ON THE THEORY OF A RHEONOMIC CARTAN SPACE

By

Michiaki KAWAGUCHI, Jr.

Introduction.

A Cartan space having its origin in the Cartan's paper "Les espace de métriques fondés sur la notion d'aire" was developed by many people, especially, L. Berwald. In the present paper we attempt to build up the geometry in this space from standpoint of the rheonomic theory. In this space, its \((n-1)\)-dimensional area is assumed to be given \(a \text{ priori}\) in such a way that it depends on a variation of time. The geometrical quantities of this space depend on \(x^a, t, u_a, u_0\). If the time-area is independent of \(u_0\), then in every moment this space reduces to a Cartan space in ordinary sense, i.e. then this area is nothing but that of a Cartan space. Since \(u_0\) may be interpreted as a velocity of a small piece of hypersurface-element, we shall call \(u_0\) a velocity of the hypersurface-element. As \(u_0\) is not invariant under a rheonomic transformation, we introduce an invariant parameter \(v\) in place of \(u_0\). This parameter \(v\) plays an important rôle in our theory. The form of fundamental function \(L(x^a, t, u_a, u_0)\) is rewritten in \(G(x^a, t, u_a, v)\) which is homogenous of degree one in \(u_a\) and lets us decide the base connection, the connection-parameters, the curvature tensors and identities of Bianchi in our space.

§ 1. Fundamental function.

In an \(n\)-dimensional rheonomic manifold \(X_n\) with coordinates \(x', t\), we consider a rheonomic hypersurface \(X_{n-1}\) given by

\[
(1.1) \quad x^a = x^a(v^1, v^2, \ldots, v^{n-1}, t) \quad a = 1, \ldots, n
\]

and suppose that its measure of \((n-1)\)-dimensional time-area in a

certain region and time interval is defined by the integral

\[(1.2) \quad 0 = \int_{(n)} \psi \left( x^{a}, \frac{\partial x^{a}}{\partial \nu^{i}}, \frac{\partial x^{a}}{\partial \tau}, t \right) d\nu^{1} d\nu^{2} \cdots d\nu^{n-1} dt, \quad i = 1, \ldots, n - 1 \]

extended over the region and time interval. Then we shall call the manifold a rheonomic Cartan space. Since (1.2) must be invariant under the rheonomic transformation of parameters \( \bar{v}^{\prime} = \bar{v}^{\prime}(v^{1}, \ldots, v^{n-1}, t) \), by using of Radon's(1) and Vivanti's(2) theorem, \( \phi \) may be written in the form \( L(x^{a}, t, u_{a}, u_{0}) \), which is positive and homogenous of degree one in \( u_{a}, u_{0} \), where \( u_{a} = (-1)^{a+1} \left| \frac{\partial x^{1}}{\partial v^{1}}, \ldots, \frac{\partial x^{a-1}}{\partial v^{a}}, \frac{\partial x^{a+1}}{\partial v^{a-1}}, \ldots, \frac{\partial x^{n}}{\partial v^{n}} \right| \) and \( u_{0} = - \frac{\partial x^{a}}{\partial \nu \nu} u_{a} \).

Under a rheonomic transformation \( \bar{x} = \bar{x}(x, t) \), \( u_{a} \) and \( u_{0} \) vary in the rule

\[(1.3) \quad \bar{u}_{a} = \left| \frac{\partial \bar{x}}{\partial x} \right| \frac{\partial x^{\beta}}{\partial \bar{x}^{\beta}} u_{\beta}, \quad \bar{u}_{0} = \left| \frac{\partial \bar{x}}{\partial x} \right| \left( u_{0} + \frac{\partial x^{\beta}}{\partial \nu \nu} u_{\beta} \right) , \]

that is, \( u_{a} \) is a vector density of weight \(-1\) and \( u_{0} \) is not a invariant. We consider therefore the quantities \( u_{a} \) and \( u_{0} \) which depend on a rheonomic hypersurface element but not necessarily on the hypersurface (1.1) itself, and which vary in the rule (1.3) under a rheonomic transformation. \( L \) must be there a scalar under a rheonomic transformation and analytic in a certain region \( \delta \). \( L \) is called the fundamental function.

\[\S 2. \quad \text{Metric functions } a^{\beta}.\]

Since under a rheonomic transformation \( \frac{\partial L}{\partial u_{a}}, \frac{\partial L}{\partial x^{\beta}} \) vary in the rule

\[\frac{\partial \bar{L}}{\partial \bar{u}_{a}} = \left( \frac{\partial L}{\partial u_{a}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\beta}} + \frac{\partial L}{\partial x^{\beta}} \frac{\partial \bar{x}^{\beta}}{\partial \bar{u}_{a}} \right) \left| \frac{\partial x}{\partial \bar{x}} \right|, \quad \frac{\partial \bar{L}}{\partial \bar{u}_{0}} = \left| \frac{\partial x}{\partial \bar{x}} \right| \frac{\partial L}{\partial u_{0}} , \]

then

\[\bar{a}^{\beta} = \frac{\partial \bar{x}^{\beta}}{\partial \bar{x}^{\beta}} \left( a^{\beta} + \frac{\partial x^{\beta}}{\partial \nu \nu} \right) , \]

---


On the Theory of a Rheonomic Cartan Space

§ 3. Parameter $v$ and metric functions $g^{\alpha\beta}$.

