ON THE THEORY OF A RHEONOMIC CARTAN SPACE

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Introduction.

A Cartan space having its origin in the Cartan’s paper “Les espace de métriques fondés sur la notion d’aire” was developed by many people, especially, L. Berwald. In the present paper we attempt to build up the geometry in this space from standpoint of the rheonomic theory. In this space, its \((n-1)\)-dimensional area is assumed to be given \textit{a priori} in such a way that it depends on a variation of time. The geometrical quantities of this space depend on \(x^{a}, t, u_{a}, u_{0}\). If the time-area is independent of \(u_{0}\), then in every moment this space reduces to a Cartan space in ordinary sense, i.e. then this area is nothing but that of a Cartan space. Since \(u_{0}\) may be interpreted as a velocity of a small piece of hypersurface-element, we shall call \(u_{0}\) a \textit{velocity of the hypersurface-element}. As \(u_{0}\) is not invariant under a rheonomic transformation, we introduce an invariant parameter \(v\) in place of \(u_{0}\). This parameter \(v\) plays an important rôle in our theory. The form of fundamental function \(L(x^{a}, t, u_{a}, u_{0})\) is rewritten in \(G(x^{a}, t, u_{a}, v)\) which is homogenous of degree one in \(u_{a}\) and lets us decide the base connection, the connection-parameters, the curvature tensors and identities of Bianchi in our space.

§ 1. Fundamental function.

In an \(n\)-dimensional rheonomic manifold \(X_{n}\) with coordinates \(x', t\), we consider a rheonomic hypersurface \(X_{n-1}\) given by

\[
(1.1) \quad x^{a} = x^{a}(v^{1}, v^{2}, \ldots, v^{n-1}, t) \quad a = 1, \ldots, n
\]

and suppose that its measure of \((n-1)\)-dimensional time-area in a

\(1\) L. Berwald: Über die \(n\)-dimensionalen Cartanschen Räume und eine Normalform der zweiten Variation eines \((n-1)\)-fachen Oberflächenintegrals, Acta Mathematica, 71 (1939), 191-248.
certain region and time interval is defined by the integral

\[(1.2) \quad 0 = \int_{(n)} \int \phi \left( x^a, \frac{\partial x^a}{\partial v^i}, \frac{\partial x^a}{\partial t}, t \right) dv^1 dv^2 \ldots dv^{n-1} dt, \quad i = 1, \ldots, n - 1\]

extended over the region and time interval. Then we shall call the manifold a rheonomic Cartan space. Since (1.2) must be invariant under the rheonomic transformation of parameters \( \bar{v}' = \bar{v}'(v^1, \ldots, v^{n-1}, t) \), by using of Radon's(1) and Vivanti's(2) theorem, \( \phi \) may be written in the form \( L(x^a, t, u_a, u_0) \), which is positive and homogenous of degree one in \( u_a, u_0 \), where \( u_a = (-1)^{a+1} \left| \frac{\partial x^1}{\partial v^i}, \ldots, \frac{\partial x^{a-1}}{\partial v^i}, \frac{\partial x^{a+1}}{\partial v^i}, \ldots, \frac{\partial x^n}{\partial v^i} \right| \) and \( u_0 = -\frac{\partial x^a}{\partial v^i} u_a \).

Under a rheonomic transformation \( \bar{x} = \bar{x}(x, t), u_a \) and \( u_0 \) vary in the rule

\[(1.3) \quad \bar{u}_a = \left| \frac{\partial \bar{x}}{\partial x} \frac{\partial x^a}{\partial \bar{x}} \right| u_a, \quad \bar{u}_0 = \left| \frac{\partial \bar{x}}{\partial x} \right| \left( u_0 + \frac{\partial x^a}{\partial \bar{v}} u_a \right), \]

that is, \( u_a \) is a vector density of weight \(-1\) and \( u_0 \) is not a invariant. We consider therefore the quantities \( u_a \) and \( u_0 \) which depend on a rheonomic hypersurface element but not necessarily on the hypersurface (1.1) itself, and which vary in the rule (1.3) under a rheonomic transformation. \( L \) must be there a scalar under a rheonomic transformation and analytic in a certain region \( \vartheta \). \( L \) is called the fundamental function.

§ 2. Metric functions \( a^g \).

Since under a rheonomic transformation \( \frac{\partial L}{\partial u_a}, \frac{\partial L}{\partial u_0} \) vary in the rule

\[ \frac{\partial \bar{L}}{\partial \bar{u}_a} = \left( \frac{\partial L}{\partial u_a} \frac{\partial \bar{\bar{x}}^a}{\partial \bar{x}} + \frac{\partial L}{\partial u_0} \frac{\partial \bar{\bar{x}}^a}{\partial \bar{v}} \right) \frac{\partial \bar{x}}{\partial \bar{x}}, \quad \frac{\partial \bar{L}}{\partial \bar{u}_0} = \frac{\partial \bar{x}}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial \bar{u}_0}, \]

then

\[ \bar{a}^r = \frac{\partial \bar{x}^r}{\partial \bar{x}^g} \left( a^g + \frac{\partial x^g}{\partial \bar{v}} \right), \]

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Putting \( a^\beta = -\frac{\partial L}{\partial u_\beta} / \frac{\partial L}{\partial u_0} \).

