PARTIALLY ORDERED ABELIAN SEMIGROUPS

I. ON THE EXTENSION OF THE STRONG PARTIAL ORDER DEFINED ON ABELIAN SEMIGROUPS

By

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Definition 1. A set $S$ is said to be a partially ordered abelian semigroup (p.o. semigroup), when in $S$ are satisfied the following conditions:

I) $S$ is an abelian semigroup under the multiplication, that is:
   1) A single-valued product $ab$ is defined in $S$ for any pair $a, b$ of $S$,
   2) $ab = ba$ for any $a,b$ of $S$,
   3) $(ab)c = a(bc)$ for any $a, b, c$ of $S$.

II) $S$ is a partially ordered set under the relation $\geq$, that is:
   1) $a \geq a$,
   2) $a \geq b, b \geq a$ imply $a = b$,
   3) $a \geq b, b \geq c$ imply $a \geq c$.

III) Homogeneity: $a \geq b$ implies $ac \geq bc$ for any $c$ of $S$.

A partial order which satisfies the condition III) is called a partial order defined on an abelian semigroup.

If $S$ is an abelian group, then $S$ is said to be a partially ordered abelian group (p.o. group).

Moreover, if a partial order defined on an abelian semigroup (group) $S$ is a linear order, then $S$ is said to be a linearly ordered abelian semigroup (group) (l.o. semigroup (l.o. group)).

We write $a > b$ for $a \geq b$ and $a \neq b$.

Definition 2. A partial order defined on an abelian semigroup $S$ (or a p.o. semigroup $S$) is called strong, when the following condition is satisfied: $ac \geq bc$ implies $a \geq b$.

Theorem 1. A partial order defined on an abelian group $G$ is always strong.

Proof. Since $G$ is a group, there exists an inverse element $c^{-1}$ of $c$. By the homogeneity $ac \geq bc$ implies $(ac)c^{-1} \geq (bc)c^{-1}$. Therefore $a \geq b$. 
Theorem 2. In the strong p. o. semigroup $S$ the following properties are held:
1) $ac = bc$ implies $a = b$ (product cancellation law).
2) $ac > bc$ implies $a > b$ (order cancellation law).
3) $a > b$ implies $ac > bc$ for any $c$ of $S$.

Proof. 1): If $ac = bc$, or, what is the same, if $ac \geq bc$ and $bc \geq ac$, then $a \geq b$ and $b \geq a$, that is, $a = b$.
2): If $ac > bc$ implies $a = b$, then $ac = bc$, which is absurd.
3): If $a > b$ implies $ac = bc$ for some $c$ of $S$, then by 1) we have $a = b$ which contradicts the hypothesis $a > b$.

Theorem 3. In the l. o. semigroup $S$ the following properties are held:
1) $ac > bc$ implies $a > b$,
2) $a^n > b^n$ for some positive integer $n$ implies $a > b$.

Proof. 1): If, under the hypothesis $ac > bc$, $a \geq b$, then by the linearity of $S$, $b \geq a$. By the homogeneity we have $bc \geq ac$, this contradicts the hypothesis. 2): Similarly, if $a^n > b^n$ implies $a \geq b$, then we have $b^n \geq a^n$.

Theorem 4. In the l. o. semigroup $S$ the following conditions are equivalent to each other:
1) $ac \geq bc$ implies $a \geq b$ (strong),
2) $ac = bc$ implies $a = b$,
3) $a > b$ implies $ac > bc$ for all $c$ of $S$.

Proof. 1) $\rightarrow$ 2): See Theorem 2, 1). 2) $\rightarrow$ 3): Suppose that $a > b$ implies $ac = bc$ for some $c$ of $S$. By 2) we have $a = b$. 3) $\rightarrow$ 1): Suppose that $ac \geq bc$ implies $a \geq b$. By the linearity we have $b > a$, therefore we have $bc > ac$ by 3).