Under a rheonomic transformation, the quantities

\[ \mathcal{A}^{\alpha\beta} = \frac{1}{2} \left[ \frac{\partial L}{\partial u_{\beta}} \left( \frac{\partial L}{\partial u_{\alpha}} - \frac{\partial L}{\partial u_{\alpha}} \frac{\partial x}{\partial t} \right) - \frac{\partial x}{\partial t} \left( \frac{\partial L}{\partial u_{\alpha}} - \frac{\partial L}{\partial u_{\alpha}} \frac{\partial x}{\partial t} \right) \right] \]

vary as a contravariant tensor density of weight 2. Let us put $\mathcal{A}^{\alpha\beta} = aa^{\alpha\beta}$ where $a = |a^{\alpha\beta}|^{-1}$, from which we can find $a = \sqrt{|\mathcal{A}|}$ where $\mathcal{A} = |\mathcal{A}|$. Putting

\[ v = \frac{1}{\sqrt{a}} \frac{\partial L}{\partial u_{0}} \]

we have an absolute scalar which is homogenous zero in $u_{\alpha}, u_{0}$. Assume that $u_{0}$ can be solved from the above equation (3.2) in the form

\[ u_{0} = F(x^{\alpha}, t, u_{\alpha}, v) \]

then we see $\rho u_{0} = F(x^{\alpha}, t, \rho u_{\alpha}, v)$, that is, the function $F$ is homogenous of degree one in $u_{\alpha}$. Putting (3.3) into $L(x^{\alpha}, t, u_{\alpha}, u_{0})$, the obtained function $G(x^{\alpha}, t, u_{\alpha}, v)$ is homogenous of degree one in $u_{\alpha}$. In the same manner $a^{\alpha}$ can be brought to functions of $x^{\alpha}, t, u_{\alpha}, v$, which are homogenous of degree zero in $u_{\alpha}$ too. Afterwards we shall represent these functions by the same letters $a^{\alpha}$.

Differentiating $G$ with respect to $u_{\beta}$, we have a contravariant tensor density of weight 2: $\mathcal{G}^{\alpha\beta} = \frac{1}{2} \frac{\partial G}{\partial u_{\alpha}} \frac{\partial G}{\partial u_{\beta}}$. Put $\mathcal{G}^{\alpha\beta} = gg^{\alpha\beta}$ where $g = 1/|g^{\alpha\beta}|$, from which we can find $g = \sqrt{|\mathcal{G}|}$, where $\mathcal{G} = |\mathcal{G}|$, then we have a contravariant tensor

\[ g^{\alpha\beta} = \mathcal{G}^{\alpha\beta} = \mathcal{G}^{\alpha\beta} \]

which are homogenous of degree zero in $u_{\alpha}$ and may take as the fundamental metric tensor in our space. Then $g_{\alpha\beta}, a_{\alpha}$ are defined by $g^{\alpha\beta} g_{\alpha\gamma} = \delta_{\gamma}^{\beta}, \ a_{\alpha} = g_{\alpha\beta} a^{\beta}$.

When $L$ does not contain $u_{0}$, this space reduces to a Cartan space for $t = \text{const.}$. 


We shall denote the covariant differentiation in the space with the symbol $D$. Since the covariant differentiation is necessary to satisfy that
1. $Dp = dp$, where $p$ is a strong scalar,
2. $D(X + Y) = DX + DY$, where $X$, $Y$ are any strong vectors,
3. $D(X \cdot Y) = X \cdot DY + DX \cdot Y$,
we define

\[(4.1) \quad DX^i = dX^i + \Gamma_{\mu\nu}^i X^\mu dx^\nu + \Gamma_{\mu}^i X^\mu dt + C_{\mu^y}^i X^\mu du_{\nu} + C_{\mu}^i X^\mu dv.\]

In consequence of the geometrical meaning of $u_\alpha$, this differentiation must be invariant under the transformation $\bar{u}_\alpha = \rho u_\alpha$. By this reason we suppose that $\Gamma_{\mu}^i$, $C_{\mu}^i$ are homogenious function of degree zero in $u_\alpha$ and $C_{\mu}^i$ of degree $-1$ and that $C_{\mu}^i u_\nu$ vanish.

Let this connection be a euclidean connection, that is, when a strong vector is transported parallel to itself, the length of this vector be invariant. Then it must be that

\[
\frac{\partial g_{\lambda\mu}}{\partial x^\nu} = g_{\alpha\mu} \Gamma_{\lambda}^\alpha + g_{\lambda\alpha} \Gamma_{\mu}^\alpha, \quad \frac{\partial g_{\lambda\mu}}{\partial u_\nu} = g_{\alpha\mu} C_{\lambda}^\alpha + g_{\lambda\alpha} C_{\mu}^\alpha,
\]

\[(4.2) \quad \frac{\partial g_{\lambda\mu}}{\partial t} = g_{\alpha\mu} \Gamma_{\lambda}^\alpha + g_{\lambda\alpha} \Gamma_{\mu}^\alpha, \quad \frac{\partial g_{\lambda\mu}}{\partial v} = g_{\alpha\mu} C_{\lambda}^\alpha + g_{\lambda\alpha} C_{\mu}^\alpha.
\]

Supposing that $C_{\lambda\mu}, C_{\lambda\mu}$ are symmetry with respect to $\lambda, \mu$, we have

\[(4.3) \quad C_{\lambda\mu} = \frac{1}{2} \frac{\partial g_{\lambda\mu}}{\partial u_\nu}, \quad C_{\lambda\mu} = \frac{1}{2} \frac{\partial g_{\lambda\mu}}{\partial v}.\]

§ 5. The normal unit vector.