§ 3. Parameter \( v \) and metric functions \( g^{\alpha\beta} \).

Under a rheonomic transformation, the quantities

\[
\mathfrak{R}^{\alpha\beta} = \frac{1}{2} \left[ \frac{\partial}{\partial u_\beta} \left( \frac{\partial L^2}{\partial u_\alpha} - \frac{\partial L^2}{\partial u_0} \frac{\partial x^\alpha}{\partial t} \right) - \frac{\partial x^\beta}{\partial t} \frac{\partial}{\partial u_0} \left( \frac{\partial L^2}{\partial u_\alpha} - \frac{\partial L^2}{\partial u_0} \frac{\partial x^\alpha}{\partial t} \right) \right]
\]

vary as a contravariant tensor density of weight 2. Let us put \( \mathfrak{R}^{\alpha\beta} = aa^{\alpha\beta} \) where \( a = |a^{\alpha\beta}|^{-1} \), from which we can find \( a = \mathfrak{R}^{\alpha\beta} \) where \( \mathfrak{R} = |\mathfrak{R}| \). Putting

\[
v = \frac{1}{\sqrt{a}} \frac{\partial L}{\partial u_0},
\]

we have an absolute scalar which is homogenous zero in \( u_\alpha, u_0 \). Assume that \( u_0 \) can be solved from the above equation (3.2) in the form

\[
u_0 = F(x^\alpha, t, u_\alpha, v)
\]

then we see \( \rho u_0 = F(x^\alpha, t, \rho u_\alpha, v) \), that is, the function \( F \) is homogenous of degree one in \( u_\alpha \). Putting (3.3) into \( L(x^\alpha, t, u_\alpha, u_0) \), the obtained function \( G(x^\alpha, t, u_\alpha, v) \) is homogenous of degree one in \( u_\alpha \). In the same manner \( a^\alpha \) can be brought to functions of \( x^\alpha, t, u_\alpha, v \), which are homogenous of degree zero in \( u_\alpha \) too. Afterwards we shall represent these functions by the same letters \( a^\alpha \).

Differentiating \( G \) with respect to \( u_\beta \), we have a contravariant tensor density of weight 2:

\[
G^{\alpha\beta} = \frac{1}{2} \frac{\partial^2 G^2}{\partial u_\alpha \partial u_\beta}.
\]

Put \( G^{\alpha\beta} = gg^{\alpha\beta} \) where \( g = 1/|g^{\alpha\beta}| \), from which we can find \( g = G^{\alpha\beta} \), where \( G = |G^{\alpha\beta}| \), then we have a contravariant tensor

\[
g^{\alpha\beta} = G^{\alpha\beta} \]

which are homogenous of degree zero in \( u_\alpha \) and may take as the fundamental metric tensor in our space. Then \( g_{\alpha\beta}, a_\alpha \) are defined by

\[
g^{\alpha\beta} g_{\alpha\gamma} = \delta^\beta_\gamma, \quad a_\alpha = g_{\alpha\beta} a^\beta.
\]

When \( L \) does not contain \( u_0 \), this space reduces to a Cartan space for \( t = \text{const} \).

We shall denote the covariant differentiation in the space with the symbol $D$. Since the covariant differentiation is necessary to satisfy that

1. $Dp = dp$, where $p$ is a strong scalar,
2. $D(X + Y) = DX + DY$, where $X$, $Y$ are any strong vectors,
3. $D(X \cdot Y) = X \cdot DY + DX \cdot Y$,

we define

$$ DX^\lambda = dX^\lambda + \Gamma_{\mu\nu}^\lambda X^\mu \, dx^\nu + \Gamma_{\mu}^\lambda X^\mu \, dt + C_{\mu\nu}^\lambda X^\mu \, du_\nu + C_{\mu}^\lambda X^\mu \, dv. $$

In consequence of the geometrical meaning of $u_\alpha$, this differentiation must be invariant under the transformation $\bar{u}_\alpha = \rho u_\alpha$. By this reason we suppose that $\Gamma_{\mu\nu}^\lambda$, $\Gamma_{\mu}^\lambda$, $C_{\mu}^\lambda$ are homogenous function of degree zero in $u_\alpha$ and $C_{\mu\nu}^\lambda$ of degree $-1$ and that $C_{\mu\nu}^\lambda \, u_\nu$ vanish.

Let this connection be a euclidean connection, that is, when a strong vector is transported parallel to itself, the length of this vector be invariant. Then it must be that

$$ \frac{\partial g_{\lambda\mu}}{\partial x^\nu} = g_{\alpha\beta} \, \Gamma_{\lambda\nu}^{\alpha} + g_{\lambda\beta} \, \Gamma_{\mu\nu}^{\alpha}, \quad \frac{\partial g_{\lambda\mu}}{\partial u_\nu} = g_{\alpha\beta} \, C_{\lambda\beta}^{\alpha} + g_{\lambda\beta} \, C_{\mu\beta}^{\alpha}, $$

$$ \frac{\partial g_{\lambda\mu}}{\partial t} = g_{\alpha\beta} \, \Gamma_{\lambda}^{\alpha} + g_{\lambda\beta} \, \Gamma_{\mu}^{\alpha}, \quad \frac{\partial g_{\lambda\mu}}{\partial u_\nu} = g_{\alpha\beta} \, C_{\lambda\beta}^{\alpha} + g_{\lambda\beta} \, C_{\mu\beta}^{\alpha}. $$

Supposing that $C_{\lambda\mu\nu}$, $C_{\lambda\mu}$ are symmetry with respect to $\lambda$, $\mu$, we have

$$ C_{\lambda\mu\nu} = \frac{1}{2} \frac{\partial g_{\lambda\mu}}{\partial u_\nu}, \quad C_{\lambda\mu} = \frac{1}{2} \frac{\partial g_{\lambda\mu}}{\partial u_\nu}. $$

§ 5. The normal unit vector.

