Pr.** By induction on $\lambda$. Put $X_i' = x_1 \cup x_2 \cdots \cup x_{i-1} \cup x_{i+1} \cdots \cup x_{\lambda-1}$, $i = 1, 2, \cdots \lambda-1$, $X_\lambda = x_1 \cup x_2 \cdots \cup x_{\lambda-1}$, then $X_i \cap x_i = 0$ by the induction-hypothesis and $X_i \cap x_i = (X_i' \cup x_\lambda) \cap x_i = X_i' \cap x_i = 0$, $i = 1, 2, \cdots \lambda-1$, $X_\lambda \cap x_\lambda = 0$ by Lemma 1, since $(X_i' \cup x_i) \cap x_\lambda = 0$.

**Corollary.** If $x_1, x_2, \cdots x_\lambda$ are independent in a modular lattice, then, putting $X_i = x_1 \cap x_2 \cap \cdots \cap x_i \cap x_{i+1} \cap \cdots \cap x_\lambda$, holds $X_1 \cap X_2 \cap \cdots \cap X_\lambda = 0$.

**Pr.** It suffices to prove $X_1 \cap X_2 \cap \cdots \cap X_i = x_{i+1} \cap x_{i+2} \cap \cdots \cap x_\lambda$; by induction on $i$. The case $i = 1$ is evident. From $X_1 \cap X_2 \cap \cdots \cap X_s = x_{s+1} \cap x_{s+2} \cap \cdots \cap x_\lambda$ follows $X_1 \cap X_2 \cap \cdots \cap X_s \cap x_{s+1} = (x_{s+1} \cup x_{s+2} \cup \cdots \cup x_\lambda) \cap x_{s+1} = (x_{s+1} \cap x_{s+1} \cap x_{s+1}) = x_{s+2} \cup \cdots \cup x_\lambda$. q.e.d.

Any three elements $x$, $y$, $z$ of a modular lattice are said to be distributive, if the relation $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$ holds.

**Theorem 1.** The distributivity relation is symmetric and dual, i.e. if $x$, $y$, $z$ are distributive, then we have

\[
x \cap (y \cup z) = (x \cap y) \cup (x \cap z) \quad \cdots \cdots \cdots \cdots \cdots (1)
\]

\[
y \cup (z \cap x) = (y \cup z) \cap (y \cap x) \quad \cdots \cdots \cdots \cdots \cdots (2)
\]

\[
z \cap (x \cup y) = (z \cap x) \cup (z \cap y) \quad \cdots \cdots \cdots \cdots \cdots (3)
\]

and the three corresponding dual relations (4), (5), (6).

**Pr.** From modular identity and (1) we have $(y \cap z) \cup (y \cap x) = \{x \cap (y \cap z)\} \cup y = (x \cap y) \cap (x \cap z) \cap y = y \cap (z \cap x)$, that is (5), and $(z \cap x) \cap (z \cap y) = z \cap \{(y \cap (z \cap x)\} = z \cap (y \cap z) \cap (y \cap x) = z \cap (x \cap y)$, that is (3). Similarly we obtain (4), (2) from (3) and (6) from (2).

**Corollary.** If $x \geq y$, then $x$, $y$, $z$ are distributive for any $z$. 
If an element $a$ of a modular lattice $L$ is distributive with any two elements in $L$, then $a$ is called neutral. The elements $I$ and $O$ in $L$ is evidently neutral, precisely improper neutral. Other neutral elements are said to be proper neutral.

**Theorem 2.** Every element of a chain is neutral. Generally, if there exists only one element with a given dimension in a modular lattice of finite dimension, then it is neutral.

Pr. Suppose $x$ is the unique element of dimension $\lambda$ and $y$ an arbitrary element. If $\text{Dim } y > \lambda$ then the quotient $\frac{y}{O}$ contains at least an element of dimension $\lambda$, i.e. the element $x$. If $\text{Dim } y < \lambda$, then the quotient $\frac{I}{y}$ contains at least an element of dimension $\lambda - \text{Dim } y$. Since its dimension equals to $\lambda$ in $L$, it is nothing else than the element $x$. Hence $x$ is neutral by the corollary to theorem 1.

Suppose that a lattice $L$ have the elements $I$ and $O$. If for an element $a$ in $L$ an element $a'$ exists, such that $a \sim a' = I$, $a \cap a' = O$, then $a$ is called complemented and $a'$ its complement.

**Theorem 3.** If an element $a$ of a modular lattice $L$ is complemented and neutral, then the complement $a'$ is also neutral and for every element $x$ of $L$ holds the identity $x = (a \cap x) \cup (a' \cap x)$. Conversely, if an element $a$ is complemented and the above identity holds for every element $x$, then $a$ is neutral.

Pr. The neutrality of the element $a$ yields

$$(a \cap x) \cup (a' \cap x) = \{a \cap (a' \cap x)\} \cup \{x \cap (a' \cap x)\} = \{a \cap (a' \cap x)\} \cup x = (a \cap a') \cup (a \cap x) \cap x = x.$$ 

Conversely $x = (a \cap x) \cup (a' \cap x)$ and $y = (a \cap y) \cup (a' \cap y)$ yield $a \cap (x \cap y) = \{(a \cap x) \cup (a \cap y) \cup (a' \cap x) \cup (a' \cap y)\} \cap a = (a \cap x) \cup (a \cap y) \cup \{(a' \cap x) \cup (a' \cap y)\} \cap a = (a \cap x) \cup (a \cap y)$, whence the neutrality of the element $a$ follows. But $a$ is the complement of $a'$ and the identity is symmetric with respect to $a$, $a'$. Therefore $a'$ is also neutral.

**Theorem 4.** If $a$ is a complemented neutral element of a modular lattice $L$ and $a'$ its complement, then every element of $L$ is uniquely representable as the join of two elements from the quotients $\frac{a}{O}$ and $\frac{a'}{O}$.
respectively. Conversely, if every element of a modular lattice $L$ is representable as the join of elements from two sublattices $L_1$, $L_2$, which have only $O$ as its common element, then we have $I = e_1 \lor e_2$, $O = e_1 \land e_2$, where $e_1, e_2$ are the greatest elements of $L_1, L_2$ respectively, and moreover every element $x$ from $L$ is uniquely representable as $x = (e_1 \land x) \lor (e_2 \land x)$, $e_1, e_2$ being neutral.

Pr. In order to prove the converse statement, let $f_1$ be the greatest element in $L_1$, then $f_1 = f_1 \land (e_1 \lor e_2) = e_1 \lor (e_2 \land f_1) = e_1$. Similarly the greatest element in $L_2$ is identical with $q$. From $S$ $x = x_1 \land x_2$ follows $x \land e_1 = (x_1 \land x_2) \land e_1 = x_1 \lor (x_2 \land e_1) = x_1$ and $x \land e_2 = x_2$.

**Theorem 5.** If in a modular lattice $L$ the relation $I = e_1 \lor e_2 \lor \cdots \lor e_\lambda$ holds with independent neutral elements $e_i$, $i = 1, 2, \cdots \lambda$, then every element $x$ of $L$ is representable as $x = (x \land e_1) \lor (x \land e_2) \lor \cdots \lor (x \land e_\lambda)$.

Pr. By induction on $\lambda$.

**Corollary and definition.** In the above theorem, the quotient $L_i$ is a sublattice of $L$ (also an ideal in $L$) and shall be denoted by $O$.

Every element $x$ of $L$ is uniquely representable as the join of elements $x_i = x \land e_i$ from $L_i$. Then the lattice $L$ is said to be a direct union of lattices $L_i$, $i = 1, 2, \cdots \lambda$, $x_i$ being the $L_i$-component of $x$.

Pr. If $x = x_1 \land x_2 \land \cdots \land x_\lambda$, $x_i \in L_i$, then $x \land e_i = x_i \land \{x_1 \land x_2 \land \cdots \land x_i \land x_{i+1} \land \cdots \land x_\lambda \} \land e_i = x_i$, since $(x_1 \land x_2 \land \cdots \land x_i \land x_{i+1} \land \cdots \land x_\lambda) \land e_i \leq (e_1 \land e_2 \land \cdots \land e_i \land e_{i+1} \land \cdots \land e_\lambda) \land e_i = 0$.

**Theorem 6.** If $L$ is the direct union of $L_i$ and $x_i, y_i$ the $L_i$-components of $x, y$ respectively, then $L_i$-components of $x \land y$ and $x \lor y$ are $x_i \land y_i$ and $x_i \lor y_i$ respectively.

Pr. $(x \land y) \land e_i = (x \land e_i) \land (y \land e_i) = x_i \land y_i$.

$(x \lor y) \land e_i = (x \land e_i) \land (y \land e_i) = x_i \land y_i$.

A lattice, in which every element is complemented and neutral, is called Boolean algebra.

**Theorem 7.** The set of all complemented neutral elements in a modular lattice $L$ is a sublattice of $L$ and is a Boolean algebra, which is called the center of $L$. 
Pr. Suppose \( a_1, a_2 \) are complemented and neutral in \( L \). Then we have

\[
(a_1 \lor a_2 \lor x) \land (a_1 \lor a_2 \lor y) = a_1 \lor \{a_2 \lor x \land (a_2 \lor y)\} = a_1 \lor \{a_2 \lor (x \land y)\} = (a_1 \lor a_2) \lor (x \land y).
\]

\[
(a_1 \lor a_2) \lor (a'_1 \lor a'_2) = a_1 \lor \{a_2 \lor a'_1 \lor (a_2 \lor a'_2)\} = a_1 \lor a_2 \lor a'_1 = I,
\]

\[
(a_1 \lor a_2) \lor (a'_1 \lor a'_2) = \{(a_1 \lor a'_1) \lor (a_2 \lor a'_2)\} \lor a'_2 = a_2 \lor a'_1 \lor a'_2 = O.
\]

Therefore \( a_1 \lor a_2 \) is complemented neutral and similarly for \( a_1 \lor a_2 \), if dually considered. \( \Box \).

If the center of \( L \) consists of only \( I \) and \( O \), then \( L \) cannot be a direct union of two sublattices and is called indecomposable.

**Theorem 8.** If a modular lattice \( L \) is the direct union of sublattices \( L_i, i = 1, 2, \ldots \lambda \), then the center \( C \) of \( L \) is the direct union of the centers \( C_i \) of \( L_i \).

Pr. For \( a \in C_i \) we have \( a = a \lor e_i \) and for arbitrary \( x, y \) in \( L \),

\[
a \land (x \lor y) = a \lor (x \lor y) = a \lor (x \lor y) = a \lor (x \lor y) = a \land y.
\]

From \( a \lor a' = e_i \), \( a \lor a' = O \) follows \( a \lor (a' \lor e_1 \lor \ldots \lor e_{i-1} \lor e_i) = I \) and \( a \lor (a' \lor e_1 \lor \ldots \lor e_{i-1} \lor e_i) = a \lor (a \lor e_1 \lor \cdots \lor e_{i-1} \lor e_i) \lor e_i = a \lor [a \lor (e_1 \lor e_2 \lor \cdots \lor e_i)] \lor e_i = a \lor a' = O \). Hence \( a \) belongs to \( C \) and \( C_i \subseteq C \). Further, if \( x \) is an element in \( C \), then \( (x \lor e_i) \lor (x' \lor e_i) = (x \lor x') \lor e_i = e_i \), and therefore \( x \lor e_i \in C_i \). Now it follows from \( x = (x \lor e_i) \lor (x' \lor e_i) \lor e_i \), that \( x \) belongs to the direct union of \( C_i \). \( \Box \).

**Theorem 9.** If \( L \) is the direct union of sublattices \( L_i^{(j)}, i = 1, 2, \ldots m \), each \( L_i^{(j)} \) being the direct union of \( L_j^{(i)}, j = 1, 2, \ldots \lambda_i \), then \( L \) is the direct union of all \( L_i^{(j)}, i = 1, 2, \ldots m ; j = 1, 2, \ldots \lambda_i \).

Pr. Let \( I = e^{(1)} \lor e^{(2)} \lor \cdots \lor e^{(m)}, e^{(i)} \in L_i \), and \( e^{(i)} = e_i^{(1)} \lor e_i^{(2)} \lor \cdots \lor e_i^{(n_i)}, e_i^{(j)} \in L_i^{(j)} \). It suffices to prove that \( e_i^{(j)} \) are independent. This can be easily proved by lemma 1.

**Theorem 10.** If the dimension of the center \( C \) of a modular lattice \( L \) is finite and equals to \( n \), then \( C \) consists of \( 2^n \) elements and \( L \) is the direct union of \( n \) indecomposable sublattices.

Pr. By induction on \( n \). In case \( n = 1 \) evident. If \( n > 1 \), let \( I = e_1 \lor e_2, e_1 \lor e_2 = O, e_i \) being proper neutral. Then \( L = L_1 + L_2, C = C_1 + C_2 \), where the dimensions \( n_1, n_2 \) of \( C_1, C_2 \) are smaller than \( n \).
with $n = n_1 + n_2$, since $\frac{I}{e_1} \simeq \frac{e_1 \cup e_2}{e_1} \simeq \frac{e_2}{O}$. Then $C_i$ consists of $2^n$ elements by induction-hypothesis. Therefore $C$ consists of $2^n \cdot 2^n = 2^n$ elements, and is the direct union of $n$ indecomposable sublattices, since $C_i$ is the direct union of $n_i$ indecomposable sublattices. q.e.d.

An element $x$ of a lattice $L$ is said to be reducible, if $x = y \cup z$, $x > y$, $x > z$. If this is not the case, then we shall say $x$ irreducible. For example every element from a chain is irreducible. Every element from a lattice of finite dimension can be represented as the join of irreducible elements. The independent irreducible elements $x_i$ with $x = x_1 \cup x_2 \cup \cdots \cup x_i$ are called a basis of the element $x$. If the center $C$ of a modular lattice consists of $2^n$ elements, then $I = e_1 \cup e_2 \cup \cdots \cup e_n$ uniquely with irreducible elements $e_i$ from $C$ and every element from $C$ is the join of some $e_i$. Hence it follows

**Theorem 11.** Any modular lattice of finite dimension is uniquely representable as the direct join of indecomposable sublattices.

**Theorem 12.** Every element of the center of a modular lattice has an unique complement. Conversely an element of a complemented modular lattice belongs to the center, if its complement is unique.

Pr. Suppose $x'$, $x''$ are complements of an element $x$ in the center. Then $I = x \cup x'$, $x'' = (x'' \cap x) \cup (x'' \cap x') = x'' \cap x'$, whence $x'' \leq x'$. Similarly we obtain $x' \leq x''$ and therefore $x' = x''$. Next let $a'$ be the unique complement of an element $a$, and $u$ be an arbitrary element, which is independent with $a$. Then $u, a, (u \cup a)'$ are independent and consequently $u \cup (u \cup a)' = a'$, whence $u \leq a'$. Since $(a \cup x)' \cup x$ is independent with $a$, where $x$ is arbitrary, so we have $(a \cup x)' \cup x \leq a'$. Now we infer

\[ x = (a \cup x) \cup \{(a \cup x)' \cup x\} \leq (a \cup x) \cup (a' \cup x) \leq x \]

Hence $a$ is neutral by theorem 3. q.e.d.

The converse statement in the above theorem does not hold for arbitrary modular lattices. For example, we consider the lattice of seven elements $O, u_1, u_2, v_1, v_2, v_3, I$, where $u_1 \cup u_2 = v_1, u_1 \cup u_2 = O$, $v_1 \cup v_2 = v_1 \cup v_3 = v_2 \cup v_3 = I$, $v_1 \cap v_2 = v_1 \cap v_3 = v_2 \cap v_3 = u_2, u_1 \cup v_2 = u_1 \cup v_3 = I, u_1 \cup v_2 = u_1 \cup v_3 = O$. This is modular and the element $v_2$ has its unique complement, but however not neutral, since $(v_2 \cup v_3) \cup (u_1 \cup v_3) = u_2 \neq v_3$. 

On Primary Lattices
Theorem 13. $C$ being the center of a modular lattice $L$, the set $W$ of all elements $(b \cap c) \cdot a$, where $c \in C$ and $a \leqq b$, is a sublattice of the center of the quotient $\frac{b}{a}$.

Pr. First we prove, that $W$ is a sublattice of $L$.

\[
\{(b \cap c_1) \cdot a\} \cup \{(b \cap c_2) \cdot a\} = (b \cap c_1) \cup (b \cap c_2) \cup a = \{b \cap (c_1 \cup c_2)\} \cdot a,
\]
\[
\{(b \cap c_1) \cdot a\} \cap \{(b \cap c_2) \cdot a\} = b \cap (c_1 \cdot a) \cap (c_2 \cdot a)
\]

\[
= b \cap \{c_1 \cap c_2\} \cdot a = \{b \cap (c_1 \cap c_2)\} \cdot a.
\]

Since we have further

\[
\{(b \cap c) \cdot a\} \cup \{(b \cap c') \cdot a\} = b, \quad \{(b \cap c) \cdot a\} \cap \{(b \cap c') \cdot a\} = a,
\]
every element $(b \cap c) \cdot a$ is complemented in $\frac{b}{a}$. From $a \leqq x \leqq b$, $x = (c \cap x) \cap (c' \cap x)$ follows

\[
[x \cap \{(b \cap c) \cdot a\}] \cup [x \cap \{(b \cap c') \cdot a\}] = (x \cap (c \cdot a) \cup (x \cap (c' \cdot a) = a.
\]

Hence $(b \cap c) \cdot a$ is neutral in the quotient $\frac{b}{a}$ by theorem 3.

Theorem 14. If a modular lattice $L$ is a direct union of $L_i$, $i = 1, 2, \cdots n$, then a quotient $\frac{b}{a}$ of $L$ is isomorphic with a direct union of quotients in $L_i$.

Pr. Suppose $I = e_1 \cup e_2 \cup \cdots \cup e_n$ and $L_i \simeq \frac{e_i}{O}$. We infer the independence of $(b \cap e_i) \cdot a$ in $\frac{b}{a}$ from the independence of $e_i$ in $L$, as follows

\[
(b \cap e_i) \cdot a \cup \ldots \cup (b \cap e_{i-1}) \cdot a \cup (b \cap e_{i+1}) \cdot a \ldots
\]

\[
= b \cap \{e_i \cup (e_1 \cup \cdots e_{i-1} \cup e_{i+1} \cdots)\} \cdot a = a.
\]

Now we have, for $a \leqq x \leqq b$, $x = a \cup \sum (e_i \cap x) = \sum [(b \cap e_i) \cdot a \cap x]$ and the quotient $\frac{b}{a}$ becomes the direct union of $\frac{(b \cap e_i) \cdot a}{a}$, $i = 1, 2, \cdots n$, which are isomorphic with the quotients $\frac{b \cap e_i}{a \cap e_i}$ respectively(1).

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(1) The sign $\sum A_i$ means the join of all elements $A_i$, $i = 1, 2, \ldots$. 

PART 2

Definition. A modular lattice, whose quotients are always chains or sublattices with no proper neutral element, is called primary. For instance, chains, indecomposable complemented modular lattices of finite dimension (projective geometries) and continuous geometries are all primary. Since the inversion of the order within elements has no influence upon the neutrality of an element, so we have

Theorem 15. A lattice, which is dually isomorphic with a given primary lattice, is primary. Every quotient of a primary lattice is also primary.

Definition. A lattice, which is a direct union of a finite number of primary lattices, is called semi-primary. In particular a complemented modular lattice of finite dimension is semi-primary.

Theorem 16. A lattice, which is dually isomorphic with a given semi-primary lattice, is semi-primary. Any arbitrary quotient of a semi-primary lattice is also semi-primary.

Pr. Let \( \overline{L} \) be dually isomorphic with a given semi-primary lattice \( L \) and denote the corresponding element in \( \overline{L} \) of an element \( x \) in \( L \) with \( \overline{x} \). If \( I = e_1 \lor e_2 \lor \ldots \lor e_n \), where \( e_i \) irreducible elements from the center of \( L \), then, putting \( E_i = e_1 \lor e_2 \lor \ldots \lor e_{i-1} \lor e_{i+1} \lor \ldots \lor e_n \), we have \( \overline{I} = \overline{E}_i \cap (\overline{E}_1 \lor \ldots \lor \overline{E}_{i-1} \lor \overline{E}_{i+1} \lor \ldots \lor \overline{E}_n) \) and \( \overline{E}_1 \lor \overline{E}_2 \lor \ldots \lor \overline{E}_n = \overline{O} \), since \( I = E_i \lor (E_1 \lor \ldots \lor E_{i-1} \lor E_{i+1} \lor \ldots \lor E_n) \) and \( E_1 \lor E_2 \lor \ldots \lor E_n = O \) hold. Hence \( \overline{E}_i \), \( i = 1, 2, \ldots n \), are independent and neutral. The quotient \( \frac{\overline{E}_i}{\overline{I}} \) is now dually isomorphic with \( \frac{I}{E_i} \simeq \frac{e_i}{O} \), whence it follows that \( \overline{L} \) is a direct union of primary lattices \( \overline{E}_i \). The proof of the second part of this theorem is immediate in virtue of theorem 14. q. e. d.

As regards theorem 13 we can sharpen this as follows.

Theorem 17. The center \( W \) of the quotient \( \frac{b}{a} \) of a semi-primary lattice \( L \) is the set of all elements \( (b \land c) \lor a \) with elements \( c \) from the center \( C \) of \( L \). In particular, if \( c \) irreducible in \( C \), then \( (b \land c) \lor a \) is irreducible in \( W \).
Pr. It suffices to prove, that every element $d$ in $W$ is representable as $(b \wedge c) \vee a$ with $c \in C$. Let $I = e_1 \vee e_2 \vee \cdots \vee e_n$ with irreducible elements $e_i$ in $C$, then we have $d = \sum \{a \cup (e_i \wedge d)\}$. Since $a \cup (e_i \wedge d) = (b \wedge e_i) \cup a \wedge d$ belongs to $W$, and since $a \cup (e_i \wedge b)$ is irreducible in $W$ follows from the fact, that the quotient $(b \wedge e_i) \cup a$ is primary, if $e_i$ is irreducible in $C$.

**Definition.** If a quotient $\frac{a}{O}$ is a chain, then it is called a chain in $L$ with $a$ as its summit. If moreover it is not contained in another chain in $L$, then it is called a maximal chain in $L$.

**Theorem 18.** In a primary lattice $L$ of finite dimension, which is not a chain, there are at least three elements, which cover the element $O$, and the element $I$ covers at least three elements. Further any element $d$, which does not belong to a chain in $L$, covers at least three elements.

Pr. Suppose there exists only one atom in $L$. Then it would become proper neutral by theorem 2, contradictory to the assumption. Next suppose there are two atoms $a$, $b$, then the quotient $\frac{a \cup b}{O}$ must have another atom $c$, which is different from both $a$, $b$, since otherwise the quotient would not be primary. The dual consideration asserts that $I$ covers at least three elements. The last part of the theorem is now obvious, since $\frac{d}{O}$ is primary, and not a chain.

**Theorem 19.** A semi-primary lattice, whose elements are all neutral, is a direct join of chains.

---

(1) Hereafter by primary or semi-primary lattices are meant always only those of finite dimension.

(2) An element, which covers $O$, is called an atom.
Pr. If a primary lattice is not a chain, then by the preceding theorem there exists at least three atoms, which are not neutral.

**Theorem 20.** If a modular lattice $L$ is a direct union of $L_i$, $i = 1, 2, \cdots n$, then any chain in $L$ is a chain in some $L_i$.

Pr. It suffices to prove only in the case $n = 2$. Let $I = \frac{e_1}{O}$, $L_1 = \frac{e_1}{O}$, $L_2 = \frac{e_2}{O}$, then the relation $a = (a \cap e_1) \cup (a \cap e_2)$ holds for any chain $\frac{a}{O}$, whence $a \cap e_1 = a$ or $a \cap e_2 = a$. The chain is therefore contained in $L_1$ or $L_2$.

**Theorem 21.** An element $a$ in a semi-primary lattice $L$ is irreducible, if and only if it is the summit of a chain in $L$.

Pr. That $a$ is irreducible, if $\frac{a}{O}$ is a chain, is evident. Suppose that $a$ is irreducible, then $a$ is contained in some primary sublattices $L_i$ of $L$, for the relation $a = \sum (a \cap e_i)$ yields $a \leq e_i$ for some $i$. If $\frac{a}{O}$ were not a chain, then it would cover at least two elements $b, c$, such that $a = b \cup c$, contrary to the assumption. q.e.d.

Since any element of a lattice of finite dimension is representable as the join of irreducible elements, so we have

**Theorem 22.** Any element of a semi-primary lattice $L$ can be represented as the join of irreducible elements, which are summits of chains in $L$.

**Theorem 23.** Any element of a complemented modular lattice $L$ of finite dimension can be represented as the join of independent atoms in $L$, that is, any element from $L$ has its basis.

Pr. We will prove by induction on the dimension of the given element $a$. If, dim $a > 1$, $a$ covers an element $b$, then $a = b \cup (b' \land a)$, $b \land (b' \land a) = O$. In virtue of the dimension-relation the element $b' \land a$ must be an atom. But $b$ is a join of independent atoms by induction-hypothesis, and these together with $b' \land a$ make a basis of $a$.

**Lemma 3.** If in a complemented modular lattice $L$ $a > b$ and if the atoms $a_i, i = 1, 2, \cdots m$, $b_j, j = 1, 2, \cdots n$ are bases of $a, b$ respectively, then we have a new basis of $a$, such that it consists of whole $b_j, j = 1, 2, \cdots n$, and a part of $a_i$.
Pr. Let \( a_i \) be an atom in \( \frac{a}{O} \), which is not contained in \( b \) and put \( b^{(1)} = b \cup a_i \). Putting \( b^{(\nu+1)} = b^{(\nu)} \cup a_{\nu+1} \) successively, such that the atom \( a_{\nu+1} \) is not contained in \( \frac{b^{(\nu)}}{O} \), we have \( \dim b^{(\nu+1)} = \dim b^{(\nu)} + 1 \leq \dim a \), whence we conclude \( b^{(m-n)} = a \).

**Theorem 24.** Any modular lattice, whose elements are all representable as the join of a finite number of atoms, is complemented.

Pr. Let \( I = c_1 \cup c_2 \cup \cdots \cup c_m \) with independent atoms \( c_i \) and let \( a \) be an arbitrary element with \( a = a_1 \cup a_2 \cup \cdots \cup a_n \), \( a_i \) being independent atoms. Then by the preceding lemma we have \( I = a \cup (c_1 \cup c_2 \cup \cdots \cup c_{m-n}) \), where \( a \cup (c_1 \cup c_2 \cup \cdots \cup c_{m-n}) = O \). q. e. d.

In theorem 22 we have shown, that every element of a semi-primary lattice \( L \) can be represented as the join of irreducible elements. We shall call the set of all elements in \( L \), which are representable as the join of irreducible elements of dimension not greater than \( \nu \), the \( \nu \)-th derivative of \( L \) and denote it with \( L^{(\nu)} \). Evidently \( L^{(0)} \) has only the element \( O \). If \( L^{(h-1)} < L^{(h)} = L \), then we shall say, that \( L \) has a height \( h \). The greatest element in \( L^{(\nu)} \) is obviously unique and shall be denoted with \( e^{(\nu)} \).

**Theorem 25.** The \( \nu \)-th derivative \( L^{(\nu)} \) of a semi-primary lattice \( L \) is an ideal in \( L \) and the dimension of any arbitrary chain in \( \frac{e^{(\nu)}}{O} \) is not greater than \( \nu \).(*)

Pr. By induction on \( \nu \). First we prove the case \( \nu = 1 \). For this purpose we have only to prove, that the meet of an element in \( L^{(1)} \) with any element in \( L \) belongs to \( L^{(1)} \). But this verification can be achieved, if we show, that no irreducible element \( b \) of dimension 2 can be contained in the quotient \( \frac{e^{(1)}}{O} \). Let \( e^{(1)} = a_1 \cup a_2 \cup \cdots \cup a_t \) with independent atoms \( a_i \), and, suppose \( b \) covers the atom \( a_s \). Then putting \( A_s = a_1 \cup \cdots \cup a_{s-1} \cup a_{s+1} \cdots \cup a_t \), we have

(*) By an ideal in \( L \) is meant a subset \( S \) of \( L \), where the join of any two elements from \( S \) and the meet of any element from \( S \) with an arbitrary element from \( L \) belong always to \( S \). That \( L^{(\nu)} \) is an ideal in \( L \), means therefore, \( L^{(\nu)} \) is identical with the quotient \( \frac{e^{(\nu)}}{O} \).
\[ e^{(1)} = A_s \cup b, \quad \frac{e^{(1)}}{A_s} \simeq \frac{b}{O} \] in contradiction with the modularity of \( L \).

If \( b \) covers none of the atoms \( a_i \), but covers the atom \( c \), then there exists an \( A_s \), such that \( A_s \cap b = O \), since otherwise \( c \leq A_1 \cap A_2 \cap \cdots \cap A_i = O \) would follow. We then have \( \frac{e^{(1)}}{A_s} \simeq \frac{b}{O} \), which is however not the case. In case \( \nu > 1 \), suppose \( x \) is an irreducible element in \( L \) with \( \dim x = \nu + 1 \). Then we have \( \frac{x \cup e^{(\nu-1)}}{e^{(\nu-1)}} \simeq \frac{x}{x \cap e^{(\nu-1)}} \), which is however not the case. In case \( v > 1 \), suppose \( x \) is an irreducible element in \( L \) with \( \dim x = v + 1 \).

