INFINITESIMAL TRANSFORMATION IN A LINE ELEMENT SPACE

By

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Introduction. In a paper on geometry in an n-dimensional space with arc length $s = \int [A_i x''^i + B]^{\frac{1}{p}} = dt$, Prof. A. KAWAGUCHI introduced a line element space with C'-connection. The structure of this space with C'-connection is determined by the *a priori* given functions $\Gamma^i(x,x')$, $C^i_{jk}(x,x')$ and this space has similar properties to those of a Finsler space. At first §1, the out line of known results in this space is explained. Introducing an infinitesimal transformation, then LIE derivatives of various geometric objects are obtained in §2. Next, §3 is devoted to define infinitesimal motions and to find the conditions for which the space admits the infinitesimal motions. And §4 gives an application of the above stated results to an infinitesimal transformation in a space with arc length $s = \int [A_i x''^i + B]^{\frac{1}{p}} = dt$.

The notations and terminology employed here are those of the paper by A. KAWAGUCHI[1] and the book of K. YANO[2].

§1. Space with C′-connection. In a manifold of line elements (x, x'), let us consider the functions $\Gamma^i(x,x')$, $C^i_{jk}(x,x')$ which have the following properties:

[I] The function $\Gamma^i(x,x')$ are (a) homogeneous of degree 2 in the x'^i and (b) $\Gamma^i_{jk} = \dfrac{\partial^2 \Gamma^i}{\partial x'^j \partial x'^k}$ are transformed as parameters of an affine connection;

[II] The functions $C^i_{jk}(x,x')$ are (a) homogeneous of degree -1 in the x'^i and (b) components of a tensor and (c) they satisfy the equations $C^i_{jk}\, x'^j = C^i_{jk}\, x'^k = 0$.

(1) A. KAWAGUCHI: Geometry in an n-dimensional space with arc length $s = \int [A_i x''^i + B]^{1/p} dt$, Trans. A.M.S., 44 (1938), 153–167.

(2) K. YANO: Groups of transformations in generalized spaces, Akademia Press Co. Ltd., Tokyo (1949).

Then the structure of this space is completely determined by virtue of these quantities. In order to obtain the connections of the space, we shall introduce a natural reference $R_i(x,x')$ at the element (x,x') and let (P,P') be the element of support (x,x') considered in the tangent space, where P is a point and P' is a vector. Then we define the displacement of the element of support and that of the natural reference in the following manners;

$$(1)\qquad dP = \delta x^i R_i\,,\quad dP' = \omega^i R_i\,,\quad dR = \omega_i^j R_j\,,$$

where $\delta x^i = dx^i$, $\omega^i = \delta x'^i = dx'^i + \Gamma^i_j dx^j$, $\omega^i_j = \Gamma^i_{jk}dx^k + C^i_{jk}\omega^k$.

The torsion and curvature tensors which are obtained from (1) by exterior differentiations are

$$(2)\qquad \begin{aligned} K^i_{jk} &= \Gamma^i_{j,k} - \Gamma^i_{k,j} + \Gamma^h_j\Gamma^i_{hk} - \Gamma^h_k\Gamma^i_{hj}\,,\\ R^i_{jkl} &= \Gamma^i_{jk,l} - \Gamma^i_{jl,k} + \Gamma^h_{jk}\Gamma^i_{hl} - \Gamma^h_{jl}\Gamma^i_{hk} + \Gamma^h_k\Gamma^i_{jl,(h)} - \Gamma^h_l\Gamma^i_{jk,(h)}\,,\\ {}^*B^i_{jkl} &= \Gamma^i_{jk,(l)} - C^i_{jl,k} + \Gamma^h_{jk}C^i_{hl} - C^h_{jl}\Gamma^i_{hk} + C^i_{jh}\Gamma^h_{kl} + \Gamma^h_k C^i_{jl,(h)}\,.\\ P^i_{jkl} &= C^i_{jk,(l)} - C^i_{jl,(k)} + C^h_{jk}C^i_{hl} - C^h_{jl}C^i_{hk}\,,\\ {}^*R^i_{jkl} &= R^i_{jkl} + C^i_{jh}K^h_{kl}\,. \end{aligned}$$

And Ricci and Bianchi's identities are

$$(3)\qquad \begin{aligned} &R^i_{[jkl]} = 0\,,\quad K^i_{[jk/l]} = 0\,,\quad {}^*R^i_{j[kl/h]} + {}^*B^i_{j[k|m|}K^m_{lh]} = 0\,,\\ &2{}^*B^i_{j[l|h|k]} + {}^*R^i_{jkl/(h)} + P^i_{jhm}K^m_{kl} + 2{}^*R^i_{j[k|m|}C^m_{l]h} = 0\,,\\ &2{}^*B^i_{jh[k/(l)]} + P^i_{jkl/h} + 2{}^*B^i_{jhm}C^m_{[kl]} + 2{}^*B^i_{jm[k}C^m_{|h|l]} = 0\,,\\ &P^i_{j[kl/(h)]} + 2P^i_{j[k|m|}C^m_{lh]} = 0\,. \end{aligned}$$

