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ON THE EXTENDED CONNECTION PARAMETERS IN A SPACE WITH AFFINE CONNECTION AND IN A RIEMANNIAN SPACE

By

Yoshie KATSURADA

Introduction. The extensor introduced first by H. V. CRAIG [1] appears as an extension of the tensor concept and holds tensor member as a part of its components, for example, let $T^a_\beta_\delta$ be the components of a second order mixed extensor of range $G$, then the quantities $T^a_\alpha_\beta$ are components of its tensor member. From this fact it has seemed to the present author to be desirable to find such an operation of an extensor that it applied to tensor member of the extensor just coincides with the similar well-known tensor operation. As the first step towards the desire, the principal purpose of the present paper is to establish an excovariant derivation, in the above stated sense, of an extensor in a space with an affine connection as well as in a RIEMANNian space, using the extended connection parameter $\Gamma^a_\beta_\delta_\epsilon$ formed by the affine connection parameter $\Gamma^a_\beta_\epsilon$ and CHRISTOFFEL symbol by means of the metric extensor $g_{a_\alpha_\beta}$ induced by the metric tensor $g_{ab}$ respectively.

In the present paper we use certain of the ideas, notations and results given in the previous paper [2] without explanation. The present author wishes to express to Prof. A. KAWAGUCHI her appreciation for his helpful guidance.

§ 1. The connection of extensor in the space with an affine connection. We shall proceed to find what is so called the extended connection parameter and to build up such the excovariant differential that contains the covariant differential of a tensor as a part of it, in the space with a symmetric affine connection.

At each point on a parameterized arc of class $M: x^a = x^a(t)$, where $t$ is a fixed essential parameter, the quantities defined by differentiating the connection parameter $\Gamma^a_\beta_\epsilon(t)$ successively by $t$:

(1) Numbers in brackets refer to the references at the end of the paper.
\begin{align*}
\Gamma_{\beta b\gamma c}^{\alpha a} & \equiv \left( \begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array} \right) \Gamma_{\beta c}^{\alpha (a-\beta-\gamma)} \\
& \equiv 0 \quad \text{for } \beta + \gamma > \alpha
\end{align*}

are considered along the arc, where \( \alpha \) is an integer not exceeding \( M \), and

\[
\left( \begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array} \right) \equiv \frac{\alpha!}{\beta! \gamma! (\alpha-\beta-\gamma)!} \\
\equiv 0 \quad \text{for } \beta + \gamma > \alpha.
\]

We can then state the following theorems:

**Theorem 1.1.** The quantities \( \Gamma_{\beta b\gamma c}^{\alpha a} \) are changed by

\[
\Gamma_{\beta b\gamma c}^{\alpha a} = \sum_{\varepsilon = r}^{\alpha-\beta} \sum_{\varphi = \beta}^{a-\theta} \sum_{\ddot{r} = \varphi j}^{\mathcal{B}^{bfc}} X_{\beta b r_{c}}^{\theta_{f}} + \sum_{\theta = \alpha}^{\beta + \gamma} X_{\beta b\gamma c}^{\theta \dot{a}} X_{\beta b\gamma c}^{\alpha a}
\]

under a transformation of the exponent ([1], p. 765), where \( X_{\beta b\gamma c}^{\theta \dot{a}} = \frac{\partial x_{\beta}^{\theta}}{\partial x_{\beta}^{\dot{a}}} \).

**Proof.** Differentiating the transformation equation of the affine connection parameter:

\[
\Gamma_{\beta b\gamma c}^{\alpha a} = \Gamma_{\varepsilon f k}^{\alpha a} X_{\beta}^{\dot{a}} X_{\varepsilon}^{\alpha} X_{\beta}^{\varepsilon} X_{\gamma}^{k} + X_{\beta b\gamma c}^{\theta a} \quad \left( X_{\beta b\gamma c}^{\theta a} \equiv \frac{\partial x_{\beta}^{\theta}}{\partial x_{\beta}^{a}} \right)
\]