From (3.4) we obtain

$$ gg^{a\beta} \, u_a = \left( G \, \frac{\partial G}{\partial u_\alpha} \, \frac{\partial G}{\partial u_\beta} + \frac{\partial G}{\partial u_\alpha} \, \frac{\partial G}{\partial u_\beta} \right) u_a = G \, \frac{\partial G}{\partial u_\beta}. $$

Multipling $u_\beta$ and summing with respect to $\beta$, then we get

$$ gg^{a\beta} \, u_a \, u_\beta = G^2. $$
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(5.2) \[ l_\lambda = \frac{u_\lambda}{\sqrt{g^{\alpha\beta} u_\alpha u_\beta}} = \frac{\sqrt{g}}{G} u_\lambda \]
gives therefore a normal unit vector for the hypersurface \( u_\alpha \). Putting

(5.3) \[ A^\mu_\lambda = \frac{G}{\sqrt{g}} C^\mu_\lambda , \]
which satisfy the relation

(5.4) \[ A^\mu_\lambda l_\nu = 0 , \]
we obtain

(5.5) \[ A^\mu_\lambda dl_\nu = C^\mu_\lambda du_\nu . \]
Thus the covariant differentiation (4.1) becomes

(5.6) \[ DX^\lambda = \xi X^\lambda + \Gamma^\mu_\lambda X^\mu dx^\nu + \Gamma^\mu_\lambda X^\mu dt + A^\mu_\lambda X^\mu dl_\nu + C^\mu_\lambda X^\mu dv. \]

The corresponding equation for the covariant components is

(5.7) \[ DX_\lambda = dX_\lambda - \Gamma^\mu_\lambda X^\mu dx^\nu - \Gamma^\mu_\lambda X^\mu dt - A^\mu_\lambda X_\mu dl_\nu - C^\mu_\lambda X_\mu dv \]
and for \( l_\lambda \) we obtain

(5.7') \[ Dl_\lambda = dl_\lambda - \Gamma^\mu_\lambda dx^\nu - \Gamma^\mu_\lambda dt - A^\mu_\lambda l_\mu dl_\nu - C^\mu_\lambda dv , \]
where \( \Gamma^\mu_\lambda l_\alpha = \Gamma^\mu_\lambda, \Gamma^\mu_\lambda l_\alpha = \Gamma^\mu_\lambda, \) etc.. Substituting (5.7') in (5.7) and using (5.4), the covariant differentiation becomes

(5.8) \[ DX_\lambda = dX_\lambda - X_\mu (\tilde{\Gamma}^\mu_\nu dx^\nu + A^\mu_\nu dl_\nu + \tilde{C}^\mu_\nu dv)^{(1)} \]
where we put

(5.9) \[ \tilde{\Gamma}^\mu_\lambda = \Gamma^\mu_\lambda + \Gamma^\beta_\nu A^\mu_\beta, \tilde{\Gamma}^\mu_\lambda = \Gamma^\mu_\lambda + A^\mu_\beta \Gamma^\beta_\lambda, \tilde{C}^\mu_\lambda = C^\mu_\lambda + A^\mu_\beta C^\beta_\lambda . \]
For \( X_\lambda = l_\lambda \) we have

(5.10) \[ Dl_\alpha (\delta^\alpha_\mu + l_\mu A^\alpha) = dl_\mu - \tilde{\Gamma}^\mu_\alpha dx^\alpha - \tilde{\Gamma}^\mu_\alpha dt - \tilde{C}^\mu_\alpha dv . \]
Using \( (\delta^\alpha_\mu + l_\mu A^\mu) (\delta^\beta_\nu - l_\nu A^\nu) = \delta^\alpha_\beta, Dl_\alpha \) is represented by

(5.11) \[ Dl_\alpha = (\delta^\alpha_\mu - l_\alpha A^\mu) (dl_\mu - \tilde{\Gamma}^\mu_\beta dx^\beta - \tilde{\Gamma}^\mu_\alpha dt - \tilde{C}^\mu_\alpha dv) . \]

(1) Since (3.4) and (4.3) give us the relation \( A^\mu_\nu = \xi A^\mu_\nu \), where \( A^\mu_\nu = A_0^{\mu\nu} \), we see easily \( A^\mu_\nu A^\nu_\beta = 0 \).
§ 6. Other two postulates of the covariant differentiation.

