PARTIALLY ORDERED ABELIAN SEMIGROUPS

I. ON THE EXTENSION OF THE STRONG PARTIAL ORDER DEFINED ON ABELIAN SEMIGROUPS

By

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Definition 1. A set S is said to be a *partially ordered abelian semigroup* (p. o. semigroup), when in S are satisfied the following conditions:

I) S is an abelian semigroup under the multiplication, that is:
   1) A single-valued product ab is defined in S for any pair a, b of S,
   2) $ab = ba$ for any a,b of S,
   3) $(ab)c = a(bc)$ for any a, b, c of S.

II) S is a partially ordered set under the relation $\geqq$, that is:
   1) $a \geqq a$,
   2) $a \geqq b$, $b \geqq a$ imply $a = b$,
   3) $a \geqq b$, $b \geqq c$ imply $a \geqq c$.

III) Homogeneity: $a \geqq b$ implies $ac \geqq bc$ for any c of S.

A partial order which satisfies the condition III) is called a *partial order defined on an abelian semigroup.*

If S is an abelian group, then S is said to be a *partially ordered abelian group* (p. o. group).

Moreover, if a partial order defined on an abelian semigroup (group) S is a linear order, then S is said to be a *linearly ordered abelian semigroup* (*group*) (l. o. semigroup (l. o. group)).

We write $a > b$ for $a \geqq b$ and $a \neq b$.

Definition 2. A partial order defined on an abelian semigroup S (or a p. o. semigroup S) is called *strong*, when the following condition is satisfied: $ac \geqq bc$ implies $a \geqq b$.

Theorem 1. A partial order defined on an abelian group G is always strong.

Proof. Since G is a group, there exists an inverse element c^{-1} of c. By the homogeneity $ac \geqq bc$ implies $(ac)c^{-1} \geqq (bc)c^{-1}$. Therefore $a \geqq b$.

Theorem 2. In the strong p. o. semigroup S the following properties are held:

1) $ac = bc$ implies $a = b$ (product cancellation law).

2) $ac > bc$ implies $a > b$ (order cancellation law).

3) $a > b$ implies $ac > bc$ for any c of S.

Proof. 1): If $ac = bc$, or, what is the same, if $ac \geqq bc$ and $bc \geqq ac$, then $a \geqq b$ and $b \geqq a$, that is, $a = b$.

2): If $ac > bc$ implies $a = b$, then $ac = bc$, which is absurd.

3): If $a > b$ implies $ac = bc$ for some c of S, then by 1) we have $a = b$ which contradicts the hypothesis $a > b$.

Theorem 3. In the l. o. semigroup S the following properties are held:

1) $ac > bc$ implies $a > b$,

2) $a^n > b^n$ for some positive integer n implies $a > b$.

Proof. 1): If, under the hypothesis $ac > bc$, $a \not> b$, then by the linearity of S, $b \geqq a$. By the homogeneity we have $bc \geqq ac$, this contradicts the hypothesis. 2): Similarly, if $a^n > b^n$ implies $a \not> b$, then we have $b^n \geqq a^n$.

Theorem 4. In the l. o. semigroup S the following conditions are equivalent to each other:

1) $ac \geqq bc$ implies $a \geqq b$ (strong),

2) $ac = bc$ implies $a = b$,

3) $a > b$ implies $ac > bc$ for all c of S.

Proof. 1) $\rightarrow$ 2): See Theorem 2, 1). 2) $\rightarrow$ 3): Suppose that $a > b$ implies $ac = bc$ for some c of S. By 2) we have $a = b$. 3) $\rightarrow$ 1): Suppose that $ac \geqq bc$ implies $a \ngeqq b$. By the linearity we have $b > a$, therefore we have $bc > ac$ by 3).

Definition 3. Two p. o. semigroups S and S' will be called *order-isomorphic* if there exists an algebraic isomorphism $x \longleftrightarrow x'$ between them which preserves order: if $a \longleftrightarrow a'$, $b \longleftrightarrow b'$, then $a \geqq b$ if and only if $a' \geqq b'$.

A p. o. semigroup S will be said to be *order-embedded* in a p. o. semigroup S', if there exists an order-isomorphism of S into S'.

Theorem 5. A p. o. semigroup S can be order-embedded in a p. o. group if and only if S is strong.