\((\alpha-\beta-\gamma)\) times by \textsc{leibnitz} rule and multiplying the generalized binomial coefficient \( \left( \begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array} \right) \), the left-hand member is equal to \( \Gamma_{\beta b\gamma c}^{\alpha a} \), and the first and second terms of the right-hand member become as follows:

\[
\left( \begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array} \right) (\Gamma_{\varepsilon f k}^{\alpha a} X_{\beta}^{\dot{a}} X_{\varepsilon}^{\alpha} X_{\beta}^{\varepsilon} X_{\gamma}^{k}) \equiv \sum_{\varepsilon = r}^{\alpha-\beta} \left( \begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array} \right) \left( \Gamma_{\varepsilon f k}^{\alpha a} X_{\beta}^{\dot{a}} X_{\varepsilon}^{\alpha} X_{\beta}^{\varepsilon} X_{\gamma}^{k} \right)
\]

and
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$\left(\begin{array}{c}
\alpha \\
\beta
\end{array}\right) (X^i_{\delta\tau} X^a_\tau)^{(\alpha - \beta - \gamma)} = \sum_{\delta = a}^{\beta + \gamma} \left(\begin{array}{c}
\alpha \\
\beta
\end{array}\right) X^i_{\delta\tau} (\beta - \gamma) X^a_\tau$

$= \sum_{\delta = a}^{\beta + \gamma} \left(\begin{array}{c}
\beta \\
\gamma
\end{array}\right) X^i_{\delta\tau} (\beta - \gamma) X^a_\tau$

$= \sum_{\delta = a}^{\beta + \gamma} X^i_{\delta\tau} X^a_\tau$

respectively, accordingly (1.2) is obtained.

**Theorem 1.2.** $\Gamma_{\beta \delta \tau \epsilon}$ is symmetric with respect to the doublet indices $\beta \delta$ and $\tau \epsilon$.

**Theorem 1.3.** When the index $\alpha$ of $\Gamma_{\beta \delta \epsilon \tau}$ is equal to zero, both the indices $\beta$ and $\tau$ are also equal to zero and $\Gamma_{\beta \delta \epsilon \tau}$ coincides with $\Gamma_{\beta \delta \epsilon \tau}$.

The two theorems are self-evident from (1.1).

Such the quantities $\Gamma_{\beta \delta \epsilon \tau}$ are called the extended connection parameter in the space with an affine connection.

Now we shall suppose such a continuous family of curves that one and only one curve belonging to it passes through every one point in the considered domain of the space. In order to define the excovariant differential of an extensor field defined on each curve belonging to it, for example, let us consider an extensor $T^{a a}_{\beta \delta \epsilon \tau}$ of the type indicated by the indices (range with respect to $\alpha$ and $\beta$ being $G_1$ and $G_2$ respectively, and of functional order $M$; $G_1, G_2 \leq M$), then the quantities given by

(1.3) $\delta^* T^{a a}_{\beta \delta \epsilon \tau} = dT^{a a}_{\beta \delta \epsilon \tau} + \sum_{\beta \delta \epsilon \tau}^{G_2} \Gamma_{\beta \delta \epsilon \tau}^{a a} T^{a a}_{\beta \delta \epsilon \tau} dx^{(\beta \delta \epsilon \tau)}$

are the components of an extensor of the same kind as the original extensor $T^{a a}_{\beta \delta \epsilon \tau}$, where the displacement $dx^{(\beta \delta \epsilon \tau)}$ means difference of the line elements at any two infinitesimally near points lying on any two infinitesimally near curves belonging to the family respectively. Such the quantities $\delta^* T^{a a}_{\beta \delta \epsilon \tau}$ and the quantities $T^{a a}_{\beta \delta \epsilon \tau}$ defined by

(1.4) $T^{a a}_{\beta \delta \epsilon \tau} = T^{a a}_{\beta \delta \epsilon \tau} + \sum_{\beta \delta \epsilon \tau}^{G_2} \Gamma_{\beta \delta \epsilon \tau}^{a a} T^{a a}_{\beta \delta \epsilon \tau} dx^{(\beta \delta \epsilon \tau)}$

are called the excovariant differential and the excovariant derivative of the extensor $T^{a a}_{\beta \delta \epsilon \tau}$ respectively and when $\delta^* T^{a a}_{\beta \delta \epsilon \tau} = 0$, we say that the
extensor $T^{a b}$ is displaced parallelly. From the definition we can get the following theorems.

**Theorem 1.4.** The excovariant derivative of $T^{a b}$ : $T^{a b}$ is an extensor of the type indicated by the indices, of ranges with respect to $a$, $b$ and $\delta$ being $G_1$, $G_2$ and $M$ respectively, and of functional order $M$.