The quotients \( \frac{e^{(\nu)}}{e^{(\nu-1)}} \) of \( L \) are complemented modular lattices by theorem 24. If in particular \( L \) is primary, then they are moreover indecomposable. We shall call the system \( (a^{(1)}, a^{(2)}, \cdots a^{(h)}) \), where \( a^{(i)} = (a \cap e^{(i)}) \cup e^{(i-1)} = e^{(i)} \cap (a \cup e^{(i-1)}) \), \( h \) being the height of \( L \), system of representatives for an element \( a \), and denote the dimension of the element \( a^{(i)} \) in the quotient \( \frac{e^{(i)}}{e^{(i-1)}} \) with \( \gamma_i(a) \).

**Theorem 26.** \( \sum_{i=1}^{h} \gamma_i(a) = \dim a \).

**Pr.** By adding the both sides of the following relations

\[ \dim (a \cap e^{(i)}) = \gamma_i(a) + \dim (a \cap e^{(i-1)}), \quad i = 1, 2, \cdots h. \] q. e. d.

By the invariant of an element \( a \) of a semi-primary lattice \( L \) is meant the system \( (\gamma_1(a), \cdots \gamma_h(a)) \). The invariant of the element \( I \) is called the invariant of \( L \) and denoted with \( (\gamma_1, \gamma_2, \cdots \gamma_h) \).

**Theorem 27.** If \( a \succeq b \) and \( a^{(i)} = b^{(i)}, \quad i = 1, 2, \cdots h \), then \( a = b \).

**Pr.** The assumption \( a^{(h)} = b^{(h)} \) yields \( a \cup e^{(h-1)} = b \cup e^{(h-1)} \). Now we will show, that \( a \cup e^{(i-1)} = b \cup e^{(i-1)} \) follows from \( a \cup e^{(i)} = b \cup e^{(i)} \). Since \( \frac{a \cup e^{(i-1)}}{a^{(i)}} = \frac{a \cup e^{(i-1)} \cap e^{(i)}}{e^{(i)}} \simeq \frac{b \cup e^{(i-1)}}{b^{(i)}} \), and \( a \cup e^{(i-1)} \succeq b \cup e^{(i-1)} \), we have \( a \cup e^{(i-1)} = b \cup e^{(i-1)} \) and finally \( a = b \).

**Definition.** A homomorphism of a lattice \( L \) into a lattice \( W \)
is called \textit{proper}, if it is not an isomorphism and moreover the image $W$ has at least two elements. A lattice $L$ is called \textit{simple}, if there exists no proper homomorphism of $L$.

\textbf{Theorem 28.} Every homomorphism of a complemented modular lattice $L$ of finite dimension is equivalent with the endomorphism $x \to x \wedge a$, where $a$ is neutral, and the homomorphic image of $L$ is then isomorphic with the quotient $\frac{I}{a}$.

Pr. Suppose $L$ is homomorphic with a lattice $W$. The set of antecedents in $L$ of the least element in $W$ is evidently an ideal in $L$. Let $a$ be the greatest element in this ideal. Then we can prove that two elements $x$ and $y$ in $L$ have the same image $\bar{x}$ in $W$, if and only if $x \wedge a = y \wedge a$. $x \to \bar{x}$, $y \to \bar{x}$ yield namely $(x \wedge y)' \wedge (x \wedge y) \to \bar{O}$, whence $(x \wedge y)' \wedge (x \wedge y) \leq a$. In virtue of the relation $x \wedge y = (x \wedge y) \wedge \{x \wedge y)' \wedge (x \wedge y)\}$ we conclude $x \wedge y \leq (x \wedge y) \wedge a$ and therefore $x \wedge a = y \wedge a$. Since for any arbitrary two elements $x$ and $y$ $(x \wedge a) \wedge (y \wedge a)$ and $x \wedge y$ have the same image in $W$, we obtain $(x \wedge a) \wedge (y \wedge a) = (x \wedge y) \wedge a$. $W$ is therefore isomorphic with the quotient $\frac{I}{a}$, if we make correspond to $\bar{x}$ the element $x \wedge a$.

\textbf{Corollary.} An indecomposable complemented modular lattice of finite dimension is simple.

This is however not the case for a chain. We can establish homomorphisms of a chain of dimension $n$ into a chain of dimension $r \leq n$ in $\frac{(n-1)!}{(r-1)!(n-r)!}$ different ways.

\textbf{Theorem 29.} A primary lattice $L$ is simple, if it is not a chain.

Pr. By induction on the height $h$ of $L$. In case $h = 1$ it is evident by the preceding corollary. In case $h > 1$, $\overline{e(i)}$, $i = 1, 2, \ldots h$, can not be all different for a homomorphism of $L$, unless it is an isomorphism. Indeed, if $a \to \bar{a}$, $b \to \bar{a}$, $a \neq b$, then either $a \wedge b > a$ or $a \wedge b > b$. Supposing $a \wedge b > a$, $(a \wedge b) \wedge (a \wedge b) > a$ holds for some $i$ by \textbf{Theorem 27}. But $(a \wedge b)^{(i)} = a^{(i)}$ belongs to the quotient $\frac{\overline{e(i)}}{e^{(i-1)}}$ and we have $\overline{e(i)} = \overline{e(i-1)}$, since the quotient $\frac{e^{(i)}}{e^{(i-1)}}$ is simple. Now we have only to treat the following two cases
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(i) \( \bar{O} = \overline{e^{(1)}} = \cdots = \overline{e^{(h-1)}} < \overline{e^{(h)}} = \overline{I} \).

(ii) \( \bar{O} < \overline{e^{(1)}} < \cdots < \overline{e^{(h-1)}} = \overline{e^{(h)}} = \overline{I} \).

For other cases do not occur by the induction-hypothesis, if we consider the quotient \( \frac{e^{(h-1)}}{O} \), which is not a chain by theorem 18.

In case (i), \( x, y \) being arbitrary from \( L \), we have

\[
(e^{(h-1)} \cap x) \cap (e^{(h-1)} \cap y) = (\bar{0} \cap x) \cap (\bar{0} \cap y) = \bar{0} \cap (\bar{x} \cap \bar{y}) = \overline{e^{(h-1)}} \cap (x \cap y),
\]

where \( e^{(h-1)} \cap x, e^{(h-1)} \cap y \) and \( e^{(h-1)} \cap (x \cap y) \) are elements from the quotient \( \frac{I}{e^{(h-1)}} \). Since this quotient is simple by the corollary to theorem 28 the present homomorphism of \( L \) induces an isomorphism of this quotient. Hence \( (e^{(h-1)} \cap x) \cap (e^{(h-1)} \cap y) = e^{(h-1)} \cap (x \cap y) \) holds and \( e^{(h-1)} \) is neutral, contrary to the hypothesis, that \( L \) is primary.

In case (ii), the homomorphism induces an isomorphism of the quotient \( \frac{e^{(h-1)}}{O} \), because \( \overline{e^{(i)}}, i = 1, 2, \cdots h-1, \) are all different.

Now, \( x, y \) being arbitrary from \( L \), we have

\[
(e^{(h-1)} \cap x) \cap (e^{(h-1)} \cap y) = (\bar{I} \cap x) \cap (\bar{I} \cap y) = \bar{I} \cap (\bar{x} \cap \bar{y}) = \overline{e^{(h-1)}} \cap (x \cap y),
\]

and consequently \( (e^{(h-1)} \cap x) \cap (e^{(h-1)} \cap y) = e^{(h-1)} \cap (x \cap y) \) contrary to the assumption. q.e.d.

We will show here an example of a modular lattice, which has no proper neutral element, and which however not simple. Consider the modular lattice \( L \), which consists of ten elements \( O, E_{1}^{(1)}, i = 1, 2, 3; E_{j}^{(2)}, j = 1, 2; E_{k}^{(3)}, k = 1, 2, 3 \) and \( \bar{I} \) such that \( E_{1}^{(1)} \cap E_{2}^{(1)} = E_{2}^{(1)} \cap E_{3}^{(1)} \), \( E_{1}^{(1)} \cap E_{2}^{(1)} = E_{2}^{(1)} \cap E_{3}^{(1)} = E_{3}^{(1)} \cap E_{1}^{(1)} = O \), \( E_{1}^{(2)} \cap E_{2}^{(2)} = E_{1}^{(3)} \), \( E_{1}^{(2)} \cap E_{2}^{(2)} = E_{2}^{(3)} \cap E_{3}^{(3)} = E_{3}^{(3)} \cap E_{1}^{(3)} = I \), \( E_{1}^{(3)} \cap E_{2}^{(3)} = E_{2}^{(3)} \), \( E_{3}^{(3)} \), \( E_{3}^{(3)} \) and the modular lattice \( L' \), which consists of eight elements \( \bar{O}, F_{i}^{(1)}, i = 1, 2, 3; F_{j}^{(2)}, j = 1, 2, 3 \) and \( \bar{I} \) such that \( F_{1}^{(1)} \cap F_{2}^{(1)} = F_{2}^{(1)} \cap F_{3}^{(1)} = F_{3}^{(1)} \cap F_{1}^{(1)} = F_{1}^{(2)}, F_{1}^{(1)} \cap F_{2}^{(1)} = F_{2}^{(1)} \cap F_{3}^{(1)} = F_{3}^{(1)} \cap F_{1}^{(1)} = O, F_{1}^{(1)} \cap F_{2}^{(2)} = F_{2}^{(2)} \cap F_{3}^{(2)} = F_{3}^{(2)} \cap F_{1}^{(2)} = \bar{I} \), \( F_{1}^{(2)} \), \( F_{2}^{(2)} = F_{2}^{(2)} \), \( F_{3}^{(2)} \), \( F_{3}^{(2)} \) and \( F_{1}^{(2)} \). \( L \) is then proper homomorphic with \( L' \) by the following correspondence:

\[
E_{i}^{(1)} \rightarrow F_{i}^{(1)}, i = 1, 2, 3, E_{1}^{(2)} \rightarrow F_{1}^{(2)}, E_{k}^{(3)} \rightarrow F_{k}^{(2)}, k = 1, 2, 3.
\]
Definition. If an element $a$ in a semi-primary lattice $L$ has a basis $a_i$, $i = 1, 2, \cdots r$, then we call the number $r$ of the components $a_i$ rank of the element $a$. If further $\nu_i$ is the dimension of $a_i$, where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$, then we call the system $(\nu_1, \nu_2, \cdots \nu_r)$ type of the element $a$. By basis, rank or type of $L$ is to be meant the basis, the rank or the type of the element $I$ of $L$ respectively.

Lemma 4. If in a semi-primary lattice $r$ irreducible elements $a_i$, $i = 1, 2, \cdots r$, are independent, among which $a_1, a_2, \cdots a_{r_\nu}$ are of dimensions greater than $\nu - 1$, then $a_i \cup e^{(\nu - 1)}$, $i = 1, 2, \cdots r_\nu$, are independent over $e^{(\nu - 1)}$.

Pr. Suppose $a_i \cup e^{(\nu - 1)}$ are not independent over $e^{(\nu - 1)}$, then $(a_1 \cup a_2 \cup \cdots \cup a_n \cup e^{(\nu - 1)}) \cap (a_{s+1} \cup e^{(\nu - 1)}) \supset e^{(\nu - 1)}$ for some $s < r_\nu$ and hence $a_{s+1} \cap e^{(\nu)} \supset a_1 \cap a_2 \cup \cdots \cup a_n \cap e^{(\nu - 1)}$. Putting $A_s = a_1 \cup a_2 \cup \cdots \cup a_s$, we have $A_s \cup (a_{s+1} \cap e^{(\nu)}) \leq A_s \cup e^{(\nu - 1)}$ and

$$\frac{A_s \cup (a_{s+1} \cap e^{(\nu)})}{A_s} \supset \frac{A_s \cup e^{(\nu - 1)}}{A_s} \supset \frac{e^{(\nu - 1)}}{A_s \cap e^{(\nu - 1)}}.$$

The quotient $\frac{A_s \cup e^{(\nu - 1)}}{A_s}$ would then contain a chain of dimension $\nu$, which contradicts with theorem 25.

Lemma 5. If $a_i$, $i = 1, 2, \cdots r_\nu$ are of dimension greater than $\nu - 1$ among the components of a basis of $L$ and $\lambda \geq \nu$, then $a_i \cap e^{(\lambda)} \cup e^{(\nu - 1)}$, $i = 1, 2, \cdots r_\nu$ make a basis of the quotient $\overline{e^{(\nu - 1)}}$.

Pr. Since $a_i \cup e^{(\nu - 1)}$ are independent over $e^{(\nu - 1)}$ by the preceding lemma, so $(a_i \cap e^{(\nu)}) \cup e^{(\nu - 1)}$ are independent atoms in the quotient $\frac{e^{(\nu)}}{e^{(\nu - 1)}}$, whence we have $\gamma_\nu = \dim \frac{e^{(\nu)}}{e^{(\nu - 1)}} \geq r_\nu$. On the other hand it holds $\dim I = \frac{\sum_i \gamma_i}{\lambda - \nu} = \frac{\sum_i \gamma_i}{\nu - \nu}$, whence $\gamma_\nu = r_\nu$ follows and $(a_i \cap e^{(\nu)}) \cup e^{(\nu - 1)}$, $i = 1, 2, \cdots r_\nu$, become a basis of $\frac{e^{(\nu)}}{e^{(\nu - 1)}}$. Next, in order to prove the case $\lambda > \nu$, we can assume $\nu = 1$ without loss of generality, since $a_i \cup e^{(\nu - 1)}$, $i = 1, 2, \cdots r_\nu$, are a basis of $\frac{I}{e^{(\nu - 1)}}$. Putting $a_i \cap e^{(\lambda)} = a_i^{(\lambda)}$, $i = 1, 2, \cdots r$, we now have $\dim (a_i^{(\lambda)} \cup a_i^{(\nu - 1)} \cup \cdots \cup a_i^{(1)}) = \sum_i \gamma_i$ in virtue of the independence of $a_i^{(\lambda)}$. The relation $\dim e^{(\lambda)} = \sum_i \gamma_i$. 


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$\sum_{i=1}^{\lambda} r_i$ yields then $e^{(\lambda)} = a_1^{(\lambda)} \cup a_2^{(\lambda)} \cup \cdots \cup a_r^{(\lambda)}$.  

Corollary. If $a_i$, $i = 1, 2, \cdots m$, are independent and irreducible in a semi-primary lattice $L$, then $a_i \cap e^{(\nu)}$, $i = 1, 2, \cdots m$, are independent and $(a_1 \cap a_2 \cup \cdots \cup a_m) \cap e^{(\nu)} = (a_1 \cap e^{(\nu)}) \cup (a_2 \cap e^{(\nu)}) \cup \cdots \cup (a_m \cap e^{(\nu)})$.

Pr. Putting $(a_1 \cap a_2 \cup \cdots \cup a_m) \cap e^{(\nu)} = \overline{e}^{(\nu)}$, it becomes the greatest element of the $\nu$-th derivative of the quotient $\frac{a_1 \cup a_2 \cdots \cup a_m}{O}$ and $a_i \cap e^{(\nu)} = a_i \cap \overline{e}^{(\nu)}$. Hence $a_i \cap e^{(\nu)}$ are a basis of the quotient $\frac{\overline{e}^{(\nu)}}{O}$.

Theorem 30. An element $a$ of a semi-primary lattice $L$ is supposed to have bases. The number $r_\nu$ of the basis-components with dimensions greater than $\nu - 1$ and the type of $a$ are then uniquely determined irrespective of the choice of bases.

Pr. In the proof of lemma 4 we have showed $r_\nu = \gamma_\nu$, where $\gamma_\nu$ is clearly irrespective of the choice of bases.

Lemma 6. If a quotient $\frac{g}{k}$ is a chain, then there exists a chain, $\frac{l}{O}$, such that $g = l \cup k$.

Pr. Let $g = a_1 \cup a_2 \cup \cdots \cup a_s$, where $a_i$, $i = 1, 2, \cdots s$, are irreducible, and let $\frac{a_\nu \cup k}{O}$ be with the highest dimension among the quotients $\frac{a_i \cup k}{O}$. Then we have $a_i \cup k \leq a_\nu \cup k$, $i = 1, 2, \cdots s$, since otherwise the sublattice $\frac{a_i \cup a_\nu \cup k}{k}$ of the quotient $\frac{g}{k}$ would not be a chain. Therefore we have $g = a_\nu \cup k$.

Theorem 31. A primary lattice $L$ is supposed to have a basis, whose components are all of dimension greater than $\nu$. Then every maximal chain in $L$ has a dimension greater than $\nu$.

Pr. Let $r$ be the rang of $L$. The case $r = 1$ being trivial, we first consider the case $r = 2$ and make its proof by induction on $\nu$. Given a chain $\frac{b}{O}$ in $L$ with dimension $\lambda < \nu + 1$, let $a_1$, $a_2$ be a basis of $e^{(\lambda+1)}$ and let $\frac{d}{O}$ be a maximal chain in $\frac{e^{(\nu)}}{O}$, which is in-
dependent with \( \frac{b}{O} \). By induction-hypothesis \( \dim d = \lambda \). Then

\[ e^{(\lambda)} = b \cup d, \quad \text{and} \quad \frac{e^{(\lambda)}}{b} \simeq \frac{d}{O} \]

is a chain of dimension \( \lambda \), which is contained in the quotient \( \frac{e^{(\lambda+1)}}{b} \).

This quotient is however not a chain, since the quotient \( \frac{e^{(\lambda+1)}}{e^{(\lambda)}} \) is not a chain. Hence there exists an atom \( \frac{c}{b} \) in \( \frac{e^{(\lambda+1)}}{b} \), which is independent with \( e^{(\lambda)} \) over \( b \). We have now by lemma 6 \( c = b \cup f \), where \( f \) is a chain, which is not contained in \( \frac{e^{(\lambda)}}{O} \). Henceforth we have \( \dim c = \dim f = \lambda + 1 \) and \( c = f, \quad b < f \). Therefore \( \frac{b}{O} \) is not maximal. In case \( r > 2 \) we prove by induction on \( r \). Given a chain \( \frac{b}{O} \) in \( L \) with dimension \( \lambda \), let \( a_i, \quad i = 1, 2, \cdots r \), be a basis of \( e^{(\lambda+1)} \) and put \( A = a_1 \cup a_2 \cup \cdots a_{r-1} \), where we can assume \( b \cup a_r = O \) without loss of generality, since otherwise we can replace \( a_r \) by a new component \( a'_r \), such that \( a'_r \cup a_1 = a_r \cup a_1, \quad a'_r \cap b = O \). Since the relation \( \frac{a_r \cup A}{A} = \frac{(b \cup a_r) \cup A}{A} \simeq \frac{b \cup a_r}{(b \cup a_r) \cap A} \) holds, and \( \dim (b \cup a_r) = 2\lambda + 1 \), we have, putting \( (b \cup a_r) \cap A = d, \quad \frac{d}{O} = \frac{d}{d \cap a_r} \simeq \frac{a_r \cup d}{a_r} \)

\[ = \frac{a_r \cup b}{a_r} \simeq \frac{b}{O} \].

The quotient \( \frac{b}{O} \) is therefore a chain in \( \frac{A}{O} \) with dimension \( \lambda \). Now by induction-hypothesis there exists a chain \( \frac{c}{O} \) of dimension \( \lambda + 1 \) in \( \frac{A}{O} \), which contains \( d \) and is independent with \( a_r \). The quotient \( \frac{c \cup a_r}{O} \) is primary of type \((\lambda + 1, \lambda + 1)\) and moreover contains the chain \( \frac{b}{O} \) of dimension \( \lambda \). Therefore \( \frac{b}{O} \) is not maximal.

**Theorem 32.** Every element of a primary lattice \( L \) has a basis.
On Primary Lattices

Pr. Since every quotient \( \frac{a}{O} \) is primary, we need only prove, that the element \( I \) has a basis. Suppose \( L \) is not a chain, we choose an irreducible element \( a_1 \) of highest dimension in \( L \), and next an irreducible element \( a_2 \) of highest dimension, which is independent with \( a_1 \). Suppose we have thus successively \( \nu \) irreducible elements \( a_i, i = 1, 2, \ldots \nu \), such that \( a_i \) is independent with \( A_{i-1} = a_1 \cup a_2 \cup \ldots \cup a_{i-1} \) and moreover of highest dimension. We will prove, that, whenever \( A_v < I \), we can find an irreducible element \( a_{v+1} \), which is independent with \( A_v \). Since \( A_v < I \), there exists certainly a chain, which is not contained in \( \frac{A_v}{O} \). We choose among those a chain \( \frac{b_{v+1}}{O} \), such that \( A_v \cup b_{v+1} = d_{v+1} \) is of least dimension.

Suppose \( d_{v+1} > O \) and \( c_{v+1} \) cover \( d_{v+1} \) in \( \frac{b_{v+1}}{O} \) and let \( \frac{e_{v+1}}{O} \) be a maximal chain in \( \frac{A_v}{O} \), which contains \( d_{v+1} \). If \( e_{v+1} > d_{v+1} \), then \( \frac{e_{v+1} \cup c_{v+1}}{O} \) is not a chain and therefore contains an atom \( h \), which does not belong to \( \frac{d_{v+1}}{O} \), such that \( h \cup e_{v+1} = e_{v+1} \cup e_{v+1} \). Then we have \( h \cap A_v = O \), which contradicts with the assumption. Now suppose \( e_{v+1} = d_{v+1} \), then, putting \( \mu = \dim e_{v+1} = \dim d_{v+1} \), holds the inequality \( \mu < \dim b_{v+1} \leq \dim a_1 \). By the way of selection of the elements \( a_i, i = 1, 2, \ldots \nu \), we have evidently \( \dim a_v \leq \dim a_{v-1} \leq \cdots \leq \dim a_1 \). The preceding theorem then asserts that there exists certainly a place, where \( \dim a_p > \mu \) and \( \dim a_{p+1} \leq \mu \) hold. If \( (a_1 \cup a_2 \cup \cdots \cup a_p) \cap e_{v+1} = O \), then \( (a_1 \cup a_2 \cup \cdots \cup a_p) \cap b_{v+1} = O \), contrary to the assumption, that \( a_{p+1} \) is independent with \( a_1 \cup a_2 \cup \cdots \cup a_p \) and moreover of highest dimension. Now let \( k_{v+1} = (a_1 \cup a_2 \cup \cdots \cup a_p) \cap e_{v+1} > O \). Then there exists a chain \( \frac{f_{v+1}}{O} \) of dimension \( \mu + 1 \) in \( \frac{a_1 \cup a_2 \cup \cdots \cup a_p}{O} \), which contains \( k_{v+1} \). The quotient \( \frac{f_{v+1} \cup c_{v+1}}{O} \), being not a chain, contains an atom \( \theta \), which is contained neither in \( \frac{f_{v+1}}{O} \) nor in \( \frac{c_{v+1}}{O} \). Since \( f_{v+1} \cup c_{v+1} = k_{v+1} \), the
quotient $\frac{f_{\nu+1} \cup c_{\nu+1}}{k_{\nu+1}}$ has a basis, whose components are both of dimension $= \mu + 1 - \dim k_{\nu+1}$. Let $\frac{\phi}{k_{\nu+1}}$ be a maximal chain in this quotient with $\dim \phi = \mu + 1$, such that $\phi$ contains $\theta \cup k_{\nu+1}$. Then $\phi = l \cup k_{\nu+1}$, where $l$ is a chain and $\dim l \leq \mu + 1$. If $\dim l = \mu + 1$, then $l \subseteq k_{\nu+1}$, whence $l \geq \theta$. The chain $\frac{k_{\nu+1}}{O}$ then contains the atom $\theta$ and henceforth $\frac{f_{\nu+1}}{O}$ contains $\theta$ contrary to the hypothesis, that $\theta$ is not contained in $\frac{f_{\nu+1}}{O}$. Therefore $\dim l \leq \mu$ and $l$ is not contained in $\frac{A_{\nu}}{O}$, since otherwise we would have $c_{\nu+1} \subseteq A_{\nu}$ from the relation $\phi \cup f_{\nu+1} = f_{\nu+1} \cup c_{\nu+1} = l \cup f_{\nu+1}$. On the other hand we have $\dim (l \cap A_{\nu}) = \dim l + \dim A_{\nu} - \dim (l \cup A_{\nu})$. $l \cup A_{\nu} = l \cup f_{\nu+1} \cup A_{\nu} = c_{\nu+1} \cup A_{\nu}$, $\dim (c_{\nu+1} \cup A_{\nu}) = \dim c_{\nu+1} + \dim A_{\nu} - \dim (c_{\nu+1} \cap A_{\nu})$, $c_{\nu+1} \cap A_{\nu} = b_{\nu+1} \cap A_{\nu} = d_{\nu+1}$, whence $\dim (l \cap A_{\nu}) = \dim l - \dim c_{\nu+1} + \mu = \dim l - 1 \leq \mu - 1$, yielding a contradiction $\dim (l \cap A_{\nu}) < \dim (b_{\nu+1} \cap A_{\nu})$. q.e.d.

**Theorem 33.** Every element of a semi-primary lattice has a basis, its type being uniquely determined irrespective of the choice of the bases.

We now fix the order of types of elements in a semi-primary lattice lexicographically; namely $(\nu_1, \nu_2, \cdots) > (\nu'_1, \nu'_2, \cdots)$, if $\nu_1 = \nu'_1 \cdots \nu_s = \nu'_s$ and $\nu_{s+1} > \nu'_{s+1}$.

**Theorem 34.** If $a \geq b$, then holds the relations following: (1) range of $a \geq$ range of $b$, (2) type of $a \geq$ type of $b$, and (3) $m_{\nu}(a) \geq m_{\nu}(b)$, where $m_{\nu}(a)$ denotes the number of basis-components with dimensions greater than $\nu - 1$ for the element $a$.

Pr. If we denote with $a^{(i)}, i = 1, 2, \cdots h$ the representatives of the element $a$, we have $a^{(i)} \geq b^{(i)}$ and consequently $\gamma_{\nu}(a) \geq \gamma_{\nu}(b)$. It follows then $m_{\nu}(a) \geq m_{\nu}(b)$ by virtue of the relation $r_{\nu} = \gamma_{\nu}$. q.e.d.

The summit of a maximal chain in $L$ is not always a basis-component of $L$. For example, consider the lattice of all subgroups in an abelian group, which is generated by an element $a_1$ of order 8 and an element $a_2$ of order 2, where $a_2 \neq a_1^4$. Then the cyclic subgroup $\{a_1^2 a_2\}$ is of order 4. This subgroup being the summit
of the maximal chain, which contains \( \{a_1^2a_2\}, \{a_1^4\} \) and \( \{e\} \), can not become a basis-component of the group. Under what conditions the summit of a maximal chain becomes a basis-component, shall be answered in the following.

**Theorem 35.** The summit of a maximal chain in a semi-primary lattice \( L \) can become a basis-component of \( L \), if and only if it is complemented.

**Definition.** A chain of highest dimension among those chains, which have a given atom commonly in possession, is called absolutely maximal.

**Theorem 36.** An element \( a \) in a semi-primary lattice \( L \) is complemented, if and only if every absolutely maximal chain in the quotient \( \frac{a}{O} \) is always absolutely maximal in \( L \).

**Pr.** By induction on the rang \( s \) of the quotient \( \frac{a}{O} \). In case \( s = 1 \) the quotient \( \frac{a}{O} \) itself is an absolutely maximal chain in \( L \) by hypothesis. Let \( c_i, i = 1, 2, \ldots r \) be a basis of \( L \) with \( \dim c_i = \nu_i \), \( \nu_1 \geq \nu_2 \geq \ldots \geq \nu_r \), and suppose \( \dim a = \nu, \nu_1 > a \geq \nu_{l+1}, \nu_o = \infty \).

By theorem 31 we have \( (c_1 \cup c_2 \cup \ldots \cup c_l) \cap a = O \) and hence \( \frac{a}{O} \) must be a chain of highest dimension, which is independent with \( c_1 \cup c_2 \cup \ldots \cup c_l \). Then \( a = \nu_{l+1} \) and the element \( a \) becomes a component of a basis together with \( c_1, c_2, \ldots c_l \). If conversely \( a \) is complemented and if \( \frac{a}{O} \) were not absolutely maximal, such that there exists a chain

\[
\frac{b}{O} \quad \text{with} \quad \dim a < \dim b, \quad a \cap b > O,
\]

then we have \( b \cap a' = O \) and

\[
\frac{a}{O} = \frac{a \cap a'}{a'} = \frac{b \cap a'}{a'} \simeq \frac{b}{O}
\]

in contradiction. Next suppose \( s > 1 \) and \( a_i, i = 1, 2, \ldots s \) are a basis of \( \frac{a}{O} \) with \( \dim a_1 \geq \dim a_2 \geq \ldots \geq \dim a_s \). First we consider the case \( \nu_1 = \dim a_1 = \dim a_2 = \ldots = \dim a_s \).