Proof. Necessity: By Theorem 1.

Sufficiency: By Theorem 2, the product cancellation law is held in

S. Let G be the set of all symbols (a, a'), $a, a' \in S$. We introduce the equality of the elements of G as follows: (a, a') is equal to (b, b') if and only if $ab' = a'b$. As we can then prove, the above-defined equality fulfils the equivalence relation. In particular $(ax, a'x) = (a, a')$ for any x of S. Next, we define the multiplication of the elements in G as follows: $(a, a')(b, b') = (ab, a'b')$. If $(a, a') = (c, c')$ and $(b, b') = (d, d')$, then $(ab, a'b') = (cd, c'd')$. One can easily verify the commutative and associative laws of multiplication. Moreover, (x, x) is the unit element of G and (a', a) is an inverse element of (a, a'). Therefore G is an abelian group under the multiplication introduced above.

Now let us define an order in G as follows: $(a, a') \geqq (b, b')$ if and only if $ab' \geqq a'b$ in S. By the strongness of S it follows immediately that if $(a, a') = (c, c')$, $(b, b') = (d, d')$ and $(a, a') \geqq (b, b')$, then $(c, c') \geqq (d, d')$. Moreover, it is easy to see that the above-defined order $\geqq$ fulfils the conditions II) 1), 2), 3) and III). Therefore G becomes a p. o. group. The correspondence $a \longleftrightarrow (ax, x)$ is the order-isomorphism of S into G.

Such an obtained group $G = Q(S)$, which is the minimal p. o. group containing S and uniquely determined by S apart from its order-isomorphism, will be called the *quotient group* of the p. o. semigroup S.

Corollary. A l. o. semigroup S can be order-embedded in a l. o. group if and only if S is strong.

Theorem 6. Let S be a p. o. semigroup with the unit element e. $e \geqq a$ for any a of S if and only if $a \geqq ab$ for any a, b of S.

Proof. Necessity: $e \geqq b$ for any b of S implies $ae = a \geqq ab$ for any a, b of S.

Sufficiency: If $a \geqq ab$ for any a, b of S, then we put $a = e$. Thus we have $e \geqq b$ for any b of S. Moreover, if S has the zero element, i. e., the element 0 such that $0a = 0$ for any a of S, then $a \geqq 0$ for any a of S.

Corollary. Let S be a p. o. semigroup order-embedded in a p. o. group G. $e \geqq a$ for any a of S, where e is the unit element of G, if and only if $a \geqq ab$ for any a, b of S.

Theorem 7. Let S be a strong p. o. semigroup, G be the quotient group of S and e the unit element of G. $e \geqq a$ for any a of S and $e > a$ $(a \in G)$ implies $a \in S$ if and only if $a \geqq ab$ for any a, b of S and if $a > b$, then there exists an element c of S such that $b = ac$.

Proof. Necessity: By Corollary of Theorem 6, $a \geqq ab$ for any a, b of S. If $a > b$, then $e > a^{-1}b$, and hence $a^{-1}b = c \in S$. Therefore $b = ac$.

Sufficiency: It is clear that $e \geqq a$ for any a of S. Moreover, let x be any element of G such that $e > x$. We can put $x=a^{-1}b$, $a, b \in S$. Thus we obtain $a > b$. Hence there exists an element c of S such that $b = ac$, therefore $x = a^{-1}b = c \in S$.

Definiton 4. Let S be a p.o.semigroup. An element a of S is called *positive* or *negative*, when $a^2 \geqq a$ or $a \geqq a^2$ respectively. In a p. o. group these coincide with the usual definition.

A partial order defined on S is called *directed*, when to any a, b of S there exists an element c of S such that $a \geqq c$ and $b \geqq c$.

Theorem 8. Let G be a p. o. group and S be the p. o. semigroup of all negative elements of G. Then $G = Q(S)$ if and only if G is directed.(1)

Proof. Necessity: By Theorem 7, $a \geqq ab$ for any a, b of S. Therefore S is directed. Let x, y be any elements of G. One can write $x = ac^{-1}$, $y = bc^{-1}$, $a, b, c \in S$. Since S is directed, there exists an element d of S such that $a \geqq d$ and $b \geqq d$. And hence if we put $z = dc^{-1}$, we have $x \geqq z$ and $y \geqq z$. Therefore G is directed.