**Theorem 1.5.** Let $v^a$ be an contravariant extensor of characteristic $(1 + 0, 0, R, R) R \leqq M$, then the following relations are observed

$$\partial v^a = dv^a + \Gamma_{bc}^a v^b dx^c (\equiv \partial v^a)$$

for the tensor member of $v^a$, and $S$ being an integer not exceeding $R$,

$$\delta^a v_z^a = dv_z^a + \sum_{\beta, \tau=0}^{S} \Gamma_{\beta \tau}^a v_{z \tau}^a dx_{(\tau)c} (\equiv \delta^a v_z^a)$$

for the member $v_z^a (z = 0, \cdots, S)$ of range $S$ which is a part of the extensor $v^a (a = 0, \cdots, R)$.

The theorems are verified without difficulty by virtue of (1.1) and (1.3).

**Theorem 1.6.** If $v^a$ is a contravariant vector, then it follows that

$$(\delta^a v^a)^{(a)} = \delta^a v^a.$$

**Proof.** From Theorem 1.5, there exists the relation

$$\delta^a v^a = \partial v^a = dv^a + \Gamma_{\beta a}^b v^b dx^a.$$ Differentiating the above equation $a$ times, we have

$$(\delta^a v^a)^{(a)} = dv^{(a)} + (\Gamma_{\beta a}^b v^b dx)^{(a)}$$

$$= dv^{(a)} + \sum_{\beta, \tau=0}^{S} \left(\frac{\alpha}{\beta} \right) \Gamma_{\beta a}^{(a-\beta+\tau)} v^{(\beta)} b dx^{(\tau)c}$$

and

$$(\delta^a v^a)^{(a)} = \delta^a v^a.$$ in consequence of (1.1) and (1.3).

**Corollary.** If a vector $v^a$ is displaced parallelly, that is, $\partial v^a = 0$, then the extensor $v^a$ induced by differentiating $v^a$ by $t$ is displaced parallelly too.

These facts state that the components of the excovariant differential $\delta^a v^a (a = 0, \cdots, R)$ hold as a part those of the excovariant differential $\delta^a v^a (z = 0, \cdots, S)$ of the subextensor $v^a$ whose components are a part of those of the extensor $v^a$, particularly the excovariant differential of the tensor member becomes the covariant differential of the tensor in the space with an affine connection, and when an extensor $v^a$ is displaced parallelly, that is, $\delta^a v^a = 0$, the tensor member $v^a$ is displaced.
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parallely by means of usual sense. It should be specially noticed that
the connection theory of extensors by means of the above-stated ex-
covariant differential contains the theory referred to tensors in the
space of the affine connection as a part.

The similar statements for general extensors of higher order will
be indicated also.

§ 2. The metric extensor of a RIEMANNian space. Let us suppose
a one-parameter system of the metric tensors \( g_{ab}(t) \) along a para-
meterized arc of class \( M: x^a = x^a(t) \), in an \( n \)-dimensional RIEMANNian
space, differentiate successively by \( t \), and construct the quantities \( g_{\alpha a\beta b} \)

\[
M
\begin{align*}
g_{\alpha a\beta b} & \equiv \left( \begin{array}{c} M \\ \alpha \beta \end{array} \right) g_{ab}^{(M-\alpha-\beta)} \\
\alpha, \beta & = 0, \ldots, M,
\end{align*}
\]

then, as proved by CRAIG ([3], p. 336), the quantities \( g_{\alpha a\beta b} \) are the com-
ponents of an extensor of characteristic \((0+2,0,M,M)\). We shall call
such the quantities \( g_{\alpha a\beta b} \) the components of the metric extensor in the
RIEMANNian space of \( M \)-th order line-elements: \( R^{(M)}_n \). The properties
of \( g_{\alpha a\beta b} \) are stated by the following theorems.

**Theorem 2.1.** The metric extensor \( g_{\alpha a\beta b} \) is symmetric.

**Proof.** From \( g_{ab} = g_{ba} \) and (2.1), we have \( g_{\alpha a\beta b} = g_{\beta b\alpha a} \).

**Theorem 2.2.** If \( G \) is \((M+1)\) \( n \)-rowed determinant constructed of
\( g_{\alpha a\beta b} \) with respect to the pairs \((\alpha a)\) and \((\beta b)\), then

\[
G \equiv [g_{\alpha a\beta b}] = |g_{ab}|^{M+1} \neq 0.
\]

**Proof.** The statement follows from the fact that \( g_{\alpha a\beta b} = 0 \) for \( \beta > M - \alpha \) and \( g_{M-\beta a\beta b} = g_{ab} \) for \( \beta = 0, 1, \ldots, M \) in consequence of (2.1).