Definition 3. Two p. o. semigroups $S$ and $S'$ will be called order-isomorphic if there exists an algebraic isomorphism $x \leftrightarrow x'$ between them which preserves order: if $a \leftrightarrow a'$, $b \leftrightarrow b'$, then $a \geq b$ if and only if $a' \geq b'$.

A p. o. semigroup $S$ will be said to be order-embedded in a p. o. semigroup $S'$, if there exists an order-isomorphism of $S$ into $S'$.

Theorem 5. A p. o. semigroup $S$ can be order-embedded in a p. o. group if and only if $S$ is strong.

Sufficiency: By Theorem 2, the product cancellation law is held in
It follows: $(a, a')$ is equal to $(b, b')$ if and only if $ab' = a'b$. As we can then prove, the above-defined equality fulfils the equivalence relation. In particular $(ax, a'x) = (a, a')$ for any $x$ of $S$. Next, we define the multiplication of the elements in $G$ as follows: $(a, a')(b, b') = (ab, a'b')$. If $(a, a') = (c, c')$ and $(b, b') = (d, d')$, then $(ab, a'b') = (cd, c'd')$. One can easily verify the commutative and associative laws of multiplication. Moreover, $(x, x)$ is the unit element of $G$ and $(a', a)$ is an inverse element of $(a, a')$. Therefore $G$ is an abelian group under the multiplication introduced above.

Now let us define an order in $G$ as follows: $(a, a') \geq (b, b')$ if and only if $ab' \geq a'b$ in $S$. By the strongness of $S$ it follows immediately that if $(a, a') = (c, c')$, $(b, b') = (d, d')$ and $(a, a') \geq (b, b')$, then $(c, c') \geq (d, d')$. Moreover, it is easy to see that the above-defined order fulfils the conditions II) 1), 2), 3) and III). Therefore $G$ becomes a p. o. group. The correspondence $a \leftrightarrow (ax, x)$ is the order-isomorphism of $S$ into $G$.

Such an obtained group $G = Q(S)$, which is the minimal p. o. group containing $S$ and uniquely determined by $S$ apart from its order-isomorphism, will be called the quotient group of the p. o. semigroup $S$.

**Corollary.** A l. o. semigroup $S$ can be order-embedded in a l. o. group if and only if $S$ is strong.

**Theorem 6.** Let $S$ be a p. o. semigroup with the unit element $e$. $e \geq a$ for any $a$ of $S$ if and only if $a \geq ab$ for any $a, b$ of $S$.

**Proof.** Necessity: $e \geq b$ for any $b$ of $S$ implies $a \geq ab$ for any $a, b$ of $S$.

Sufficiency: If $a \geq ab$ for any $a, b$ of $S$, then we put $a = e$. Thus we have $e \geq b$ for any $b$ of $S$. Moreover, if $S$ has the zero element, i.e., the element 0 such that $0a = 0$ for any $a$ of $S$, then $a \geq 0$ for any $a$ of $S$.

**Corollary.** Let $S$ be a p. o. semigroup order-embedded in a p. o. group $G$. $e \geq a$ for any $a$ of $S$, where $e$ is the unit element of $G$, if and only if $a \geq ab$ for any $a, b$ of $S$.

**Theorem 7.** Let $S$ be a strong p. o. semigroup, $G$ be the quotient group of $S$ and $e$ the unit element of $G$. $e \geq a$ for any $a$ of $S$ and $e > a$ ($a \in G$) implies $a \in S$ if and only if $a \geq ab$ for any $a, b$ of $S$ and if $a > b$, then there exists an element $c$ of $S$ such that $b = ac$.

**Proof.** Necessity: By Corollary of Theorem 6, $a \geq ab$ for any $a, b$ of $S$. If $a > b$, then $e > a^{-1}b$, and hence $a^{-1}b = c \in S$. Therefore $b = ac$. 