Sufficiency: Let x be any element of G. If a be chosen such that $x \geqq a$ and $e \geqq a$ (e is the unit element of G), then

$$x = a\,(ax^{-1})^{-1}, \quad e \geqq a, \quad e \geqq ax^{-1}.$$

Definition 5. An element of a semigroup S is said to be of *infinite order* if all its powers are different. If there exists a positive integer n such that $a^i \neq a^j$ for $1 \leqq i < j \leqq n$ and $a^n = a^k$ for all integers $k \geqq n$, then a is called *quasi-idempotent* and such positive integer n is called the *length* of a. If the length of a is 1 then a is idempotent in the usual sense.

Theorem 9. An element of a l. o. semigroup S is of infinite order or quasi-idempotent.

Proof. Let a be not of infinite order. There exist positive integers n, m such that $a^n = a^m$, $m > n$, and n is the least. Since S is a l. o. semigroup,

$$a > a^2 > \cdots > a^{n-1} > a^n \geqq a^{n+1} \geqq \cdots \geqq a^m = a^n \text{ (or its dual).}$$

Therefore $a^n = a^k$ for all $k \geqq n$.

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(1) Cf. A. H. Clifford: Partially ordered abelian groups, Ann. Math., vol. 41 (1940), pp. 465–473.

Theorem 10. Let S be a strong l. o. semigroup. Then $a^n = b^n$ implies $a = b$. And if there exists a quasi-idempotent element e, then e is the unit element.

Proof. Since S is strong, $a > b$ implies $a^2 > ab > b^2$. Hence for all positive integers n, $a^n > b^n$. Next, the length of e must be 1. Hence $e^2 = e$. For every x of S, $ex = e^2x$ and hence $x = ex$, that is, e is the unit element. Therefore S has at most one quasi-idempotent element.

Definition 6. A partial order defined on an abelian semigroup S is called *normal*, when the following condition is satisfied:[2]

$$a^n \geqq b^n \text{ for some positive integer } n \text{ implies } a \geqq b.$$

Theorem 11. A strong l. o. semigroup S is always normal.

Proof. Suppose that $a \ngeqq b$. Then we have, by the linearity of S, $b > a$, which implies $b^n > a^n$ for every positive integer n.

Corollary. A l. o. group G is always normal.

Theorem 12. In the normal p. o. semigroup the following properties are held: 1) $a^n > b^n$ implies $a > b$, 2) $a^n = b^n$ implies $a = b$.

Proof. 1): By the normality, $a^n > b^n$ implies $a \geqq b$. If $a = b$, then we have $a^n = b^n$. 2): The normality means that if $a^n = b^n$, or what is the same $a^n \geqq b^n$ and $b^n \geqq a^n$, then $a \geqq b$ as well as $b \geqq a$, that is, $a = b$.

Corollary. An element of a normal p. o. group has an infinite order, except the unit element.

Definition 7. Suppose that two partial orders P and Q are defined on the same semigroup S and that the relation $a > b$ in P implies $a > b$ in Q; then Q will be called an *extension* of P. An extension which defines a linear order on S will be called a *linear extension.*

In the set $\mathfrak{P}$ of all partial orders defined on the same semigroup S, we put $Q \succ P$ if and only if Q is an extension of P. Then $\mathfrak{P}$ is a partially ordered set under this relation $\succ$.

Theorem 13. Let P be a strong partial order defined on an abelian semigroup S and x and y are any two elements non-comparable in P. Then there exists an extension Q, which is strong, of P such that $x > y$ in Q if and only if P is normal.[3]

Proof. Sufficiency: Let P be a normal strong partial order defined

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(2) Cf. L. Fuchs: On the extension of the partial order of groups, Amer. Journ. Math., vol. 72 (1950), pp. 191–194.

(3) Cf. L. Fuchs: l. c.

on S and the elements x and y are not comparable in P. Let us define a relation Q as follows:

We put $a > b$ in Q if and only if $a \neq b$ and there are two non-negative integers n, m, such that not both zero and

$$(\S) \qquad a^n y^m \geqq b^n x^m \quad \text{in} \quad P,$$

where if $m = 0$ or $n = 0$ (§) means that $a^n \geqq b^n$ or $y^m \geqq x^m$ in P respectively.