From (3.4) we obtain

\[gg^{\alpha\beta} u_\alpha = \left(G \frac{\partial G}{\partial u_\alpha} \partial u_\beta + \frac{\partial G}{\partial u_\alpha} \frac{\partial G}{\partial u_\beta}\right) u_\alpha = G \frac{\partial G}{\partial u_\beta}.\]

Multipling $u_\beta$ and summing with respect to $\beta$, then we get

\[(5.1) \quad gg^{\alpha\beta} u_\alpha u_\beta = G^2.\]
$l_{\lambda} = \frac{u_{\lambda}}{\sqrt{g^{\alpha\beta} u_{\alpha} u_{\beta}}} = \frac{\sqrt{g}}{G} u_{\lambda}$
gives therefore a normal unit vector for the hypersurface $u_{\alpha}$. Putting

$$A_{\lambda}^{\nu} = \frac{G}{\sqrt{g}} C_{\lambda}^{\nu},$$

which satisfy the relation

$$A_{\lambda}^{\nu} l_{\nu} = 0,$$

we obtain

$$A_{\lambda}^{\nu} dl_{\nu} = C_{\lambda}^{\nu} du_{\nu}.$$

Thus the covariant differentiation (4.1) becomes

$$DX^{\lambda} = \mathcal{S} X^{\lambda} + \Gamma^{\nu}_{\lambda\nu} X^{\alpha} dx^{\nu} + \Gamma^{\nu}_{\lambda} X^{\mu} dt + A_{\lambda}^{\mu\nu} X^{\mu} dl_{\nu} + C_{\lambda}^{\nu} X^{\nu} dv.$$

The corresponding equation for the covariant components is

$$DX_{\lambda} = dX_{\lambda} - \Gamma_{\lambda\nu}^{\mu} X_{\mu} dx^{\nu} - \Gamma_{\lambda}^{\mu} X_{\mu} dt - A_{\lambda}^{\mu\nu} X_{\mu} dl_{\nu} - C_{\lambda}^{\mu} X_{\mu} dv$$

and for $l_{\lambda}$ we obtain

$$dl_{\lambda} = dl_{\lambda} - \Gamma_{\lambda\nu}^{\mu} dx^{\nu} - \Gamma_{\lambda}^{\mu} dt - A_{\lambda}^{\mu\nu} l_{\mu} dl_{\nu} - C_{\lambda}^{\mu} dv,$$

where $\Gamma_{\lambda\mu}^{\nu} l_{\nu} = \Gamma_{\lambda\mu}^{\nu}, \Gamma_{\lambda}^{\mu} l_{\mu} = \Gamma_{\lambda}^{\mu}$, etc. Substituting (5.7') in (5.7) and using (5.4), the covariant differentiation becomes

$$DX_{\lambda} = dX_{\lambda} - X_{\mu} (\hat{\Gamma}_{\lambda\nu}^{\mu} dx^{\nu} + A_{\lambda}^{\mu\nu} dl_{\nu} + \hat{C}_{\lambda}^{\nu} dv)$$

where we put

$$\hat{\Gamma}_{\lambda\nu}^{\mu} = \Gamma_{\lambda\nu}^{\mu} + \Gamma_{\beta\nu}^{\mu}, \hat{\Gamma}_{\lambda}^{\mu} = \Gamma_{\lambda}^{\mu} + A_{\lambda}^{\mu\alpha} \Gamma_{\alpha}, \hat{C}_{\lambda}^{\mu} = C_{\lambda}^{\mu} + A_{\lambda}^{\mu\alpha} C_{\alpha}.$$

For $X_{\lambda} = l_{\lambda}$ we have

$$dl_{\lambda} (\delta_{\mu}^{\alpha} + l_{\mu} A^{\alpha}) = dl_{\lambda} - \hat{\Gamma}_{\lambda\mu}^{\alpha} dx^{\alpha} - \hat{\Gamma}_{\lambda}^{\alpha} dt - \hat{C}_{\lambda}^{\alpha} dv.$$

Using $(\delta_{\mu}^{\alpha} + l_{\mu} A^{\alpha}) (\delta_{\nu}^{\beta} - l_{\nu} A^{\beta}) = \delta_{\nu}^{\beta}$, $dl_{\lambda}$ is represented by

$$dl_{\lambda} = (\delta_{\mu}^{\alpha} - l_{\alpha} A^{\alpha}) (dl_{\lambda} - \hat{\Gamma}_{\lambda\beta}^{\mu} dx^{\beta} - \hat{\Gamma}_{\lambda}^{\beta} dt - \hat{C}_{\lambda}^{\beta} dv).$$

(1) Since (3.4) and (4.3) give us the relation $A_{\nu}^{0\mu} = l_{\nu} A^{\mu}$, where $A^{\mu} = A_{0}^{\mu\nu}$, we see easily $A_{\mu}^{\lambda\nu} A_{\nu}^{0\beta} = 0$. 
§ 6. Other two postulates of the covariant differentiation.