**Proof.** Necessity: By Corollary of Theorem 6, $a \geq ab$ for any $a, b$ of $S$. If $a > b$, then $e > a^{-1}b$, and hence $a^{-1}b = c \in S$. Therefore $b = ac$. 

**Proof.**
Sufficiency: It is clear that $e \geq a$ for any $a$ of $S$. Moreover, let $x$ be any element of $G$ such that $e > x$. We can put $x = a^{-1}b$, $a, b \in S$. Thus we obtain $a > b$. Hence there exists an element $c$ of $S$ such that $b = xc$, therefore $x = a^{-1}b = c \in S$.

**Definition 4.** Let $S$ be a p.o.semigroup. An element $a$ of $S$ is called positive or negative, when $a^i \geq a$ or $a \geq a^i$ respectively. In a p. o. group these coincide with the usual definition.

A partial order defined on $S$ is called directed, when to any $a, b$ of $S$ there exists an element $c$ of $S$ such that $a \geq c$ and $b \geq c$.

**Theorem 8.** Let $G$ be a p. o. group and $S$ be the p. o. semigroup of all negative elements of $G$. Then $G = Q(S)$ if and only if $G$ is directed.

*Proof.* Necessity: By Theorem 7, $a \geq ab$ for any $a, b$ of $S$. Therefore $S$ is directed. Let $x, y$ be any elements of $G$. One can write $x = ax^{-1}$, $y = bx^{-1}$, $a, b, c \in S$. Since $S$ is directed, there exists an element $d$ of $S$ such that $a \geq d$ and $b \geq d$. And hence if we put $z = dax^{-1}$, we have $x \geq z$ and $y \geq z$. Therefore $G$ is directed.

Sufficiency: Let $x$ be any element of $G$. If $a$ be chosen such that $x \geq a$ and $e \geq a$ ($e$ is the unit element of $G$), then

$$x = a((ax^{-1})^{-1}, e \geq a, e \geq ax^{-1}.$$  

**Definition 5.** An element of a semigroup $S$ is said to be of infinite order if all its powers are different. If there exists a positive integer $n$ such that $a^i \approx a^j$ for $1 \leq i < j \leq n$ and $a^m = a^k$ for all integers $k \geq n$, then $a$ is called quasi-idempotent and such positive integer $n$ is called the length of $a$. If the length of $a$ is 1 then $a$ is idempotent in the usual sense.

**Theorem 9.** An element of a l. o. semigroup $S$ is of infinite order or quasi-idempotent.

*Proof.* Let $a$ be not of infinite order. There exist positive integers $n, m$ such that $a^n = a^m$, $m > n$, and $n$ is the least. Since $S$ is a l. o. semigroup,

$$a > a^2 > \cdots > a^{n-1} > a^n \geq a^{n+1} \geq \cdots \geq a^m = a^n \text{ (or its dual).}$$

Therefore $a^n = a^k$ for all $k \geq n$.

Theorem 10. Let $S$ be a strong l. o. semigroup. Then $a^n = b^n$ implies $a = b$. And if there exists a quasi-idempotent element $e$, then $e$ is the unit element.

Proof. Since $S$ is strong, $a > b$ implies $a^2 > ab > b^2$. Hence for all positive integers $n$, $a^n > b^n$. Next, the length of $e$ must be 1. Hence $e = e$. For every $x$ of $S$, $ex = e'x$ and hence $x = ex$, that is, $e$ is the unit element. Therefore $S$ has at most one quasi-idempotent element.

Definition 6. A partial order defined on an abelian semigroup $S$ is called normal, when the following condition is satisfied: (2)

$$a^n \geq b^n \text{ for some positive integer } n \text{ implies } a \geq b.$$ 

Theorem 11. A strong l. o. semigroup $S$ is always normal.

Proof. Suppose that $a \geq b$. Then we have, by the linearity of $S$, $b > a$, which implies $b^n > a^n$ for every positive integer $n$.