Then \( a_i, i = 1, 2, \ldots s \), are all able to become components of a basis of \( L \), as in the proof of theorem 32 and therefore complemented. Next consider the case \( \nu_1 < \dim a_1 = \dim a_2 = \ldots = \dim a_s \). Then
there exists $\nu_t$, such that $\nu_t > \dim a_1 \geq \nu_{t+1}$. Putting $C_t = c_1 \cup c_2 \cup \cdots \cup c_t$, we have $a \cap C_t = O$, since otherwise the absolutely maximal chain in $\frac{a}{O}$ which contains an atom in $a \cap C_t$ would not be absolutely maximal in $L$ by theorem 31. Hence we have $a_1 \cap C_t = O$ and necessarily $\dim a_1 = \nu_{t+1}$. Then $a_1$ can become a basis-component of $L$ together with $c_1, c_2, \cdots, c_t$. Since further $(c_1 \cup c_2 \cup \cdots \cup c_t \cup a_1 \cdots \cup a_{i-1}) \cap a_i = O$ and $\dim a_i = \nu_{t+1}$, $i = 1, 2, \cdots, s$, so $a_i$ are all basis-components of $L$ and therefore $a$ is complemented. Finally consider the case $\dim a_t > \dim a_{t+1} = \cdots = \dim a_s$. Putting $a_1 \cup a_2 \cup \cdots \cup a_t = A_1$, $a_{t+1} \cup \cdots \cup a_s = A_2$, it follows by the induction-hypothesis, that every absolutely maximal chain in $\frac{A_1}{O}$ or $\frac{A_2}{O}$ is always absolutely maximal in $\frac{a}{O}$ and consequently absolutely maximal in $L$. $A_1$ is therefore complemented in $L$ by the induction-hypothesis and $L$ has a basis, which contain all $a_i$, $i = 1, 2, \cdots, l$ as its components. Let it be $b_i$, $i = 1, 2, \cdots, r$ with $\dim b_i = \nu_i$. Since $\nu_1 \geq \dim a_1 > \dim a_s$, there exists $\nu_t$ such that $\nu_t > \dim a_s \geq \nu_{t+1}$. Putting $B_t = b_1 \cup b_2 \cup \cdots \cup b_t$, we have $B_t \cap A_2 = O$ and $\dim a_s = \nu_{t+1}$. The elements $a_{t+1}, \cdots, a_s$ become then basis-components of $L$ together with $b_1, \cdots, b_t$. Since $a_1, a_2, \cdots, a_t$ are part of $b_1, \cdots, b_t$, so $a_i$ are all basis-components of $L$, whence $a$ is complemented.

The converse statement of the last two cases follows immediately from the fact, that if $a$ is complemented, then every basis-component $a_i$ of $a$ becomes a basis-component of $L$. In fact the quotient $\frac{a_i}{O}$ will then be absolutely maximal in $L$ according to the converse statement of the first case.

**Corollary.** A chain in $L$ can become a basis-component of $L$ if and only if it is absolutely maximal in $L$.

**Theorem 37.** If a complemented element $c$ in a primary lattice $L$ is of rang $r$, then we can find $r + 1$ complements of $c$, such that the join of them equals to $e^{(h)}$, where by $h$ is meant the height of the complements.

Pr. The cases $c = I$ or $c = O$ being trivial, we prove by induction on $r$. In case $r = 1$, let $c'$ be a complement of $c$ and let
be an absolutely maximal chain of highest dimension in $\frac{c'}{O}$. Further let $\dim c = t$, $\dim d = h$ and $d'$ be a complement of $d$ in $\frac{c'}{O}$. If $t \geq h$, there exists a maximal chain $\frac{c'''}{O}$ in $\frac{c \cup d}{O}$, which does not contain the atom commonly with $\frac{c}{O}$ and $\frac{d}{O}$. Then we have $c \cap c''' = \Theta$, $c \cup c''' = c \cup d$, $\dim c''' = h$, $c \cup (c'' \cup d') = O$, $c \cup (c'' \cup d') = I$. Hence $c'' \cup d'$ is a complement of $c$ and $(c'' \cup d') \cup c' = (c'' \cup d') \cup d' = (c \cup d) \cup (e^{(h)}) \cup d' = (c \cup e^{(h)}) \cup c' = e^{(h)}$. If $t < h$, let $g$ be the element of dimension $h - t$ in $\frac{d}{O}$. There exists a maximal chain $\frac{c''}{O}$ of dimension $t$ in $\frac{c \cup d}{g}$, which does not possess the atom commonly with $\frac{c}{O}$ and $\frac{d}{g}$. We have then $c'' \cap c = O$ and $\frac{c''}{O}$ is a chain of dimension $h$ in $\frac{c \cup d}{O}$. For, if $\frac{c''}{O}$ were not a chain, $(c \cup d) \cap e^{(h)}$ and $c'' \cap e^{(h)}$ would be of rang 2 and $c'' \cap c > O$. Hence we obtain $c'' \cap c = d \cap c'' = c \cap d$ and consequently $c'' \cap d'$ is a complement of $c$ and $c \cap (c'' \cup d') = I = e^{(h)}$. In case $r > 1$, let $c_i$, $i = 1, 2, \cdots r$, be a basis of $c$ and let $\frac{b}{O}$ be an absolutely maximal chain of highest dimension in $\frac{c'}{O}$, where $c'$ is a complement of $c$, and $\dim b = h$. We choose a complement $b'_i$ of $c_i$ in $\frac{c_i \cup b}{O}$, such that $b \cap b'_i = (c_i \cap b) \cup e^{(h)}$. Further let $c'_i$, $f$ be complements of $c_i$, $b$ in $\frac{c}{O}$, $\frac{c'}{O}$ respectively. Then we have $(b'_i \cap f) \cap c = (b'_i \cap f) \cup (c_i \cup c'_i)$ $= (b'_i \cap c_i) \cup (f \cap c'_i) = (c_i \cap b) \cup (f \cap c'_i) = c \cap c' = I$, $(b'_i \cap f) \cup c = (b'_i \cap f) \cap c_i = b'_i \cap c_i = O$. Therefore $b'_i \cap f$, $i = 1, 2, \cdots r$, are all complements of $c$ and we have $c' \cup \sum_{i=1}^{r} (b'_i \cap f) = \sum_{i=1}^{r} (b \cap b'_i \cap f) = b \cap f \cup \sum_{i=1}^{r} (c_i \cap e^{(h)})$ $= \sum_{i=1}^{r} (c_i \cap c') \cap e^{(h)} = e^{(h)}$.

**Corollary 1.** Every complemented element in a primary lattice, which is different from $O$ and $I$, has at least two complements.

**Corollary 2.** An element in a semi-primary lattice, which has
an unique complement, is neutral and consequently belongs to the center.

Pr. Let $c$ be such an element and $I = e_1 \uplus e_2 \uplus \cdots \uplus e_n$, where $e_i$ independent and irreducible in the center. Since $c \cap e_i$ must have an unique complement in $\frac{e_i}{O}$, we have necessarily $c \cap e_i = e_i$ or $O$ and hence $c$ belongs to the center.

**Corollary 3.** If $\frac{c}{O}$ is a chain of highest dimension in a primary lattice $L$ of rang $r$, then we can choose two of its complements, such that their meet is of rang $r-2$ and complemented, and that their join equals to $e^{(h)}$, where by $h$ is meant the height of the complements.

Pr. Considering the case $t \geq h$ in the proof of the above theorem, we have $(c' \cap \overline{d'}) \cap c' = d'$ and $d'$ is evidently complemented.

**Theorem 38.** If $a \uplus b$ and $a \cap b$ are both complemented in a semi-primary lattice $L$, then $a$ and $b$ are complemented in $L$.

Pr. Since $a \cap b$ is complemented in the quotients $\frac{a}{O}$ and $\frac{b}{O}$, we have from $a = (a \cap b) \uplus \overline{a}$, $b = (a \cap b) \cap \overline{b}$, where $\overline{a} \cap b = O$, $b \cap a = O$, the relation $a \cap b = a \uplus b = b \uplus \overline{a}$. Hence $a$, $b$ are both complemented in $\frac{a \cap b}{O}$ and consequently also in $L$.

**Lemma 7.** If an element $a$ of a semi-primary lattice can be represented as the join of $m$ irreducible elements, then rang of $a \leq m$.

Pr. By induction on $m$. Suppose $m \geq 1$ and $a = a_1 \uplus a_2 \uplus \cdots \uplus a_m$, where $a_i$ is of the highest dimension among the irreducible elements $a_i$. Since the dimension of any chain in $\frac{a}{O}$ is not greater than that of $a_1$, so $a_1$ becomes a component of a basis and consequently complemented. Putting $a_2 \uplus \cdots \uplus a_m = A$, we have $\frac{a}{a_1} = a_1 \uplus A \simeq \frac{A}{a_1} \simeq a'_1$, where $a'_1$ is a complement of $a_1$. Since $A$ is the join of $a_i \uplus (A \cap a_i)$, $i = 2, 3, \cdots m$, where $\frac{a_i \uplus (A \cap a_i)}{A \cap a_i}$ are chains, so we have rang of $a'_1 \leq m-1$ by induction-hypothesis. Hence rang of $a \leq m$.

**Theorem 39.** If $a \geq b$, then $m_V(a) \geq m_V\left(\frac{a}{b}\right)$ and type of $a \geq$
type of $\frac{a}{b}$. In particular, rang of $a \geq$ rang of $\frac{a}{b}$.

Pr. In case $\nu = 1$, let $a_i, i = 1, 2, \ldots r$, be a basis of $a$. Then $a = \sum (a_i \cdot b)$ and by the above lemma rang of $\frac{a}{b} \leq$ rang of $a$.

Now, denoting with $e^{(i)}, E^{(i)}$ the greatest element of the $i$-th derivative of the quotient $\frac{a}{O}, \frac{a}{b}$ respectively, we have the inequality

$$m(\frac{a}{b}) = \text{rang of } \frac{a}{E^{(\nu-1)}} \leq \text{rang of } \frac{a}{E^{(\nu-1)}} \leq \text{rang of } \frac{a}{e^{(\nu-1)}} = m(a),$$

since $E^{(\nu-1)}$ contains $b \cdot e^{(\nu-1)}$.

Lemma 8. A basis $a_i, i = 1, 2, \ldots m$, of a complemented element $a$ in a semi-primary lattice $L$ and a basis $a'_i, i = 1, 2, \ldots n$, of its complement make together a basis of $L$.

Pr. From lemma 1 and lemma 2 follows the independency of $a_i, a'_i$.

Lemma 9. If an element $a$ in a semi-primary lattice $L$ is complemented, then $a \cdot e^{(i)}$ are complemented in $\frac{I}{e^{(i)}}$.

Pr. This follows directly from Lemma 4 and lemma 8.

Lemma 10. Given an element $a$ with the basis $a_i, i = 1, 2, \ldots r$ in a semi-primary lattice $L$ and given another element $b$ with $b \leq a$, rang $b = s < r$, we can choose $r-s$ components $a_i, a_i', \ldots a_{i_{r-s}}$ among $a_i$, such that $(a_i \cap a_{i'} \cap \cdots \cap a_{i_{r-s}}) \cap b = O$.

Pr. Suppose $b_i, i = 1, 2, \ldots s$, is a basis of $b$, then $b^{(1)} = b_i^{(1)} \cup b_2^{(1)} \cup \cdots \cup b_s^{(1)}$, where $b_i^{(1)} = b_i \cap e^{(1)}$. Since the first derivative $\frac{a^{(1)}}{O}$ of $a$ is complemented modular, so we have by lemma 3 a new basis of $a^{(1)}$, which consists of $b_i^{(1)}, i = 1, 2, \ldots s$ and $a_i^{(1)}, a_i'^{(1)}, \ldots, a_{i_{r-s}}^{(1)}$, such that $b \cap (a_i \cap a_{i'} \cap \cdots \cap a_{i_{r-s}}) = O$.

Theorem 40. If $a \geq b$ and $b$ is complemented in $\frac{a}{O}$, then

$$m_i(\frac{a}{b}) = m_i(a) - m_i(b), \ i = 1, 2, \ldots h.$$
equalities does not hold.

Pr. It suffices to prove in the case $a = I$. Suppose $b$ has a complement $b'$, then we have by lemma 8

\[ m_i\left(\frac{I}{b}\right) = m_i(b') = m_i(I) - m_i(b). \]

If $b$ is not complemented, then there exists certainly an absolutely maximal chain $\frac{d}{O}$ in $\frac{b}{O}$, which is not absolutely maximal in $\frac{I}{O}$ by theorem 36. So let $\frac{f}{O}$ be a chain in $\frac{I}{O}$, which contains the atom in $\frac{b}{O}$, such that $\dim f > \dim d$. Further we can find a basis of $b$, to which $d$ belongs as its component. Suppose $r, s$ are the ranges of $I$ and $b$ respectively. In the case $r = s$, we have $b \equiv I$, $m_i\left(\frac{I}{b}\right) > O$, and $m_i(I) = m_i(b)$. In case $r > s$, we can find $A = a_{i_1} \cup a_{i_2} \cup \cdots \cup a_{i_{i-r}}$, where $a_i$ are the basis-components of $\frac{I}{O}$, such that $b \cap A = O$ by lemma 10. If $d \ll f$, we have

\[
(b \cup f) \cap (b \cup A) = b \cup \{f \cap (b \cup A)\} = b \cup \{f \cap (d \cap d' \cup A)\}
\]

\[ = \left[d \cup \{(d' \cup A) \cap f\}\right] \cup b = d \cup b = b,
\]

where $d'$ is a component of $d$ in $\frac{b}{O}$. $b \cup f$ and $b \cup A$ are therefore independent over $b$ and consequently we have

\[
m_i\left(\frac{I}{b}\right) \geq m_i\left(\frac{b \cup f \cup A}{b}\right) = m_i\left(\frac{b \cup f}{b}\right) + m_i\left(\frac{b \cup A}{b}\right) = 1 + m_i(A) > m_i(I) - m_i(b).
\]

If $d \ll f$ does not hold, then $\frac{d \cup f}{O}$ is not a chain and consequently of rang two by lemma 7. We have then a chain $\frac{h}{O}$ such that $d \cup f = f \cup h$, $h \cap f = O$ and $\dim h < \dim d$, whence follows, putting $\dim d = i$, $e^{(i-1)} \cup d < e^{(i-1)} \cup f$. Since $d$ is in $\frac{b}{O}$ complemented, so $d \cup e^{(i-1)}$ is complemented in $\frac{b \cup e^{(i-1)}}{e^{(i-1)}}$ by lemma 4 and
henceforth the chain $\frac{d \cdot e^{(i-1)}}{e^{(i-1)}}$ is absolutely maximal in $\frac{b \cup e^{(i-1)}}{e^{(i-1)}}$ but not in $\frac{I}{e^{(i-1)}}$, whence follows

$$m_1\left(\frac{I}{b \cup e^{(i-1)}}\right) > m_1\left(\frac{I}{e^{(i-1)}}\right) - m_1\left(\frac{b \cup e^{(i-1)}}{e^{(i-1)}}\right).$$

Considering the relation $\dim \frac{I}{b \cup e^{(i-1)}} = \dim \frac{I}{e^{(i-1)}} - \dim \frac{b \cup e^{(i-1)}}{e^{(i-1)}}$, it then holds for some $j$ the relation

$$m_j\left(\frac{I}{b \cup e^{(i-1)}}\right) < m_j\left(\frac{I}{e^{(i-1)}}\right) - m_j\left(\frac{b \cup e^{(i-1)}}{e^{(i-1)}}\right) = m_{i+j-1}(I) - m_{i+j-1}(b).$$

Since further, for the $(i-1)$-th derivative $E^{(i-1)}$ of $\frac{I}{b}$ the relation $b \cup e^{(i-1)} \leq E^{(i-1)}$ holds, so we have

$$m_{i+j-1}\left(\frac{I}{b}\right) = m_j\left(\frac{I}{E^{(i-1)}}\right) \leq m_j\left(\frac{I}{b \cup e^{(i-1)}}\right) < m_{i+j-1}(I) - m_{i+j-1}(b).$$

**PART 3**

**Definition.** Two elements $a$ and $b$ in a modular lattice are called **quasi-perspective**, if there exists an element $c$, which is called the **axis**, such that $a \cap c = O$ and $a \cup c = b \cup c$. If furthermore $b \cap c = O$, then $a$ and $b$ are called **perspective** with respect to the axis $c$ and denoted with the symbol $a \sim b$.

**Theorem 41.** If $a$ and $b$ are quasi-perspective, then $\frac{a}{O}$ and $\frac{b}{b \cap c}$ are isomorphic by the correspondence $u \rightarrow v$ with $O \leq u \leq a$, $b \cap c \leq v = b \cap (u \cap c) \leq b$, where $u$ and $v$ are also quasi-perspective with respect to the axis $c$.

**Pr.** $v \cap c = (b \cup c) \cap (u \cap c) = (a \cap c) \cap (u \cap c) = u \cap c$, $u \cap c = a \cap c = O$, whence the quasi-perspective of $u$ and $v$ with the axis $c$ follows. We see from the relation $(v \cap c) \cap a = (u \cap c) \cap a = u \cap (c \cap a) = u$ that the correspondence $u \rightarrow v = b \cap (u \cap c)$ is one-to-one. It remains to prove that this induces moreover an isomorphism. From $u_1 \rightarrow v_1 = (u_1 \cup c) \cap b$, $u_2 \rightarrow v_2 = (u_2 \cup c) \cap b$, we obtain

$$v_1 \cup v_2 = \{(u_1 \cup c) \cap b\} \cup \{(u_2 \cup c) \cap b\} \leq (u_1 \cup u_2 \cup c) \cap b.$$
$u_1 \cdot u_2 = \{(v_1 \cdot c) \cap a\} \cdot \{(v_2 \cdot c) \cap a\} \leq (v_1 \cdot v_2 \cdot c) \cap a$,

$(u_1 \cdot u_2 \cdot c) \cap b \leq (v_1 \cdot v_2 \cdot c) \cap b = v_1 \cdot v_2$,

which yields $v_1 \cdot v_2 = (u_1 \cdot u_2 \cdot c) \cap b$. Further we have

$v_1 \cdot v_2 = \{(u_1 \cdot c) \cap b\} \cdot \{(u_2 \cdot c) \cap b\} \geq \{(u_1 \cap u_2) \cdot c\} \cap b = v_1 \cap v_2$.

We can choose the axis of quasi-perspectivity in such a manner, that it is contained in the quotient $\frac{a \cdot b}{O}$. Put $c' = c \cap (a \cdot b)$. Then

$u_1 \cdot c = b \cdot c$

$a \cap c = b \cap c = O$

$(b \cap c') \cap (a \cdot c') = (v_1 \cdot c) \cap (a \cdot c) = (v_1 \cap c) \cap b$.

Further from $v = (u \cdot c) \cdot (c \cdot e^{(i)})$ we obtain

$v \cdot e^{(i)} = (u \cdot c) \cdot (c \cdot e^{(i)}) \cap b = \{(u \cdot e^{(i)}) \cdot (c \cdot e^{(i)})\} \cap (b \cdot e^{(i)})$.

**Theorem 42.** If $a$ and $b$ are perspective with the axis $c$, then $a \cdot e^{(i)}$ and $b \cdot e^{(i)}$ are perspective with the axis $e \cdot e^{(i)}$ by the correspondence $u \cdot e^{(i)} \rightarrow v \cdot e^{(i)}$, where $O \leq u \leq a$, $O \leq v \leq b$, and $v = (u \cdot c) \cap b$.

**Pr.** From $a \cdot c = b \cdot c$ and $a \cap c = b \cap c = O$ we have by the corollary to lemma 4, the following relations

$(a \cdot e^{(i)}) \cdot (c \cdot e^{(i)}) = (a \cdot c) \cdot e^{(i)} = (b \cdot c) \cdot e^{(i)} = (b \cdot e^{(i)}) \cdot (c \cdot e^{(i)})$,

$(a \cdot e^{(i)}) \cap (c \cdot e^{(i)}) = O$, 

$(b \cdot e^{(i)}) \cap (c \cdot e^{(i)}) = O$.

Further from $v = (u \cdot c) \cap b$ we obtain

$v \cdot e^{(i)} = (u \cdot c) \cdot (c \cdot e^{(i)}) \cap b = \{(u \cdot e^{(i)}) \cdot (c \cdot e^{(i)})\} \cap (b \cdot e^{(i)})$.

**Theorem 43.** Two irreducible elements $e_1$, $e_2$ of the same dimension in a primary lattice are perspective.

**Pr.** Suppose $e_1 = e_2$ and dim $e_1 = \dim e_2$. The quotient $\frac{e_1 \cdot e_2}{O}$ being not a chain, there exists certainly an element $e_3$, such that $e_1 \cdot e_2 = e_1 \cdot e_3$, $e_1 \cap e_3 = O$. For the case $e_1 \cap e_2 = O$ we may take for $e_3$ the summit of a maximal chain in $\frac{e_1 \cdot e_2}{O}$, which contains neither the atom in $\frac{e_1}{O}$ nor the atom in $\frac{e_2}{O}$. Then $e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3$. 

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$e_1 \cap e_3 = e_2 \cap e_3 = O$, proving $e_1 \sim e_2$. For the case $e_1 \cap e_2 > O$, $e_2 \cap e_3 = O$ follows from $e_1 \cap e_3 = O$, whence we have $e_1 \cup e_2 = e_2 \cup e_3$, and consequently $e_1 \sim e_2$.

**Theorem 44.** If in a primary lattice two quotients $\frac{a}{a \cap d}$ and $\frac{b}{d}$ are isomorphic, and if $(a \cap b) \cap d$ is complemented in $\frac{I}{d}$, then $a$ and $b$ are quasi-perspective in $\frac{I}{a \cap d}$.

**Pr.** Put $a \cap d = a'$, $a' \cap b = (d \cap a) \cap b = d \cap (a \cap b) = d'$ and let $c_i, i = 1, 2, \ldots s$, be a basis of $\frac{d'}{d}$. Since by hypothesis $d'$ is complemented in $\frac{a'}{d}$, we can find a basis of $\frac{a'}{d}$, such that it consists of $c_i, i = 1, 2, \ldots s$ and $e_j, j = 1, 2, \ldots m$. Since $d'$ is also complemented in $\frac{b}{d}$, and since by hypothesis $\frac{b}{d}$ is isomorphic with $\frac{a'}{d}$, we may assume that $c_i, i = 1, 2, \ldots s$, $f_i, j = 1, 2, \ldots m$, is a basis of $\frac{b}{d}$ with $(e_1 \cup \cdots \cup e_m) \cap (f_1 \cup \cdots \cup f_m) = d$, where $\frac{e_i}{d}$ and $\frac{f_i}{d}$ are chains of the same dimension. Now $e_i$ and $f_i$ are perspective in $\frac{I}{d}$ by the preceding theorem and hence there exists $g_i$, such that $e_i \cup g_i = f_i \cup g_i = e_i \cap g_i = f_i \cap g_i = d$. Then we have $a \cup \sum g_i = a' \cup \sum g_i = \sum c_i \cup \sum e_i \cup \sum g_i = \sum c_i \cup \sum f_i \cup \sum g_i = b \cup \sum g_i$, $(a \cap \sum g_i) \cap d = a' \cap \sum g_i = d = b \cap \sum g_i$. Hence $a \cap \sum g_i = d \cap a$ and therefore $a$ is quasiperspective to $b$ in $\frac{I}{a \cap d}$ with the axis $\sum g_i$.

**Corollary 1.** If $\frac{a}{O}$ and $\frac{b}{O}$ are isomorphic and if $a \cap b$ is complemented, then $a$ and $b$ are perspective.

**Corollary 2.** If $a$ and $b$ are perspective, then they are projective. Conversely, if $a$ and $b$ are projective and further if $a \cap b$ is complemented, then they are perspective.

**Pr.** From $a \cap c = b \cap c$, $a \cap c = b \cap c = O$ follows immediately

$$\frac{a}{O} = \frac{a}{a \cap c} \sim \frac{a \cap c}{c} = \frac{b \cap c}{c} \sim \frac{b}{O}.$$

The converse statement follows from the preceding corollary.
Theorem 45. A modular lattice $L$ is primary if and only if all atoms in any quotient of $L$ are pairwise perspective.

Pr. It is evident by theorem 43, that any two atoms in a quotient of a primary lattice are perspective. We will show conversely, that there exists no proper neutral element in $L$, if $L$ is not a chain, and if $L$ satisfies the above mentioned condition. Let $a$ be an arbitrary element distinct from $O$ and $I$. For the case, $\frac{a}{O}$ is not a chain, let $\frac{b}{O}$ be simply a chain in $L$, which is not contained in $\frac{a}{O}$. For the case, $\frac{a}{O}$ is a chain, the chain $\frac{b}{O}$ must be moreover so chosen, that it does not contain the element $a$. Then we have $b \cap a < a$, $b \cap a < b$ and, denoting the atoms in $\frac{a}{b \cap a}$, $\frac{b}{b \cap a}$ with $c$, $d$ respectively, we can find $f$ such that $c \cup f = d \cup f = c \cup d$, $c \cap f = d \cap f = b \cap a = c \cap d$, since by hypothesis $c$ and $d$ are perspective in $\frac{a \cup b}{a \cap b}$. Then we have $(d \cup f) \cap a = (c \cup d) \cap a \geq c$ and $(d \cap a) \cap (f \cap a) = \{(f \cap a) \cap d\} \cap a = \{(b \cap a) \cup d\} \cap a = d \cap a = b \cap a$, whence $(d \cap a) \cap (f \cap a) < (d \cup f) \cap a$, and therefore $a$ is not neutral. q.e.d.

In a primary lattice $L$ let $a^*$ be quasi-perspective to $b$ with axis $x$ and $b$ quasi-perspective to $a$ with axis $y$, where $O < a^* \leq a$. Then $\frac{a^*}{O}$ is isomorphic with a quotient $\frac{a}{\kappa}$. We shall call this isomorphism a projection of $a^*$ into $a$, and $\kappa$, $a \cap b$, and $(x \cup y) \cap a = C$ respectively base, pole and center of the projection. Here the base $\kappa = \{(x \cap b) \cup y\} \cap a$ is obviously contained in the center $C$. In the particular case $\kappa = O$ we have automatically $a^* = a$ and then the projection is called a projective automorphism of $\frac{a}{O}$.

Theorem 46. In a projection of $a^*$ into $a$ with base $\kappa$, pole $P$, center $C$ to every element $u$ in $P \cap a^*$ corresponds $u \cup \kappa$ and to every element in $C \cap a^*$ corresponds an element in $C$.

Pr. If $u \leq a^* \cap b$, then $v = (u \cup x) \cap b = u \cap (x \cap b)$, $(v \cup y) \cap a = \{u \cap (x \cap b) \cup y\} \cap a = u \cap \{(x \cap b) \cap y\} \cap a = u \cap \kappa$. If $u \leq (x \cap y) \cap a^*$, then $v = (u \cup x) \cap b \leq [(x \cap y) \cap a^*] \cup x \cap b = (x \cap y) \cap (a^* \cup x) \cap b = (x \cap y) \cap b$, and, $\{(x \cap y) \cap b\} \cup y \cap a = (x \cap y) \cap (b \cup y) \cap a = (x \cap y) \cap a = C$, whence $(v \cup y) \cap a \leq C$. 

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Theorem 47. If in a projection of $a^*$ into $a$ with pole $P$, $u \rightarrow v$ with $v = (u \cdot x) \cap b$ and $v \rightarrow w$ with $w = (v \cdot y) \cap a$, then $u \cap P = v \cap P = w \cap P$.

Pr. From $(u \cap P) \cap x = O$ follows $v \cap P = (u \cdot x) \cap P = u \cap P$ by lemma 1. Further we have $(v \cap P) \cap y = b \cap y = O$, whence $w \cap P = (v \cdot y) \cap P = v \cap P$ follows by lemma 1.

Theorem 48. If $u \cap C = O$, then $v \cap (x \cdot y) = b \cap x$, $w \cap C = \kappa$.

Pr. From the hypothesis $u \cap (x \cdot y) = O$, we obtain $v \cap (x \cdot y) = (u \cdot x) \cap b \cap (x \cdot y) = x \cap b$ and $w \cap C = (v \cdot y) \cap a \cap (x \cdot \iota J) = \kappa$.

Definition. If the axes of a projection of $a^*$ into $a$ are both irreducible, then it is called normal.

Theorem 49. In a normal projection the base $\kappa$ and the center $C$ are both irreducible.

Pr. $C = (x \cdot y) \cap a$ and $C \cap x = O$ yield

$$\frac{C}{O} \simeq \frac{C \cap x}{x} \leq \frac{x \cap y}{x} \simeq \frac{y}{x \cap y}.$$

Theorem 50. If a normal projection of $a^*$ into $a$ has the base $\kappa$ with dimension $\nu$, then to every irreducible element $g$ of dimension $\lambda$ in $a^*$, which is independent with $P$, corresponds an irreducible element $h$ of dimension $\lambda + \nu$ in $\frac{a}{O}$, which is independent with $P$.

Pr. $h \cap P = O$ follows immediately from $g \cap P = O$ by theorem 47. From $\frac{a}{P} = \frac{a \cap b}{b} \simeq \frac{b \cap y}{b} \leq \frac{y}{O}$ follows, that $a = P \cup b$ with an irreducible element $l$ by lemma 6. Hence we obtain from $v = (g \cdot x) \cap b$ and $(v \cdot y) \cap a = h$ the relation

$$h \cap P = (v \cap P \cup y) \cap a = (v \cap P \cup y) \cap (P \cup l) = P \cup \{l \cap (v \cap P \cup y)\}.$$

Putting $l' = l \cap (v \cap P \cup y)$, we have $\frac{h}{O} \simeq \frac{h \cap P}{P} \simeq \frac{P \cup l'}{P} \simeq \frac{l'}{l' \cap P'}$ proving that $\frac{h}{O}$ is a chain of dimension $\lambda + \nu$, since $\frac{g}{O}$ and $\frac{h}{\kappa}$ are isomorphic in the projection.

Lemma 11. In a primary lattice $L$ with height $h$, there exists
an irreducible element of dimension $h$, which is independent with two
given elements, whose ranges are less than $m_h$.

Pr. Let $a_i$, $i = 1, 2, \cdots m_h$, be all the basis-components with
dimensional $h$ of $L$, and $c_1, c_2$ two given elements, whose ranges are
less than $m_h$. The theorem is evident for the case, where among
the atoms $a_i^{(1)} = a_i \cap e^{(1)}$ at least such an one exists, that it is con-
tained neither in $\frac{c_1}{O}$ nor in $\frac{c_2}{O}$. If it is not the case, there exists
certainly an atom $a_x^{(1)}$ contained in $\frac{c_1}{O}$ but not contained in $\frac{c_2}{O}$
and also an atom $a_y^{(1)}$ contained in $\frac{c_2}{O}$ but not contained in $\frac{c_1}{O}$.