We consider the invariant differential form

\[ \phi = D (g_{\lambda \mu} \delta x^\lambda \delta x^\mu) + D (g_{\lambda \mu} \delta x^\lambda \delta x^\mu) - D (g_{\lambda \mu} \delta x^\lambda \delta x^\mu) \]

where the index 1, 2, 3 under the differential symbol \( D \), \( \delta \) denote the directions of the differentiations. They are here supposed to be interchangeable, that is, \( \delta x^a = \delta x^b \) \((a, b = 1, 2, 3)\). If we choose \( dt = dt = 0 \), \( DL_a = 0 \) and \( dv = 0 \), we have

\[
\phi = 2g_{\lambda \nu} \left[ \frac{ddx^a}{\partial l_a} + \left( \left\{ \frac{\lambda}{\phi} \right\} + A^a_{\nu} \Gamma_{\alpha \psi}^0 \right) dx^\alpha \right] \delta x^\nu,
\]

where we put

\[
\left\{ \frac{\lambda}{\phi} \right\} = g^{\lambda \nu} \left( \partial_t g_{\nu \phi} + \partial_{\omega} \alpha_{\nu} - \partial_{\nu} \alpha_{\omega} \right), \quad A^a_{\nu} = \frac{1}{2} \frac{\partial a_{\nu}}{\partial l_a}, \quad A^a_{\phi} = g^{\lambda \nu} A^\phi_{\nu}.
\]

Since \( g_{\lambda \mu} \) is the strong tensor of degree 2 and \( \delta x^a \) is a strong vector, the term held in the bracket of above equation is a strong contravariant vector.

In order that we desire that the covariant differentiation does not change the property "strong" and the order of tensors, we shall define

\[
\Gamma_{\omega^\mu}^a = \left\{ \frac{\lambda}{\phi} \right\} + A^a_{\nu} \Gamma_{\alpha \psi}^0 \left( A^a_{\nu} \Gamma_{\alpha \psi}^0 - A^a_{\phi} \Gamma_{\alpha \psi}^0 \right) \delta x^\alpha \delta x^\psi,
\]

(6.3)

\[
\Gamma_{\omega}^a = \left\{ \frac{\lambda}{\phi} \right\} + A^a_{\nu} \Gamma_{\alpha \psi}^0 \left( A^a_{\nu} \Gamma_{\alpha \psi}^0 - A^a_{\phi} \Gamma_{\alpha \psi}^0 \right) \delta x^\alpha \delta x^\psi.
\]

(6.4)

Then we can verify \( Dg_{\lambda \mu} = 0 \). Putting \( \delta x^a = \delta x^a \) in (6.1), we have

\[
\phi = D \left( g_{\lambda \omega} \delta x^\lambda \delta x^\omega \right) = Dg_{\lambda \omega} \delta x^\lambda \delta x^\omega + 2g_{\lambda \omega} D \delta x^\lambda \delta x^\omega.
\]

(6.5)

On the other hand (6.2) becomes

\[
\phi = 2g_{\lambda \omega} D \delta x^\lambda \delta x^\omega.
\]

(6.6)

From (6.5) and (6.6) we obtain
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\[ D g_{\lambda \omega} \frac{\partial x^1}{\partial x^2} = 0. \]

Hence arbitrariness of \( \frac{\partial x^1}{\partial x^2} \) leads us to

(6.7) \[ D g_{\lambda \omega} = 0. \]

§ 7. The parameters of connection \( \Gamma_{\lambda \mu \nu}^1 \).

From (4.2) and (5.9) we have immediately

(7.1) \[ \Gamma_{\lambda \mu \nu} + \Gamma_{\mu \lambda \nu} = \frac{\partial g_{\lambda \mu}}{\partial x^k} + 2A_{\lambda \mu} \Gamma_{\omega \nu \lambda}, \]

From this the two following equations are obtained by cyclic permutation of \( \lambda, \mu, \nu \)

(7.2) \[ \Gamma_{\lambda \mu \nu} + \Gamma_{\nu \mu \lambda} = \frac{\partial g_{\mu \nu}}{\partial x^k} + 2A_{\mu \nu} \Gamma_{\lambda \nu \lambda}, \]

(7.3) \[ -\Gamma_{\nu \lambda \mu} - \Gamma_{\lambda \nu \mu} = -\frac{\partial g_{\mu \nu}}{\partial x^k} - 2A_{\nu \lambda} \Gamma_{\omega \mu \lambda}. \]

Now we shall assume that \( \Gamma_{\lambda \mu \nu} \) are symmetry with respect to \( \lambda, \nu \). Then summing up (7.1), (7.2), and (7.3), we have

\[ \Gamma_{\lambda \mu \nu} = g_{\mu \nu} + A_{\lambda \mu} \Gamma_{\omega \nu \lambda} + A_{\mu \nu} \Gamma_{\lambda \nu \lambda} - A_{\nu \lambda} \Gamma_{\omega \mu \lambda}, \]

where we put \( g_{\mu \nu} \Gamma_{\lambda \mu \nu} = g_{\mu \nu} \left\{ \begin{array} {l} \omega \\ \lambda \end{array} \right\}. \) Putting