Corollary. A l. o. group $G$ is always normal.

Theorem 12. In the normal p. o. semigroup the following properties are held: 1) $a^n > b^n$ implies $a > b$, 2) $a^n = b^n$ implies $a = b$.

Proof. 1): By the normality, $a^n > b^n$ implies $a \geq b$. If $a = b$, then we have $a^n = b^n$. 2): The normality means that if $a^n = b^n$, or what is the same $a^n \geq b^n$ and $b^n \geq a^n$, then $a \geq b$ as well as $b \geq a$, that is, $a = b$.

Corollary. An element of a normal p. o. group has an infinite order, except the unit element.

Definition 7. Suppose that two partial orders $P$ and $Q$ are defined on the same semigroup $S$ and that the relation $a > b$ in $P$ implies $a > b$ in $Q$; then $Q$ will be called an extension of $P$. An extension which defines a linear order on $S$ will be called a linear extension.

In the set $\mathfrak{P}$ of all partial orders defined on the same semigroup $S$, we put $Q > P$ if and only if $Q$ is an extension of $P$. Then $\mathfrak{P}$ is a partially ordered set under this relation $\succ$.

Theorem 13. Let $P$ be a strong partial order defined on an abelian semigroup $S$ and $x$ and $y$ are any two elements non-comparable in $P$. Then there exists an extension $Q$, which is strong, of $P$ such that $x > y$ in $Q$ if and only if $P$ is normal. (3)

Proof. Sufficiency: Let $P$ be a normal strong partial order defined

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(3) Cf. L. Fuchs: l. c.
on $S$ and the elements $x$ and $y$ are not comparable in $P$. Let us define a relation $Q$ as follows:

We put $a > b$ in $Q$ if and only if $a \equiv b$ and there are two non-negative integers $n$, $m$, such that not both zero and

$$a^n y^m \geqq b^n x^m \quad \text{in } P,$$

where if $m = 0$ or $n = 0$ ($\S$) means that $a^n \geqq b^n$ or $y^m \geqq x^m$ in $P$ respectively.

First, we note that $n$ is never zero, for otherwise we should have $y^m \geqq x^m$ in $P$, whence (by the normality) we have $y \geqq x$ in $P$ against the hypothesis.

i ) We begin with verifying that $a > b$ and $b > a$ in $Q$ are contradictory. Suppose that $a > b$ and $b > a$, namely $a^n y^m \geqq b^n x^m$ and $b^j y^j \geqq a^j x^j$ in $P$ for some non-negative integers $n$, $m$, $i$, $j$. By multiplying $i$ times the first, $n$ times the second inequality, one obtains $(ab)^{ni} y^{mi+nj} \geqq (ab)^{ni} x^{mi+nj}$ in $P$. By the strongness of $P$ we have $y^{mi+nj} \geqq x^{mi+nj}$ in $P$. If $mi + nj$ does not vanish, by the normality we have $y \geqq x$, this contradicts the hypothesis. On the other hand, if $mi + nj$ is zero, i.e., both $m$ and $j$ vanish, then $a^n \geqq b^n$ and $b^i \geqq a^i$ in $P$. Therefore we have $a \geqq b$ and $b \geqq a$ in $P$, that is, $a = b$ which is absurd.

ii ) We show the transitivity of $Q$. Assume that $a > b$ and $b > c$ in $Q$, i.e., for some non-negative integers $n$, $m$, $i$, $j$, $a^n y^m \geqq b^n x^m$ and $b^j y^j \geqq a^j x^j$ in $P$. By multiplying as in i ) we get $a^j y^j \geqq x^j$. Here $ni$ is not zero, and $a = c$ is by i ) impossible, so that $a > c$ in $Q$.