Here we can define such the unique reciprocal excontravariant metric
extensor \( g_{\alpha a\beta b}^M \) of the excovariant metric extensor \( g_{\alpha a\beta b}^M \) that satisfies the
following equation:

\[
M
\begin{align*}
g_{\alpha a\beta b} g_{\alpha a\gamma c}^M & = \delta^\gamma_b \delta^\delta_a, \\
\end{align*}
\]

where symbol \( \delta \) denotes KRONECKER delta. The structure of \( g_{\alpha a\beta b}^M \) is given
by the next theorem:

**Theorem 2.3.** The quantities \( \left[ \begin{array}{c} \alpha \\ \beta \\ M \end{array} \right] g^{(\alpha + \beta - M)} \) that are the components
of an extensor \([3], p. 335\) are equal to \(g^{aa\beta b}_{M}\), where

\[
\begin{bmatrix}
\alpha & M \\
\beta & M
\end{bmatrix} = \left(M - \alpha M - \beta\right) \left(M^{-1}\right)^{-1} \left(M^{-1}\right) = 0
\] for \(\alpha + \beta - M \geq 0\)

\[= 0\] for \(\alpha + \beta - M < 0\).

Proof. Putting \(\tilde{a}_{M}^{aa\beta b} \equiv \left[a_{M}^{\alpha\beta}\right] g^{aa\beta b}_{M}(\alpha + \beta M)\), we get

\[
\sum_{\mathfrak{U}}^{M} a_{\tau c}^{aa\beta b} = \sum_{\alpha \Rightarrow Q}^{M} \left(\alpha + \beta M\right) g^{ab_{\gamma}^{(\beta - \gamma)}} g_{ac}^{(M - \beta - \gamma)}
\]

(putting \(\mu = \alpha + \beta - M\)).

consequently \(\tilde{a}_{M}^{aa\beta b} = g^{aa\beta b}_{M}\), since \(g^{aa\beta b}_{M}\) is determined uniquely by equation (2.2).

If we look for the relations between the metric extensor \(g_{aa\beta b}^{R}\) (or \(g^{aa\beta b}_{M}\)) defined by (2.1) in the space of \(R\)-th order line-elements and \(g_{aa\beta b}^{R}\) (or \(g^{aa\beta b}_{M}\)) of \(R_{R}^{(M)}\), \(R\) being any fixed integer not exceeding \(M\), we can find the following theorem:

Theorem 2.4. The following relation holds good:

\[
(M + \sigma)_{M}^{R} g_{\lambda a M - R + \sigma b} = (M + \sigma)_{R}^{M} g_{\lambda a \sigma b} \quad \sigma, \lambda = 0, 1, \cdots, R.
\]

Proof. From (2.1), we obtain that \(g_{\lambda a M - R + \sigma b}^{M} = (M + \sigma)_{R}^{M} g_{\lambda a \sigma b}^{M} \), from which the statement follows at once.

Theorem 2.5. There exists the relation

\[
\frac{M!}{M - R + \sigma!} g^{aaM - R + \sigma b}_{M} = \frac{R!}{\sigma!} \sigma^{aaM_{R}} g^{aa\sigma b}_{R} \quad \alpha, \sigma = 0, 1, \cdots, R.
\]

We define the quantities \(v^{aa}_{b} \equiv g^{aa\beta b}_{M} v^{a}_{b}\) determined by components of an extensor \(v^{aa}\) of characteristic \((1 + 0, 0, M, G)\) as the excovariant components of the extensor, then we have

Theorem 2.6. If \(v_{a}\) are covariant components of a contravariant vector \(v^{a}\), i.e., \(v_{b} = g_{ba} v^{a}\), then the excovariant components of the extensor \(v^{a}_{b}\) are equal to the quantities \(\left(M\beta\right) v_{b}^{(M - \beta)}\) that are the components of an excovariant
extensor in virtue of the theorem of Kawaguchi ([4], p. 22), that is,

\[ v_{\beta b} \equiv g_{\alpha a \beta b} v^{(a)a} = \left( \frac{M}{\beta} \right) v_{b}^{(M-\beta)}. \]