(7.4) \[ H_{\lambda \mu} = g_{\lambda \mu} + A_{\omega} A_{\omega \lambda \mu} = g_{\lambda \mu} + A_{\lambda} A_{\mu} - \frac{1}{4} \frac{G^2}{\sqrt{g}} A_{\omega} \frac{G}{\omega \mu} \frac{\partial^3 G}{\omega \nu \mu \omega}. \]

Under the assumption that the rank of \( |H_{\lambda \mu}| \) be \( n \), we can determine \( \Gamma_{\lambda \mu \nu} \) by the following relations

(7.5) \[ \Gamma_{\lambda \mu \nu} = \left\{ \begin{array} {l} \omega \\ \lambda \nu \end{array} \right\} + A_{\lambda} \left( \gamma_{\alpha \nu \lambda} - l_{\nu} \gamma_{\alpha \nu \lambda} \right) + A_{\nu} \left( \gamma_{\alpha \lambda \nu} - l_{\lambda} \gamma_{\alpha \lambda \nu} \right) \\
- A_{\nu} \left( \gamma_{\alpha \lambda} - l_{\alpha} \gamma_{\alpha \lambda} \right) + (A_{\nu} \left( l_{\nu} + A_{\nu} \right) - A_{\nu} \left( l_{\alpha} \right) - A_{\lambda} \left( l_{\alpha} \right) + A_{\lambda} \right) K_{\nu}^\alpha \left( \gamma_{\alpha \nu \lambda} + A_{\lambda} \gamma_{\nu \lambda} \right) \]

where

\[ \gamma_{\alpha \nu \lambda} = g_{\beta \nu \alpha} \gamma_{\alpha \nu \beta}, \quad H_{\lambda}^\mu K_{\nu}^\mu = H_{\lambda}^\nu K_{\nu}^\mu = \delta_{\lambda}^\nu. \]
§ 8. The parameters of connection $\tilde{\Gamma}_{\lambda}^{\omega}$.

Multiplying (6.4) by $l_{\lambda}$ and summing with respect to $\lambda$, we have

\[ \tilde{\Gamma}_{\alpha}^{\omega} = \left\{ \begin{array}{c} 0 \\ \alpha \end{array} \right\} + A_{\alpha}^{\beta} \tilde{\Gamma}_{\beta}^{\omega} + A_{\alpha}^{\beta} \tilde{\Gamma}_{\beta}^{\omega} - A_{\alpha}^{\beta} \tilde{\Gamma}_{\beta}^{\omega} \]

or

\[ (\delta_{\omega}^{\alpha} - l_{\omega} A^{\alpha}) \tilde{\Gamma}_{\alpha}^{\omega} = \left\{ \begin{array}{c} 0 \\ \alpha \end{array} \right\} + A_{\alpha}^{\omega} \tilde{\Gamma}_{\omega}^{\alpha} - A_{\omega}^{\alpha} \tilde{\Gamma}_{\omega}^{\alpha} . \]

Since $(\delta_{\omega}^{\alpha} - l_{\omega} A^{\alpha})(\delta_{\omega}^{\alpha} + l_{\omega} A^{\omega}) = \delta_{\alpha}^{\omega}$, we obtain:

\[ \tilde{\Gamma}_{\alpha}^{\omega} = \left( \delta_{\omega}^{\alpha} + l_{\omega} A^{\omega} \right) \left( \left\{ \begin{array}{c} 0 \\ \omega \end{array} \right\} + A_{\omega}^{\alpha} \tilde{\Gamma}_{\alpha}^{\omega} - A_{\alpha}^{\omega} \tilde{\Gamma}_{\alpha}^{\omega} \right) . \]

In use of (8.3) and (7.5), we can determine $\tilde{\Gamma}_{\lambda}^{\omega}$ from (6.4).

§ 9. The stretch tensor $W_{ij}$.

We consider the invariant differential form $\phi = \delta \left( g_{j \omega} \delta x^{j} \delta x^{\omega} \right)$ and choose the two (infinitesimal) variations $\delta$, $\delta$ in the following manner:

\[ \begin{array}{llll} \delta x^{i} = 0 \, , & \delta t = 0 \, , & DL_{k} = 0 \, , & d\nu = 0 \, , \\
\delta v^{i} = 0 \, , & \delta t = 0 \, , & DL_{k} = 0 \, , & d\nu = 0 . \end{array} \]

In words, $\delta$ and $\delta$ mean the variation of time and that of the virtual space respectively. We suppose that $\delta$, $\delta$ are interchangeable. In this case $\phi$ shows us the stretching of the space for the time variation $dt$. Calculating $\phi$, we have

\[ \phi = 2 \left[ \omega_{k \omega} + \tilde{\Gamma}_{\alpha}^{\omega} A_{k \omega}^{\alpha} - A_{k \omega}^{\alpha} \tilde{\Gamma}_{\alpha}^{\omega} + A_{k \omega}^{\alpha} \tilde{\Gamma}_{\alpha}^{\omega} \right] \delta x^{i} \delta x^{\omega} dt , \]

where we put

\[ \omega_{k \omega} = {1 \over 2} \left( \partial_{w} g_{k \omega} - \alpha^{\nu} \partial_{\nu} g_{k \omega} - 2 g_{k \alpha} \partial_{\omega} \alpha^{\alpha} \right) , \quad A_{k}^{\alpha} = \delta \left( A_{k}^{\mu} \alpha^{\nu} - A_{k}^{\nu} \right) . \]

Using the last quantity we get a strong tensor

\[ W_{k \omega} = \omega_{k \omega} + \tilde{\Gamma}_{\alpha}^{\omega} A_{k \omega}^{\alpha} - A_{k \omega}^{\alpha} \tilde{\Gamma}_{\alpha}^{\omega} + A_{k \omega}^{\alpha} \tilde{\Gamma}_{\alpha}^{\omega} , \]

which is symmetry with respect to $\lambda$, $\omega$ and shall be called the stretch tensor of our space.
§ 10. The curvature strong tensors.