We consider the invariant differential form

\[
\phi = D(g_{\lambda \mu} \delta x^\lambda \delta x^\mu) + D(g_{\lambda \mu} \delta x^\lambda \delta x^\mu) - D(g_{\lambda \mu} \delta x^\lambda \delta x^\mu)
\]

where the index 1, 2, 3 under the differential symbol \( D \), \( \delta \) denote the directions of the differentiations. They are here supposed to be interchangeable, that is, \( dd\delta x^a = d\delta x^a (a, b = 1, 2, 3) \). If we choose \( dt = d\delta t = 0, Dl_a = 0 \) and \( dv = 0 \), we have

\[
\phi = 2g_{\lambda \nu} \left[ d\delta x^\lambda + \left( \left\{ \lambda \omega \mu \right\} + A^\lambda_{\nu} \tilde{r}^0_{\nu \omega} + A^\lambda_{\mu} \tilde{r}^0_{\mu \omega} - A^\lambda_{\nu \omega} \tilde{r}^0_{0 \nu} \right) dx^\omega dv^\mu \right]
\]

\[
+ \left( \left\{ \lambda \omega \right\} + A^\lambda_{\nu} \tilde{r}^0_{\nu} + A^\lambda_{\mu} \tilde{r}^0_{\mu} - A^\lambda_{\nu \omega} \tilde{r}^0_{0 \nu} \right) dx^\omega dt^\mu \right], \tag{6.2}
\]

where we put

\[
\left\{ \begin{array}{l} \lambda \\ \omega \end{array} \right\} = g^\lambda \nu \left( \partial_\nu g_{\nu \omega} + \partial_\omega g_{\nu \nu} - \partial_\nu g_{\nu \omega} \right), \quad A^\nu_{\nu} = \frac{1}{2} \frac{\partial a_{\nu}}{\partial l_{\nu}}, \quad A^\lambda_{\nu} = g^\lambda \nu A^\nu_{\nu}.
\]

Since \( g_{\lambda \mu} \) is the strong tensor of degree 2 and \( \delta x^a \) is a strong vector, the term held in the bracket of above equation is a strong contravariant vector.

In order that we desire that the covariant differentiation does not change the property "strong" and the order of tensors, we shall define

\[
\tilde{r}^0_{\nu \omega} = \left\{ \begin{array}{l} \lambda \\ \nu \omega \end{array} \right\} + A^\lambda_{\nu} \tilde{r}^0_{\nu \omega} + A^\lambda_{\mu} \tilde{r}^0_{\mu \omega} - A^\lambda_{\nu \omega} \tilde{r}^0_{\nu \omega}, \tag{6.3}
\]

\[
\tilde{r}^0_{\nu} = \left\{ \begin{array}{l} \lambda \\ \nu \end{array} \right\} + A^\lambda_{\nu} \tilde{r}^0_{\nu} + A^\lambda_{\mu} \tilde{r}^0_{\mu} - A^\lambda_{\nu \omega} \tilde{r}^0_{\nu \omega}, \tag{6.4}
\]

Then we can verify \( Dg_{\lambda \mu} = 0 \). Putting \( \delta x^a = \delta x^a \) in (6.1), we have

\[
\phi = D \left( g_{\lambda \omega} \delta x^\lambda \delta x^\omega \right) = Dg_{\lambda \omega} \delta x^\lambda \delta x^\omega + 2g_{\lambda \omega} D \delta x^\lambda \delta x^\omega. \tag{6.5}
\]

On the other hand (6.2) becomes

\[
\phi = 2g_{\lambda \omega} D \delta x^\lambda \delta x^\omega. \tag{6.6}
\]

From (6.5) and (6.6) we obtain
On the Theory of a Rheonomic Cartan Space

\[ D g_{\lambda\omega} \partial x^i \partial x^\omega = 0 . \]

Hence arbitrariness of \( \partial x^i \) leads us to

\[ (6.7) \quad D g_{\lambda\omega} = 0 . \]

\section{7. The parameters of connection \( \hat{\Gamma}^1_{\lambda\omega} \).}

From (4.2) and (5.9) we have immediately

\[ (7.1) \quad \hat{\Gamma}^\mu_{\lambda\nu} + \hat{\Gamma}^\nu_{\mu\lambda} = \frac{\partial g_{\lambda\mu}}{\partial x^\nu} + 2A_{\lambda\mu}^\omega \Gamma^\omega_{\nu\lambda} . \]

From this the two following equations are obtained by cyclic permutation of \( \lambda, \mu, \nu \)

\[ (7.2) \quad \hat{\Gamma}^\nu_{\mu\lambda} + \hat{\Gamma}^\lambda_{\nu\mu} = \frac{\partial g_{\nu\mu}}{\partial x^\lambda} + 2A_{\nu\mu}^\omega \Gamma^\omega_{\lambda\nu} , \]

\[ (7.3) \quad -\hat{\Gamma}^\mu_{\lambda\nu} - \hat{\Gamma}^\nu_{\mu\lambda} = -\frac{\partial g_{\nu\lambda}}{\partial x^\mu} - 2A_{\nu\lambda}^\omega \Gamma^\omega_{\mu\nu} . \]

Now we shall assume that \( \hat{\Gamma}^1_{\lambda\nu} \) are symmetry with respect to \( \lambda, \nu \). Then summing up (7.1), (7.2), and (7.3), we have

\[ \hat{\Gamma}^1_{\lambda\nu} = \gamma^1_{\lambda\mu\nu} + A_{\lambda\mu}^\omega \Gamma_{\omega\nu\lambda} + A_{\mu\nu}^\omega \Gamma_{\omega\lambda\nu} - A_{\nu\lambda}^\omega \Gamma_{\omega\mu\nu} , \]

where we put \( \gamma^1_{\lambda\mu\nu} = g_{\lambda\mu} \left\{ \begin{array}{l} \omega \\ \lambda
\end{array} \right\} \). Putting

\[ (7.4) \quad H^\lambda_{\mu} = g^\lambda_{\mu} + A_{\omega} A^\omega_{\mu\lambda} = g^\lambda_{\mu} + A^\lambda_{\mu
} - \frac{1}{4} \frac{G}{\sqrt{g}} A_{\omega} \frac{G^2}{\partial u_\lambda \partial u_\mu \partial u_\omega} . \]