Then, the quotient $\frac{a_x^{(1)} \cup a_y^{(1)}}{O}$ being not a chain, contains an atom $a$,
which is distinct from both $a_x^{(1)}$ and $a_y^{(1)}$. The absolutely maximal
chain, which contains $a$, is of dimension $h$ by theorem 31 and is
independent with both $c_1$, $c_2$. q. e. d.

Two projections are said to be identical, if their correspondence
relations are same in both projections.

**Theorem 51.** In a primary lattice $L$ with the height $h$ and
$h_m \geq 4$, let $\mathfrak{P}_1, \mathfrak{P}_2$ be two normal projections of $a^*$ into a with centers
$C_i$ and poles $P_i$, $i = 1, 2$, where it is assumed that rang $a \leq m_h - 2$, $P_1 \leq P_2$, and $C_1 \geq C_2$ or $C_1 \leq C_2$. If for a given element $t_0$, which
satisfies the relations $t_0 \cap C_1 = O$, $t_0 \cap C_2 = O$, $a^* = (P_1 \cap a^*) \cap t_0$, holds
$\mathfrak{P}_1(t_0) = \mathfrak{P}_2(t_0)$, then $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are identical.

Pr. Suppose that $\mathfrak{P}_i$ is generated by the quasi-perspectivities
$\frac{a^*}{O} \sim \frac{b_i}{\tau_i}$ with axis $x_i$ and $\frac{b_i}{O} \sim \frac{a}{\eta_i}$ with axis $y_i$, where we can
assume that $a \cap y_i = b_i \cap y_i = a \cap b_i$, $a^* \cap x_i = b_i \cap x_i = a^* \cap b_i$. If we put ($t_0 \cap x_i) \cap b_i = v_0^{(1)}$, $(v_0^{(2)} \cap y_i) \cap a = w_0$, ($t_0 \cap x_2) \cap b_2 = v_0^{(2)}$, then we have $(v_0^{(2)} \cap y_2) \cap a = w_0$ by hypothesis. It is required to prove, using
these relations, that for any element $t$ the relation $(v_2 \cap y_2) \cap a = w$
follows from $(t \cap x_1) \cap b_1 = v_1$, $(v_1 \cap y_1) \cap a = w$, $(t \cap x_2) \cap b_2 = v_2$. We
shall now distinguish two cases: case (i), where $x_2 \cap (a \cap x_1) = O$ and
case (ii), where $x_2 \cap (a \cap x_1) > O$.

Since $(t \cap x_2) \cap b_2 \subseteq (a^* \cap x_2) \cap x_1 = O$, we have by lemma 1

\begin{equation}
(1) \quad v_2 = (t \cap x_1 \cap x_2) \cap b_2.
\end{equation}

Since $a \cap x_2 = a \cap b_2 = a \cap y_2 = b_2 \cap y_2$ and $(w \cap y_2 \cap b_2) \cap y_1 \subseteq (a \cap x_2) \cap y_1$
\[ = a \cap y_1 \leq w, \text{ we have also by lemma 1} \]

\[ (2) \quad \overline{v}_2 = (w \cup y_2) \cap b_2 = (w \cup y_2 \cup y_1) \cap b_2. \]

Then (1) and (2) yield

\[ (3) \quad \overline{v}_2 \cap v_2 = (t \cup x_1 \cup x_2) \cap (w \cup y_1 \cup y_2) \cap b_2. \]

Putting \( u = (x_1 \cup x_2) \cap (y_1 \cup y_2) \) and considering \( v_1 \leq w \cup y_1, v_1 \leq t \cup x_1 \), we have \( u \cup v_1 \leq (t \cup x_1 \cup x_2) \cap (w \cup y_1 \cup y_2) \). Hence from (3) follows

\[ (4) \quad (u \cup v_1) \cap b_2 \leq v_2 \cap \overline{v}_2. \]

Now we will show that \( (u \cup v_1) \cap b_2 = v_2 \). From \( v_1 = (t \cup x_1) \cap b_1 \)

\[ = (t \cup x_1 \cup x_2) \cap b_1, \text{ we have} \]

\[ (5) \quad u \cup v_1 = (u \cap b_1) \cap (t \cup x_1 \cup x_2). \]

\[ (6) \quad (t_0 \cup x_1 \cup x_2) \cap (w_0 \cup y_1 \cup y_2) = (v_0^{(1)} \cup x_1 \cup x_2) \cap (v_0^{(1)} \cup y_1 \cup y_2) \]

\[ = v_0^{(1)} \cup \{(y_1 \cup y_2) \cap (v_0^{(1)} \cup x_1 \cup x_2)\} = v_0^{(1)} \cup u \]

where the last inequality follows by lemma 1, since

\[ v_0^{(1)} \cap (x_1 \cup x_2 \cup y_1 \cup y_2) = v_0^{(1)} \cap (C_i \cup x_1) \cap (t_0 \cup x_1) \cap (C_i \cup x_1) \cap b_1 \]

\[ = b_1 \cap [x_1 \cap \{t_0 \cap (C_i \cup x_1)\}] = b_1 \cap x_1 \leq x_1 \cup x_2, \]

where \( i = 1 \) or 2 according as \( C_1 \geq C_2 \) or \( C_2 \geq C_1 \). Similarly we obtain

\[ (7) \quad (t_0 \cup x_1 \cup x_2) \cap (w_0 \cup y_1 \cup y_2) = v_0^{(0)} \cup u. \]

But \((a^* \cap P_1) \cup v_0^{(1)} = (a^* \cap P_1) \cup \{t_0 \cup x_1 \cap b_1\} = \{(a^* \cap P_1) \cap t_0 \cup x_1\} \cap b_1 \]

\[ = (a^* \cup x_1) \cap b_1 = b_1 \text{ and } (a^* \cap P_1) \cup v_0^{(0)} = b_2. \]

Hence we have \( b_1 \cup u = b_2 \cup u \) in virtue of (6) and (7). Substituting this in (5) we have

\[ u \cup v_1 = (u \cap b_1) \cap (t \cup x_1 \cup x_2), \text{ whence} \]

\[ (u \cup v_1) \cap b_2 = b_2 \cap (t \cup x_1 \cup x_2) = b_2 \cap (t \cup x_2) = v_2. \]

Then from (4) follows \( v_2 \leq v_2 \cap \overline{v}_2 \) and this implies \( \overline{v}_2 \geq v_2 \). Hence

\[ (v_2 \cup y_2) \cap a \leq (v_2 \cup y_2) \cap a = [(w \cup y_2) \cap b_2] \cup y_2 = (w \cup y_2) \cap (b_2 \cup y_2) \cap a = w. \]

If we further regard, that \( (v_2 \cup y_2) \cap a \) and \( w \) are of the same dimension, then we have finally the required result.

Proof of the case (ii). By lemma 11 we can find an irreducible element \( x_3 \), such that \( x_3 \cap (a \cup x_1) = O, x_3 \cap (a \cup x_2) = O \) and \( \dim x_3 = \dim y_1 \). Since \( \frac{a}{P_1} = \frac{a}{a \cap b_1} = \frac{a \cup b_1}{b_1} = \frac{b_1 \cup y_1}{b_1} \simeq \frac{y_1}{O} \), so the uqo-


\[ \frac{a \cdot x_3}{P_1} \text{ is of type } (\dim y_1, \dim y_1) \text{ and consequently there exists} \]

a complement \( b'_3 \) of \( x_3 \cdot P_1 \) in \( \frac{a \cdot x_3}{P_1} \), such that \( a \cap b'_3 = P_1 \), \( b'_3 \cap (x_3 \cdot P_1) = P_1 \), and \( a \cdot b'_3 = a \cdot x_3 = b'_3 \cdot x_3 \). Putting \( b_3 = b'_3 \cdot (a^* \cdot x_3) \), we have \( b_3 \cap x_3 = a^* \cdot x_3 \), \( b_3 \cdot x_3 = O \). Therefore \( a^* \) and \( b_3 \) are perspective with respect to the axis \( x_3 \). Next, put \( v_3 = (t_0 \cdot x_3) \cap b_3 \), \( (w_0 \cdot v_3) \cap (C_1 \cdot x_3) = y_3 \), and we have

\[ (8) \quad v_3 \cap (P_1 \cap a^*) = (a^* \cdot x_3) \cap b_3 = b_3 \]

Now it holds \( t_0 \cdot v_1 = t_0 \cdot (b_1 \cdot (t_0 \cdot x_1)) = (t_0 \cdot b_1) \cap (t_0 \cdot x_1) = t_0 \cdot x_1 \), since \( t_0 \cdot b_1 = t_0 \cdot (a^* \cdot P_0) \cap b_1 = a^* \cdot b_1 = b_1 \cdot x_1 \). Hence it follows \( t_0 \cdot w_0 = t_0 \cdot \{(v_1 \cdot y_1) \cap a\} = (t_0 \cdot v_1 \cdot y_1) \cap a = (t_0 \cdot x_1 \cdot y_1) \cap a = t_0 \cdot \{(x \cdot y_1) \cap a\} = t_0 \cdot C_1 \). Now from \( w_0 \cap C_1 = \kappa_1 \), we have \( w_0 \cdot t_0 = t_0 \cdot C = w_0 \cap C_1 \) and by virtue of this relation

\[ (9) \quad v_3 \cdot y_3 = (w_0 \cdot v_3) \cap (C_1 \cdot x_3) = w_0 \cdot v_3 = w_0 \cdot y_3 \cdot y_3 \]

From (8) and (9) we have

\[ (10) \quad b_3 \cdot y_3 = (P_1 \cdot a^*) \cap v_3 \cdot y_3 = (P_1 \cdot a^*) \cap w_0 \cdot y_3 = a \cdot y_3 \]

where \( (P_1 \cdot a^*) \cap w_0 = a \) follows from \( (P_1 \cdot a^*) \cap t_0 = a^* \) by the isomorphism \( \frac{a^*}{a} \simeq \frac{O}{\kappa_1} \). Further we have

\[
\begin{align*}
\frac{b_3 \cdot y_3}{y_3} &= (w_0 \cdot v_3) \cap (C_1 \cdot x_3) \cap b_3 = \{v_3 \cup (w_0 \cdot b'_3)\} \cap (C_1 \cdot x_3) \\
&= \{v_3 \cup (w_0 \cdot P_1)\} \cap (C_1 \cdot x_3) = v_3 \cap (C_1 \cdot x_3) \quad \text{(by theorem 47)} \\
&= (t_0 \cdot x_3) \cap b_3 \cap (C_1 \cdot x_3) = x_3 \cap b_3 = O,
\end{align*}
\]

whence \( b_3 \cdot y_3 = O \) and so \( b_3 \) is quasiperspective to \( a \) with the axis \( y_3 \) by the relation (10). Since \( v_3 \cdot y_3 = (t_0 \cdot x_3) \cap (C_1 \cdot x_3) \cap b_3 = x_3 \cap b_3 = O \) and \( y_3 \simeq \frac{y_3 \cdot y_3}{y_3} = \frac{w_0 \cdot y_3}{y_3} = \frac{w_0}{v_3} \), so \( y_3 \) is irreducible and we have a normal projection \( \mathfrak{P}_3 \) of \( a^* \) into \( a \) by means of \( x_3, y_3, b_3 \). Next by virtue of \( v_3 \cdot x_3 = t_0 \cdot x_3 \) we have \( C_3 = (x_3 \cdot y_3) \cap a = (w_0 \cdot t_0 \cdot x_3) \cap (C_1 \cdot x_3) \cap a = (t_0 \cdot C_1 \cdot x_3) \cap (C_1 \cdot x_3) \cap a = C_1 \cap a = C_1 \) and \( P_3 = a \cap b_3 = P_1 \cap a^* \leq P_1 \), whence \( (P_3 \cap a^*) \cap t_0 = a^* \). From (9) follows \( (v_3 \cdot y_3) \cap a = (w_0 \cdot v_3) \cap a = w_0 \cdot (v_3 \cap a) = w_0 \cdot (t_0 \cdot P_1) = w_0 \) and
therefore $\mathfrak{P}_{3}(t_{0}) = \mathfrak{P}_{1}(t_{0}) = \mathfrak{P}_{2}(t_{0})$. Now, using the result in the first case, we obtain $\mathfrak{P}_{3} \equiv \mathfrak{P}_{1}$ and $\mathfrak{P}_{3} \equiv \mathfrak{P}_{2}$.

**PART 4.**

Hereafter $L$ is supposed to be primary and $m_{h} \geq 4$ with the height $h$. $a_{1}, a_{2}$ being two independent irreducible elements in $L$, the quotient $\frac{a_{1} \cup a_{2}}{O}$ is called a straight line in $L$. In a given straight line $l$ we choose a complemented irreducible element as its origin and one of its complements in $l$ shall be fixed as the infinite point of $l$. Every irreducible element in $l$, which is independent with the infinite point, is called a finite point. The dimension of any finite point is evidently not greater than that of the origin. In what follows, we denote with $a_{1}$ the origin and with $a_{2}$ the infinite point. Now we will establish a composition among finite points of dimension $\lambda$ by means of a normal projective automorphism in the sublattice $\frac{(a_{1} \cap e^{(\lambda)} \cup a_{2})}{O}$. If the infinite point contains its center and is contained in its pole, then we say such an automorphism addition. If in an addition the image of the point, which is contained in the origin, is the finite point $t$, we denote this addition with the symbol $A_{t}$ and the image of any point $u$ with $A_{t}(u)$.

**Theorem 52.** For any finite point $t$ of a straight line there exists the uniquely determined addition $A_{t}$.

**Pr.** According to the assumption $m_{h} \geq 4$, we can find certainly an irreducible element $x$ of dimension $\lambda$, which is independent with $a_{1} \cup a_{2}$. We put $a_{i}^{(\lambda)} = a_{i} \cap e^{(\lambda)}$, $i = 1, 2$, and choose an irreducible element $a_{3}$ of dimension $\lambda$ in $\frac{a_{1} \cup x}{O}$, which is independent with both $a_{1}$ and $x$. Next we put $a_{3} \cup a_{2} = b$, $a^{(\lambda)} = a_{1}^{(\lambda)} \cup a_{2}$, then we have $a^{(\lambda)} \cap x = b \cap x = O$, $a^{(\lambda)} \cap x = b \cap x = a^{(\lambda)} \cap b$. Hence $a^{(\lambda)}$ is perspective to $b$ with the axis $x$. Putting further $(a_{1}^{(\lambda)} \cap x) \cap b = v_{0}$, $(t \cap v_{0}) \cap (a_{2} \cap x) = y$, we have $a_{2} \cap y = (a_{2} \cap t \cap v_{0}) \cap (a_{2} \cap x) = (a^{(\lambda)} \cap v_{0}) \cap (a_{2} \cap x) = (a_{1}^{(\lambda)} \cap x \cap a_{2}) \cap (a_{2} \cap x) = a_{2} \cap x$, whence $a^{(\lambda)} \cap y = a^{(\lambda)} \cap x = b \cap x = b \cap y$. From $v_{0} \cap (t \cap a_{2}) = v_{0} \cap a^{(\lambda)} = O$ follows $a^{(\lambda)} \cap y = a_{2} \cap (t \cap v_{0}) = a_{2} \cap t = O$ and $b \cap y = O$ by virtue of the relation $\dim b = \dim a^{(\lambda)}$. Hence
$a^{(1)} \sim b$ with the axis $y$ and $y$ is irreducible by \( \frac{y}{O} \simeq \frac{a^{(1)} \cup y}{a^{(1)}} \).

Thus we obtain a normal projective automorphism of $\frac{a^{(1)}}{O}$ by means of $x, y, b$. That this is the required addition $A_t$, can be seen as follows. $(x \cup y) \cap a^{(1)} = (x \cup t \cup v_0) \cap (a_2 \cup x) \cap a^{(1)} = (x \cup t \cup v_0) \cap a_2 \leq a_2$ and $C \leq a_2 = a^{(1)} \cap b = P$. Furthermore $v_0 \cap y = (t \cup v_0) \cap (a_2 \cup x \cup v_0) = t \cup v_0$, whence $(v_0 \cup y) \cap a^{(1)} = t \cup (v_0 \cap a^{(1)}) = t$.

By theorem 51 we can conclude further, that such an addition is unique.

**Theorem 53.** (commutative law of addition). $t, s$ being any two finite points of the same dimension in a straight line, $A_t(s) = A_s(t)$ holds.

Pr. In the proof of the preceding theorem, put $v_0 = x'$, $a_2 \cup x = b'$, $(x' \cup a_2) \cap (s \cup x) = y'$. Then $x'$ is obviously an irreducible element of dimension $\lambda = \dim t = \dim s$. We have $a_2 \cup v_0 = b$ and $a_2 \cup y' = (a_2 \cup \cap (x' \cup a_2) = (a^{(1)} \cup x) \cap b = b$. Hence $a^{(1)} \cup y' = a^{(1)} \cup a^{(1)} \cup x' = a^{(1)} \cap x$ and $b' \cup y' = x \cup a_2 \cup y' = x \cup b = b' \cup x' = a^{(1)} \cap y'$. Since $a^{(1)} \cap y' = (x' \cap s) \cap s = b \cap s = O$, $b' \cap y' = (x' \cap a_2) \cap x = b \cap x = O$, $a^{(1)} \cup x' = a^{(1)} \cap v_0 = a^{(1)} \cup b = O$ and $b' \cap x' = (a_2 \cap x) \cap v_0 = O$, so we have $a^{(1)} \sim b'$ with axis $x'$ and $a^{(1)} \sim b'$ with axis $y'$, where $y'$ is irreducible by the relation \( \frac{y'}{O} \simeq \frac{a^{(1)} \cup y'}{a^{(1)}} = \frac{a^{(1)} \cup x}{a^{(1)}} \simeq \frac{x}{O} \). Thus we obtain a normal projective automorphism of $\frac{a^{(1)}}{O}$ by means of $b', x', y'$. It holds furthermore $a^{(1)} \cap b' = a^{(1)} \cap (a_2 \cap x) = a_2$, $x' \cap y' = (x' \cap a_2) \cap (s \cap x \cap x') = b \cap (s \cap x \cap x')$, $(x' \cap y') \cap a^{(1)} = a_2 \cap (s \cap x \cap x') \leq a_2$ and $(a^{(1)} \cup x') \cap b' = (a^{(1)} \cup x) \cap (a_2 \cap x) = x$, $(x \cap y') \cap a^{(1)} = (x \cap v_0 \cup a_2) \cap (s \cap x) \cap a^{(1)} = a^{(1)} \cap (s \cap x) = s$. This automorphism is therefore the addition $A_s$. Since $(t \cap x') \cap b' = (t \cap v_0) \cap (a_2 \cap x) = y$ and $(y \cap y') \cap a^{(1)} = [y \cup \{v_0 \cup a_2) \cap (s \cap x)\}] \cap a^{(1)} = [y \cup \{b \cap (s \cap x)\}] \cap a^{(1)}$, so we have the required result $A_s(t) = A_t(s)$. q. e. d.

This theorem justifies the adoption of the symbol $t + s$ for the point $A_t(s) = A_s(t)$, if dim $s = \dim t$.

**Theorem 54.** (associative law of addition). $t, s, u$ being any three finite points of the same dimension, holds the identity

\[ t + (s + u) = (t + s) + u. \]
Pr. Suppose we have obtained $A_t$ by means of $x, y, b$ just as in the proof of theorem 52. We put further $(s \cup x) \cap b = v_1$, $(a_1 \cup v_1) \cap (a_2 \cup x) = z$. Then $a_2 \cup v_1 = (a_2 \cup s \cup x) \cap b = b$.

\begin{align*}
(1) & \quad a^{(\lambda)} \cup z = a_2 \cup a^{(\lambda)} \cup z = a_2 \cup \{(a^{(\lambda)} \cup v_1) \cap (a^{(\lambda)} \cup x)\} \\
& \quad = a_2 \cup a^{(\lambda)} \cup v_1 = a^{(\lambda)} \cup b = a^{(\lambda)} \cup x \\
(2) & \quad a^{(\lambda)} \cap z = (a^{(\lambda)} \cup v_1) \cap a_2 = a^{(\lambda)} \cup a_2 = O
\end{align*}

by lemma 1, since $v_1 \cup a^{(\lambda)} = (s \cup x) \cap b \cap a^{(\lambda)} = s \cup a_2 = O$. Now $z$ is irreducible and of dimension $\lambda$. For in virtue of (1) and (2)

\[ \frac{z}{O} \simeq \frac{a^{(\lambda)} \cup z}{a^{(\lambda)}} \simeq \frac{a^{(\lambda)} \cup x}{O} \]

Further $b \cup z = v_0 \cup a_2 \cup z = v_0 \cup \{(a^{(\lambda)} \cup a_2 \cup v_1) \cap (a_2 \cup x)\} = v_0 \cup a_2 \cup x = b \cup z = a^{(\lambda)} \cup x$. This, together with (1), yields $b \cup z = a^{(\lambda)} \cup z$. Hence $a^{(\lambda)} \sim b$ with the axis $z$ and we obtain $A_{t+s}$ by means of $z, y, b$. For we have $a^{(\lambda)} \cup b = a_2$, $z \cup y \leq a_2 \cup x$, whence $(z \cup y) \cap a^{(\lambda)} \leq a_2$ and $(a^{(\lambda)} \cup z) \cap b = (a^{(\lambda)} \cup v_1) \cap b = v_1$, $(v_1 \cup y) \cap a^{(\lambda)} = A_t(s)$. We see further, that $A_*$ can be obtained by means of $z, x, b$. In fact $(z \cup x) \cap a^{(\lambda)} \leq (a_2 \cup x) \cap a^{(\lambda)} \leq a_2$ and $(a^{(\lambda)} \cup z) \cap b = v_1$, $(v_1 \cup y) \cap a^{(\lambda)} = A_t(s)$. It holds then $(A_s(u) \cup x) \cap b = (u \cup z) \cap b$, whence we have $\left\{((A_s(u) \cup x) \cap b) \cup y\right\} \cap a^{(\lambda)} = \left\{((u \cup z) \cup b) \cup y\right\} \cap a^{(\lambda)}$, proving the identity $A_t(A_s(u)) = A_{t+s}(u)$. 

q. e. d.

Theorem 55. (possibility and uniqueness of substraction). Given any two finite points $s, t$ of dimension $\lambda$, there exists the addition $A_*$ uniquely, such that $A_*(u) = s$, i. e. $u + t = s$.

Pr. We choose $x, b$ as in the proof of theorem 52 and put $(x \cup t) \cap b = v$, $(v \cup s) \cap (a_2 \cup x) = y$. Then $a_2 \cup v = b$ and $a_2 \cup y = (a_2 \cup v \cup s) \cap (a_2 \cup x) = (a^{(\lambda)} \cup v) \cap (a_2 \cup x) = a_2 \cup x$. Hence $a^{(\lambda)} \cup y = a^{(\lambda)} \cup x$ and $b \cup y = b \cup x$, whence $a^{(\lambda)} \cup y = b \cup y$. Further we have $a^{(\lambda)} \cup y = a_2 \cap (v \cup s)$, and $(a_2 \cup s) \cap a^{(\lambda)} = O$, since $v \cap (a_2 \cup s) = v \cap a^{(\lambda)} = (x \cup t) \cap a_2 = t \cap a_2 = O$, and $b \cap y = (v \cup s) \cap b \cap (a_2 \cup x) = v \cap (a_2 \cup x) = O$. Therefore $a^{(\lambda)}$ is perpective to $b$ with the axis $y$. Furthermore holds the relation $\frac{y}{O} \simeq \frac{a^{(\lambda)} \cup y}{a^{(\lambda)}} \simeq \frac{a^{(\lambda)} \cup x}{O}$ and consequently $y$ is irreducible. Since $a^{(\lambda)} \cap b = a_2$ and $(x \cup y) \cap a^{(\lambda)} = (v \cap v \cup s) \cap a^{(\lambda)} \leq a_2$, we have now an addition by means of $x, y, b$ and, since $v \cap y = (v \cap a_2 \cup x) \cap (v \cup s) = (b \cup x) \cap (v \cup s) = v \cup s$, $(v \cap y) \cap a^{(\lambda)} = (v \cup s) \cap a^{(\lambda)} = s \cap (v \cap a^{(\lambda)}) = s$ holds, the image of $t$ is indeed the point $s$. Since from
$A_t(u) = A_t(u')$ follows $u = u'$, so the substraction is unique. q. e. d.

The dimension of the meet of a finite point $t$ and the origin is called order of the point $t$ and denoted with $O(t)$. Every finite point, whose order is zero, is called unit. In a given straight line $l = \frac{a_1 \cup a_2}{O}$, where $a_1$ is the origin and $a_2$ the infinite point, put $a_1 \cap e^{(i)} = a_1^{(i)}$, $a_1^{(j)} \cup a_2 = a^{(i)}$. A normal projection of $a^{(i)}$ into $a^{(j)}$, where $j \leq i$, is called multiplication, if $P \geq a_2$, $\kappa \leq C \leq a_1$ holds, where $\kappa$, $P$, $C$ are base, pole, and center of the projection. We choose one of complemented units in $l$ as a fixed unit point of $l$ and denote it with $e_{12}$.

**Theorem 56.** Given a finite point $t$ of dimension $i$ with order $\nu$, there exists uniquely such a multiplication, that the image of the point $e^{(j)}_{12} = e_{12} \cap e^{(j)}$ is the point $t$, where $j = i - \nu$.

**Pr.** We choose an irreducible element $x$ of dimension $j$ in $L$, which is independent with $a_1 \cup a_2$, and also an irreducible element $a_3$ of dimension $j$ in $\frac{a_1^{(j)} \cup x}{O}$, which is independent with both $a_1$ and $x$. Putting $a^{(j)} = a_1^{(j)} \cup a_2$, $b = a_2 \cup a_3$, we have $a^{(j)} \cup x = b \cup x$, $a^{(j)} \cap x = b \cap x = O$. Hence $a^{(j)}$ is perspective to $b$ with the axis $x$. Next we put $(e^{(j)}_{12} \cup x) \cap b = v_{12}$, $(t \cap v_{12}) \cap (a_1^{(i)} \cup x) = y$. Then it follows

\[
(b \cap y) = (t \cap v_{12}) \cap (a_1^{(i)} \cup x) \cap b = \{v_{12} \cap (t \cap b)\} \cap (a_1^{(i)} \cup x) = v_{12} \cap (a_1^{(i)} \cup x) = (e^{(j)}_{12} \cup x) \cap b = O,
\]

\[
a_2 \cup v_{12} = (a_2 \cup e^{(j)}_{12} \cup x) \cap b = (a^{(j)} \cup x) \cap b = b,
\]

\[
v_{12} \cap y = (t \cap v_{12}) \cap (a_1^{(i)} \cup x \cup v_{12}) = (t \cap v_{12}) \cap (a_1^{(i)} \cup e^{(j)}_{12} \cup x) = t \cap v_{12} = (t \cap v_{12}) \cap (a_1^{(i)} \cup t \cap x) = t \cap y,
\]

whence $b \cap y = a_2 \cap v_{12} \cap y = a_2 \cup t \cap y = a^{(i)} \cup y$. By these relations we see that $b$ is quasiperspective to $a^{(i)}$ and $y$ is irreducible, since $v_{12} \cap y = O$, $v_{12} \cup y = t \cap v_{12}$ and $y \simeq \frac{v_{12} \cup y}{v_{12}} = \frac{t \cap v_{12}}{v_{12}} \simeq \frac{t}{O}$. Further $a^{(i)} \cup b = a_2$, $(x \cap y) \cap a^{(i)} \leq a_1$ and $\kappa = a^{(i)} \cap y = a_1^{(i)}$, so we have a multiplication by means of $x$, $y$, $b$. Next it follows

\[
(v_{12} \cap y) \cap a^{(i)} = (t \cap v_{12}) \cap a^{(i)} = t \cap (v_{12} \cap a^{(j)}) = t
\]

and therefore the image of $e^{(j)}_{12}$ is the point $t$. q. e. d.
We denote the above multiplication with the symbol $M_t$ and the image of a point $u$ with $M_t(u)$. That such multiplication is unique, is the result of theorem 51, since $e_1^{(j)} \cap C = O$, $e_2^{(j)} \cup (P \cap a^{(j)}) = e_2^{(j)} \cup a_2 = a^{(j)}$ hold. Given a finite point $t$ with $\dim t = i$, $O(t) = \nu$ and an arbitrary finite point $s$, we define product of $t$ and $s$ as the point $M_t(s^{(i-\nu)}), \text{ where } s^{(i-\nu)} = s \cap e^{(i-\nu)}$, and denote with the symbol $ts$. The present definition yields then $\dim ts = i$, if $\dim s \geq i - \nu$, and $\dim ts = \dim s + \nu$, if $\dim s < i - \nu$.

**Theorem 57.** If $\dim t = i$, $O(t) = \nu$, $\dim s = \mu < i - \nu$, then $t \cdot s = t^{(\nu + \mu)}s$, where $t^{(\nu + \mu)} = t \cap e^{(\nu + \mu)}$.