The curvature of our space is defined by

\[(DD - DD) X^\iota = \Omega_\mu^{\iota} X^\mu.\]  

Then this bilinear form \(\Omega_\mu^{\iota}\) can be decomposed into following form

\begin{align*}
\Omega_\mu^{\iota} &= R_\mu^{\nu\omega} \, dx^\nu \, dx^\omega + \tilde{P}_\mu^{\iota\omega} (dx^\nu \, Dl^\omega - dx^\nu \, Dl^\omega) + \tilde{S}_\mu^{\nu\omega} \, Dl^\nu \, Dl^\omega \\
&\quad + \tilde{P}_\mu^{\nu\omega} (dx^\nu \, dt - dx^\nu \, dt) + \tilde{P}_\mu^{\nu\iota} (Dl^\nu \, dt - Dl^\nu \, dt) + \tilde{P}_\mu^{\nu\eta} (dv \, dt - dv \, dt) \\
&\quad + P_\mu^{\nu\iota} (dx^\nu \, dv - dx^\nu \, dv) + \tilde{P}_\mu^{\nu\iota} (Dl^\nu \, dv - Dl^\nu \, dv). 
\end{align*}

Let \(f\) be a field in our space, we have the differential

\[df = \left(\frac{\partial f}{\partial x^\iota} + f \parallel^\nu \tilde{\Gamma}_\iota^{\nu\omega} \right) dx^\nu + f \parallel\omega \, Dl^\omega \\
+ \left(\frac{\partial f}{\partial t} + f \parallel^\nu \tilde{\Gamma}_\iota^{\nu} \right) dt + \left(\frac{\partial f}{\partial v} + f \parallel \omega \tilde{\Gamma}_\iota^{\omega} \right) dv \]

where \(f \parallel^\iota = \frac{G}{\sqrt{g}} \frac{\partial f}{\partial u^\iota} \). In § 5 we have had the covariant differential for the contravariant vector \(X^\iota\), that is,

\[(DX)^\iota = dX^\iota + \tilde{P}_\iota^{\nu\mu} X^\mu \, dx^\nu + \tilde{P}_\iota^{\nu\iota} X^\iota \, dt + A_\mu^{\nu\iota} X^\mu \, Dl^\nu + \tilde{C}_\iota^{\iota} X^\iota \, dv.\]

Then we shall rewrite (10.4) in the strong tensor form

\[DX^\iota = X^\iota \parallel^\nu \, dx^\nu + X^\iota \parallel\iota \, Dl^\iota + X^\iota \parallel^\nu \, dt + X^\iota \parallel \iota \, dv, \]

where we put

\[X^\iota \parallel^\nu = \frac{\partial X^\iota}{\partial x^\nu} + \tilde{P}_\iota^{\nu\mu} X^\mu + X^\iota \parallel^\iota \tilde{\Gamma}_\iota^{\nu\omega}, \quad X^\iota \parallel\iota = X^\iota \parallel\nu + A_\mu^{\nu\iota} X^\mu, \]

and shall call them the covariant differential coefficients. We make use of (5.8) and (10.3) to calculate the left-hand member of (10.1) and compare the obtained result with (10.2), then we get
\[ R^\mu_{\nu\omega} = \left( \frac{\partial \hat{\Gamma}^k_{\mu\nu}}{\partial x^\omega} + \hat{\Gamma}^k_{\mu\nu} \right)_{\omega} - \left( \frac{\partial \hat{\Gamma}^k_{\mu\nu}}{\partial x^\omega} + \hat{\Gamma}^k_{\mu\omega} \right)_{\nu} - \left( \frac{\partial \hat{\Gamma}^k_{\mu\nu}}{\partial x^\omega} + \hat{\Gamma}^k_{\mu\omega} \right)_{\nu} - \left( \frac{\partial \hat{\Gamma}^k_{\mu\nu}}{\partial x^\omega} + \hat{\Gamma}^k_{\mu\omega} \right)_{\nu} + \hat{\Gamma}^k_{\mu\nu} \right)_{\omega} \]

\[ P^\mu_{\nu\omega} = \hat{\Gamma}^k_{\mu\nu} \right)_{\omega} - A^k_{\mu\nu} \right)_{\nu} - A^k_{\mu\nu} \right)_{\nu} - A^k_{\mu\nu} \right)_{\nu} + \hat{\Gamma}^k_{\mu\nu} \right)_{\omega} \]

\[ S^\mu_{\nu\omega} = A^k_{\mu\nu} A_{\mu\nu} - A^k_{\mu\nu} A_{\mu\nu} \]