Under the assumption that the rank of \( |H^1_{\mu}| \) be \( n \), we can determine \( \hat{\Gamma}^\nu_{\nu} \) by the following relations

\[ \hat{\Gamma}^\nu_{\nu} = \left\{ \begin{array}{l} \omega \\ \lambda
\end{array} \right\} + A_{\omega} \left( \gamma_{\lambda\nu\nu} - l_{\nu} \gamma_{\lambda\omega\nu} \right) + A^\mu_{\omega} \left( \gamma_{\alpha\nu\lambda} - l_{\lambda} \gamma_{\alpha\omega\nu} \right) \]

\[ (7.5) \quad -A_{\nu\lambda}^\omega \left( \gamma_{\lambda\omega\nu} - l_{\nu} \gamma_{\lambda\omega\nu} \right) + (A_{\gamma}^\omega_{\nu} l_{\nu} + A_{\gamma}^\omega_{\nu} l_{\nu} - A_{\nu\lambda}^\omega l_{\nu} \gamma_{\nu\omega} \gamma_{\lambda\omega} + A_{\gamma}^\omega_{\nu} l_{\nu} - A_{\mu\lambda}^\omega l_{\nu} \gamma_{\nu\omega} \gamma_{\lambda\omega} - A_{\mu\lambda}^\omega l_{\nu} \gamma_{\nu\omega} \gamma_{\lambda\omega} + A_{\mu\lambda}^\omega l_{\nu} \gamma_{\nu\omega} \gamma_{\lambda\omega}) \]

where

\[ \gamma_{\alpha\omega\nu} = g^\beta_{\omega} \gamma_{\alpha\beta\nu} , \quad H^\nu_{\nu} K^\nu_{\mu} = H^\nu_{\nu} K^\nu_{\mu} = \delta^\nu_{\mu} . \]
§ 8. The parameters of connection $\tilde{\Gamma}^i_\alpha$.

Multiplying (6.4) by $l_\lambda$ and summing with respect to $\lambda$, we have

\begin{equation}
\tilde{\Gamma}^o_\alpha = \left\{ \begin{array}{c} 0 \\ \alpha \end{array} \right\} + A^\alpha_\alpha \tilde{\Gamma}^0_\beta + A^\alpha_\beta \tilde{\Gamma}^0_\alpha \end{equation}

or

\begin{equation}
(\delta^\alpha_\alpha - l_\omega A^\alpha) \tilde{\Gamma}^0_\alpha = \left\{ \begin{array}{c} 0 \\ \alpha \end{array} \right\} + A^\alpha_\alpha \tilde{\Gamma}^0_\alpha - A^\alpha_\alpha \tilde{\Gamma}^0_\alpha.
\end{equation}

Since $(\delta^\alpha_\alpha - l_\omega A^\alpha)(\delta^\alpha + l_\beta A^\beta) = \delta^\alpha$, we obtain

\begin{equation}
\tilde{\Gamma}^0_\alpha = (\delta_\alpha + l_\beta A^\beta) \left( \left\{ \begin{array}{c} 0 \\ \alpha \end{array} \right\} + A^\alpha_\alpha \tilde{\Gamma}^0_\alpha - A^\alpha_\alpha \tilde{\Gamma}^0_\alpha \right).
\end{equation}

In use of (8.3) and (7.5), we can determine $\tilde{\Gamma}^i_\lambda$ from (6.4).

§ 9. The stretch tensor $W_{ij}$.

We consider the invariant differential form $\phi = \delta (g_{\lambda \omega} \delta x^\lambda \delta x^\omega)$ and choose the two (infinitesimal) variations $\delta, \bar{\delta}$ in the following manner:

\begin{align*}
\delta x^i &= 0, \quad dt = 0, \quad Dl_\lambda = 0, \quad dv = 0, \\
\bar{\delta} x^i &= 0, \quad dt = 0, \quad Dl_\lambda = 0, \quad dv = 0.
\end{align*}

In words, $\delta$ and $\bar{\delta}$ mean the variation of time and that of the virtual space respectively. We suppose that $\delta, \bar{\delta}$ are interchangeable. In this case $\phi$ shows us the stretching of the space for the time variation $dt$. Calculating $\phi$, we have

\[ \phi = 2 \left[ \omega_{\lambda \omega} + \tilde{\Gamma}^0_\alpha A^\alpha_\lambda \omega - A^\alpha_\lambda \omega \tilde{\Gamma}^0_\alpha \beta \beta \omega + A^\alpha_\lambda \omega \tilde{\Gamma}^0_\alpha \omega \right] \omega x^i \omega x^o dt, \]

where we put

\[ \omega_{\lambda \omega} = \frac{1}{2} (\partial \partial g_{\lambda \omega} - a^\alpha \partial \partial g^\omega \partial \partial a^\alpha), \quad A^\alpha = 2 (A^\alpha_\mu a^\mu - A^\alpha). \]

Using the last quantity we get a strong tensor

\[ W_{\lambda \omega} = \omega_{\lambda \omega} + \tilde{\Gamma}^0_\alpha A^\alpha_\lambda \omega - A^\alpha_\lambda \omega \tilde{\Gamma}^0_\alpha \beta \beta \omega + A^\alpha_\lambda \omega \tilde{\Gamma}^0_\alpha \omega, \]

which is symmetry with respect to $\lambda, \omega$ and shall be called the stretch tensor of our space.
§ 10. The curvature strong tensors.