**Pr.** We choose $x$, $a_3$, $y$, $b$ just as in theorem 56. Putting $b^{(\mu)} = a_2 \cup a_3^{(\mu)}$, $x^{(\mu)} = x \cap e^{(\mu)}$, $(e_1^{(j)} \cup x^{(\mu)}) \cap (a_3^{(\mu)}) = v^{(\mu)}$, $y^{(\mu)} = (t^{(\nu + \mu)} \cup v^{(\mu)}) \cap (a_3^{(\mu)})$, we have $M_t(v^{(\mu)})$ by means of $x^{(\mu)}$, $y^{(\mu)}$, $b^{(\mu)}$. Since $t^{(\nu + \mu)} = [(s \cup x^{(\mu)}) \cap b^{(\mu)}] \cup y^{(\mu)} \cap a^{(\mu)} = ts$ and $\dim t^{(\nu + \mu)} = \dim ts$, so we obtain $t^{(\nu + \mu)} = ts$.

**Theorem 58.** $s$, $t$ being two finite points with $O(s) = \mu$, $O(t) = \nu$, $\dim ts = j$, then $O(ts) = O(t) + O(s)$, if $j > \mu + \nu$, and $O(ts) = j$, if $j \leq \mu + \nu$.

**Pr.** Suppose $M_t$ is generated by means of $x$, $y$, $b$ just as in theorem 56. Then, putting $\dim t = i$, $(s^{(i-\nu)} \cup x) \cup b = v_1$, $(v_1 \cup y) \cap a^{(i)} = ts$ and $a_1^{(i)} \cup y = a_1^{(i)} \cup x^{(i)}$. Hence we obtain

$$(3) \quad a_1^{(i)} \cup ts = (a_1^{(i)} \cup v_1 \cup y) \cap a^{(i)} = (a_1^{(i)} \cup v_1 \cup x) \cap a^{(i)}$$

$$(3) \quad a_1^{(i)} \cup ts = (a_1^{(i)} \cup s^{(i-\nu)} \cup x) \cap a^{(i)} = a_1^{(i)} \cup s^{(i-\nu)}.$$

In case $\dim s \geq i - \nu$, we have $\dim ts = j = i$. Then from (3) it follows

$$\dim (a_1^{(i)} \cup ts) = i + (i - \nu - \mu), \text{ if } \nu + \mu < i,$$

$$\dim (a_1^{(i)} \cup ts) = i, \text{ if } \nu + \mu \geq i,$$

whence

$$O(ts) = \nu + \mu, \text{ if } \nu + \mu < j$$

$$O(ts) = j, \text{ if } \nu + \mu \geq j$$

In case $\dim s < i - \nu$, we have $j = \dim s + \nu \geq \mu + \nu$ and from (3) $\dim (a_1^{(i)} \cup ts) = i + \dim s - \mu$, whence follows

(1) This follows from $a_1^{(i)} \cup y \leq a_1^{(i)} \cup x$, $a_1^{(i)} \cup y = a_1^{(i)}$ and $\dim y = \dim x + \nu$. 


Lemma 12. The product of a point in the origin with any finite point is in the origin.

Pr. Suppose $t = a_1^{(i)}$ and $M_t$ is generated by $x, y, b$. Then $(O \cdot x) \cap b = O, (O \cdot y) \cap a^{(i)} = \kappa = a_1^{(i)} = t$.

Theorem 59. (associative law of multiplication). $(ts)u = t(su)$.

Pr. Suppose $\dim t = i, O(t) = \nu, O(s) = \mu$. If $\mu + \nu \geq i$, then $ts = a_1^{(i)}$ by theorem 58 and $(ts)u = a_1^{(i)}$ by the preceding lemma. Since $\dim (su) \geq \mu \geq i - \nu$, we have $\dim t(su) = i, O(su) \geq \mu$. Hence $\dim t(su) \leq O(t) + O(su)$ and by theorem 58 $t(su) = a_1^{(i)}$.

Now we have only to treat the case $\mu + \nu < i$. Put $i - \nu = \lambda$, if $\dim s \geq i - \nu$, and $\dim s = \lambda$, if $\dim s < i - \nu$. Theorem 57 and 58 yield then the results $ts = t^{(\lambda + \nu)}s$, $\dim ts = \lambda + \nu, O(ts) = \mu + \nu$. Since further $\dim su \leq \dim s$, we have $t(su) = t^{(\lambda + \nu)}(su)$ by theorem 57. Therefore we can assume $\dim t = i = \lambda + \nu$ without loss of generality and we see that

\[(4)\quad (ts)u = M_t(u^{(\lambda - \nu)}), \quad (5)\quad t(su) = M_t\{M_s(u^{\lambda - \nu})\}\]

by virtue of $M_s(u^{(\lambda - \nu)}) = (su)^{\lambda - \nu}$. We will now prove that the right hand sides of both (4) and (5) are equal. Suppose $M_t$ is generated by $x, y, b$, where $b = a_2 \cdot a_3^{(i)}, (s^{(i)} \cdot x) \cap b = v_1$ and $(v_1 \cdot y) \cap a^{(i)} = ts$ as in theorem 56. Putting $(e_2^{(i)} \cdot v_1) \cap (x \cdot a_1^{(i)}) = z$, we have $a^{(i)} \cap z = (e_2^{(i)} \cdot v_1) \cap a^{(i)} = e_2^{(i)} \cap a_1^{(i)} = O$, where $(a_2^{(i)} \cdot e_2^{(i)}) \cdot v_1 \leq a^{(i)} \cap v_1 = O$.

From the relations $z \cap e_2^{(i)} = O, z \cap e_2^{(i)} = (e_2^{(i)} \cdot v_1) \cap (x \cdot a_1^{(i)} \cdot e_2^{(i)}) = e_2^{(i)} \cdot v_1$ follows $\frac{z}{O} \simeq \frac{z \cdot e_2^{(i)}}{e_2^{(i)}} = \frac{e_2^{(i)} \cdot v_1}{e_2^{(i)}} \simeq \frac{v_1}{O}$ and therefore $z$ is irreducible. Further we have

\[(6)\quad a_2 \cdot v_1 = (a_2 \cdot s^{(i)} \cdot x) \cap b = (a^{(i)} \cdot x) \cap b = b\]

\[(7)\quad v_1 \cdot z = (e_2^{(i)} \cdot v_1) \cap (x \cdot v_1 \cdot a_1^{(i)}) = (e_2^{(i)} \cdot z) \cap (x \cdot s^{(i)} \cdot a_1^{(i)}) = e_2^{(i)} \cdot z.\]

From (6) and (7) follows $b \cdot z = a_2 \cdot v_1 \cdot z = a^{(\lambda - \nu)} \cdot z$ and hence $a^{(\lambda - \nu)}$ is quasi-perspective to $b$ with the axis $z$. Since further $(x \cdot z) \cap a^{(i)} = (e_2^{(i)} \cdot v_1 \cdot x) \cap a^{(i)} \leq a_1$, so we have a multiplication by means of $z, x, b$. That this multiplication is indeed $M_s^{(i)}$, can be seen from
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\[(e^{2}_{3}^{(\lambda-\mu)} \bowtie z) \cap b = (x \bowtie a^{(i)} \bowtie e^{2}_{3}^{(\lambda-\mu)}) \cap v_{1} = v_{1},\]
\[(v_{1} \bowtie x) \cap a^{(i)} = (s^{(i)} \bowtie x) \cap a^{(i)} = s^{(i)}.\]

Next the quasi-perspectivity of \(a^{(\lambda-\mu)}\) and \(b\) with axis \(z\) and the quasi-perspectivity of \(b\) and \(a^{(i)}\) with axis \(y\) yield a normal projection of \(a^{(\lambda-\mu)}\) into \(a^{(i)}\). Further \((y \bowtie z) \cap a^{(i)} \leq (x \bowtie a^{(i)} \cap a^{(i)} \leq a_{1},\)
\((e^{2}_{3}^{(\lambda-\mu)} \bowtie z) \cap b = v_{1}, (v_{1} \bowtie y) \cap a^{(i)} = t \cdot s,\)
Hence this projection is \(M_{ts}\).
Then we have \((u^{(\lambda-\mu)} \bowtie z) \cap b = v_{2}, (v_{2} \bowtie x) \cap a^{(i)} = M_{s}(u^{(\lambda-\mu)}),\)
\(\{M_{s}(u^{(\lambda-\mu)}) \bowtie x\} \cap b = (v_{2} \bowtie x) \cap b = v_{2}, (v_{2} \bowtie y) \cap a^{(i)} = M_{sa}(u^{(\lambda-\mu)}).\) q.e.d.

**Theorem 60.** If \(O(t) \leq O(s)\) and \(\dim s \leq \dim t\) for any two finite points \(t, s\), then there exists uniquely a finite point \(q\), such that \(s = tq\) and \(\dim q = \dim s - O(t)\).

**Pr.** Put \(\dim t = i\), \(O(t) = \nu, a^{(i-\nu)} = a^{(i-\nu)} \bowtie a_{2}, b = a_{3} \bowtie a_{2},\)
\((e^{2}_{3}^{(\lambda-\mu)} \bowtie z) \cap b = v_{12}, (v_{22} \bowtie t) \cap (a^{(i)} \bowtie x) = y\) just as in theorem 56, then we have \(M_{t}\) by means of \(x, y, b\). Now in this projection to \(s\) corresponds the element \(q = (v \bowtie x) \cap a^{(i-\nu)},\) where \(v = (s \bowtie y) \cap b\). Since \(q \bowtie x = v \bowtie x, q \cap a^{(i-\nu)} \bowtie x = O\) and \(\frac{v}{O} \simeq \frac{v \bowtie x}{x} = \frac{q \bowtie x}{x} = \frac{q}{O},\)
and since \(v \bowtie y = s \bowtie y, v \bowtie y = b \bowtie y = O, \frac{v}{O} \simeq \frac{v \bowtie y}{y} = \frac{s \bowtie y}{y} \simeq \frac{s}{a^{(i)}},\)
so \(q\) is irreducible and of dimension \(\dim s - \nu\). Further, by virtue of \(q \bowtie a_{2} = v \bowtie a_{2} = (s \bowtie y) \bowtie a_{2} = O, q\) is finite and \(s = tq\).

**Theorem 61.** If \(O(s) = \mu, O(t) = \nu, \mu \geq \nu\) and \(\dim s \leq \dim t + (\mu - \nu)\) for any two finite points \(s, t\), then there exists a finite point \(q\) such that \(s = qt\) and \(\dim q = \dim s\).

**Pr.** First we consider the case \(\mu = \nu\) and put \(\dim s = i\). Since the case \(i = \nu\) is trivial, we assume that \(i > \nu\). We determine \(x, b\) as in theorem 56 and put \((t^{(i)} \bowtie x) \cap b = v, (s \bowtie v) \cap (a^{(i)} \bowtie x) = y',\)
Then we have \(a^{(i)} \bowtie y' = (a^{(i)} \bowtie s \bowtie v) \cap (a^{(i)} \bowtie x) = (a^{(i)} \cap a^{(i-\nu)} \bowtie v) \cap (a^{(i)} \bowtie x) = a^{(i)} \bowtie x\) and \(a^{(i)} \cap y' = (s \bowtie v) \cap a^{(i)} = a^{(i)}\). Hence \(\frac{y'}{a^{(i)}} \simeq \frac{a^{(i)} \bowtie y'}{a^{(i)}} = \frac{x}{O}\) and \(\frac{y'}{a^{(i)}}\) is a chain of dimension \(i\). By lemma 6 we have then \(y' = l \bowtie a^{(i)}\), where \(l\) is irreducible and of dimension not less than \(i\). Since \(l \leq a^{(i)} \bowtie x\), the dimension of \(l\) must be equal to \(i\) and consequently \(l \bowtie a^{(i)} = O\). The quotient \(\frac{y'}{O}\) is then of type \((i, \nu)\). If we put further \(v \bowtie (a^{(i)} \bowtie x) = v_{0},\) then \(v_{0} = (x \bowtie a^{(i)}) \cap b\) and
\[
dim v_0 = \dim b + \dim (x \circ a_{1}^{(\nu)}) - \dim (b \circ x) = \nu.
\]

Now \( v_0 \) is contained in \( \frac{y'}{O} \) and \( \frac{y'}{v_0} \) is a chain of dimension \( i \). For \( \frac{y'}{v_0} = \frac{(s \circ v) \cap (a_{1}^{(\nu)} \circ x)}{v \cap (s \circ v) \cap (a_{1}^{(\nu)} \circ x)} = \frac{s \circ v}{v} = \frac{s}{O}. \)

Since \( m_j\left(\frac{y'}{v_0}\right) = m_j(y') - m_j(v_0) \) holds for every \( j \), so \( v_0 \) is complemented in \( \frac{y'}{O} \) by theorem 40. The chain \( \frac{a_{1}^{(\nu)}}{O} \) is absolutely maximal in \( \frac{y'}{O} \). Therefore a complement \( y \) of \( v_0 \) in \( \frac{y'}{O} \) is irreducible with dimension \( i \) and independent with \( a_{1} \) by the assumption \( i > \nu \).

Hence \( y \) is also a complement of \( a_{1}^{(\nu)} \) in \( \frac{y'}{O} \). From \( b \circ y = v \circ (a_{1}^{(\nu)} \circ x) \)

\[
= (t^{(i)} \circ x) \cap (a_{1}^{(\nu)} \circ x) \cap b = (x \circ a_{1}^{(\nu)}) \cap b = v_0 \quad \text{follows} \quad b \circ y = b \circ y' \circ y = v_0 \circ y = O \quad \text{and from} \quad a^{(i)} \circ y = a^{(i)} \circ y' = a^{(i)} \circ x \quad \text{follows} \quad b \leq a^{(i)} \circ y, \quad \text{whence} \quad b \circ y = a^{(i)} \circ y \quad \text{and consequently} \quad a^{(i)} \sim b \quad \text{with axis} \quad y. \quad \text{Since further} \quad a^{(i)} \circ b = a_{2}, \quad (x \circ y) \cap a^{(i)} \leq (x \circ y') \cap a^{(i)} \leq a_{1}^{(\nu)} \quad \text{holds, so we have a multiplication by means of} \quad x, \quad y, \quad b. \quad (t^{(i)} \circ x) \cap b = v \quad \text{and}
\]

\[
(v \circ y) \circ a^{(i)} = (v \circ v_0 \circ y) \circ a^{(i)} = (v \circ y') \circ a^{(i)}
\]

\[
= (s \circ v) \cap (a_{1}^{(\nu)} \circ t^{(i)} \circ x) \cap a^{(i)} = s \circ \{v \cap (a_{1}^{(\nu)} \circ t^{(i)})\} = s
\]

show that this is the required multiplication.

In case \( \mu > \nu \), we choose an arbitrary finite point \( q' \) with \( \dim q' = i, \quad O(q') = \mu - \nu \) and put \( s' = q't. \) Then \( O(s') = O(s) \) and \( \dim s' = \dim s. \) Hence there exists a finite point \( q'' \), such that \( s = q''s' = q''(q't). \) Finally, applying the associative law, we obtain \( s = (q''q')t. \)

**Theorem 62.** \( t, s \) being two finite points of the same dimension, it holds \( O(t+s) \geq O(t) = O(s) \), if \( O(t) = O(s) \), and \( O(t+s) = \text{Min} \{O(t), O(s)\} \), if \( O(t) \neq O(s). \)

Pr. Put \( O(s) = \mu, \quad O(t) = \nu \) and assume \( \nu \geq \mu. \) In virtue of \( A_t(s^{(\mu)}) = A_t(a_{1}^{(\mu)}) = t^{(\mu)} \) and \( A_t(s^{(\mu+1)}) = t^{(\mu+1)} \) we have \( \dim \{(t+s) \cap t\} = \mu \), whence follows \( (t+s) \cap a_{1} \geq s \circ a_{1}, \) that is \( O(t+s) \geq \mu. \) In particular \( O(t+s) = \mu, \) if \( \nu > \mu. \)

**Theorem 63.** (first distributive law). If \( s, u \) are two finite
points of the same dimension, then holds $t(s+u) = ts+tu$ for every finite point $t$.

Pr. Suppose $\dim t = i$, $O(t) = \nu$. Then we can assume $\dim s = \dim u = i-\nu$ without loss of generality. Next suppose we obtain $M_t$ by means of $x$, $y$, $b$, such that, putting $\lambda = i-\nu$, $(e^{(i)} \odot x) \triangleq b = v$, $(v \triangleright t) \odot (a_1^{(i)} \odot x) = y$. Next we choose an irreducible element $x'$ of dimension $\lambda$, which is independent with $a_1 \odot a_2 \odot x$ and an irreducible element $a_4$ of dimension $\lambda$ in $O(a_1^{(i)} \odot x')$, which is independent with both $x'$ and $a_1$. Putting $b' = a_4 \odot a_2$, $(a_1^{(i)} \odot x') \triangleq b' = v'$, $(v' \odot s) \triangleq (a_2 \odot x') = y'$, we have $a^{(i)} \sim b'$ with axis $x'$ and $b' \sim a^{(i)}$ with axis $y'$. Since $a^{(i)} \triangleq b' = a_2$, $(x' \odot x') \triangleq a^{(i)} \leq a_2$ and $(v' \odot y') \triangleq a^{(i)} = s$ hold, so we obtain $A_s$ by means of $x'$, $y'$, $b'$. Now the perspectivity $a^{(i)} \odot x' \sim b \odot x'$ with axis $x$ and the quasi-perspectivity of $b \odot x'$ and $a^{(i)} \odot x'$ with axis $y$ generates a normal projection $\mathfrak{P}$ of $a^{(i)} \odot x'$ into $a^{(i)} \odot x'$, where the pole is $(a^{(i)} \odot x') \triangleq (b \odot x') = x' \odot a_2$. Since $v' \odot (x' \odot a_2) = O$, $\mathfrak{P}(v') = q$ is irreducible and of dimension $i$ with $q \triangleq (x' \odot a_2) = O$, $q \geq \lambda = a^{(i)}$ by theorem 50. Putting $b'' = a_2 \odot q$, we have

$$b'' \odot x' = a_2 \odot \mathfrak{P}(v') \odot \mathfrak{P}(x') = a_2 \odot \mathfrak{P}(v' \odot x') = a_2 \odot \mathfrak{P}(a_1^{(i)} \odot x'),$$

where

$$\mathfrak{P}(a^{(i)}) = [(a_1^{(i)} \odot x) \odot (b \odot x')] \odot y \cap (a_1^{(i)} \odot x').$$

Therefore we have $b'' \odot x' = a^{(i)} \odot x'$, $a^{(i)} \cap x' = O$ and $b'' \odot x' = O$, whence $a^{(i)} \sim b''$ with axis $x'$. Further we have

$$b'' \odot y' = a_2 \odot y' \odot \mathfrak{P}(v') = a_2 \odot x' \odot \mathfrak{P}(v') = b'' \odot x' = a^{(i)} \odot x' = a^{(i)} \odot y'.$$

and $b'' \odot y' = a^{(i)} \odot y' = (v' \odot s) \odot a_2 = v' \odot a_2 = O$, whence $a^{(i)} \sim b''$ with axis $y'$. It holds furthermore $a^{(i)} \triangleq b'' \geq a_2$, $(x' \odot y') \triangleq a^{(i)} \leq a_2$ and $(a_1^{(i)} \odot x') \triangleq b'' = (a_1^{(i)} \odot x') \cap (a_2 \odot q) = \mathfrak{P}\{(a_1^{(i)} \odot x') \cap (a_2 \odot v')\} = \mathfrak{P}(v') = q$, $(q \odot y') \triangleq a^{(i)} = \mathfrak{P}\{(v' \odot y') \cap a^{(i)}\} = \mathfrak{P}(s)$.

Since $\mathfrak{P}$ induces a multiplication $M_t$ in $a^{(i)} \odot O$, so $\mathfrak{P}(s) = ts$. Consequently we have $A_{ts}$ by means of $x'$, $y'$, $b''$, and

$$t(s+u) = \mathfrak{P}(s+u) = \mathfrak{P}\{(u \odot x') \odot b'' \odot y'\} \cap \mathfrak{P}a^{(i)}$$

$$= [(\mathfrak{P}(v') \odot x') \odot b'' \odot y'] \cap a^{(i)} = ts + \mathfrak{P}(u) = ts + tu.$$

**Theorem 64.** (second distributive law). If $u, s$ are two $a^{(i)}$-
points of the same dimension, then \((s+u)t = st + ut\) for every finite point \(t\).

Pr. Putting \(\dim s = \dim u = i\), we assume \(\lambda = \dim t = i - \nu > O\). First we consider the case \(O(s) = O(u) = \nu\), \(O(t) = 0\). Then \(\mu = O(s+u) \geq \nu\) by theorem 62. We choose \(a_3, x, b\) as usual such that \(a^{(i)} \sim b\) with axis \(x\). Putting \((e^{(i)} \cdot x) \cap b = p\), \((t \cdot x) \cap b = q\) and

\[
(8) \quad (s \cdot p) \cap (a_1^{(i)} \cdot x) = y_s, \quad (9) \quad (u \cdot p) \cap (a_1^{(i)} \cdot x) = y_u,
\]

we have

\[
(10) \quad (y_s \cdot q) \cap a^{(i)} = st, \quad (11) \quad (y_u \cdot q) \cap a^{(i)} = ut.
\]

If we put \(\lambda' = i - \mu\), then \(\lambda' \leq \lambda\) and from the above perspectivity follows \(a^{(i)} \sim b^{(i)}\) with axis \(x^{(i)}\) and \((e^{(i)} \cdot x^{(i)}) \cap b^{(i)} = P^{(i)}\), \((t^{(i)} \cdot x^{(i)}) \cap b^{(i)} = q^{(i)}\).

We choose an irreducible element \(a_4\) of dimension \(i\) in \(\frac{a_1^{(i)} \cdot x'}{O}\), which is independent with both \(a_1\) and \(x'\). Putting \(a_4^{(i)} \cdot a_1^{(i)} = a'\), \(a_4^{(i)} \cdot a_4 = b'\), we have \(a' \sim b'\) with axis \(x'\). Further putting \((a_1^{(i)} \cdot x') \cap b' = v', (v' \cdot s) \cap (a_2 \cdot x') = y'\), we have \(A_s\) by means of \(x', y', b'\). Hence \([\{(u \cdot x') \cap b'\} \cap y'] \cdot a^{(i)} = s + u\) and putting \(\{(s + u) \cdot p^{(i)}\} \cap (a_1^{(i)} \cdot x) = y_{s+u}\), we have

\[
(12) \quad (y_{s+u} \cdot q^{(i)}) \cap a^{(i)} = (s + u)t.
\]

Next it holds for \(a_1^{(i)} \cdot x = d\) the relations

\[
(d \cdot x') \cdot p = a_1^{(i)} \cdot x' \cdot x \cdot p = a_1^{(i)} \cdot e^{(i)} \cdot x \cdot x' = a_1^{(i)} \cdot a_2^{(i)} \cdot x \cdot x' = (a' \cdot x') \cdot p,
\]

\[
(d \cdot x') \cap p = d \cap p = d \cap (e^{(i)} \cdot x) \cap b = x \cap b = O,
\]

\[
(a' \cdot x') \cap p = (a' \cdot x') \cap (e^{(i)} \cdot x) \cap b = a_4^{(i)} \cdot (e^{(i)} \cdot x) = O.
\]

Hence \(a' \sim x' \sim d \sim x'\) with axis \(p\). Next we have

\[
(d \cdot x') \cdot q = (a_1^{(i)} \cdot x') \cdot (q \cdot x) = (a_1^{(i)} \cdot x') \cdot (t \cdot x) = a_1^{(i)} \cdot a_2^{(i)} \cdot x = a' \cdot x' \cdot q
\]

and

\[
(d \cdot x') \cap q = (d \cdot x') \cap (t \cdot x) \cap b = x \cap b = O,
\]

\[
(a' \cdot x') \cap q = (a' \cdot x') \cap (t \cdot x) \cap b = a_4^{(i)} \cap (t \cdot x) = O.
\]
Therefore \( a' \preceq x' \sim d \preceq x' \) with axis \( q \) and we obtain a normal projective automorphism \( T \) of \( a' \preceq x' \) by means of \( p, q, d \preceq x' \), its pole being \( (a' \preceq x') \cap (d \preceq x') = a^{(0)} \preceq x' \). From (8), (9), (10), (11) and (12) we have \( st = T(s), \ ut = T(u), \ (s + u)t \leq T(s + u) \) and \( T(x') = x' \), \( T(b') \preceq x' = T(b' \preceq x') = T(a' \preceq x') \). Now we proceed as follows

\[
T(a^{(1)}_2) = \{(a^{(1)}_2 \preceq p) \cap (d \preceq x') \} \preceq q \cap (a' \preceq x')
\]

Hence \( T(b') \preceq x' = T(a' \preceq x') \preceq T(a^{(0)}_1 \preceq a^{(0)}_2 \preceq x') = a^{(0)}_1 \preceq a^{(0)}_2 \preceq x' = a' \preceq x' \),

\[
T(b') \cap T(y') = T(b' \cap y') = T(a' \cap y') = a' \cap T(y').
\]

and \( T(b') \cap x' = T(b' \cap x') = O \), \( T(b') \cap T(y') = T(b' \cap y') = O \).

Therefore \( a^{(0)} \sim T(b') \cap a_2 \) with axis \( x' \) and \( T(y') \). Since further \( T(b') \cap a' = T(a^{(2)}_2) = a^{(2)}_2 \), \( \{x' \cap T(y')\} \cap a^{(0)} \leq a_2 \), and

\[
\{(a^{(0)}_1 \cap x') \cap T(b') \} \cap T(y') = T[\{(a^{(0)}_1 \cap x') \cap b') \} \cap y'] \cap a' = T(s) = st,
\]

so we obtain \( A_{st} \) by means of \( x', T(y'), T(b') \cap a_2 \). Then

\[
st + ut \geq \{(ut \cap x') \cap T(b') \} \cap T(y') \cap a'
\]

\[
= T[\{(u \cap x') \cap b') \} \cap y'] \cap a' = T(s + u).
\]

Now we have \((s + u)t \leq st + ut\), from which we conclude \((s + u)t = st + ut\) considering the dimensions of the both sides.

In case \( O(t) = O \) and \( O(s) \geq O(u) \), it follows \( O(s + u) = O(u) = O(-u). \) Hence we have \((s + u)t + (-u)t = \{(s + u) + (-u)\}t = st.\) From \((-u)t + ut = \{(-u) + u\}t = a^{(0)}_1 \) follows then \((s + u)t = st + ut.\) In case \( O(t) > 0 \), it is evidently \( O(t + e^{(1)}) = 0 \). Hence it follows

\[
(s + u)(t + e^{(1)}) = s(t + e^{(1)}) + u(t + e^{(1)}) = st + ut + se^{(1)} + ue^{(1)} = st + ut + s + u.
\]

On the other hand we have \((s + u)(t + e^{(1)}) = (s + u)t + (s + u)\) by applying the first distributive law. Hence the uniqueness of subtraction yields \((s + u)t = st + ut.\)

(1) Here \(-u\) means a finite point, for which \((-u) + u = a^{(0)}_1\) holds.
PART 5.

As in the preceding parts let $L$ be a primary lattice with height $h$, $m_{h} \geqq 4$ and let $a_{i}, 1, 2, \cdots, r$, be its basis with $d_{i} = \dim a_{i}$, where $d_{1} = d_{2} = d_{3} = d_{4} = h$ holds. In each straight line $\frac{a_{i} \cdot a_{j}}{O}$ we choose an irreducible element $e_{ij}$ of dimension $d_{ij}$ as its unit element, which is independent with both $a_{i}$ and $a_{j}$. In the straight line $\frac{a_{i} \cdot a_{j}}{O}$, where $i, j$ distinct from 1, we fix the element $[a_{i} \cdot \{(a_{i} \cdot e_{1j}) \cap (a_{j} \cdot e_{ij})\}] \cap (a_{i} \cdot a_{j})$ as its unit element $e_{ij}$. Then we will show that (i) $e_{ij}$ is independent with both $a_{i}$ and $a_{j}$. (ii) $e_{ij}$ is irreducible and of dimension $\min (d_{i}, d_{j})$. In order to prove these, we suppose $d_{i} \geqq d_{j}$.

Since $e_{ij} \cap a_{i} = \{(a_{i} \cdot e_{ij}) \cap (a_{j} \cdot e_{ij})\} \cap a_{i} = (a_{i} \cdot e_{1j}) \cap a_{i} = e_{ij} \cap a_{i} = 0$, so lemma I yields $e_{ij} \cap a_{j} = O$. The validity of (ii) can be seen by

$$\frac{e_{ij}}{O} \simeq \frac{a_{i} \cdot e_{ij}}{a_{i}} = \frac{(a_{i} \cdot a_{j} \cdot e_{ij}) \cap (a_{j} \cdot e_{ij})}{a_{i}} = \frac{a_{i} \cdot a_{j}}{a_{i}} \simeq \frac{a_{j}}{O}.$$

Theorem 65. Among the unit elements $e_{ij}$ of straight lines $\frac{a_{i} \cdot a_{j}}{O}$ in $L$ hold the relations

$$[a_{k} \cdot \{(a_{i} \cdot e_{k}) \cap (a_{j} \cdot e_{k})\}] \cap (a_{i} \cdot a_{j}) = e_{ij}^{(dk)},$$

where $k$ is distinct from $i$ and $j$.

Pr. By definition the case $k = 1$ is evident. For the case $i = 1$ (or $j = 1$) we have

$$(a_{1} \cdot e_{ij}) \cap (a_{j} \cdot e_{ik}) = [a_{1} \cdot (a_{k} \cdot e_{ij}) \cap (a_{j} \cdot e_{ik})] \cap (a_{j} \cdot e_{ik}) = \{(a_{k} \cdot e_{ij}) \cap (a_{j} \cdot e_{ik})\} \cap (a_{j} \cdot e_{ik}) = (a_{k} \cdot e_{ij}) \cap (a_{j} \cdot e_{ik}),$$

whence $[a_{k} \cdot \{(a_{i} \cdot e_{k}) \cap (a_{j} \cdot e_{ik})\}] \cap (a_{i} \cdot a_{j}) = (a_{k} \cdot e_{ij}^{(dk)}) \cap (a_{i} \cdot a_{j}) = e_{ij}^{(dk)}$.