(10.5)

Since these coefficients are not all strong tensors, we deform (10.2) as follows:

\[ \Omega^\mu_{\nu\omega} = R^\mu_{\nu\omega} \right)_{\nu} + P^\mu_{\nu\omega} \left( \right)_{\nu} + P^\mu_{\nu\omega} \left( \right)_{\nu} + S^\mu_{\nu\omega} \right)_{\nu} \]

(10.6)

where $P^\mu_{\nu\omega} = \hat{P}^\mu_{\nu\omega} - \alpha^\mu \hat{R}^\mu_{\nu\omega}$, $P^\mu_{\nu\omega} = \hat{P}^\mu_{\nu\omega} + \alpha^\mu \hat{P}^\mu_{\nu\omega}$ and $P^\mu_{\nu\omega} = \hat{P}^\mu_{\nu\omega} + \alpha^\mu \hat{P}^\mu_{\nu\omega}$.

Thus we obtain our curvature strong tensors $R^\mu_{\nu\omega}$, $P^\mu_{\nu\omega}$, $S^\mu_{\nu\omega}$, $P^\mu_{\nu\omega}$, $P^\mu_{\nu\omega}$ and $P^\mu_{\nu\omega}$. 
§ 11. Properties of the curvature strong tensors.

In virtue of (6.7), we see

\[(DD - DD) g_{\lambda \mu} = 0.\]

Calculate the left-hand member of (11.1), then we have

\[(DD - DL) g_{\lambda \mu} = -(g_{a \mu} R_{a \mu}^{\lambda} \nu \omega + g_{a \mu} P_{a \mu}^{\lambda} \nu \omega) \delta x^\nu \delta x^\omega - (g_{a \mu} S_{a \mu}^{\lambda} \nu \omega + g_{a \mu} S_{a}^{\lambda} \nu \omega) DL^\nu DL^\omega \]

\[= -(g_{a \mu} R_{a \mu}^{\lambda} \nu \omega + g_{a \mu} P_{a \mu}^{\lambda} \nu \omega) \left( \delta x^\nu DL^\omega - \delta x^\omega DL^\nu \right) \]

\[= -(g_{a \mu} P_{a \mu}^{\lambda} \nu \omega + g_{a \mu} P_{a \mu}^{\lambda} \nu \omega) \left( \delta x^\nu dt - \delta x^\omega dt \right) \]

\[= -(g_{a \mu} P_{a}^{\lambda \nu \omega} + g_{a \mu} P_{a}^{\lambda \nu \omega}) \left( DL^\nu dt - DL^\omega dt \right) \]

\[= -(g_{a \mu} P_{a \nu \omega}^{\lambda} + g_{a \mu} P_{a \nu \omega}^{\lambda}) \left( \delta x^\nu dv - \delta x^\omega dv \right) \]

\[= -(g_{a \mu} P_{a \nu \omega}^{\lambda} + g_{a \mu} P_{a \nu \omega}^{\lambda}) \left( DL^\nu dv - DL^\omega dv \right) , \]

consequently,

\[R_{\lambda \mu \nu \omega} + R_{\mu \lambda \nu \omega} = 0, \quad P_{a \mu \nu \omega}^{\lambda} + P_{a \mu \nu \omega}^{\lambda} = 0, \quad S_{a \mu \nu \omega}^{\lambda} + S_{a \mu \nu \omega}^{\lambda} = 0, \]

\[P_{13 \mu \nu \omega}^{\lambda} + P_{15 \mu \nu \omega}^{13} = 0, \quad P_{25 \mu \nu \omega}^{13} + P_{25 \mu \nu \omega}^{13} = 0, \quad P_{34 \mu \nu \omega}^{13} + P_{34 \mu \nu \omega}^{13} = 0, \]

That is, they are skew-symmetry with respect to \(\lambda, \mu\).

From definition (10.5), we have evidently

\[R_{\mu \nu \omega}^{1} = - R_{\mu \nu \omega}^{1} , \quad S_{\mu \nu \omega}^{1} = - S_{\mu \nu \omega}^{1} . \]

§ 12. The identities of RICCI and BIANCHI.

Now we shall proceed to find the identities between the torsion strong tensors, the curvature strong tensors and their derivatives,
which correspond to the so-called identities of Ricci and Bianchi. The Pfaaffian forms

\[(12.1) \quad \Pi^k = (\bar{\omega}^k)' + [\omega^k_\mu \bar{\omega}^\mu]\]

are components of a covariant strong vector, where \(\omega, \bar{\omega}\) are Cartan's symbols. Consider the external derivative of (12.1), then we have the relation

\[(12.2) \quad (\Pi^k)' + [\omega^k_\mu \Pi^\mu] = [\Omega^k_\mu \bar{\omega}^\mu].\]