First, we note that n is never zero, for otherwise we should have $y^m \geqq x^m$ in P, whence (by the normality) we have $y \geqq x$ in P against the hypothesis.

i) We begin with verifying that $a > b$ and $b > a$ in Q are contradictory. Suppose that $a > b$ and $b > a$, namely $a^n y^m \geqq b^n x^m$ and $b^i y^j \geqq a^i x^j$ in P for some non-negative integers n, m, i, j. By multiplying i times the first, n times the second inequality, one obtains $(ab)^{ni} y^{mi+nj} \geqq (ab)^{ni} x^{mi+nj}$ in P. By the strongness of P we have $y^{mi+nj} \geqq x^{mi+nj}$ in P. If $mi + nj$ does not vanish, by the normality we have $y \geqq x$, this contradicts the hypothesis. On the other hand, if $mi + nj$ is zero, i. e., both m and j vanish, then $a^n \geqq b^n$ and $b^i \geqq a^i$ in P. Therefore we have $a \geqq b$ and $b \geqq a$ in P, that is, $a = b$ which is absurd.

ii) We show the transitivity of Q. Assume that $a > b$ and $b > c$ in Q, i. e., for some non-negative integers n, m, i, j, $a^n y^m \geqq b^n x^m$ and $b^i y^j \geqq c^i x^j$ in P. By multiplying as in i) we get $a^{ni} y^{mi+nj} \geqq c^{ni} x^{mi+nj}$ in P. Here ni is not zero, and $a = c$ is by i) impossible, so that $a > c$ in Q.

iii) We prove next the homogeneity of Q. $a \neq b$ implies $ac \neq bc$ for any c of S, since P is strong. Hence if $a > b$ in Q, namely, if $a \neq b$ and $a^n y^m \geqq b^n x^m$ in P for some n, m, then $ac \neq bc$ and $(ac)^n y^m \geqq (bc)^n x^m$ in P. Therefore $a > b$ implies $ac > bc$ in Q for any c of S.

iv) Q is an extension of P, for if $a > b$ in P, then $ay^0 > bx^0$ in P, therefore $a > b$ in Q.

v) It is clear that $x > y$ in Q. In fact, $xy \geqq yx$ in P.

vi) We may prove the normality and the strongness of Q. Indeed, supposing $a^n > b^n$ in Q for some positive integer n, i.e., $(a^n)^i y^j \geqq (b^n)^i x^j$ in P, we see at once that $a > b$ in Q. Suppose that $ac > bc$ in Q, i.e., $(ac)^n y^m \geqq (bc)^n x^m$ in P for some n, m. Then by the strongness of P we are led to the result $a > b$ in Q.

Necessity: Let us assume that there exist elements a and b such that $a^n \geqq b^n$ and $a \ngeqq b$ in P. Then a and b can not be comparable in P by the strongness of P. And hence there exists a strong extension

Q of P in which $b > a$. This is however absurd, since by the strongness of Q this would imply $b^n > a^n$ in Q, contrary to the hypothesis $a^n \geqq b^n$ in P.

Definition 8. If $P_1, P_2, \cdots, P_\alpha, \cdots$ is a well-ordered chain of partial orders defined on the same abelian semigroup S such that each of them is some extension of the preceding ones, then the *union* of the chain may be defined to be a partial order P defined on S such that $a \geqq b$ in P if and only if $a \geqq b$ in P_α holds for some one and hence for all subsequent subscripts α.

It is easy to see that P is normal or strong if all P_α are normal or strong respectively.

Theorem 14. For every normal strong partial order P defined on an abelian semigroup S and every two elements x, y non-comparable in P, there exists a normal strong linear extension L_{xy} with the property that $x > y$ in L_{xy}.

Proof. By Theorem 13 there exists a normal strong extension Q of P such that $x > y$ in Q. Let $\mathfrak{P}'$ be a set of all normal strong partial orders defined on S which are extensions of Q. $\mathfrak{P}'$ is a partially ordered set as a subset of $\mathfrak{P}$ in Definition 7. By ZORN's lemma there exists a maximal linearly ordered subset $\mathfrak{P}^*$ of $\mathfrak{P}'$. Let L_{xy} be an union of $\mathfrak{P}^*$. Then L_{xy} is a maximal order, that is, order which has no proper extension. By Theorem 13 this can happen only in case any two elements are comparable in L_{xy}, that is to say, L_{xy} is linear. Moreover, L_{xy} is strong and normal, and $x > y$ in L_{xy}.