The cases, where all $i, j, k$ are distinct from 1, can be treated as follows.
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\[ a_i \cdot e_{kj} \approx [a_1 \cdot \{ (a_i \cdot a_k \cdot e_{ij}) \cap (a_i \cdot a_j \cdot e_{1k}) \} ] \cap (a_i \cdot a_k \cdot a_j) ; \]

\[ a_j \cdot e_{ki} = [a_1 \cdot \{ (a_j \cdot a_k \cdot e_{1i}) \cap (a_i \cdot a_j \cdot e_{1k}) \} ] \cap (a_i \cdot a_k \cdot a_j) . \]

Now the relation \( a_i \cap (a_i \cdot a_j \cdot e_{1k}) = O \) yields

\[ (a_i \cdot e_{kj}) \cap (a_j \cdot e_{ki}) = [a_1 \cdot \{ (a_i \cdot a_k \cdot e_{1j}) \cap (a_j \cdot a_k \cdot e_{1i}) \} ] \cap (a_i \cdot a_k \cdot a_j) \]

\[ [a_k \cdot \{ (a_i \cdot e_{kj}) \cap (a_j \cdot e_{ki}) \} ] \cap (a_i \cdot a_j) = [a_1 \cdot \{ (a_i \cdot e_{1j}) \cap (a_j \cdot e_{1k}) \} ] \cap (a_i \cdot a_j) \leq e_{ij} . \)

For an irreducible element \( l \) of dimension \( \lambda \) in the straight line \( \frac{a_i \cdot a_j}{O} \) we choose a basis-component \( a_k \) such that \( \dim a_k \geq \lambda, \ k \neq i, \ k \neq j \). Then we can find an irreducible element \( l' \) of dimension \( \lambda \) with \( a_k \cap l = a_k \cap l', \ a_k \cap l' = O, \ (a_i \cap a_j) \cap l' = O \). We put

\[ \bar{l} = [\{(a_j \cap l') \cap (a_i \cap a_k)\} \cap \{(a_i \cap l') \cap (a_j \cap a_k)\}] \cap (a_i \cap a_j) \]

and will prove that \( \bar{l} \) is also an irreducible element of dimension \( \lambda \), uniquely determined irrespective of the choice of \( a_k, \ l' \). Suppose \( \cap a_j = O \) and take \( a_i, \ a_j \) respectively as the origin and the infinite point of the straight line. Using \( a_j \cap l = a_j \cap a_i^{(\lambda)} \), we have

\[ (a_j \cap a_i^{(\lambda)}) \cap a_k = (a_j \cap l') \cap a_k, \ (a_j \cap l') \cap a_k = l' \cap a_k = O \]

by lemma 1, since \( (a_k \cap l) \cap a_j = l \cap a_j = O \). Therefore \( a_j \cap a_i^{(\lambda)} \sim a_j \cap l' \)

with axis \( a_k \). Putting \( (a_i \cap l') \cap (a_j \cap a_k) = m \), we obtain \( m \cap a_j = (a_i \cap l') \cap a_j = a_i \cap a_j = O \) and

\[ \frac{m}{O} \approx \frac{m \cap a_j}{a_j} = \frac{a_j \cap a_i^{(\lambda)}}{a_j} \approx \frac{a_k^{(\lambda)}}{O} . \]

Hence \( m \) is irreducible and of dimension \( \lambda \). Further we have

\[ (a_j \cap a_i^{(\lambda)}) \cap m = a_j \cap a_i^{(\lambda)} \cap a_k^{(\lambda)} = a_j \cap a_k^{(\lambda)} \cap l' = (a_j \cap l') \cap m \]

and \( m \cap (a_j \cap a_i^{(\lambda)}) = O. \ m \cap (a_j \cap l') = m \cap a_j = O \) by lemma 1, since

\[ l' \cap (a_j \cap m) = l' \cap (a_j \cap a_i^{(\lambda)}) = l' \cap (a_k \cap l) \cap (a_j \cap a_i^{(\lambda)}) \]

\[ = l' \cap a_k^{(\lambda)} \cap \{ a_j \cap (a_k \cap l) \} = l' \cap a_k^{(\lambda)} = O . \]

(1) The calculation of dimension of the left-hand side yields the required result.
Therefore \( a_j \bowtie l' \) is perspective to \( a_j \bowtie a_i^{(\lambda)} \) with axis \( m \). Since further 
\((a_j \bowtie a_i^{(\lambda)}) \cap (a_j \bowtie l') = a_j, \) \((a_k \bowtie m) \cap (a_j \bowtie a_i^{(\lambda)}) \leq a_j \) holds, we have really an addition in \( \frac{a_i \bowtie a_j}{O} \) by means of \( a_k, m, a_j \bowtie l' \). Now we put 
\[ p = (a_i \bowtie a_k) \cap (a_j \bowtie l') = (a_i^{(\lambda)} \bowtie a_k) \cap (a_j \bowtie l'). \]
Then 
\[ \bar{l} = (p \bowtie m) \cap (a_i \bowtie a_j) = (p \bowtie m) \cap (a_i^{(\lambda)} \bowtie a_j). \]
Hence \( \bar{l} \) is the image of \( a_i^{(\lambda)} \) in this addition and therefore \( \bar{l} \) is irreducible with dimension \( \lambda \). Since further
\[(l \bowtie a_k) \cap (a_j \bowtie l') = l', \]
\[(l' \bowtie m) \cap (a_i^{(\lambda)} \bowtie a_j) = (a_i \bowtie a_k \bowtie l') \cap (a_i^{(\lambda)} \bowtie a_j) = a_i^{(\lambda)}, \]
so we have \( A_{\bar{l}}(l) = a_i^{(\lambda)} \), that is \( l = -\bar{l} \). Therefore \( \bar{l} \) is uniquely determined irrespective of the choice of \( a_k, l \) by theorem 51.

The element \( l \) is called conjugate to the element \( \bar{l} \).

**Theorem 66.** \( (e_{ki} \bowtie e_{kj}) \cap (a_i \bowtie a_j) = e_{ij}^{(dj)}, \) where \( e_{ij} \) is conjugate to \( e_{ij} \) and \( e_{ij} \cap a_i = O, e_{ij} \cap a_j = O \).

Pr. Putting \( (a_i \bowtie e_{kj}) \cap (a_j \bowtie e_{ki}) = e_{ij}^{(d)}, \) we have \( a_k \bowtie e_{ij}^{(d)} = a_k \bowtie e_{ij}^{(dj)} \) by theorem 65, and \( e_{ij}^{(d)} \cap a_k = O, e_{ij}^{(d)} \cap (a_i \bowtie a_j) = O. \) Now it follows
\[ e_{ij}^{(dj)} = \left[ \{(a_j \bowtie e_{ij}^{(d)}) \cap (a_i \bowtie a_k)\} \cap (a_i \bowtie a_j) \right] \]
\[ = \left[ \{(a_j \bowtie e_{ij}^{(d)}) \cap (a_i \bowtie a_k)\} \right. \left. \cap (a_i \bowtie a_j) \right] \]
\[ = (e_{ij}^{(dj)} \bowtie e_{ij}^{(d)}) \cap (a_i \bowtie a_j) = (e_{ij}^{(dj)} \bowtie e_{ij}^{(d)}) \cap (a_i \bowtie a_j) \]

**Theorem 67.** \( (e_{ij}^{(d)}) \bowtie (a_i \bowtie a_k) = e_{ik}^{(dj)}. \)

Pr. Choose \( a_p, \) such that \( d_p = h, \rho \vdash i, \rho \vdash j, \rho \vdash k. \) Then we have
\[ \bar{e}_{ij} = (e_{pi} \bowtie e_{pj} \bowtie e_{pk}) \cap (a_i \bowtie a_j) = (e_{pi} \bowtie e_{pj} \bowtie e_{pk}) \cap (a_i \bowtie a_j). \]
\[ \bar{e}_{jk} = (e_{pi} \bowtie e_{pj} \bowtie e_{pk}) \cap (a_j \bowtie a_k) \]
\[ \bar{e}_{ij} \cap \bar{e}_{jk} = \left[ \{(e_{pi} \bowtie e_{pj} \bowtie e_{pk}) \cap (a_i \bowtie a_j)\} \cap (e_{pi} \bowtie e_{pj} \bowtie e_{pk}) \right. \left. \cap (a_i \bowtie a_k) \cap (e_{pi} \bowtie e_{pj} \bowtie e_{pk}) \right] \]
\[ = (a_i^{(dj)} \bowtie a_j \bowtie a_k) \cap (e_{pi} \bowtie e_{pj} \bowtie e_{pk}) \]
\[ \bar{e}_{ij} \cap \bar{e}_{jk} \cap (a_i \bowtie a_k) = (e_{pi} \bowtie e_{pk}) \cap (a_i^{(dj)} \bowtie a_k) = e_{ik}^{(dj)}. \ q. e. d. \]

Given two different straight lines \( \frac{a_k \bowtie a_i}{O} \) and \( \frac{a_k \bowtie a_j}{O}, \) \( a_k \bowtie a_i^{(dj)} \)
is perspective to $a_k \cup a_j^{(d_i)}$ with axis $e_{ij}$. This perspectivity shall be denoted with the symbol $P^{(k \ i \ j)}$ and the image of a point $q$ with $P^{(k \ i \ j)} q$. For the sake of completeness we shall give the symbol $P^{(k \ i \ j)}$ the meaning of the identical correspondence such that $P^{(k \ i \ j)} q = q$.

Theorem 68.

(1) $P^{(k \ i \ j)} a_k = a_k$,
(2) $P^{(k \ i \ j)} a_j^{(d_i)} = a_j^{(d_i)}$,
(3) $P^{(k \ i \ j)} e_{ki}^{(d_j)} = e_{kj}^{(d_j)}$,
(4) $P^{(k \ i \ j)} e_{ki}^{(d_j)} = e_{kj}^{(d_j)}$.

Pr. Suppose $d_j \geqq d_i$. Then (1) follows from $(a_k \cup e_{ij}) \cap (a_k \cup a_j^{(d_i)}) = a_k$, and (2) from $(a_i \cup e_{ij}) \cap (a_k \cup a_j^{(d_i)}) = a_j^{(d_i)}$. Theorem 66 yields $e_{ki} \cup e_{ij}^{(d_j)} = e_{ki} \cup e_{ij}^{(d_j)}$ and consequently $(e_{ki} \cup e_{ij}) \cap (a_k \cup a_j^{(d_i)}) = e_{kj}^{(d_j)}$, which proves (3), and finally by theorem 67 we have $(e_{ki} \cup e_{ij}) \cap (a_k \cup a_j^{(d_i)}) = e_{kj}^{(d_j)}$, which proves (4).

Theorem 69. If $c$ is irreducible in $\frac{a_k \cup a_i}{O}$ and if $P^{(k \ i \ j)} c = d$,
then $P^{(k \ i \ j)} \bar{c} = \bar{d}$.

P.. Choose $a_p$, such that $d_p = h$, $p \neq k, i, j$ and an irreducible element $c'$ such that $a_p \cup c = a_p \cup c'$, $c' \cap a_p = O$, $c' \cap (a_k \cup a_i) = O$. Then by definition $\bar{c} = \{(a_k \cup c') \cap (a_p \cup a_i)\} \cup \{(a_i \cup c') \cap (a_p \cup a_k)\} \cap (a_i \cup a_k)$. The perspectivity of $a_p \cup a_k \cup a_j^{(d_i)}$ and $a_p \cup a_k \cup a_j^{(d_i)}$ with axis $e_{ij}$ being denoted with the symbol $\mathfrak{P}$, we have

$$P^{(k \ i \ j)} \bar{c} = \left[\{a_k \cup \mathfrak{P}(c') \cap (a_p \cup a_j^{(d_i)})\}\right.$$  
$$\cup \{a_j^{(d_i)} \cup \mathfrak{P}(c') \cap (a_p \cup a_k)\}] \cap (a_j^{(d_i)} \cup \mathfrak{P}(c')).$$

But $a_p \cup d = a_p \cup \mathfrak{P}(c) = a_p \cup \mathfrak{P}(c')$ and $\mathfrak{P}(c') \cap a_p = O$, $\mathfrak{P}(c') \cap (a_k \cup a_j^{(d_i)}) = O$. Hence $P^{(k \ i \ j)} \bar{c} = \bar{d}$. q.e.d.

The image of a given element $t$ exists for $P^{(k \ i \ j)}$, when $t$ belongs to $a_k \cup a_j^{(d_i)}$. Every element, whose image exists, is said to be proper with respect to the perspectivity. If for every element $t$, which is proper with respect to $\mathfrak{P}_2$ and $\mathfrak{P}_3$, $\mathfrak{P}_2(t)$ being proper with
respect to $\Psi_1$, the relation $\Psi_3(\Psi_2(t)) = \Psi_3(t)$ holds, then we say that $\Psi_3$ is product of $\Psi_1$ and $\Psi_2$ and write $\Psi_1\Psi_2 = \Psi_3$: Further an element $t$ is said to be proper with respect to the product of perspectivities $\Psi_1$, $\Psi_2$, $\cdots \Psi_\lambda$, if $\Psi_1(\Psi_2 \cdots \Psi_\lambda(t))$ exists. Two products are said to be equivalent, if for every element, which is proper with respect to both, its image is the same for both products.

**Theorem 70.** If $i$, $j$, $k$, $l$ are all different, then

$$P(kj\overline{k}jl)P(ki\overline{k}ij) = P(ki\overline{k}jl).$$

**Pr.** Put $\text{Min} (d_i, d_j) = \nu$. Then we have

$$P(kj\overline{k}jl)P(ki\overline{k}ij)t = \{(t \cdot e_{ik}) \cap (a_k \cup a_i^{(d_k)}) \} \cap (a_k \cup a_i^{(d_j)})$$

$$= (t \cdot e_{ik} \cup e_{ij}) \cap (a_k \cup a_i^{(d_j)}).$$

But from theorem 67 follows $e_{ik}^{(d_i)} \cup e_{ij} = e_{ij}^{(d_j)} \cup e_{ik}$, which yields

$$P(kj\overline{k}jl)P(ki\overline{k}ij)t \geqq (t \cdot e_{ik}^{(d_j)}) \cap (a_k \cup a_i^{(d_j)}) = P(ki\overline{k}jl)t.$$

**Theorem 71.** If $i$, $j$, $k$, $\rho$ are all different, then

$$P(kj\overline{k}\rho\rho)P(ki\overline{k}\rho\rho) = P(ki\overline{k}\rho\rho)P(kj\overline{k}\rho\rho).$$

**Pr.** $P(kj\overline{k}\rho\rho)P(ki\overline{k}\rho\rho)t = \{(t \cdot e_{ik}) \cap (a_k \cup a_i^{(d_k)}) \} \cap (a_k \cup a_i^{(d_j)})$ 

$$= (t \cdot e_{ik} \cup e_{ij}) \cap (a_k \cup a_i^{(d_j)}) .$$

On the other hand we have

$$P(kj\overline{k}\rho\rho)P(ki\overline{k}\rho\rho)t = \{(t \cdot e_{ij}) \cap (a_k \cup a_i^{(d_j)}) \} \cap (a_k \cup a_i^{(d_j)})$$

$$= (t \cdot e_{ij} \cup e_{ik}) \cap (a_k \cup a_i^{(d_j)}) .$$

**Theorem 72.** If $i$, $j$, $\rho$, $k$ are all different, then

$$P(ki\overline{k}i)P(kj\overline{k}j)P(ki\overline{k}j)P(kj\overline{k}i) = P(ki\overline{k}j)P(kj\overline{k}i)P(ki\overline{k}j)P(kj\overline{k}i).$$
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Pr. \( P(\rho i_j)P(\rho j_i)P(\rho i_j)P(\rho j_i) = P(k i_j)P(\rho j_i)P(\rho j_i)P(\rho j_i) \) by theorem 70

\[ = P(k i_j)P(k j_i)P(\rho j_i)P(\rho j_i) \quad \text{by theorem 71} \]

\[ = P(k i_j)P(k j_i)P(\rho j_i)P(k j_i) \quad \text{again by theorem 70.} \]

q.e.d.

Now we define \( P(\rho i_j) = P(k i_j)P(k i_j) \), if \( k \neq j, i \neq j, k \neq \rho \).

If moreover \( i, j, k, \rho \) are all different, then this is the perspectivity of \( a_4(d_i) \sim a_4(d_j) \) and \( a_k(d_i) \sim a_k(d_j) \) with axis \( e_i \sim e_k \) by theorem 71. Further we define \( P(\rho i_j) = P(k i_k)P(k i_j) \), where \( d_k = h \) and \( k \neq i, j \).

**Theorem 73.** \( P(k i_k)P(\rho i_j) = P(k i_j) \).

Pr. The case \( i = k \) or \( j = \rho \) is evident. For the case \( i \neq k, j \neq \rho \) we can assume \( \sigma \neq j \) and

\[ P(k i_k)P(\rho i_j) = P(k i_k)P(\rho i_j)P(\rho i_j) = P(k i_k)P(\rho i_j) = P(k i_j) \]

For the case \( i \neq k, j \neq \rho, j = k, i = \rho \) we can assume \( \sigma \neq i \) and

\[ P(j \rho)P(i j) = P(j \rho)P(i j) = P(j \rho)P(i j) = P(j \rho)P(i j) = P(j \rho)P(i j) = P(j \rho)P(i j) \]

where \( \sigma \neq i, j, \sigma \). For the case \( i \neq k, j \neq \rho, j = k, i \neq \rho, i = \sigma \) we have

\[ P(j \rho)P(i j) = P(j \rho)P(i j)P(i j) = P(j \rho)P(i j) \]

Finally in the case \( i \neq k, j \neq \rho, j = k, i \neq \rho, i \neq \sigma \),

\[ P(j \rho)P(i j) = P(j \rho)P(i j)P(i j) = P(j \rho)P(i j) \]

**Theorem 74.** \( P(k i_k)P(\rho i_j) = P(k i_j) \).

Pr. The case \( i = k \) or \( j = \rho \) is evident. Suppose \( i \neq k, j \neq \rho, j \neq k \). By the preceding theorem we have
\[ P(\begin{array}{l}i\rho\varepsilon_i\varepsilon_j\end{array}) P(\begin{array}{l}i\sigma\varepsilon_i\varepsilon_j\end{array}) = P(\begin{array}{l}j\rho\varepsilon_i\varepsilon_j\end{array}) P(\begin{array}{l}j\sigma\varepsilon_i\varepsilon_j\end{array}) = P(\begin{array}{l}i\sigma\varepsilon_i\varepsilon_j\end{array}). \]

If \( i \neq k, j \neq \rho \) and \( j = k \), then we have for the case \( i \neq \rho \)

\[ P(\begin{array}{l}j\rho\varepsilon_i\varepsilon_j\end{array}) P(\begin{array}{l}i\sigma\varepsilon_i\varepsilon_j\end{array}) = P(\begin{array}{l}j\rho\varepsilon_i\varepsilon_j\end{array}) P(\begin{array}{l}i\sigma\varepsilon_i\varepsilon_j\end{array}) = P(\begin{array}{l}i\sigma\varepsilon_i\varepsilon_j\end{array}) \]

and for the case \( i = \rho \)

\[ P(\begin{array}{l}j\rho\varepsilon_i\varepsilon_j\end{array}) P(\begin{array}{l}i\sigma\varepsilon_i\varepsilon_j\end{array}) = P(\begin{array}{l}j\rho\varepsilon_i\varepsilon_j\end{array}) P(\begin{array}{l}i\sigma\varepsilon_i\varepsilon_j\end{array}) = P(\begin{array}{l}i\sigma\varepsilon_i\varepsilon_j\end{array}) \]

by the preceding theorem. q. e. d.

Theorem 73 and 74 show that through successive perspectivities the final image of a given point depends only on the final straight line, whereupon the process of perspectivities ends, and never depends on the choice of intermediating straight lines.

**Theorem 75.** \( P(\begin{array}{c}i\varepsilon_j\varepsilon_i\varepsilon_j\end{array}) t = t^{-1} \) for every unit \( t \) in \( \frac{a_i \cdot a_j}{O} \), where \( t^{-1} \) means a point, for which \( M_k(t^{-1}) \leq e_i j \) holds.

Pr. We put \( \dim t = \lambda \). From \( t \cdot a_i \leq a_i \cdot a_j, t \cdot a_j \leq a_i \cdot a_j \) follows \( \dim a_i \geq \lambda, \dim a_j \geq \lambda \). Suppose \( P(\begin{array}{c}i\varepsilon_j\varepsilon_i\varepsilon_j\end{array}) = \frac{P(\begin{array}{l}k\varepsilon_j\varepsilon_i\varepsilon_j\end{array}) P(\begin{array}{l}j\varepsilon_i\varepsilon_j\end{array})}{\varepsilon_i \varepsilon_j} \)

where \( \dim a_k \geq \lambda \), and put

\[
(t \cdot \overline{e_i}^{(\lambda)}) \cap (a_k^{(\lambda)} \cdot a_j^{(\lambda)}) = t', \quad (t' \cdot \overline{e_j}^{(\lambda)}) \cap (a_i^{(\lambda)} \cdot a_k^{(\lambda)}) = t'',
\]

\[
(t'' \cdot \overline{e_k}^{(\lambda)}) \cap (a_i^{(\lambda)} \cdot a_j^{(\lambda)}) = t'''.
\]

Then we have obviously \( t''' = P(\begin{array}{c}i\varepsilon_j\varepsilon_i\varepsilon_j\end{array}) t \) and

\[
a_i^{(\lambda)} \cdot t'' = a_i^{(\lambda)} \cdot a_k^{(\lambda)}, \quad a_k^{(\lambda)} \cdot t'' = a_i^{(\lambda)} \cdot a_k^{(\lambda)},
\]

whence \( (a_i^{(\lambda)} \cdot a_j) \cdot t'' = (a_k^{(\lambda)} \cdot a_j) \cdot t'' \). Further

\[
(a_i^{(\lambda)} \cdot a_j) \cdot t'' = (t' \cdot \overline{e_i}^{(\lambda)}) \cap a_i^{(\lambda)} = \overline{e_j}^{(\lambda)} \cdot a_i^{(\lambda)} = O,
\]

\[
(a_k^{(\lambda)} \cdot a_j) \cdot t'' = (t' \cdot \overline{e_j}^{(\lambda)}) \cap a_k^{(\lambda)} = t' \cdot a_k^{(\lambda)} = O.
\]

Therefore \( a_i^{(\lambda)} \cdot a_j \sim a_k^{(\lambda)} \cdot a_j \) with axis \( t'' \) and we have a projective automorphism of \( a_k^{(\lambda)} \cdot a_j \) by means of \( t'', \overline{e_i}^{(\lambda)} \) and \( a_k^{(\lambda)} \cdot a_j \). If we take \( a_j \) as the infinite point and \( a_i \) as the origin of the straight line.
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\[
\frac{a_i \cdot a_j}{O},
\]

then this automorphism becomes the multiplication \( M_{-t} \).

In fact \((a^{1\lambda)} \cap (a^{2\lambda)} \cap a_j) = a_j\), \((t'' \cap e^{(2\lambda)}_k) \cap (a^{(3\lambda)}_i \cap a_j) \leq a_i\), and

\[
(e^{(1\lambda)}_k \cap t'') \cap (a^{(2\lambda)}_i \cap a_j) = (t' \cap e^{(2\lambda)}_k) \cap (a^{(3\lambda)}_i \cap a_j) = t.
\]

The image of the point \( e^{(1\lambda)}_i = -e^{(1\lambda)}_i \) being \( t \), we see that the image of \( e^{(1\lambda)}_i \) is the point \(-t\) by the use of the first distributive law. Now it follows \((t'' \cap t'') \cap (a^{(3\lambda)}_k \cap a_j) = e^{(2\lambda)}_k\) and \((e^{(1\lambda)}_k \cap e^{(2\lambda)}_k) \cap (a^{(3\lambda)}_i \cap a_j) = e^{(3\lambda)}_k\) by theorem 67. Hence \((-t) \cdot t''' = -e^{(3\lambda)}_k\), whence \(t''' = t^{-1}\). q. e. d.

The quotient \( \frac{a_j \cap a_k}{O} \), where \( a_i, a_j, a_k \) are any three basis-components of \( L \), is called a plane in \( L \). The perspectivity of two planes \( \frac{a_i \cap a_j \cap a_k}{O} \) and \( \frac{a_i \cap a_j \cap a_k}{O} \) with axis \( e_{kk'} \) shall be denoted by

\[ P(i, j, k) \] \( \frac{a_j \cap a_k}{O} \). Then we define further

\[
P(i, j, k) \quad P(i, j, k') \quad P(i, j, k) \quad P(i, j, k') \quad P(i, j, k') \quad P(i, j, k') \quad P(i, j, k') \quad P(i, j, k')
\]

... and so forth.

**Theorem 76.** The perspectivity \( P(i, j, k) \) in \( \frac{a_j \cap a_k}{O} \) induces

\[
P(i, j, k) \quad P(i, j, k') \quad P(i, j, k') \quad P(i, j, k') \quad P(i, j, k') \quad P(i, j, k') \quad P(i, j, k') \quad P(i, j, k')
\]

... in \( \frac{a_i \cap a_j \cap a_k}{O} \).

Pr. In case \( i = i', j = j', k = k' \) we have for \( t \leq a_j \cap a_k \)

\[
P(i, j, k) = (t \cap e_{kk'}) \cap (a_i \cap a_j \cap a_k)
\]

The proofs of the other cases can be readily obtained by virtue of theorem 73 and 74.

**PART 6.**

As in the preceding parts we consider a primary lattice \( L \) with a basis \( a_1, a_2, \cdots a_r \), where \( r \geq 4 \) and \( \dim a_1 = \dim a_2 = \dim a_r \)
$\text{E. Inaba} = \dim a_{\underline{4}} = h$ holds, $h$ being the height of $L$, and we denote with $A_i$ the join of all $a_v, v = 1, 2, \cdots i - 1, i + 1, \cdots r$ except $a_i$, with $A_{ij}$ the join of all $a_v$ except $a_i$ and $a_j$, etc. If $l$ is an irreducible element of dimension $\lambda$ in $L$, we put $l_{ij} = (A_{ij} \cap l) \cap (a_i \cap a_j)$, and will prove

**Theorem 77.** $l_{ij}$ is irreducible and, if, putting $\dim (l \cap A_i) = \lambda_i$, $\lambda_j \geq \lambda_i$, then $l_{ij} \cap a_j = O$, $\dim l_{ij} = \lambda - \lambda_i$, $\dim (l_{ij} \cap a_i) = \lambda_j - \lambda_i$.

Pr. Since $\frac{l_{ij}}{O} \simeq \frac{A_{ij} \cap l_{ij}}{A_{ij}} \simeq \frac{A_{ij}}{l_{ij} \cap A_{ij}} \simeq \frac{l_{ij} \cap a_i}{A_{ij}} \cap a_i \simeq \frac{(A_{ij} \cap l) \cap (a_i \cap a_j)}{A_{ij}} \simeq \frac{l_{ij} \cap a_i}{A_{ij}} \cap a_i \simeq \frac{A_{ij} \cap (l \cap A_j)}{l \cap A_{ij}} \simeq \frac{l \cap A_j}{l \cap A_{ij}}$ holds, $l_{ij}$ is irreducible and $\dim l_{ij} = \dim l - \dim (l \cap A_{ij}) = \lambda - \lambda_i$. Similarly we have $\frac{l_{ij} \cap a_j}{O} \simeq \frac{l \cap A_i}{l \cap A_{ij}}$ whence $l_{ij} \cap a_j = O$ follows in virtue of $\lambda_j \geq \lambda_i$.

**Theorem 78.** If $\lambda_i \geq \min (\lambda_j, \lambda_k)$, then $\overline{l_{kj}} \leq l_{ki} \cap l_{ji}$. If moreover $\lambda_i \geq \lambda_j \geq \lambda_k$ holds, then $\overline{l_{kj}} \cap l_{ij} = l_{ji} \cap l_{ij}$.

Pr. Choose $a_\rho$, such that $\rho \neq i, j, k$ and $\dim a_\rho = h$ and an irreducible element $l'$, such that $a_\rho \cap l_{kj} = a_\rho \cap l'$, $l' \cap a_\rho = O$, $l' \cap (a_k \cap a_j) = O$. Putting $t = (l' \cap a_i) \cap (A_{ijk} \cap l)$, we have

$$A_{ijk} \cap t = (A_{kj} \cap l') \cap (A_{ijk} \cap l) = (A_{kj} \cap l') \cap (A_{ijk} \cap l) = A_{ijk} \cap l' ,$$

and $t \cap a_i = (l' \cap a_i) \cap (A_{ijk} \cap l) = (l' \cap a_i) \cap (A_{ijk} \cap l') = a_i \cap l'$.