The curvature of our space is defined by

\[(DD - DD) X^\lambda = \Omega^\lambda_{\mu} X^\mu .\]  

Then this bilinear form \(\Omega^\lambda_{\mu}\) can be decomposed into following form

\[
\begin{align*}
\Omega^\lambda_{\mu} &= R^\lambda_{\mu \nu \omega} dx^\nu dx^\omega + \dot{P}^\lambda_{\mu \nu} (dx^\nu Dl^\omega - dx^\omega Dl^\nu) + \dot{S}^\lambda_{\mu \nu} Dl^\nu Dl^\omega \\
&\quad + \ddot{P}^\lambda_{\mu \omega} (dx^\omega dt - dx^\omega dt) + \dddot{P}^\lambda_{\mu \nu} (Dl^\omega dt - Dl^\nu dt) + \dddot{P}^\lambda_{\mu \omega} (Dl^\nu dt - Dl^\omega dt) \\
&\quad + P^\lambda_{\mu \nu} (dx^\nu dv - dx^\nu dv) + P^\lambda_{\mu \omega} (Dl^\omega dv - Dl^\omega dv) .
\end{align*}
\]

Let \(f\) be a field in our space, we have the differential

\[
\begin{align*}
\text{df} = (\frac{\partial f}{\partial x^\nu} + f^a \Gamma^a_{\nu} a^\nu) dx^\nu + f^a Dl^\nu \\
&\quad + (\frac{\partial f}{\partial t} + f^a \Gamma^a_{\nu} a^\nu) dt + (\frac{\partial f}{\partial v} + f^a \Gamma^a_{\nu} a^\nu) dv,
\end{align*}
\]

where \(f^a = \frac{G}{\sqrt{g}} \frac{\partial f}{\partial u^a}\). In § 5 we have had the covariant differential for the contravariant vector \(X^i\), that is,

\[
DX^i = dX^i + \dot{f}^i_{\mu} X^\mu dx^\nu + \ddot{f}^i_{\mu} X^\mu dt + A^i_{\mu} X^\mu Dl^\nu + \dddot{f}^i_{\mu} X^\mu dv .
\]

Then we shall rewrite (10.4) in the strong tensor form

\[
DX^i = X^i |_{\nu} \delta x^\nu + X^i |^\nu Dl^\nu + X^i |_{\nu} dt + X^i |_{\nu} dv ,
\]

where we put

\[
\begin{align*}
X^i |_{\nu} &= \frac{\partial X^i}{\partial x^\nu} + \dot{f}^i_{\mu} X^\mu + X^i |^\nu a^\nu , \\
X^i |_{\nu} &= \frac{\partial X^i}{\partial t} + \ddot{f}^i_{\mu} X^\mu + X^i |_{\nu} \delta X^\nu , \\
X^i |_{\nu} &= \frac{\partial X^i}{\partial v} + \dddot{f}^i_{\mu} X^\mu + X^i |_{\nu} \delta X^\nu ,
\end{align*}
\]

and shall call them the covariant differential coefficients. We make use of (5.8) and (10.3) to calculate the left-hand member of (10.1) and compare the obtained result with (10.2), then we get
$$R_{\mu\nu\omega}^{\lambda} = \left( \frac{\partial \Gamma_{\mu\nu}^{\lambda}}{\partial x^\omega} + \Gamma_{\mu\nu}^{\alpha} \Gamma_{\alpha\omega}^{\lambda} \right) - \left( \frac{\partial \Gamma_{\mu\omega}^{\lambda}}{\partial x^\nu} + \Gamma_{\mu\omega}^{\alpha} \Gamma_{\alpha\nu}^{\lambda} \right)$$

$$+ \dot{\Gamma}_{\mu\nu}^{\alpha} \dot{\Gamma}_{\alpha\omega}^{\lambda} - \dot{\Gamma}_{\mu\omega}^{\alpha} \dot{\Gamma}_{\alpha\nu}^{\lambda} - A_{\mu\omega}^{\lambda} \left( \frac{\partial \Gamma_{\alpha\nu}^{\lambda}}{\partial x^\omega} + \Gamma_{\alpha\nu}^{\rho} \dot{\Gamma}_{\rho\omega}^{\lambda} \right)$$

$$- \left( \frac{\partial \Gamma_{\alpha\omega}^{\lambda}}{\partial x^\nu} + \Gamma_{\alpha\omega}^{\rho} \dot{\Gamma}_{\rho\nu}^{\lambda} \right),$$

$$P_{\mu\nu}^{i\omega} = \dot{\Gamma}_{\mu\nu}^{i\omega} - A_{\mu\omega}^{i\nu} - A_{\mu\omega}^{i\lambda} \Gamma_{\lambda\nu}^{\omega},$$

$$S_{\mu}^{i\nu\omega} = A_{\mu\nu}^{i\omega} A_{\mu\omega}^{i\nu} - A_{\mu\lambda}^{i\omega} A_{\mu\omega}^{i\nu}.$$