Theorem 15. A strong linear order may be defined on an abelian semigroup S if and only if in S are satisfied the following conditons: 1) $ax = bx$ implies $a = b$, 2) $a^n = b^n$ for some positive integer n implies $a = b$.

Proof. The necessity is obvious by Theorems 2 and 12. If we consider a vacuous partial order P of S in the sense of TUKEY, then P is the partial order defined on S. And conditions 1) and 2) say that P is strong and normal. Therefore, by Theorem 14 for any x, y of S there exists a strong linear extension L_{xy} of P in which $x > y$.

Corollary. A linear order may be defined on an abelian group if and only if all its elements, except the unit element, are of infinite order.(4)

(4) F. LEVI: Arithmetische Gesetze im Gebiete diskreter Gruppen, Rendiconti Palermo, vol. 35 pp. (1913), 225–236.

G. BIRKHOFF: Lattice Theory, second edition, Theorem 14, p. 224.

Definition 9. Let $\mathfrak{S} = \{P_\alpha\}$ be any set of partial orders, each defined on the same abelian semigroup S. We define the new partial order P on S as follows: For any two elements a, b we put $a \geqq b$ in P if and only if $a \geqq b$ in every P_α of the set $\mathfrak{S}$. Indeed, P is again a partial order defined on S, moreover P is normal or strong if all P_α of $\mathfrak{S}$ are normal or strong respectively. The partial order P is said to be the *product* of the P_α or to be *realized* by the set $\mathfrak{S}$ of partial orders, written $P = \Pi P_\alpha$.

Let $\mathfrak{G} = \{G_\alpha\}$ be a set of l. o. groups and G the (restricted or complete) direct product of G_α. Then one can introduce a partial order defined on G as usual, so that G becomes a p. o. group. We shall call G a *vector-group*. It is clear that a vector-group is always strong and normal.

Theorem 16. A strong partial order P defined on an abelian semigroup S may be realized by a certain set of strong linear orders if and only if P is normal.

Proof. The necessity is obvious, since by Theorem 11 a strong linear order, and hence every product of strong linear orders, is normal. On the other hand, if P is not linear, then there exist to any pair of elements x, y non-comparable in P the corresponding linear extensions L_{xy} and L_{yx} described in Theorem 14. It is easy to see that these linear orders realize P.

Theorem 17. A p. o. semigroup S can be order-embedded in a vector-group if and only if S is normal and strong.

Proof. Let P be a partial order defined on S. If P is strong and normal, then by Theorem 16 P is realized by a certain set of strong linear orders, which are extensions of P, defined on the semigroup S; $P = \Pi P_\alpha$. Let S_α be the strong l. o. semigroup when we consider that P_α is the strong linear order defined on S. And let G_α be the quotient group of S_α. G_α is a l. o. group. Then S is order-embedded in the direct product G of G_α. The necessity is obvious.

Corollary. A p. o. group G can be order-embedded in a vector-group if and only if G is normal[(5)].

Theorem 18. Let $\mathfrak{T} = \{S_\alpha\}$ be a set of strong l. o. semigroups and S the (restricted or complete) direct product of S_α. Then one can introduce a linear order defined on S, so that S becomes a strong l. o.

(5) A. H. Clifford: l.c., Theorem 1.

semigroup.(6)

Proof. We may consider that the S_α are well-ordered. Elements of S are then given by their components: $x = \{x_\alpha\}$, $x_\alpha \in S_\alpha$.

Let us define a relation P in S as follows:

We put $x > y$ in P if and only if $x \neq y$ and

$$x_\alpha = y_\alpha \quad \text{for all} \quad \alpha < \beta \quad \text{and} \quad x_\beta > y_\beta .$$

We see readily that P is a strong linear order defined on S.

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(6) Cf. K. Iwasawa: On linearly ordered groups, Journ. Math. Soc. Japan, vol. 1 (1948), pp. 1–9.