iii ) We prove next the homogeneity of $Q$. $a \equiv b$ implies $ac \equiv bc$ for any $c$ of $S$, since $P$ is strong. Hence if $a > b$ in $Q$, namely, if $a \equiv b$ and $a^n y^m \geqq b^n x^m$ in $P$ for some $n$, $m$, then $ac \equiv bc$ and $(ac)^n y^m \geqq (bc)^n x^m$ in $P$. Therefore $a > b$ implies $ac > bc$ in $Q$ for any $c$ of $S$.

iv ) $Q$ is an extension of $P$, if for $a > b$ in $P$, then $a y^0 > b x^0$ in $P$, therefore $a > b$ in $Q$.

v ) It is clear that $x > y$ in $Q$. In fact, $xy \geqq yx$ in $P$.

vi ) We may prove the normality and the strongness of $Q$. Indeed, supposing $a^n > b^n$ in $Q$ for some positive integer $n$, i.e., $(a^n)^i y^j \geqq (b^n)^j x^j$ in $P$, we see at once that $a > b$ in $Q$. Suppose that $ac > bc$ in $Q$, i.e., $(ac)^n y^m \geqq (bc)^n x^m$ in $P$ for some $n$, $m$. Then by the strongness of $P$ we are led to the result $a > b$ in $Q$.

Necessity: Let us assume that there exist elements $a$ and $b$ such that $a^n \geqq b^n$ and $a \equiv b$ in $P$. Then $a$ and $b$ can not be comparable in $P$ by the strongness of $P$. And hence there exists a strong extension
$Q$ of $P$ in which $b > a$. This is however absurd, since by the strong-
ness of $Q$ this would imply $b^n > a^n$ in $Q$, contrary to the hypothesis $a^n
\geq b^n$ in $P$.

**Definition 8.** If $P_1, P_2, \ldots, P_a, \ldots$ is a well-ordered chain of partial
orders defined on the same abelian semigroup $S$ such that each of them
is some extension of the preceding ones, then the union of the chain
may be defined to be a partial order $P$ defined on $S$ such that $a \geq b$ in
$P$ if and only if $a \geq b$ in $P_a$ holds for some one and hence for all sub-
sequent subscripts $a$.

It is easy to see that $P$ is normal or strong if all $P_a$ are normal or
strong respectively.

**Theorem 14.** For every normal strong partial order $P$ defined on
an abelian semigroup $S$ and every two elements $x, y$ non-comparable in
$P$, there exists a normal strong linear extension $L_{xy}$ with the property
that $x > y$ in $L_{xy}$.

**Proof.** By Theorem 13 there exists a normal strong extension $Q$ of
$P$ such that $x > y$ in $Q$. Let $\mathfrak{P}'$ be a set of all normal strong partial
orders defined on $S$ which are extensions of $Q$. $\mathfrak{P}'$ is a partially ordered
set as a subset of $\mathfrak{P}$ in Definition 7. By Zorn's lemma there exists a
maximal linearly ordered subset $\mathfrak{P}^*$ of $\mathfrak{P}'$. Let $L_{xy}$ be an union of $\mathfrak{P}^*$.
Then $L_{xy}$ is a maximal order, that is, order which has no proper ex-
tension. By Theorem 13 this can happen only in case any two elements
are comparable in $L_{xy}$, that is to say, $L_{xy}$ is linear. Moreover, $L_{xy}$ is
strong and normal, and $x > y$ in $L_{xy}$.

**Theorem 15.** A strong linear order may be defined on an abelian
semigroup $S$ if and only if in $S$ are satisfied the following conditons:
1) $ax = bx$ implies $a = b$, 2) $a^n = b^n$ for some positive integer $n$ im-
plies $a = b$.

**Proof.** The necessity is obvious by Theorems 2 and 12. If we con-
sider a vacuous partial order $P$ of $S$ in the sense of Tukey, then $P$ is
the partial order defined on $S$. And conditions 1) and 2) say that $P$ is
strong and normal. Therefore, by Theorem 14 for any $x, y$ of $S$ there
exists a strong linear extension $L_{xy}$ of $P$ in which $x > y$.