Now it follows

$$t \cap a_i = (l' \cap a_i) \cap (A_{ijk} \cap l) = (l' \cap a_i) \cap (A_{ijk} \cap l') = a_i \cap l' .$$

But $A_{jk} \cap t \cap a_k \cap a_j = A_{ki} \cap t$ and $(A_{pi} \cap t) \cap a_i = A_{pi} \cap a_i = O$, where $(A_{pi} \cap a_i) \cap t = A_p \cap t = a_i \cap (A_{ijk} \cap l) = a_i \cap A_{ijk} = O$, since
$l \cap (a_i \cup A_{ijk}) = l \cap A_{jk} \leq A_{ijk}$ by hypothesis. Hence we have by lemma 1

$$l_{kj} = [(A_{j;i}\cup t) \cap (a_j \cup a_i \cup a_k)] \cup [(A_{ik} \cup t) \cap (a_i \cup a_k \cup a_p)] \cap (a_k \cup a_j)$$

$$\leq [(A_{ji} \cup t) \cap (a_j \cup a_i)] \cup [(A_{ik} \cup t) \cap (a_i \cup a_k)] \cap (a_k \cup a_j)$$

$$= [(A_{ji} \cup t) \cap (a_j \cup a_i)] \cup [(A_{ik} \cup l) \cap (a_i \cup a_k)] \cap (a_k \cup a_j)$$

$$= (l_{ij} \cup l_{ki}) \cap (a_k \cup a_j),$$

whence

$$l_{kj} \leq l_{ji} \cup l_{ki}.$$
formula of the theorem. The second one follows from this by observing that \( \ell = \overline{\ell} \).

**Theorem 80.** If \( \lambda_i \geq \lambda_j \geq \lambda_k \), and if we put \( P_{ij}^k g_{kj} = h_{kj} \), \( P_{ij}^{(i)} g_{ji} = h_{ji} \), \( P_{ij}^{(i)} g_{ik} = h_{ki} \), then \( h_{ki} \), \( h_{ij} \) and \( h_{kj} \) are finite with the relation \( h_{ki} = h_{kj}h_{ji} \).

Pr. We put \( P_{ij}^{(i)} g_{kj} = l_{kj} \), \( P_{ij}^{(i)} g_{ji} = l_{ji} \) yielding \( P_{ij}^{(i)} l_{kj} = h_{kj} \), \( P_{ij}^{(i)} l_{ji} = h_{ji} \). By theorem 77 we have, \( g_{kj} \cap a_i = O \), \( g_{ji} \cap a_i = O \), \( g_{kj} \cap a_j = O \), \( g_{ji} \cap a_j = O \), \( g_{kj} \cap a_3 = O \), \( g_{ji} \cap a_3 = O \), \( h_{ki} \cap a_3 = O \), \( h_{kj} \cap a_3 = O \), \( h_{ji} \cap a_3 = O \). Putting \( \lambda = \dim g \), we have \( \dim l_{kj} = \dim l_{ki} = \dim h_{kj} = \dim h_{ki} = \lambda - \lambda_k \), \( \dim h_{ji} = \dim l_{ji} = \lambda - \lambda_j \). We put further \( a = a_1^{(\lambda - \lambda_k)} \cap a_2 \), \( b = a_3^{(\lambda - \lambda_j)} \cap a_2 \), \( a^* = a_1^{(\lambda - \lambda_j)} \cap a_2 \). Then \( a^* \backslash b \) with axis \( e_{13}^{(\lambda - \lambda_j)} \). Since \( \dim (l_{kj} \cap a_1) = \dim (g_{kj} \cap a_1) = \lambda - \lambda_k \), and \( a_1^{(\lambda - \lambda_k)} \cap l_{kj} = a_1^{(\lambda - \lambda_k)} \cap a_2 \cap a_3 = a_3^{(\lambda - \lambda_k)} \cap a_2 \), so we have \( a \cap l_{kj} = b \cap l_{kj} \), where \( b \cap l_{kj} = a_3^{(\lambda - \lambda_j)} \cap l_{kj} = O \). Thus we obtain \( M_{h_{kj}} \) by means of \( e_{13}^{(\lambda - \lambda_j)} \), \( l_{kj} \), \( b \). For \( a \cap b = a_2 \), \( (e_{13}^{(\lambda - \lambda_j)} \cap l_{kj}) \cap a \leq a_1 \), and \( (e_{13}^{(\lambda - \lambda_j)} \cap l_{kj}) \cap b = \widehat{e}_{13}^{(\lambda - \lambda_j)} \cap b \), \( P_{ij}^{(i)} l_{kj} = h_{kj} \). But theorem 79 yields \( (h_{kj} \cap e_{13}^{(\lambda - \lambda_j)}) \cap b = (h_{kj} \cap e_{13}^{(\lambda - \lambda_j)} \cap (a_1 \cap a_2) = P_{ij}^{(i)} h_{ki} = P_{ij}^{(i)} h_{kj} = l_{ji}. \) Since moreover from \( g_{ji} \cap g_{kj} = g_{ji} \cap g_{ki} \) follows \( l_{ji} \cap l_{kj} = l_{ji} \cap h_{ki} \) by \( P_{ij}^{(i)} l_{kj} = h_{kj} \), so we have \( (l_{ji} \cap l_{kj}) \cap a = h_{ki} \), that is \( h_{ki} = h_{kj}h_{ji} \). q. e. d.

In the following we use the notation \( P_{ij}^{(i)} g_{ij} = h_{ij} \), if \( \lambda_i \leq \lambda_j \), and \( P_{ij}^{(i)} g_{ij} = h_{ij} \), if \( \lambda_i \geq \lambda_j \). If \( \lambda_i = \lambda_j \), then \( h_{ij} \) is an unit and \( h_{ij} = h_{ij}^{-1} \) by theorem 75. Given an arbitrary irreducible element \( g \) in \( L \), we can find \( A_p \) such that \( g \cap A_p = O \) by the corollary to lemma 2. Then \( \lambda = \dim g \leq \dim a_\sigma \) and \( h_{pi} \), \( i = 1, 2, \ldots r \) are all finite points of dimension \( \lambda \), where \( h_{pp} \) means \( e_{13}^{(\lambda - \lambda_j)} \). Every system \( \{h_{pi}\} \), \( i = 1, 2, \ldots r \) with arbitrary unit \( \varepsilon \) is now called a system of coordinates of the irreducible element \( g \). If \( A_\rho \cap g = O \) and \( A_\sigma \cap g = O \) \( (\sigma \neq \rho) \), then \( h_{pp} \) is an unit and we have \( h_{pi} = h_{pi} h_{oi} \), if \( i = \sigma \), \( i \neq \rho \), by the preceding theorem. Furthermore \( h_{pp} = h_{po} h_{ao} \), \( h_{po} = h_{po} h_{ao} \). Hence two systems of coordinates \( \{h_i\} \) and \( \{h_i'\} \) of an irreducible element are in a relation \( h_i = e h_i' \), \( i = 1, 2, \ldots r \) with an unit \( e \). Among the coordinates \( h_i \) there exists always at least an unit \( h_\rho \) such that \( d_\rho \geq \lambda \) and, since from \( g_{pi} \cap a_\rho \leq a_\rho \cap a_i \) the relation \( h_{pi} \cap a_1 \leq a_1 \cap a_2^{(d_\lambda)} \) follows, it holds \( O(h_\rho) \geq \lambda - d_\lambda \). Conversely to every system of finite points \( h_i \), \( i = 1, 2, \ldots r \) of dimension \( \lambda \) in \( \frac{a_1 \cap a_2}{O} \), where \( h_\rho \) is an unit with \( d_\rho \geq \lambda \) and \( O(h_\rho) \geq \lambda - d_\lambda \), there corresponds
an irreducible element in $L$, such that $h_i$, $i = 1, 2, \ldots r$, are its coordinates.

In order to prove this assertion, we put $h^{-1}_p h_i = h_{pi}$, $P(\rho) h_{pi} = g_{pi}$, where $h_{pi}$ is proper, since $h_{pi} \cup a^{(d_{p})} \leq a^{(d_{p})} \cup a^{(d_{i})}$, and put

$$g = \prod_{i \neq p} (A_{pi} \cup g_{pi}).$$

Then $g_{pi} \cap A_p = g_{pi} \cap a_i = O$, $A_p \cap g = \{g_{pi} \cup A_{pi} \cap (A_{pi} \cup g_{pi}) \cap A_p \cap g = O\} \cup g_{pi} \cap A_{pi} = O$, by the corollary to lemma 2, and

$$A_p \cup g = \prod_{i \neq p} \{A_{pi} \cup g_{pi}\} \cup A_p = \prod_{i \neq p} A_{pi} = O,$$

Hence it holds $\frac{g}{O} \simeq \frac{A_{pi} \cup g_{pi}}{A_{pi}} \simeq \frac{g_{pi}}{O}$. $g$ is therefore an irreducible element of dimension $\lambda$ with the relation $(A_{pi} \cup g) \cap (a_{pi} \cup a_{i}) = g_{pi}$, proving that $h_i$ are coordinates of $g$. We prove further, that for the coordinates $h^{(1)}_i, h^{(2)}_i$ of two different elements $g^{(1)}, g^{(2)}$ the relations $\epsilon h_{pi}^{(1)} = h_{pi}^{(2)}$ do not hold with an unit $\epsilon$. If $\epsilon h_{pi}^{(1)} = h_{pi}^{(2)}$, then, $h_{pi}^{(1)}$ and $h_{pi}^{(2)}$ being units, we have $\lambda_p = \lambda_i = O$ for both $g^{(1)}$ and $g^{(2)}$. Hence $h_{pi}^{(1)} = h_{pi}^{(2)} h_{pi}^{(1)}$ and consequently $h_{pi}^{(1)} = h_{pi}^{(2)}$, $g_{pi}^{(1)} = g_{pi}^{(2)}$. From this we obtain $A_{pi} \cup g^{(1)} = A_{pi} \cup g^{(2)} = A_{pi} \cup g^{(1)}$, whence $g^{(1)} = \prod_{i \neq p} (A_{pi} \cup g^{(1)}) = g^{(2)}$. q. e. d.

We shall now investigate the relation, which the coordinates of every point on a given straight line satisfy.

**Theorem 81.** The coordinates $h_i, i = 1, 2, \ldots r$ of any point $P$ on a straight line $\frac{P_1 \cup P_2}{O}$, where $P_1, P_2$ are independent points with coordinates $h_1^{(1)}$ and $h_1^{(2)}$ respectively, satisfy the relation

$$h_i = \lambda_1 h_1^{(1)} + \lambda_2 h_1^{(2)}, \quad i = 1, 2, \ldots r$$

with two fixed points $\lambda_1$ and $\lambda_2$ of dimension $\dim P$, where either $\lambda_1$ or $\lambda_2$ is an unit.

Pr. We can assume that $\dim P_1 = s \geq \dim P_2 = t$, and find $A_{kj}$ such that $A_{kj} \cap (P_1 \cup P_2) = O$ by lemma 10. For an arbitrary point $P$ in $\frac{P_1 \cup P_2}{O}$ it holds either $P \cap A_k = O$ or $P \cap A_j = O$. If we put $P_1 \cup P_2 = l$, $A_k \cap l = w$, then we have
\[ \frac{w}{O} \cong \frac{w \cup A_{kj}}{A_{kj}} = \frac{A_k \cup (l \cup A_{kj})}{A_{kj}} = \frac{(a_j \cup A_{kj}) \cap (l \cup A_{kj})}{A_{kj}} \]

\[ = \frac{A_{kj} \cap \{a_j \cap (l \cup A_{kj})\}}{A_{kj}} \cong \frac{a_j \cap (l \cup A_{kj})}{O}. \]

Hence \( w \) is a point and \( w \cap A_j = O \), whence \( w_{ij} \cap a_i = O \), \( \dim w_{ij} = \dim w \) for all \( i \neq j \). We first consider the case \( P_1 \cap A_k = O \), \( P \cap A_j = O \) and put \( P(l_{ji}^\ast)W_{ij} = W_{ij}, \) \( P(l_{kj}^\ast)P_{kj} = P_{kj}^* \), \( \lambda = \dim P = \dim P_{kj} \) = \( \dim P_{kj}^\ast \), \( \nu = \dim (P \cap A_j) = \dim (P_{kj} \cap a_k) = \dim (P_{kj}^\ast \cap a_k) \). Then \( P(l_{ji}^\ast)P_{kj}^* = P(l_{kj}^\ast)P_{kj} = h_{jk} \), where \( h_{kj} \cap a_2 = O \) and, since \( P_{kj} \cap a_k = a_k^\lambda \), we have \( \dim w = \dim P_2 = t = \dim W_{ij}^\ast \).

We choose an irreducible element \( W_{ij}^\ast \) of dimension \( \lambda \) in \( \frac{a_2 \cup a_3}{O} \), which contains \( W_{ij} \), if \( \lambda > t \), and put \( W_{ij}^\ast = W_{ij}^\nu \), if \( \lambda \leq t \). Then, putting \( P\overline{W}_{ij} = l_{ji}^\ast, P(l_{kj}^\ast)W_{ij}^\nu = l_{ji} \), we have \( \overline{l}_{ji} \geq l_{ji} \), \( \overline{l}_{ji} \cap a_2 = O \).

Next it holds obviously \( a_1^{\lambda-\nu} \cup a_2 \sim a_3^{\lambda-\nu} \cup a_2 \) with axis \( e_{13}^{\lambda-\nu} \) and \( \alpha_1^{\lambda-\nu} \cap a_2 \) is quasi-perspective to \( a_1^{\lambda} \cup a_2 \) with axis \( e_{13}^{\lambda} \).

Since, in virtue of \( P_{kj} \cap a_k = a_k^{\nu} \), the relations

\[ (a_1^{\lambda} \cup a_2) \cup P_{kj}^* = (a_3^{\lambda-\nu} \cup a_2) \cup P_{kj}^*, \quad P_{kj}^\ast \cap (a_3^{\lambda-\nu} \cup a_2) = P_{kj}^\ast \cap a_2 = O \]

hold, so \( a_3^{\lambda-\nu} \cup a_2 \) is quasi-perspective to \( a_1^{\lambda} \cup a_2 \) with axis \( P_{kj}^\ast \). Further \( (a_1^{\lambda} \cup a_2) \cap (a_3^{\lambda-\nu} \cup a_2) = a_2, (e_{13}^{\lambda-\nu} \cup P_{kj}^\ast) \cap (a_1^{\lambda} \cup a_2) = a_1 \),

\[ (e_{12}^{\lambda-\nu} \cup e_{13}^{\lambda-\nu}) \cap (a_3^{\lambda-\nu} \cup a_2) = e_{23}^{\lambda-\nu}, \quad (e_{12}^{\lambda-\nu} \cup P_{kj}^\ast) \cap (a_1^{\lambda} \cup a_2) = h_{kj}. \]

Thus we have \( M_{kj} \) by means of \( e_{13}^{\lambda-\nu}, P_{kj}^\ast \) and \( a_3^{\lambda-\nu} \cup a_2 \). If we put \( M = (W_{ij}^\nu \cap P_{kj}^\ast) \cap (a_1^{\lambda} \cup a_2), \) so we have from

\[ (l_{ij}^\ast \cap e_{13}^{\lambda-\nu}) \cap (a_3^{\lambda-\nu} \cup a_2) = P(l_{ij}^\ast)l_{ij}^\nu = W_{ij}^\nu \]

the relation \( M = h_{kj}l_{ij}^\nu = -h_{kj}l_{ij}^\nu \). It holds furthermore \( a_1^{\lambda} \cup a_2 \sim P_{kj}^\ast \cup a_2 \) with axis \( W_{ij}^\nu \). Since \( (a_1^{\lambda} \cup a_3^{\nu}) \cap (a_2 \cup P_{kj}^\ast) = P_{kj}^*, (P_{kj}^\ast \cup W_{ij}^\nu) \cap (a_1^{\lambda} \cup a_2) = M, \) so we have \( A_M \) by means of \( a_3^{\lambda}, W_{ij}^\nu \) and \( a_2 \cup P_{kj}^\ast \). Now we put \( P_{kij} = (A_{kij} \cup P) \cap (a_k \cup a_i \cup a_j) \) and obtain

\[ \frac{P_{kij}}{O} \cong \frac{P_{kij} \cup A_{kij}}{A_{kij}} = \frac{A_{kij} \cup P}{A_{kij} \cup O} \cong \frac{P}{O}. \]

Thus \( P_{kij} \) is a point of dimension \( \lambda \) and similarly for
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\[ P_{kij} = (A_{kij} \cup P_1^{(3)}) \cap (a_k \cup a_i \cup a_j)^{(3)}. \]

Then it holds

\[ w_{ij}^{(1)} = (A_{ij} \cup P^{(3)} \cup P_2^{(3)}) \cap (a_i \cup a_j). \]

\[ P_{kij} \cup w_{ij}^{(1)} = \{A_{kij} \cup P \cap (a_k \cup a_i \cup a_j)\} \cup \{(A_{kij} \cup P_1^{(3)} \cup P_2^{(3)}) \cap (a_k \cup a_i \cup a_j)\} \]
\[ = (A_k \cup P_i) \cap (A_{kij} \cup P_1^{(3)} \cup P_2^{(3)}) \cap (a_k \cup a_i \cup a_j) \]
\[ = (A_k \cup P_i) \cap (A_{kij} \cup P_1^{(3)} \cup P_2^{(3)}) \cap (a_k \cup a_i \cup a_j) = \overline{P}_{kij} \cap w_{ij}^{(1)}. \]

If we put \( P^{(kij)} \overline{P}_{kij} = Q_{kij} \) and \( P^{(kij)} P_{kij} = Q_{kij} \), then

\[ (2) \quad Q_{kij} \cap W_{ij}^{(*)} = \varpi_{kij} \cap W_{ij}^{(*)}. \]

We have further \( P_{kij} \cup a_3^{(3)} = P_{k} \cup a_3^{(3)} \) from \( P_{kij} \cup a_3 \cap (a_k \cup a_3) = P_k \), whence follows \( Q_{kij} \cup a_3^{(3)} = h_{ki} \cup a_3^{(3)} \), and similarly \( Q_{kij} \cup a_3 = P_{kij} \cup a_3 \). Hence we have

\[ (h_{ki} \cup a_3^{(3)}) \cap (P_{kij}^* \cup a_2) = (Q_{kij} \cup a_3^{(3)}) \cap (Q_{kij} \cup a_2) = Q_{kij}, \]

where \( Q_{kij} \cap (a_2 \cup a_3) = O \) follows from \( P_{kij} \cap (a_i \cup a_j) = O \), and putting \( (Q_{kij} \cap W_{ij}^{(*)}) \cap (a_3^{(3)} \cup a_2) = U \), it holds

\[ (3) \quad U = M + h_{ki} = -h_{ki}l_{ji}^{(*)} + h_{ki}. \]

In virtue of (2) we have similarly \( U = -h_{ki}l_{j}^{(*)} + h_{ki} \). This, together with (3), yields \( h_{ki} - h_{ki}^{(1)}l_{j}^{(*)} = (h_{ki} - h_{ki}^{(1)}l_{ji}^{(*)}). \) If \( \lambda > t \), then \( \dim (P \cap P_1) \geq \lambda - t \), \( O(h_{ki} - h_{ki}^{(1)}l_{ji}^{(*)}) \geq \lambda - t \), and hence

\[ (4) \quad h_{ki} - h_{ki}^{(1)}l_{ji}^{(*)} = (h_{ki} - h_{ki}^{(1)}l_{j}^{(*)}). \]

If \( \lambda \leq t \), then \( l_{ji}^{(*)} = l_{ij} \) and therefore (4) holds also in this case.

We next consider the case \( P_1 \cap A_k > O \) and \( P \cap A_k = O \). Then it holds \( P_1 \cap w > O \) and therefore \( P_2 \cap A_k = O \), \( P_1 \cap P = O \), whence \( \dim P \leq t = \dim P_2 \). Further we have

\[ w \simeq \frac{P_2 \cup w}{P_2} = \frac{(P_2 \cup A_k) \cap (P_1 \cup P_2)}{P_2} \simeq \frac{P_1 \cap (P_2 \cup A_k)}{O} \geq \frac{P_1 \cap e^{(t)}}{O} = \frac{P^{(t)}}{O}, \]

which yields \( \dim w \geq t \). Since the point \( P \) lies on the straight line \( \frac{P^{(t)} \cap P_2}{O} \), we can apply the above result for the present case, if we only relace the point \( P_1 \) with \( P_2 \). We have henceforth

(1) Here \( \overline{P}_{kij} \) does not signify the conjugate point of \( P_{kij} \).
In the similar manner we treat the case $P \cap A_j = O$, $P_1 \cap A_j = O$ and the case $P \cap A_j = O$, $P_1 \cap A_j > O$, and we have

$$h_{ji} - h_{ji}^{(1)(\lambda)} = (h_{jk} - h_{jk}^{(1)(\lambda)})m_{ki}, \quad \text{or}$$

$$h_{ji} - h_{ji}^{(0)(\lambda)} = (h_{jk} - h_{jk}^{(0)(\lambda)})m_{ki},$$

where $A_j \cap l = u$, $P_{1}(u)u_{ik} = m_{ki}$.

Now, assuming $P_1 \cap A_k = O$ and $P_2 \cap A_k = O$, it follows from (4)

$$h_{ki}^{(2)} - h_{ki}^{(1)(\lambda)} = (h_{kj}^{(2)} - h_{kj}^{(1)(\lambda)})l_{ji},$$

where $h_{kj}^{(2)} - h_{kj}^{(1)(\lambda)}$ is an unit. For

$$g_{kj}^{(2)}g_{kj}^{(1)} = (A_{kj} \cap P_1) \cap (A_{kj} \cap P_2) \cap (a_k \cap a_j)$$

$$= [A_{kj} \cap (P_1 \cap (A_{kj} \cap P_2)) \cap (a_k \cap a_j) = A_{kj} \cap (a_k \cap a_j) = O.$$

Putting $h_{kj}^{(2)} - h_{kj}^{(1)(\lambda)} = \epsilon^{-1}$, we have from (4) and (8)

$$h_{ki}^{(2)} - h_{ki}^{(1)(\lambda)} = (h_{kj}^{(2)} - h_{kj}^{(1)(\lambda)})\epsilon(h_{ki}^{(2)} - h_{ki}^{(1)(\lambda)}),$$

for the coordinates of the point $P$ of dimension $\lambda$ with $P \cap A_k = O$. If we put further $(h_{kj} - h_{kj}^{(0)(\lambda)})\epsilon = \Theta$, then $\Theta$ is a finite point of dimension $\lambda$ and $h_{ki} = \Theta h_{ki}^{(2)} + (\epsilon^{(\lambda)} - \Theta)h_{ki}^{(1)}$. Furthermore it is evident from theorem 62 that either $\Theta$ or $\epsilon^{(\lambda)} - \Theta$ is an unit.

If $P \cap A_k > O$, then $P \cap A_j = O$, $w \cap A_j = O$ and $l = P_1 \cap w$, $\lambda = \dim P \leq t = \dim w$. Hence we have from (7)

$$h_{ji} - l_{ji}^{(\lambda)} = h_{ji}m_{ki}.$$

Since further $u \cap A_k = O$, $P_1 \cap A_k = O$, we have $m_{ki} - h_{ki}^{(\lambda)} = -h_{ki}^{(\lambda)}u_{ji}$ with $\mu = \dim u$ and, substituting this in (9), we obtain

$$h_{ji} = h_{jk}h_{ki}^{(1)} + (e_{12}^{(\lambda)} - h_{jk}h_{kj}^{(1)})u_{ji}$$

$$= h_{jk}h_{ki}^{(1)} + (e_{12}^{(\lambda)} - h_{jk}h_{kj}^{(1)})\epsilon(h_{ki}^{(2)} - h_{ki}^{(1)(\lambda)}).$$

If we put further, $(e_{12}^{(\lambda)} - h_{jk}h_{kj}^{(1)})\epsilon = \Theta$, then $h_{ji} = \Theta h_{ki}^{(2)} + (h_{jk} - \Theta)h_{kj}^{(1)}$, where in case $O(h_{jk} - \Theta) > O$ it follows from $(h_{jk} - \Theta)\epsilon^{-1} = h_{jk}h_{kj}^{(2)} - e_{12}^{(\lambda)}$ that $O(h_{jk}) = O$, whence $O(\Theta) = O$.

We can treat the case $P_1 \cap A_j = O$, $P_2 \cap A_j = O$ similarly as above.

(1). That $u \geq \lambda$, follows from $P \cap u \geq (P \cap A_j) \cap (P \cap P) = P \cap P_1^{(1)}$. 
For the case $P_1 \cap A_k = O$, $P_2 \cap A_k > O$, we have $l = P_1 \cap w$ and from (10) we obtain

\begin{equation}
(11) \quad h_{ji}^{(2)} = h_{jk}^{(1)} + (e_{12}^{(1)} - h_{jk}^{(2)} h_{kj}^{(1)})l_{ji}.
\end{equation}

Since $O(h_{jk}^{(2)}) > 0$, $e_{12}^{(1)} - h_{jk}^{(2)} h_{kj}^{(1)}$ is a unit and we have from (4) and (11):

\begin{equation}
(12) \quad h_{ki} - h_{ki}^{(1)}(\lambda_{1}) = (h_{kj} - h_{kj}^{(1)}(\lambda_{2}))\epsilon(h_{ji}^{(2)} - h_{Jk}^{(2)} h_{kj}^{(1)}).
\end{equation}

where $e_{12}^{(1)} - h_{jk}^{(2)} h_{kj}^{(1)} = \epsilon^{-1}$ and $P \cap A_k = O$. It follows then

\begin{equation}
(13) \quad h_{ki} = \Theta h_{ki}^{(1)} + (e_{12}^{(1)} - \Theta h_{kj}^{(1)})h_{ki}^{(2)},
\end{equation}

with $\Theta = (h_{kj} - h_{kj}^{(1)})\epsilon$. That either $\Theta$ or $e_{12}^{(1)} - \Theta h_{kj}^{(1)}$ is a unit, can be easily seen. If $P \cap A_k > O$, then $P \cap A_j = O$ and $\dim P \leq t$. Consequently $P \leq P(1)^{t} \cap P(2) = l'$. Hence we have to consider the new straight line $l'$, where $P(1)^{t} \cap A_j > O$. $P(2) \cap A_j = O$, $P \cap A_j = O$, and, interchanging $k$ with $j$ and $P(2)$ with $P(1)^{t}$ in the above result, we obtain

\begin{equation}
(14) \quad h_{ji} = \Theta h_{ji}^{(1)} + (e_{12}^{(1)} - \Theta h_{kj}^{(1)})h_{ji}^{(2)},
\end{equation}

where $\Theta = (h_{kj} - h_{kj}^{(1)})\epsilon$, $\epsilon^{-1} = e_{12}^{(1)} - h_{kj}^{(1)} h_{kj}^{(2)}$.

Finally for the case $P_1 \cap A_k > O$, $P_2 \cap A_k = O$, we can treat as follows. If $P_2 \cap A_j = O$, then it reduces to the second case and, if $P_2 \cap A_j = O$, then we can treat in the same way as the third case.

**Theorem 82.** (Converse of theorem 81) If $h_{i}^{(1)}, h_{i}^{(2)}$ represent the coordinates of two independent points $P_1$, $P_2$ respectively, and if $\lambda_1$, $\lambda_2$ are two finite points of the same dimension $s$, where at least one is an unit and $\dim \lambda_1 h_{i}^{(1)} = \dim \lambda_2 h_{i}^{(2)} = s$, then $h_{i} = \lambda_1 h_{i}^{(1)} + \lambda_2 h_{i}^{(2)}$ are coordinates of a point of dimension $s$ on the straight line $P(1)^{t} \cap P(2)$.

Pr. We can assume here that $\lambda_1$ is an unit and $t = \dim P_2 \leq \dim P_1 = s = \dim \lambda_i$. Supposing $(P_1 \cap P_2) \cap A_{jk} = O$, we first consider the case $P_1 \cap A_k = O$, $P_2 \cap A_k = O$. Then we can assume further that $h_{i}^{(1)} = h_{ki}^{(1)}$ and $h_{i}^{(2)} = h_{ki}^{(2)}$. Since $h_{ki}^{(1)} \cap h_{ki}^{(2)} = O$, either $\lambda_1 + \lambda_2$ or $\lambda_1 h_{ki}^{(1)} + \lambda_2 h_{ki}^{(2)}$ is an unit. If $\lambda_1 + \lambda_2$ is an unit, we put

\begin{equation}
(15) \quad (\lambda_1 + \lambda_2) h_{kj} = \lambda_1 h_{kj}^{(1)} + \lambda_2 h_{kj}^{(2)}, \quad P(1)^{t} h_{kj} = P_{kj},
\end{equation}

\begin{equation}
(16) \quad P(1)^{t} h_{kj}^{(2)} = P_{kj}, \quad (A_{jk} \cap P_{kj}) \cap l = P.
\end{equation}

We shall now show that $P$ is a point of dimension $s$ on $l$. Since
$O(\lambda_{0}) \geq s - \dim P_{2}$, we obtain $O(h_{kj} - h_{kj}^{(1)}) \geq s - \dim P_{2}$, whence $\dim (P_{k_{j}} \cap P_{k_{j}}^{(1)}) \geq s - \dim P_{2}$ follows. Hence we have

$$A_{jk} \cup P_{k_{j}} \leq A_{jk} \cap P_{k_{j}}^{(1)} \cup P_{k_{j}}^{(2)} = A_{jk} \cup P_{1} \cap P_{2}$$

and then

$$\frac{P}{O} \sim \frac{A_{jk} \cup P_{k_{j}}}{A_{jk}} \sim \frac{P_{k_{j}}}{O}$$

and $(P \cup A_{jk}) \cap (a_{j} \cup a_{k}) = P_{k_{j}}$. Since the point $P$ lies on $l = P_{1} \cup P_{2}$ and $P_{1} \cap A_{k} = O$, $P_{2} \cap A_{k} = O$, $P \cap A_{k} = O$, we have for the coordinates of $P$ $h_{ki} = \theta h_{ki}^{(0)} + (e_{12}^{0} - \theta) h_{ki}^{(1)}$, where $\Theta = (h_{kj} - h_{kj}^{(1)}) \epsilon$ and \epsilon^{-1} = h_{kj}^{(0)} - h_{kj}^{(1)}. But $(\lambda_{1} + \lambda_{2}) \theta = \lambda_{2}(h_{kj}^{(0)} - h_{kj}^{(1)}) \epsilon = \lambda_{2}$ and $(\lambda_{1} + \lambda_{2})(e_{12}^{0} - \Theta) = \lambda_{1}$. Hence $(\lambda_{1} + \lambda_{2}) h_{ki} = h_{i}$ are coordinates of $P$. If $\lambda_{1} + \lambda_{2}$ is not an unit and if $h_{kj}^{(0)} + \lambda_{2} h_{kj}^{(2)}$ is an unit, then both $\lambda_{1}$ and $\lambda_{2}$ are units and $t = s$. Putting $(A_{jk} \cup P_{jk}) \cap l = P$, $\lambda_{1} + \lambda_{2} = (\lambda_{1} h_{kj}^{(1)} + \lambda_{2} h_{kj}^{(0)}) h_{jk}$ and $P_{k} h_{jk} = P_{jk}$, we have

$$\frac{P}{O} \sim \frac{A_{jk} \cup P_{jk}}{A_{jk}} \sim \frac{P_{jk}}{O}.$$  

Hence $P$ is a point of dimension $s$ with $P \cap A_{j} = O$ and $(P \cup A_{jk}) \cap (a_{j} \cup a_{k}) = P_{k_{j}}$. Its coordinates are now $h_{ji} = \theta h_{ji}^{(0)} + (h_{jk} - \theta) h_{ji}^{(1)}$ with $(e_{12}^{0} - h_{jk} h_{ji}^{(1)}) \epsilon = \theta$, $\epsilon^{-1} = h_{kj}^{(0)} - h_{kj}^{(1)}$. But $(\lambda_{1} h_{kj}^{(0)} + \lambda_{2} h_{kj}^{(2)}) \Theta = \lambda_{2}$ and $(\lambda_{1} h_{kj}^{(0)} + \lambda_{2} h_{kj}^{(2)})(h_{jk} - \Theta) = \lambda_{1}$, yielding $(\lambda_{1} h_{kj}^{(1)} + \lambda_{2} h_{kj}^{(0)}) h_{ji} = h_{i}$, proving that these are coordinates of the point $P$.