**Proof.** Differentiating the equation: \( g_{ba} v^{a} = v_{b} \) \((M - \beta)\) times by Leibnitz rule and multiplying the binomial coefficient \( \left( \frac{M}{\beta} \right) \) to the both members of the resulting equation, we get \( g_{aa \beta b} v^{(a)a} = \left( \frac{M-M}{\beta} \right) v_{b}^{(M-\beta)}. \)

**Corollary 1.** The following relation consists: \( v_{\beta b} g^{aa \beta b} = v^{(a)a}. \)

**Corollary 2.** It follows that \( \sum_{\beta = 0}^{M} \left( \frac{M}{\beta} \right) v_{b}^{(M-\beta)} g^{aa \beta b} = v^{(a)a}. \)

Let \( v^{a} \) be the components of an excontravariant extensor in \( R_{n}^{M} \) and consider the quantity \( g_{aa \beta b} v^{a} v^{b} \), then \( g_{aa \beta b} v^{a} v^{b} \) is a scalar. The square root of the scalar \( (g_{aa \beta b} v^{a} v^{b})^{2} \) is called the length of the excontravariant extensor \( v^{a} \), and the scalar: \( g_{aa \beta b} v^{a} v^{b} (g_{aa \beta b} v^{a} v^{b})^{-\frac{1}{2}} \times (g_{aa \beta b} v^{a} v^{b})^{-\frac{1}{2}} \) formed by any two excontravariant extensors \( v_{1}^{a} \) and \( v_{2}^{a} \) of the same kind is called the cosine of the angle between the two extensors. Further we define the length \( T \) of a general extensor \( T^{a_{1}a_{1} \cdots a_{A}a_{A}}_{\beta_{1}b_{1} \cdots \beta_{B}b_{B}} \) with the different ranges \( G_{r} \) and \( G_{s} \) for the indices \( a_{r} \) and \( \beta_{s} \) and the angle \( \theta \) between two extensors \( T_{1}^{a_{1}a_{1} \cdots a_{A}a_{A}}_{\beta_{1}b_{1} \cdots \beta_{B}b_{B}} \) and \( T_{2}^{a_{1}a_{1} \cdots a_{A}a_{A}}_{\beta_{1}b_{1} \cdots \beta_{B}b_{B}} \) of the same kind as follows:

\[
T^{2} = g_{a_{1}a_{1},T_{1}^{a_{1}a_{1}} \cdots T_{A}^{a_{A}a_{A}}_{G_{1}^{T_{1}}}} g_{b_{1}b_{1},T_{1}^{b_{1}b_{1}} \cdots T_{B}^{b_{B}b_{B}}_{G_{1}^{T_{2}}}} ^{T_{a_{1}a_{1} \cdots a_{A}a_{A}}_{\beta_{1}b_{1} \cdots \beta_{B}b_{B}} \Gamma_{1}^{\tau_{1}} \cdots \Gamma_{A}^{\alpha_{A}}_{\delta_{1} \cdots \delta_{B}^{T_{1}}}} \times T_{1}^{a_{1}a_{1} \cdots a_{A}a_{A}}_{\beta_{1}b_{1} \cdots \beta_{B}b_{B}} \Gamma_{1}^{\tau_{1} \cdots \tau_{A}^{\alpha_{A}}_{\delta_{1} \cdots \delta_{B}^{T_{2}}}}
\]

and

\[
\cos \theta = \frac{1}{T_{1}T_{2}} g_{a_{1}a_{1},T_{1}^{a_{1}a_{1}} \cdots T_{A}^{a_{A}a_{A}}_{G_{1}^{T_{1}}}} g_{b_{1}b_{1},T_{1}^{b_{1}b_{1}} \cdots T_{B}^{b_{B}b_{B}}_{G_{1}^{T_{2}}}} ^{T_{a_{1}a_{1} \cdots a_{A}a_{A}}_{\beta_{1}b_{1} \cdots \beta_{B}b_{B}} \Gamma_{1}^{\tau_{1}} \cdots \Gamma_{A}^{\alpha_{A}}_{\delta_{1} \cdots \delta_{B}^{T_{1}}}} \times T_{1}^{a_{1}a_{1} \cdots a_{A}a_{A}}_{\beta_{1}b_{1} \cdots \beta_{B}b_{B}} \Gamma_{1}^{\tau_{1} \cdots \tau_{A}^{\alpha_{A}}_{\delta_{1} \cdots \delta_{B}^{T_{2}}}}
\]

respectively. By means of this definition, the following theorem is stated:

**Theorem 2.7.** If \( v \) is the length of an excontravariant extensor \( v^{a} \), then we get: \( v^{2} = g^{a} v_{aa} v_{bb} \).