(10.5)

Since these coefficients are not all strong tensors, we deform (10.2) as follows:

$$\Omega_{\mu}^{i} = R_{\mu\nu\omega}^{i} \delta x^\nu \delta x^\omega + P_{\mu\nu}^{i\omega} \left( \delta x^\nu Dl_{\omega} - \delta x^\nu Dl_{\omega} \right) + S_{\mu}^{i\nu\omega} Dl_{\nu} Dl_{\omega}$$

(10.6)

$$+ P_{\mu\nu}^{i\omega} \left( \delta x^\nu dt - \delta x^\nu dt \right) + P_{\mu\omega}^{i\nu} \left( Dl_{\omega} dt - Dl_{\omega} dt \right) + P_{\mu}^{i\omega} \left( Dl_{\omega} dv - Dl_{\omega} dv \right)$$

$$+ P_{\mu}^{i\nu} \left( Dl_{\nu} dv - Dl_{\nu} dv \right) + P_{\mu\nu}^{i\nu} \left( Dl_{\nu} dv - Dl_{\nu} dv \right),$$

where

$$P_{\mu\nu}^{i\omega} = \dot{P}_{\mu\nu}^{i\omega} - a^\beta R_{\mu\nu\beta}^{i\omega},$$

Thus we obtain our curvature strong tensors $R_{\mu\nu\omega}^{i}, P_{\mu\nu}^{i\omega}, S_{\mu}^{i\nu\omega}, P_{\mu\omega}^{i\nu}, P_{\mu\nu}^{i\nu}$ and $P_{\mu\nu}^{i\nu}\omega$.


§ 11. Properties of the curvature strong tensors.

In virtue of (6.7), we see

\[(DD - DD) g_{\lambda\mu} = 0.\]

Calculate the left-hand member of (11.1), then we have

\[
(DD - DL) g_{\lambda\mu} = -(g_{a\mu} R^{a}_{\lambda \nu \omega} + g_{a\mu} P^{a}_{\lambda \nu \omega}) \delta x^{\nu} \delta x^{\omega} - (g_{a\mu} S^{a}_{\lambda \nu \omega} + g_{a\mu} S^{a}_{\mu \nu \omega}) DL_{\nu} DL_{\omega}
\]

\[
- (g_{a\mu} R_{\lambda \nu \omega}^{a} + g_{a\mu} P_{\lambda \nu \omega}^{a}) (\delta x^{\nu} DL_{\omega} - \delta x^{\omega} DL_{\nu})
\]

\[
- (g_{a\mu} P_{\lambda \nu \omega}^{a} + g_{a\mu} P_{\mu \nu \omega}^{a}) (\delta x^{\nu} DL_{\omega} - \delta x^{\omega} DL_{\nu})
\]

\[
(11.1)
\]

\[
(DD - DL) g_{\lambda\mu} = -(g_{a\mu} P^{a}_{\lambda \nu \omega} + g_{a\mu} P_{\lambda \nu \omega}^{a}) (\delta x^{\nu} DL_{\omega} - \delta x^{\omega} DL_{\nu})
\]

\[
- (g_{a\mu} P_{\lambda \nu \omega}^{a} + g_{a\mu} P_{\mu \nu \omega}^{a}) (\delta x^{\nu} DL_{\omega} - \delta x^{\omega} DL_{\nu})
\]

\[
- (g_{a\mu} P_{\lambda \nu \omega}^{a} + g_{a\mu} P_{\mu \nu \omega}^{a}) (\delta x^{\nu} DL_{\omega} - \delta x^{\omega} DL_{\nu})
\]

\[
- (g_{a\mu} P_{\lambda \nu \omega}^{a} + g_{a\mu} P_{\mu \nu \omega}^{a}) (\delta x^{\nu} DL_{\omega} - \delta x^{\omega} DL_{\nu})
\]

\[
(11.2)
\]

consequently,  

\[
R_{\lambda \mu \nu \omega} + R_{\mu \lambda \nu \omega} = 0, \quad P^{a}_{\lambda \mu \nu \omega} + P^{a}_{\lambda \mu \nu \omega} = 0, \quad S^{a}_{\lambda \mu \nu \omega} + S^{a}_{\mu \lambda \nu \omega} = 0,
\]

\[
(11.3)
\]

That is, they are skew-symmetry with respect to \(\lambda, \mu\).

From definition (10.5), we have evidently

\[
(11.4) \quad R^{\lambda}_{\mu \nu \omega} = - R^{\lambda}_{\mu \omega \nu}, \quad S^{\lambda}_{\mu \nu \omega} = - S^{\lambda}_{\mu \omega \nu}.
\]

§ 12. The identities of RICCI and BIANCHI.