By help of (11.4), this relation offers us the required identities

\[
\begin{align*}
R^\nu_\mu R^\alpha_\beta + A^\nu_\mu R^\alpha_\beta & = 0, & P^\nu_\mu R^\alpha_\beta & = 0, & A^\nu_\mu R^\alpha_\beta & = 0, \\
2P^\nu_\mu & - 2a^\nu_\mu |_v + 2A^\nu_\mu R^\alpha_\beta & = 0, & P^\nu_\mu & = a^\nu_\mu |_v + A^\nu_\mu R^\alpha_\beta & = 0, \\
S^\nu_\mu |_v & + A^\nu_\mu R^\alpha_\beta & = 0, & S^\nu_\mu |_v & = a^\nu_\mu R^\alpha_\beta & = 0, \\
P^\nu_\mu - A^\nu_\mu |_v & = a^\nu_\mu R^\alpha_\beta & = 0, & P^\nu_\mu & = A^\nu_\mu R^\alpha_\beta & = 0, \\
\end{align*}
\]

where we put

\[
\begin{align*}
a^\nu_\mu & = \Gamma^\nu_\mu - \frac{\partial a^\nu_\mu}{\partial x^a} - \frac{\partial a^\nu_\mu}{\partial x^\beta} \Gamma^\alpha_\beta, & a^\nu_\mu & = \frac{\partial a^\nu_\mu}{\partial x^\alpha} - \frac{\partial a^\nu_\mu}{\partial x^\beta} A^0_\beta, \\
a^\nu_\mu & = \frac{\partial a^\nu_\mu}{\partial x^\alpha} - \frac{\partial a^\nu_\mu}{\partial x^\beta} \frac{\partial a^\nu_\mu}{\partial x^\alpha}. \\
\end{align*}
\]
On the Theory of a Rheonomic Cartan Space

The identities corresponding to the identities of BIANCHI, are obtained from the coefficients of crochets \([\delta x^\nu \delta x^\omega \delta x^\alpha], [Dl_{\nu} Dl_{\omega} Dl_{\alpha}], [\delta x^\nu \delta x^\omega Dl_{\alpha}], [Dl_{\nu} Dl_{\omega} \delta x^\alpha], [\delta x^\nu \delta x^\omega dt], \) etc. in the relation of PFaffian forms

\[(12.4) \quad (\Omega^{\lambda}_{\mu})' - [\omega^{\nu}_{\lambda} Dl_{\nu}] + [\Omega^{\nu}_{\lambda} \omega_{\nu}] = 0\]

and have the forms:

\[
R^{\lambda}_{\mu}[\nu \omega]_{a} + P^{\lambda}_{\mu}[\nu \beta] R_{\nu\beta\omega a} = 0, \quad S^{\lambda}_{\mu}[\nu \omega]_{a} + S^{\lambda}_{\mu}[\beta \omega] S_{\nu\beta \nu a} = 0,
\]

\[
R^{\lambda}_{\mu}[\nu \omega]_{a} + R^{\lambda}_{\mu}[\nu \beta] A_{\nu\beta}^{\alpha} + P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta\omega a} - S^{\lambda}_{\mu}[\nu \beta] R_{\nu\beta\omega a} + P^{\lambda}_{\mu}[\nu \beta] = 0,
\]

\[
S^{\lambda}_{\mu}[\nu \omega]_{a} - S^{\lambda}_{\mu}[\beta \omega] P_{\nu\beta \nu a}^{\nu} + P^{\lambda}_{\mu}[\nu \beta] S_{\nu\beta \nu a}^{\nu} + P^{\lambda}_{\mu}[\nu \beta] |^{\omega} = 0,
\]

\[
R^{\lambda}_{\mu}[\nu \omega]_{a} + R^{\lambda}_{\mu}[\nu \beta] A_{\nu\beta}^{\alpha} + P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta\omega a} - S^{\lambda}_{\mu}[\nu \beta] R_{\nu\beta\omega a} + P^{\lambda}_{\mu}[\nu \beta] = 0.
\]

(12.5) \[
S^{\lambda}_{\mu}[\nu \omega]_{a} + P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} + P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} - P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} + P^{\lambda}_{\mu}[\nu \beta] A_{\alpha}^{\omega}
\]

\[
+ P^{\lambda}_{\mu}[\nu \beta] A_{\alpha}^{\omega} = 0,
\]

\[
S^{\lambda}_{\mu}[\nu \omega]_{a} + P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} + P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} - P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} + P^{\lambda}_{\mu}[\nu \beta] A_{\alpha}^{\omega}
\]

\[
+ P^{\lambda}_{\mu}[\nu \beta] A_{\alpha}^{\omega} = 0,
\]

\[
S^{\lambda}_{\mu}[\nu \omega]_{a} + P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} + P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} - P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} + P^{\lambda}_{\mu}[\nu \beta] A_{\alpha}^{\omega}
\]

\[
+ P^{\lambda}_{\mu}[\nu \beta] A_{\alpha}^{\omega} = 0,
\]

\[
P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} + P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} = 0,
\]

\[
P^{\lambda}_{\mu}[\nu \beta] A_{\alpha}^{\omega} + P^{\lambda}_{\mu}[\nu \beta] A_{\alpha}^{\omega} = 0,
\]

\[
P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} + P^{\lambda}_{\mu}[\nu \beta] P_{\nu\beta \nu a}^{\nu} = 0,
\]

\[
P^{\lambda}_{\mu}[\nu \beta] A_{\alpha}^{\omega} + P^{\lambda}_{\mu}[\nu \beta] A_{\alpha}^{\omega} = 0.
\]