**Theorem 2.8.** When \( v \) is the length of a contravariant vector \( v^{a} \), the relation:
\[
\frac{d^M v^2}{dt^M} = g_{a a \beta b} v^{(\alpha) a} v^{(\beta) b}
\]

holds good.

Proof. Differentiating the equation \( v^2 = g_{ab} v^a v^b \) \( M \) times by Leibnitz rule, the above equation is obtained.

§ 3. Extended Christoffel symbol in the Riemannian space. Denoting Christoffel symbol of the first kind and of the second kind by means of \( g_{aa \beta b} \) and \( g^{aa \beta e} \) by

\[
\Gamma_{\beta b \gamma c, a a}^M = \frac{1}{2} \left( g_{a a \gamma e; (\beta)b} + g_{\beta b \gamma e; (\beta)c} - g_{\beta b \gamma e; (\alpha)a} \right)
\]

and

\[
\Gamma^{aa \beta e}_{\beta b \gamma c} = \sum_{\delta=0}^{M} g^{aa \gamma e} \Gamma_{\beta b \gamma c, \delta e}^M
\]

respectively, we have the following theorems:

Theorem 3.1. Christoffel symbol of the second kind \( \Gamma_{\beta b \gamma c}^M \) coincides just with the extended connection parameter: \( \left( \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right) \Gamma_{\beta c}^{\alpha a} (a - \beta - \gamma) \), where \( \Gamma_{\beta c}^{\alpha a} \) indicates Christoffel symbol of the second kind by means of \( g_{ab} \) and \( g^{ae} \).

Proof. At first let us calculate the first term in the right-hand member of (3.2), that is,

\[
\Gamma_{\beta b \gamma c}^{aa} = \frac{1}{2} \sum_{\delta=0}^{M} g^{aa \gamma e} (g_{\delta e \gamma c; (\beta)b} + g_{\beta b \delta e; (\gamma)c} - g_{\beta b \gamma e; (\alpha)a})
\]

then we have

\[
\sum_{\delta=0}^{M} g^{aa \gamma e} g_{\delta e \gamma c; (\beta)b} = \sum_{\delta=0}^{M} \left( \begin{array}{c} \alpha \delta \\ M \end{array} \right) g^{ae (a + \delta - M)} (M - \delta - \gamma) g_{ec; \beta b} (M - \beta - \gamma) = \sum_{\delta=0}^{M} \left( \begin{array}{c} \alpha \delta \\ M \end{array} \right) g^{ae (a + \delta - M)} g_{ec; \beta b} (M - \beta - \gamma)
\]

and putting \( \mu = \alpha + \beta - M \),

\[
\sum_{\delta=0}^{M} g^{aa \gamma e} g_{\delta e \gamma c; (\beta)b} = \sum_{\mu=0}^{a - \beta - M} \left( \begin{array}{c} \alpha - \beta - \gamma \\ \mu \end{array} \right) (\alpha - \beta - \gamma) g^{ae (\mu)} g_{ec; \beta b} (a - \beta - \gamma)
\]

from the reason that \( \left( \begin{array}{c} \alpha \delta \\ M \end{array} \right) (M - \beta - \gamma) = \left( \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right) (a - \beta - \gamma) \). Similarly the second and third terms in the right-hand member of (3.2) become as follows:
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\begin{align*}
\sum_{\delta=0}^{M} g^{a\delta e}_{b\delta e (\gamma)c} (g^{ae} g_{bc})^{(a-\gamma-b)}
\end{align*}
and

\begin{align*}
\sum_{\delta=0}^{M} g^{a\delta e}_{b\delta e (\beta)c} (g^{ae} g_{bc})^{(a-\beta)}
\end{align*}
respectively. Consequently we get

\begin{align*}
\Gamma_{\beta b}^{aa} = (\alpha_{\beta}) (g^{ae} g_{bc})^{(a-\beta)}
\end{align*}

Theorem 3.2. If \( R \) is an integer not exceeding \( M \) and if \( \Gamma_{aa\beta b,\sigma c}^{R} \) is the CHRISTOFFEL symbol of the first kind by means of \( g_{aa\beta b} \), then we see that

\begin{align*}
\left( \begin{array}{ll} M+ & \sigma \\
R & \end{array} \right) \Gamma_{aa\beta b,\sigma c}^{R} = \left( \begin{array}{ll} M+ & \sigma \\
M & \end{array} \right) \Gamma_{aa\beta b,\sigma c}^{R}
\end{align*}
\( \alpha, \beta, \sigma = 0, \cdots, R \).