**Corollary.** A linear order may be defined on an abelian group if
and only if all its elements, except the unit element, are of infinite order.

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(4) F. Levi: Arithmetische Gesetze im Gebiete diskreter Gruppen, Rendiconti Palermo,
vol. 35 pp. (1913), 225-236.
Definition 9. Let \( \mathfrak{S} = \{P_{\alpha}\} \) be any set of partial orders, each defined on the same abelian semigroup \( S \). We define the new partial order \( P \) on \( S \) as follows: For any two elements \( a, b \) we put \( a \geq b \) in \( P \) if and only if \( a \geq b \) in every \( P_{\alpha} \) of the set \( \mathfrak{S} \). Indeed, \( P \) is again a partial order defined on \( S \), moreover \( P \) is normal or strong if all \( P_{\alpha} \) of \( \mathfrak{S} \) are normal or strong respectively. The partial order \( P \) is said to be the product of the \( P_{\alpha} \) or to be realized by the set \( \mathfrak{S} \) of partial orders, written \( P = \Pi P_{\alpha} \).

Let \( \mathfrak{G} = \{G_{\alpha}\} \) be a set of l.o. groups and \( G \) the (restricted or complete) direct product of \( G_{\alpha} \). Then one can introduce a partial order defined on \( G \) as usual, so that \( G \) becomes a p.o. group. We shall call \( G \) a vector-group. It is clear that a vector-group is always strong and normal.

Theorem 16. A strong partial order \( P \) defined on an abelian semigroup \( S \) may be realized by a certain set of strong linear orders if and only if \( P \) is normal.

Proof. The necessity is obvious, since by Theorem 11 a strong linear order, and hence every product of strong linear orders, is normal. On the other hand, if \( P \) is not linear, then there exist to any pair of elements \( x, y \) non-comparable in \( P \) the corresponding linear extensions \( L_{xy} \) and \( L_{yx} \) described in Theorem 14. It is easy to see that these linear orders realize \( P \).

Theorem 17. A p.o. semigroup \( S \) can be order-embedded in a vector-group if and only if \( S \) is normal and strong.

Proof. Let \( P \) be a partial order defined on \( S \). If \( P \) is strong and normal, then by Theorem 16 \( P \) is realized by a certain set of strong linear orders, which are extensions of \( P \), defined on the semigroup \( S \); \( P = \Pi P_{\alpha} \). Let \( S_{\alpha} \) be the strong l.o. semigroup when we consider that \( P_{\alpha} \) is the strong linear order defined on \( S \). And let \( G_{\alpha} \) be the quotient group of \( S_{\alpha} \). \( G_{\alpha} \) is a l.o. group. Then \( S \) is order-embedded in the direct product \( G \) of \( G_{\alpha} \). The necessity is obvious.

Corollary. A p.o. group \( G \) can be order-embedded in a vector-group if and only if \( G \) is normal\(^{(5)} \).

Theorem 18. Let \( \mathcal{F} = \{S_{\alpha}\} \) be a set of strong l.o. semigroups and \( S \) the (restricted or complete) direct product of \( S_{\alpha} \). Then one can introduce a linear order defined on \( S \), so that \( S \) becomes a strong l.o.

\(^{(5)}\) A. H. Clifford: l.c., Theorem 1.
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Proof. We may consider that the $S_\alpha$ are well-ordered. Elements of $S$ are then given by their components: $x = \{x_\alpha\}$, $x_\alpha \in S_\alpha$.

Let us define a relation $P$ in $S$ as follows:

We put $x > y$ in $P$ if and only if $x \equiv y$ and

$$x_\alpha = y_\alpha \quad \text{for all } \alpha < \beta \quad \text{and} \quad x_\beta > y_\beta.$$  

We see readily that $P$ is a strong linear order defined on $S$.

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