For the case $P_{1} \cap A_{j} = O$ and $P_{2} \cap A_{j} = O$ we can treat analogously as above. In case $P_{1} \cap A_{k} = O$, $P_{2} \cap A_{k} = O$, we can assume that $h_{i}^{(0)} = h_{i}^{(1)}$ and $h_{i}^{(0)} = h_{i}^{(1)}$. Since $O(h_{kj}^{(0)}) > O$, $\lambda_{1} + \lambda_{2} h_{kj}^{(2)}$ must be an unit. Putting $(\lambda_{1} + \lambda_{2} h_{kj}^{(2)}) h_{kj} = \lambda_{1} h_{kj}^{(1)} + \lambda_{2}$, $P_{k} h_{jk} = P_{k_{j}}$ and $(A_{jk} \cup P_{k_{j}}) \cap l = P$, we see that $P$ is a point of dimension $s$ with $P \cap A_{k} = O$, $(P \cup A_{jk}) \cap (a_{j} \cup a_{k}) = P_{k_{j}}$ and its coordinates are $h_{ki} = \Theta h_{ki}^{(0)} + (e_{12}^{0} - \Theta h_{kj}^{(2)}) h_{ki}^{(1)}$ with $\Theta = (h_{kj} - h_{kj}^{(1)}) \epsilon$, $e_{12}^{0} - h_{kj}^{(1)} h_{kj}^{(1)} = \epsilon^{-1}$. Since $(\lambda_{1} + \lambda_{2} h_{kj}^{(2)}) \Theta = \lambda_{2}$ and $(\lambda_{1} + \lambda_{2} h_{kj}^{(2)})(e_{12}^{0} - \Theta h_{kj}^{(2)}) = \lambda_{1}$, we have $(\lambda_{1} + \lambda_{2} h_{kj}^{(2)}) h_{ki} = h_{i}$. Finally, if $P_{1} \cap A_{k} > O$, $P_{2} \cap A_{k} = O$, we have either the case $P_{1} \cap A_{j} = O$, $P_{2} \cap A_{j} = O$ or the case $P_{1} \cap A_{j} = O$, $P_{2} \cap A_{j} > O$. The latter case can be treated in the same way as above, if we interchange $j$ and $k$.

**Theorem 83.** Let $a^{(1)}$, $a^{(2)}$, $\cdots$, $a^{(\nu)}$ be a basis of an element $a$ in $L$ and $P$ be an irreducible element (point) of dimension $\lambda$ in $\mathfrak{a}$ with coordinates $h_{i}$, then there exists finite points $\lambda_{k}$, $k = 1, 2, \cdots, \nu$.  


of dimension \( \lambda \) in \( \frac{a_1 \cup a_2}{O} \), such that \( h_i = \sum_{k=1}^{v} \lambda_k h_i^{(k)} \), \( O(\lambda_k) \geqq \lambda - \dim a^{(k)} \) and at least one of \( \lambda_k \) is a unit, where \( h_i^{(k)} \), \( i = 1, 2, \ldots r \) are coordinates of \( a^{(k)} \). Conversely, if \( \lambda_k, k = 1, 2, \ldots \nu \) are all finite points of dimension \( \lambda \) with \( O(\lambda_k) \geqq \lambda - \dim (a^{(k)}) \), of which at least one is an unit, then \( h_i = \sum_{k=1}^{v} \lambda_k h_i^{(k)} \) are coordinates of a point of dimension \( \lambda \) in \( \frac{a}{O} \).

Pr. By induction on \( \nu \). The case \( \nu = 2 \) is evident from theorems 81 and 82. Suppose, \( \nu > 2 \) and \( P \) does not belong to \( \frac{a^{(\nu)}}{O} \). Putting

\[
A^{(\nu)} = a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(\nu-1)} , \quad (P \cup a^{(\nu)}) \cap A^{(\nu)} = d ,
\]

we have

\[
\frac{d}{O} \simeq \frac{d \cup a^{(\nu)}}{a^{(\nu)}} \simeq \frac{P}{P \cap a^{(\nu)}} .
\]

Hence \( \frac{d}{O} \) is a chain with \( O < \dim d \leqq \lambda \) and consequently the point \( P \) lies on the straight line \( \frac{a^{(\nu)} \cup d}{O} \). By theorem 81 we have

\[
h_i = \lambda_i h_i^{(\nu)} + \mu d_i , \quad \text{where } \lambda_i \text{ and } \mu \text{ are finite points of dimension } \lambda \text{, of which at least one is an unit, and where } d_i \text{ are coordinates of the point } d .
\]

But by induction-hypothesis we have \( d_i = \sum_{k=1}^{v-1} \lambda_k h_i^{(k)} \), where \( \lambda_k \) are all finite points with \( \dim \lambda_k = \dim d \) and at least one of \( \lambda_k \) is an unit. Hence it follows \( h_i = \sum_{k=1}^{v-1} \mu \lambda_k h_i^{(k)} + \lambda_i h_i^{(\nu)} \), where either \( \lambda_i \) or at least one of \( \mu \lambda_k \) is an unit.

The converse statement of the theorem can be proved also by induction on \( \nu \). If \( \lambda \) is not an unit, then at least one of \( \lambda_1, \lambda_2, \ldots \lambda_{\nu-1} \) is an unit and \( h_i' = \sum_{k=1}^{v-1} \lambda_k h_i^{(k)} \) will be coordinates of a point \( d \) with dimension \( \lambda \) in \( \frac{A^{(\nu)}}{O} \). Hence \( h_i = h_i' + \lambda_i h_i^{(\nu)} \) are coordinates of a point with dimension \( \lambda \) in \( \frac{d \cup a^{(\nu)}}{O} \). In case, \( \lambda \) is an unit, let \( O(\lambda_k) = v \) be the least one among \( O(\lambda_i), \ i = 1, 2, \ldots \nu-1 \) and put \( \lambda_k = \lambda_{P\mu_k}, k = 1, 2, \ldots \nu-1 \) with \( \dim \mu_k = \lambda - v \). Then \( h_i'' = \sum_{k=1}^{v-1} \mu \lambda_k h_i^{(k)} \) are coordinates of a point \( c \) with \( \dim c = \lambda - v \) in \( \frac{A^{(\nu)}}{O} \), since \( \mu_{\nu} = \varepsilon_{\nu}^{(\lambda - v)} \) is an unit. Therefore \( h_i = \lambda_i h_i'' + \lambda_i h_i^{(\nu)} \) are coordinates of a point with dimension \( \lambda \) in \( \frac{c \cup a^{(\nu)}}{O} \).
Let the height of the given primary lattice $L$ be $h$ and $a_i$, $i = 1, 2, \cdots, r$ be its basis with $r \geq 4$, $d_1 = d_2 = d_3 = d_4 = h$. Then all finite points of dimension $h$ in the straight line $\frac{a_1 - a_2}{O}$ generates a ring $R$ with respect to the addition and the multiplication defined in part 4. The set of all points $P$ in $R$ with $O(P) \geq i$ is now an ideal $J_i$ in $R$ by theorems 58 and 62. That conversely every left or right ideal in $R$ is such an ideal, can be seen from theorems 60 and 61. Indeed, if $u$ is a point in an ideal $J$ with the least order $O(u) = i$, then we can find by theorems 60 and 61 points $q_1$ and $q_2$ in $R$ for any point $t$ with $O(t) \geq i$, such that $t = q_1u$ or $t = uq_2$, whence $J_1 \leq J$ and hence $J = J_1$ follows. Now we see that the ideal $J_1$ is a prime ideal without divisors and consequently the residue class ring $R/J_1$ is a field. (generally non-commutative). If we select a point $\tau$ from each class of $R/J_1$ as its representant, then we can represent uniquely any finite point $t$ in $R$ as $t = \tau_0 + \tau_1\pi + \tau_2\pi^2 + \cdots + \tau_{h-1}\pi^{h-1}$ with a fixed point $\pi$ of order 1.

Now, $\tau$ being the rank of the lattice $L$, we consider a $R$-module $M$ of rank $\tau$, i.e. a module with the operator ring $R$ and with a basis $x_1, x_2, \cdots, x_r$. To each irreducible element (point) $P$ of dimension $\lambda$ in $L$ we make correspond a submodule $M_P$ of rank 1, such that $M_P = \left(\sum_{i=1}^{r} \pi^{h-i}h_ix_i\right)$, where $h_i$ are coordinates of $P$ and $\pi$ is a fixed point in $R$ with $O(\pi) = 1$. Given any submodule $\left(\sum b_ix_i\right)$ of rank 1 in $M$, such that $O(b_i) \geq h - \dim a_i$, there exists always its corresponding point $P$ in $L$. In order to prove this, let $\min O(b_i) = O(b_\lambda) = t$, then $\pi^{-t}b_\lambda = h_\lambda$ are coordinates of a point of dimension $h - t$ in $L$, since $O(h_\lambda) \geq h - t - d_i$ and $h_\lambda$ is an unit. That this correspondence is one-to-one can be proved as follows. Let $h_i$ and $h'_i$ be two coordinates of a point of dimension $\lambda$ with $h'_i = \pi h_i$, $\pi$ being an unit. Then $\pi^{h-\lambda}h'_i = (\pi^{h-\lambda}\pi^{-(h-\lambda)})\pi^{h-\lambda}h_i$ and $\pi^{h-\lambda}\epsilon\pi^{-(h-\lambda)}$ is an unit in $R$. Hence the submodule $\left(\sum \pi^{h-\lambda}h'_ix_i\right)$ is identical with the submodule $\left(\sum \pi^{h-\lambda}h_ix_i\right)$. Conversely if $\left(\sum b_ix_i\right)$ and $\left(\sum b'_ix_i\right)$ are the same submodule, then $b'_i = \epsilon b_i$ with an unit $\epsilon$ and $\min O(b'_i) = \min O(b_i) = t$, $\pi^{-t}b'_i = (\pi^{-t}\epsilon\pi^t)\pi^{-t}b_i$, where $\pi^{-t}\epsilon\pi^t$ is evidently an unit.

**Theorem 84.** Let $M_1$, $M_2$ be the corresponding submodule for
the point $P_1$, $P_2$ respectively. From $P_1 \leq P_2$ follows $M_1 \leq M_2$ and vice versa.

Pr. Suppose $P_2 \cap A_k = O$ and $h_{k1}^{(i)}$, $h_{k2}^{(i)}$, $i = 1, 2, \cdots r$ are coordinates of $P_1$, $P_2$ respectively, where $\dim P_1 = t_1$, $\dim P_2 = t_2$. We have $M_1 = \sum h_{ki}^{(1)}x_i$, $M_2 = \sum h_{ki}^{(2)}x_i$ with $h_{ki}^{(1)} = \pi^{h-t_1}h_{ki}^{(1)}$, $h_{ki}^{(2)} = \pi^{h-t_2}h_{ki}^{(2)}$. From $P_1 \leq P_2$ follows $t_1 \leq t_2$ and $h_{k1}^{(1)} = h_{k2}^{(2)}$, whence $h_{ki}^{(1)} = \pi^{t_1-t_2}h_{ki}^{(2)}$ and consequently $t_2 - t_1 = \mu$, $h_{ki}^{(1)} = \sigma h_{ki}^{(2)}$, $\sigma$ being an unit, which yields $P_1 \leq P_2$.

Lemma 13. $h_{ki}^{(1)}$, $i = 1, 2, \cdots r$ being coordinates of independent points $P_\nu$, $\nu = 1, 2, \cdots s$, it holds

$$\min \left\{ \Omega(\sum_{j=1}^{s} \lambda_j h_{\nu}^{(j)}), \cdots \right\} = \min (\Omega(\lambda_1), \cdots, \Omega(\lambda_s)).$$

Pr. By induction on $s$. First we treat the case $s = 2$, where we can suppose that $\Omega(\lambda_2) \geq \Omega(\lambda_1)$ and $h_{k1}^{(1)}$ is an unit. If $\Omega(\lambda_2 h_{k1}^{(2)}) > \Omega(\lambda_1)$, then we have $\Omega(\lambda_2 h_{k1}^{(2)} + \lambda_1 h_{k2}^{(2)}) = \Omega(\lambda_1)$ and $\Omega(\lambda_1 h_{k1}^{(1)} + \lambda_2 h_{k2}^{(1)}) \geq \Omega(\lambda_1)$; $i = 1, 2, \cdots r$. If $\Omega(\lambda_2 h_{k1}^{(2)}) = \Omega(\lambda_1)$, then $\Omega(\lambda_2) = \Omega(\lambda_1)$ and $h_{k1}^{(1)}$ is an unit. Hence we can assume $h_{k1}^{(1)} \leq e_{12}$, $h_{k2}^{(2)} \leq e_{12}$. If $\Omega(\lambda_1 h_{k1}^{(1)} + \lambda_2 h_{k2}^{(2)}) > \Omega(\lambda_1)$ for every $j$, it would follow $\Omega(h_{k1}^{(1)} - h_{k2}^{(1)}) > O$ for every $j$, which yields however the dependence of $P_1$ and $P_2$, contrary to the hypothesis. If $s > 2$, we can put by induction-hypothesis

$$\min \left\{ \Omega(\sum_{j=1}^{s} \lambda_j h_{\nu}^{(j)}), \cdots \right\} = \min (\Omega(\lambda_1), \cdots, \Omega(\lambda_{s-1})) = \nu.$$

We conclude therefore that $h_{i}^{(1)} = \pi^{-\nu_{0}}a_{i}^{(1)} + \cdots + \pi^{-\nu_{s-1}}a_{i}^{(s-1)}$ are coordinates of a point $P'$ of dimension $\lambda - \nu$, which is independent with $P_\nu$. Then it follows $\sum_{j=1}^{s} \lambda_j h_{i}^{(j)} = \pi^\nu h_{i}^{(1)} + \lambda_j h_{i}^{(s)}$ and

$$\min \left\{ \Omega(\sum_{j=1}^{s} \lambda_j h_{i}^{(j)}), \cdots, \Omega(\sum_{j=1}^{s} \lambda_j h_{i}^{(r)}) \right\} = \min (\Omega(\nu), \Omega(\lambda_0)) = \min (\Omega(\lambda_1), \cdots, \Omega(\lambda_s)).$$

Theorem 85. If $a_{i}^{(1)}$, $i = 1, 2, \cdots s$ are independent points in $L$, then the corresponding submodules $M_i = \langle \sum_{j=1}^{s} A_{j}^{(i)}x_j \rangle$, $i = 1, 2, \cdots s$ are independent. Moreover to every point in the quotient $\frac{a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(s)}}{O}$ corresponds uniquely a submodule in the module $M$ of rank $s$, which
is generated by $\sum_{j=1}^{r}A_{j}(x_{j})$, $i = 1, 2, \ldots s$. Conversely to every submodule of rank 1 in $M$ corresponds a point in $\frac{a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(s)}}{O}$.

Pr. First we consider the case $s = 2$. Since any submodule, which is contained both in $M_{1}$ and $M_{2}$, corresponds to a point in $\frac{a^{(1)} \cup a^{(2)}}{O}$ by theorem 84, so $M_{1}$ and $M_{2}$ are independent. The coordinates of a point of dimension $\lambda$ in $\frac{a^{(1)} \cup a^{(2)}}{O}$ are $\lambda_{1}a^{(1)}_{j} + \lambda_{2}a^{(2)}_{j}$, where $a^{(i)}_{j}$, $i = 1, 2$ are coordinates of the point $a^{(i)}$ respectively. Then it corresponds the submodule generated by

$$\sum_{j=1}^{r} \pi^{h-\lambda}(\lambda_{1}a^{(1)}_{j} + \lambda_{2}a^{(2)}_{j})x_{j} = \pi^{h-\lambda}\lambda_{1}\pi^{-(h-n_{1})} \sum_{j=1}^{r} A_{j}^{(1)}x_{j} + \pi^{h-\lambda}\lambda_{2}\pi^{-(h-n_{2})} \sum_{j=1}^{r} A_{j}^{(2)}x_{j},$$

where $n_{i} = \dim a^{(i)}$, $i = 1, 2$, $\lambda \leq \max(n_{1}, n_{2})$ and $O(\pi^{h-\lambda}\lambda_{1}) \geq h-n_{1}$, $O(\pi^{h-\lambda}\lambda_{2}) \geq h-n_{2}$. Hence it is contained in the submodule, generated by $M_{1}$ and $M_{2}$. Conversely, if a submodule $\{\sum_{j=1}^{r}(r_{1}A_{j}^{(1)} + r_{2}A_{j}^{(2)})x_{j}\}$ of rank 1 is given, we have by lemma 13

$$\text{Min} \{O(r_{1}A^{(1)} + r_{2}A^{(2)}), \cdots\} = \text{Min} \{O(r_{1}\pi^{h-n_{1}}), O(r_{2}\pi^{h-n_{2}})\}.$$  

Denoting this with $\nu$, it corresponds the point, whose coordinates are

$$\pi^{-\nu}(r_{1}A^{(1)}_{j} + r_{2}A^{(2)}_{j}) = \pi^{-\nu}r_{1}\pi^{h-n_{1}}a^{(1)}_{j} + \pi^{-\nu}r_{2}\pi^{h-n_{2}}a^{(2)}_{j},$$

where $\pi^{-\nu}r_{1}\pi^{h-n_{1}}$, $\pi^{-\nu}r_{2}\pi^{h-n_{2}}$ are both of dimension $h-\nu$ and at least one of them is an unit. The point is therefore of dimension $h-\nu$ in $\frac{a^{(1)} \cup a^{(2)}}{O}$.

The general case will be proved by induction on $s$. Any submodule of rank 1 in the submodule generated by $\sum_{j=1}^{r} A_{j}(x_{j})$, $\cdots \sum_{j=1}^{r} A_{j}^{(s)}x_{j}$ shall correspond a point in $\frac{a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(s)}}{O}$ by hypothesis, if $\mu \leq s$. $\sum A^{(1)}_{j}x_{j}, \cdots, \sum A^{(s)}_{j}x_{j}$ are then independent in virtue of the relations $(a^{(1)} \cup \cdots \cup a^{(s-1)}) \cap a^{(s)} = O$, $\mu = 2, \cdots s$. The coordinates of a point of dimension $\lambda$ in $\frac{a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(s)}}{O}$ are

$$h_{i} = \sum_{j=1}^{r} \lambda_{j}a^{(i)}_{j}$$

and its corresponding submodule is generated by

$$\sum_{i=1}^{s} \pi^{h-\lambda}h_{i}x_{i} = \sum_{i=1}^{s} \pi^{h-\lambda} \sum_{j=1}^{r} \lambda_{j}a^{(i)}_{j}x_{i} = \sum_{j=1}^{r} \pi^{h-\lambda}\lambda_{j}\pi^{-(h-n_{i})} \sum_{i=1}^{s} A_{j}^{(i)}x_{j},$$
where \( n_i = \dim a^{(i)} \), \( O(\lambda_j) \geq \lambda - n_j \). Conversely, if a submodule 
\( \left( \sum_{j=1}^{r} \pi^{h-n_j}r_j A^{(j)}x_i \right) \) of rank 1 is given with \( \min \left( \sum_{j=1}^{r} r_j A^{(j)} \right) = \nu \), then \( \nu = \min \left( \pi^{h-n_j}r_j A^{(j)} \right) \) and the coordinates of the corresponding point are
\( \sum_{j=1}^{r} \pi^{-\nu}r_j \pi^{h-n_j}a^{(j)}_i \), where \( \pi^{-\nu}r_j \pi^{h-n_j} \) are of dimension \( h - \nu \) in \( R \) and at least one of them is an unit. Hence the point belongs to \( \frac{a^{(1)} \cup a^{(2)} \cup \cdots \cup a^{(s)}}{O} \). q. e. d.

Now we make correspond to an element with a basis \( c^{(1)}, c^{(2)}, \ldots, c^{(s)} \) in \( L \), the submodule in \( M \), which is generated by \( \sum_{j=1}^{r} C_j^{(j)}x_j \), \( \ldots \sum_{j=1}^{r} C_j^{(s)}x_j \). That this correspondence is unique, can be seen from the preceding theorem. We see furthermore, that, if \( c \leq b \) in \( L \), then \( M_c \leq M_b \) and vice versa, where by \( M_b, M_c \) are meant the corresponding submodules of \( b, c \) respectively. Therefore we have

**Theorem 86.** A primary lattice of rank \( r \geq 4 \) with a basis \( a_1, a_2, \ldots a_r \), where \( \dim a_1 = \dim a_2 = \dim a_3 = \dim a_r = h \), \( h \) being the height of \( L \), is isomorphic with the lattice of all submodules in a submodule \( M \) of a \( R \)-module \( M' \) with basis \( x_1, x_2, \ldots x_r \), where, \( R \) being an uniserial complete primary ring, \( M \) has a basis \( \pi^\lambda x_i, \) \( i = 1, 2, \ldots r \) with \( \pi \) in \( R \) and \( \lambda = h - \dim a_i, O(\pi) = 1 \).

**Theorem 87.** If \( K \) is an uniserial complete primary ring, then the lattice of all submodules in any \( K \)-module \( M \) is primary.

Pr. Since \( K \) is complete primary and uniserial, \( K \) has only such ideals \( J_i = (\pi^i), i = 1, 2, \ldots l \), where \( J_i = (O) < J_{i-1} < \cdots < J_1 < K \) holds. Every submodule \( A \) of rank 1 in \( M \) is generated by an element \( \sum a_i x_i \), where \( x_1, x_2, \ldots \) are basis of \( M \) and \( a_i \in K \). If \( \eta \in J_1 \), then the submodule \( A_\eta = (\sum \eta a_i x_i) \) is properly contained in \( A \), because the ideal \( (\eta a_i) \) is different from the ideal \( (a_i) \) and hence \( \varepsilon \eta a_i = a_i \) with \( \varepsilon \in K \) does not hold. We have only to show by theorem 45, that, a submodule \( N \) being contained in a submodule \( N* \), two atoms in the quotient \( \frac{N*}{N} \) are always perspective. We represent these two atoms as \( A \cup N \) and \( B \cup N \), where \( A = (\sum a_i x_i) \), \( B = (\sum b_i x_i) \). We shall now show that \( A \cup N \) and \( B \cup N \) are perspective with axis \( C \cup N \) in \( \frac{N*}{N} \), where \( C \) is the submodule \( \{ \sum (a_i + b_i) x_i \} \). Since it holds evidently \( A \cup C \cup N = B \cup C \cup N \), it
remains to prove that \((A \cup N) \cap (C \cup N) = N\), \((B \cup N) \cap (C \cup N) \neq N\). Since \(A \cup N\) is an atom in \(\frac{N^*}{N}\), \(A\) would belong to \(C \cup N\), if \((A \cup N) \cap (C \cup N) = N\) does not hold. But then it would follow that a submodule \(\{\sum(a_i - \eta a_i - \eta b_i)x_i\}\) with \(\eta \in K\) is contained in \(N\). Since \(B\) cannot be contained in \(A \cup N\), so \(\eta\) belongs to the ideal \(J_1\). Then both \(A\) and \(B\) are contained in \(N\) and consequently also \(A\) in \(N\), contrary to the assumption. Similarly we can prove \((B \cup N) \cap (C \cup N) = N\).

**Theorem 88.** If \(R\) is the ring of all matrices with degree \(n\) in an uniserial complete primary ring \(K\), then the lattice \(L\) of all submodules in a \(R\)-module of rank \(m\) is isomorphic with the lattice \(L^*\) of all submodules in a \(K\)-module of rank \(mn\).

Pr. Let \(x_i, i = 1, 2, \cdots m\) be a basis of the \(R\)-module \(M\) and consider a \(K\)-module \(M'\) with basis \(e_{ij}x_i, i = 1, 2, \cdots m, j = 1, 2, \cdots n\), where \(e_{ij}\) are matrix units. To every submodule \(A\) in \(M\), which is generated by \(A^{(\nu)} = \sum a_{ijk}^{(\nu)}e_{kj}x_i, a_{ijk}^{(\nu)} \in K, \nu = 1, 2, \cdots r,\) we correspond the submodule \(A'\) in \(M'\), which is generated by \(A_{\lambda \nu} = \sum a_{ijk}^{(\nu)}e_{kj}x_i, \lambda = 1, 2, \cdots r, \mu = 1, 2, \cdots n\). Then to every submodule in \(M'\), which is generated by \(B_{\nu} = \sum b_{ij}^{(\nu)}e_{ij}x_i, b_{ij}^{(\nu)} \in K, \nu = 1, 2, \cdots s\) corresponds the submodule in \(M\), which is generated by \(B'_{\lambda} = \sum b_{ij}^{(\nu)}e_{kj}x_i\). Now, if a submodule \(C = \sum c_{ijk}e_{kj}x_i\) of rank 1 in \(M\) is contained in the submodule \(A\), it holds

\[
\sum_{j,k} c_{ijk}e_{kj} = \sum_{\nu} A_{\nu} \sum_{j,k} a_{ijk}^{(\nu)}e_{kj}, \quad i = 1, 2, \cdots m
\]

with \(A_{\nu} = \sum_{\nu} \lambda_{\nu}^{(\nu)}e_{st} \in R\), whence \(c_{ijk} = \sum_{\nu} \lambda_{\nu}^{(\nu)}a_{ijk}^{(\nu)}\). Then

\[
\sum_{j,k} c_{ijk}e_{kj}x_i = \sum_{\nu} \lambda_{\nu}^{(\nu)} \sum_{i,j} a_{ijk}^{(\nu)}e_{kj}x_i.
\]

Hence the corresponding submodule \(C'\) in \(M'\), which is generated by \(\sum c_{ijk}e_{kj}x_i, k = 1, 2, \cdots n\), is contained in \(A'\). Conversely, if a submodule \((\sum d_{ij}e_{ij}x_i), d_{ij} \in K\), is contained in \(A'\), then \(d_{ij} = \sum \lambda_{\nu}^{(\nu)}a_{ijk}^{(\nu)}\) and

\[
\sum_{k,j} d_{ij}e_{kj}x_i = \sum_{\nu} (\sum \lambda_{\nu}^{(\nu)}e_{ks}) \sum_{i,j} a_{ijk}^{(\nu)}e_{kj}x_i.
\]

Hence the corresponding submodule in \(M\) is contained in \(A\). The above consideration shows us that the correspondence is one-to-one and isomorphic. q. e. d.

By the preceding theorem we see readily that, if \(R\) is a direct
sum of uniserial primary rings $R_i$, $i = 1, 2, \cdots \lambda$ and if $R_i$ is the ring of all matrices with degree $n_i$ in a complete primary ring $K_i$, then the lattice $L$ of all submodules in a $R$-module of rank $s$ is isomorphic with the direct union of lattices $L_i$, where $L_i$ is the lattice of all submodules in a $K_i$-module of rank $n_i s$. Therefore $L$ is semi-primary. Conversely, if $L$ is a direct union of primary lattices $L_i$ and $m_{h_i}(L_i) \geq 4$, then $L$ is isomorphic with a lattice of submodules in a $R$-module, where $h_i$ being the height of $L_i$ and $R$ is a direct sum of uniserial primary rings. It is to be remarked further that by theorem 88 the lattice of all submodules in a $K$-module of rank $n$ is isomorphic with the lattice of all left ideals in the ring of all matrices with degree $n$ in $K$. Then theorem 86 asserts that a primary lattice of rank $r$ with $m_h \geq 4$ is isomorphic with a lattice of left ideals in the ring of all matrices with degree $r$ in an uniserial complete primary ring.