Proof. In virtue of Theorem 2.4, we can find

\begin{align*}
g_{aaM-R+\sigma c;\beta b}^{R} &= \left( \begin{array}{ll} M+ & \sigma \\
R & \end{array} \right) \left( \begin{array}{ll} M+ & \sigma \\
M & \end{array} \right) g_{aa\mathcal{O}c;\beta b}^{R}
\end{align*}
for \( \beta \leq R \)

\begin{align*}
= 0
\end{align*}
for \( \beta > R \),

\begin{align*}
g_{M-R+0c\beta b_{1}(a)a}^{R} &= \left( \begin{array}{ll} M+ & \sigma \\
R & \end{array} \right) \left( \begin{array}{ll} M+ & \sigma \\
M & \end{array} \right) g_{aa\mathcal{O}c;\beta b_{1}(a)a}^{R}
\end{align*}
for \( \alpha \leq R \)

\begin{align*}
= 0
\end{align*}
for \( \alpha > R \)

and

\begin{align*}
g_{aa\beta b;M-R+\sigma c}^{R} &= \left( \begin{array}{ll} M+ & \sigma \\
\alpha \beta & \end{array} \right) g_{ab}^{(M-a-R)}(M-R+\sigma)c
\end{align*}
for \( \sigma \leq R \)

\begin{align*}
= \left( \begin{array}{ll} M+ & \sigma \\
\alpha \beta & \end{array} \right) g_{ab}^{(M-a-R)}(M-R+\sigma)c
\end{align*}
for \( \sigma > R \),

consequently, it follows that

\begin{align*}
\left( \begin{array}{ll} M+ & \sigma \\
R & \end{array} \right) \Gamma_{aa\beta b,\sigma c}^{R} = \left( \begin{array}{ll} M+ & \sigma \\
M & \end{array} \right) \Gamma_{aa\beta b,\sigma c}^{R}
\end{align*}

Theorem 3.3. There exists the relation

\begin{align*}
\Gamma_{aa\beta b}^{R} = \sum_{\sigma=M-R}^{M} g^{\sigma e}_{\sigma} \Gamma_{aa\beta b,\sigma e}^{M}
\end{align*}
\( \alpha, \beta, \gamma = 0, \cdots, R \).
$\Gamma_{ab}^{a}R$ being CHRISTOFFEL symbol by means of $g_{ab}$ and $g^{ac}a_{c}$.

The theorem is verified without difficulty from Theorem 2.5 and Theorem 3.2.

Corollary 1. If $\Gamma_{ab}^{a}$ is CHRISTOFFEL symbol by means of the metric tensor $g_{ab}$, then

$$\Gamma_{ab}^{a} = g^{ac}M \Gamma_{ac}^{a}M \quad \alpha, \beta, \gamma = 0, (M: not\ summing).$$

In virtue of the theorem, we know that the quantities $\sum_{c=M-R}^{M}g^{ac}M \Gamma_{ac}^{a}M$ may be adopted as connection parameter for a member of range $R$ of an extensor and especially $g^{ac}M \Gamma_{ac}^{a}M (M: not\ summing)$ as that for a tensor member. Such a set of $(M + 1)$ CHRISTOFFEL symbols in $R^{M}$, i.e.,

$$\sum_{c=M-R}^{M}g^{ac}M \Gamma_{ac}^{a}M = \Gamma_{ac}^{a} \quad R=0, \ldots, M$$

is called extended CHRISTOFFEL symbol.

§ 4. The excovariant differentiation of the RIEMANNian space.

We shall adopt the definition of excovariant differential of an extensor given by (1.1) in §1. Then we have the following theorems:

Theorem 4.1. The extensor $g_{ab}$, $g^{ab}$ $(R = 0, \ldots, M)$ and $\delta^{*}d^{a}$ behave as constants in excovariant differentiation.