Now we shall proceed to find the identities between the torsion strong tensors, the curvature strong tensors and their derivatives,
which correspond to the so-called identities of Ricci and Bianchi. The Pfaffian forms

\[(12.1) \quad \Pi^\lambda = (\bar{\omega}^\lambda ' + [\omega^\lambda \bar{\omega}^\mu])\]

are components of a covariant strong vector, where $\omega$, $\bar{\omega}$ are Cartan's symbols. Consider the external derivative of (12.1), then we have the relation

\[(12.2) \quad (\Pi^\lambda ') ' + [\omega^\lambda \Pi^\mu] = [\Omega^\lambda \bar{\omega}^\mu].\]

By help of (11.4), this relation offers us the required identities

\[
\begin{align*}
R_{\{\mu | \nu \omega \}}^\lambda + A_{\{\mu}^\lambda \omega P_{0 | a}^{\nu} | \nu \omega} &= 0, & P_{\{\mu | \nu \} | \nu}^{\lambda \omega} - A_{\{\mu}^\lambda \omega | \nu} + A_{\{\mu}^\lambda \omega P_{0 | a}^{\nu} | \nu}^{\omega} &= 0, \\
2P_{\{\mu | \nu \} | \nu}^{\lambda \omega} - 2a_{\{\nu}^\lambda \omega | \nu} + 2A_{\{\mu}^\lambda \omega P_{0 | a}^{\nu} | \nu} - a_{\{\nu}^\lambda \omega R_{\{\mu | \nu \} | \nu} &= 0, \\
P_{\{\mu | \nu \} | \nu}^{\lambda \omega} + \bar{C}_{\{\mu | \nu \} | \nu} + A_{\{\mu}^\lambda \omega P_{0 | a}^{\nu} | \nu} &= 0, \\
S_{\mu}^{\nu | \omega} + A_{\{\mu}^\lambda \omega A_{\{\nu}^{\mu} | \omega} + A_{\{\mu}^\lambda \omega S_{\nu}^{\mu} | \omega} &= 0, \\
P_{\{\mu | \nu \} | \nu}^{\lambda \omega} + A_{\{\mu}^\lambda \omega | \nu} - a_{\{\nu}^\lambda \omega a_{\{\nu}^\lambda \omega} + A_{\{\mu}^\lambda \omega P_{0 | a}^{\nu} | \nu} + a_{\{\nu}^\lambda \omega | \mu} &= 0, \\
- a_{\{\nu}^\lambda \omega P_{0 | a}^{\nu} | \nu} &= 0, \\
2P_{\{\mu | \nu \} | \nu}^{\lambda \omega} + 2A_{\{\mu}^\lambda \omega a_{\{\nu}^{\mu} | \omega} + a_{\{\nu}^\lambda \omega S_{\nu}^{\mu} | \omega} &= 0, \\
a_{\{\nu}^\lambda \omega | \nu} - a_{\{\nu}^\lambda \omega a_{\{\nu}^{\mu} | \omega} - \bar{C}_{\{\nu}^{\lambda \omega} a_{\{\nu}^{\mu} | \omega} + a_{\{\nu}^\lambda \omega P_{0 | a}^{\nu} | \nu} &= 0.
\end{align*}
\]

where we put

\[
\begin{align*}
a_{\{\nu}^\lambda \omega &= \Gamma_{\{\nu}^{\lambda \omega} - \frac{\partial a_{\{\nu}^{\lambda \omega}}}{\partial x^\omega} - \frac{\partial a_{\{\nu}^{\lambda \omega}}}{\partial z^\nu} \Gamma_{\{\nu}^{\omega} | \omega}, & a_{\{\nu}^\lambda \omega &= - \frac{\partial a_{\{\nu}^\lambda \omega}}{\partial t^\lambda} - \frac{\partial a_{\{\nu}^\lambda \omega}}{\partial t^\omega} A_{\{\nu}^{\mu} | \omega}, \\
a_{\{\nu}^\lambda \omega &= - \frac{\partial a_{\{\nu}^\lambda \omega}}{\partial y^\lambda} \bar{C}_{\{\nu}^{\omega} - \frac{\partial a_{\{\nu}^\lambda \omega}}{\partial v^\nu}.
\end{align*}
\]
The identities corresponding to the identities of BIANCHI, are obtained from the coefficients of crotchets \(\delta x^\nu \delta x^\omega \delta x^\alpha\), \([Dl^\nu Dl^\omega Dl^\alpha]\), \([\delta x^\nu \delta x^\omega Dl^\alpha]\), \([Dl^\nu Dl^\omega \delta x^\alpha]\), \([\delta x^\nu \delta x^\omega dt]\), etc. in the relation of Pfaffian forms

\[(12.4) \quad (\Omega^\mu_\nu)' - [\omega^\mu_\nu \Omega^\nu_\mu] + [\Omega^\mu_\nu \omega^\nu_\mu] = 0\]

and have the forms:

\[
R^\mu_\nu(\Omega^\nu_\mu + P^\mu_\nu R^\nu_\mu) = 0, \quad S^\mu_\nu(\omega^\nu_\mu + S^\nu_\mu \omega^\mu_\nu) = 0,
\]

\[
R^\mu_\nu(\omega^\nu_\mu + R^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) - S^\mu_\nu \omega^\mu_\nu + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) = 0.
\]

\[
S^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) - S^\mu_\nu \omega^\mu_\nu + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) = 0.
\]

\[
R^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + R^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) - S^\mu_\nu \omega^\mu_\nu + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) = 0.
\]

\[
S^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) - S^\mu_\nu \omega^\mu_\nu + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) = 0.
\]

\[
R^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + R^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) - S^\mu_\nu \omega^\mu_\nu + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) = 0.
\]

\[
S^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) - S^\mu_\nu \omega^\mu_\nu + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) = 0.
\]

\[
R^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + R^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) - S^\mu_\nu \omega^\mu_\nu + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) = 0.
\]

\[
S^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) - S^\mu_\nu \omega^\mu_\nu + P^\mu_\nu(\delta x^\nu \delta x^\omega \delta x^\alpha) = 0.
\]