Theorem 4.2. If $T^{a_{1}a_{2}\ldots a_{A}a_{A_{B}}B_{1}\ldots B_{B}}$ is an extensor of range $R$, its excovariant differential $\delta^{*}T^{a_{1}a_{2}\ldots a_{A}a_{A_{B}}B_{1}\ldots B_{B}}$ by means of $\Gamma_{ab}^{a}$ is equal to $\delta^{*}T^{a_{1}a_{2}\ldots a_{A}a_{A_{B}}B_{1}\ldots B_{B}}$.

Corollary. When $T^{a_{1}a_{2}\ldots a_{A}a_{A_{B}}B_{1}\ldots B_{B}}$ is a tensor, the well known covariant differential $\delta T^{a_{1}a_{2}\ldots a_{A}a_{A_{B}}B_{1}\ldots B_{B}}$ in the RIEMANNian space is nothing but $\delta^{*}T^{a_{1}a_{2}\ldots a_{A}a_{A_{B}}B_{1}\ldots B_{B}}$.

Theorem 4.3. If $v^{a}$ and $v^{a}$ are two extensors of range $R$ satisfying $\delta^{*}v^{a} = 0$ and $\delta^{*}v^{a} = 0$, that is, the extensors are displaced parallelly to themselves, then it follows that $\delta^{*}v = 0$ and $\delta^{*}v = 0$, where $v$ and $\theta$ mean the length of $v^{a}$ and the angle between $v^{a}$ and $v^{a}$ respectively.

Theorem 4.4. If $v$ is the length of a vector $v^{a}$ and if $v^{a}$ is displaced parallelly, then the extensor $v^{(a)}$ $(R = 0, \ldots, R)$ formed by differentiating the vector $v^{a}$ times is displaced parallelly and the scalar $\frac{d^{*}v^{a}}{dt^{a}}$ $(R = 0, \ldots, R)$ does not change.
The theorem is proved by virtue of Theorem 1.6, Theorem 2.8 and Theorem 3.1.

**Theorem 4.5.** If $T^{a_1, a_2, \ldots, a_m, \beta_1, \beta_2, \ldots, \beta_{M}}$ is any general extensor with different range for each Greek index, $\delta^* T^2 = 0$ follows from $\delta^* T^{a_1, a_2, \ldots, a_m, \beta_1, \beta_2, \ldots, \beta_{M}} = 0$.

**Corollary.** The length of a vector and the angle between two vectors are invariant under the parallel displacement of the vectors.

In consequence of the above-mentioned result, our excovariant differential is the operation with the geometrical meaning such that the scalar $\frac{d^R v^a}{dt}$ resp. $\frac{d^R \cos \theta}{dt}$ $(R = 0, \ldots, M)$ obtained by differentiating the square of the length of a vector $v^a$ resp. cosine of the angle between two unit vectors $v_i^a$ and $v_j^a$ $R$ times becomes invariant under the parallel displacement of the extensors $v^{(a\beta}_a$ and $v^{(a\beta}_a$ of range $R$ and that the operation coincides with the ordinary covariant differential in the Riemannian space for tensor members. Thus it will be very interesting that the connection theory of extensor which holds the already known tensor theory in the Riemannian space as a part of it has been established.

§ 5. The connection of extensor in an Euclidean space. Consider an Euclidean space that is a special one of the Riemann space, then the space has the rectangular cartesian coordinate system $x^a$ in which the metric tensor $g_{ab}$ becomes equal to $\delta_{ab}$, and the metric extensor $g_{a\alpha \beta \beta}$ and $g^{a\alpha \beta \beta}$ are equal to $\delta_{a\alpha \beta \beta}$ and $\delta^{a\alpha \beta \beta}$ respectively, putting $\delta_{a\alpha \beta \beta} = (M\alpha) \delta_{M-a\alpha \beta}$ and $\delta^{a\alpha \beta \beta} = (M\alpha)^{-1} \delta^{M-a\alpha \beta}$. Evidently, we have the following theorems:

**Theorem 5.1.** The following relation holds good:

$$\sum_{a=0}^{M} \delta_{a\beta} \delta^{a\gamma} = \delta^{\gamma}_{\beta}.$$

**Theorem 5.2.** In the cartesian coordinate system, the extended Christoffel symbol $\Gamma^{R}_{\beta \gamma}$ $(R = 0, \ldots, M)$ vanishes identically and the excovariant differential becomes the ordinary differential.

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REFERENCES