Moreover we can prove the following identities:

$$(4)\qquad \begin{aligned} &K^i_{jk,(h)} = R^i_{hjk}\,,\quad R^i_{jkl}x'^j = K^i_{kl}\,,\quad {}^*B^i_{jkl}x'^j = 0\,,\quad B^i_{jkl} = \Gamma^i_{jk,(l)}\,,\\ &{}^*B^i_{jkl} = B^i_{jkl} - C^i_{jl/k}\,,\quad P^i_{jkl} = C^i_{jk/(l)} + C^h_{kl}C^i_{jh} - C^i_{jl,(k)}\,,\\ &B^i_{j[l|h|/k]} + R^i_{jkl,(h)} = 0\,,\quad R^i_{j[kl/h]} + B^i_{j[k|m|}K^m_{lh]} = 0\,. \end{aligned}$$

§ 2. **Infinitesimal transformation.** Let us consider an infinitesimal transformation

$$(5)\qquad \bar{x}^i = x^i + \xi^i(x)\,\delta t\,,$$

then Lie derivative of a geometric object Ω on the manifold of (x,x') can be defined as usual:

$$(6)\qquad D\Omega = (X\Omega)\,\delta t = \Omega(\bar{x},\bar{x}') - \bar{\Omega}(\bar{x},\bar{x}')\,.$$

A. For a contravariant vector $v^i(x,x')$,

$$Xv^i = v^i_{,h}\xi^h + v^i_{,(h)}\xi^{h'} - \xi^i_{,h}v^h$$

which can be written in

$$(7)\qquad Xv^i = \Delta v^i - \xi^i_{.h}v^h\,,$$

where we assume there that $C^i_{hk}\xi^h$ is equal to zero, that is, $\frac{\delta\xi^i}{dt} = \frac{d\xi^i}{dt} + \Gamma^i_j\xi^j$ and put

$$(8) \qquad \Delta v^i = v^i_{/h}\xi^h + v^i_{/(h)}\frac{\delta\xi^h}{dt}, \quad \xi^i_{\cdot h} = \xi^i_{/h} + C^i_{hk}\frac{\delta\xi^k}{dt} = \xi^i_{,h} + \omega^i_h(\Delta),$$

$$\omega^i_h(\Delta) = \Gamma^i_{hk}\xi^k + C^i_{hk}\frac{\delta\xi^k}{dt}.$$

B. For a scalar field $S(x,x')$, putting $\vartheta\Omega(x,x') = \Omega_{,h}\xi^h + \Omega_{,(h)}\xi^{h'}$, then we have $XS = \vartheta S$. This expression for XS can be written in

$$(9) \qquad XS = \Delta S \quad \text{where} \quad S_{/h} = S_{,h} - \Gamma^k_h S_{,(k)}, \quad S_{/(k)} = S_{,(k)}.$$

We have for the operation Δ, even if $P\cdot Q$ means inner or outer product, $\Delta(P\cdot Q) = \Delta P\cdot Q + P\cdot\Delta Q$. From this we have for a covariant vector w_i and a tensor $T^i_{\cdot j}$

$$(10) \qquad Xw_i = \Delta w_i + \xi^h_{\cdot j}w_h, \quad XT^i_{\cdot j} = \Delta T^i_{\cdot j} - \xi^i_{\cdot h}T^h_{\cdot j} + \xi^h_{\cdot j}T^i_{\cdot h}.$$

C. Lie derivative of a connection ω^i_j is, form the definition (6), given by

$$X\omega^i_j = \vartheta\omega^i_j - \xi^i_{,h}\omega^h_j + \xi^h_{,j}\omega^i_h + d\xi^i_{,j}.$$

On the other hand, it follows from (8) that

$$\xi^i_{,j} = \xi^i_{\cdot j} - \omega^i_j(\Delta), \quad d\xi^i_{,j} = d\xi^i_{\cdot j} - d\omega^i_j(\Delta) = \delta\xi^i_{\cdot j} - \omega^i_h\xi^h_{\cdot j} + \omega^h_j\xi^i_{\cdot h} - d\omega^i_j(\Delta)$$

which gives us

$$(11) \qquad X\omega^i_j = \delta\xi^i_{\cdot j} + \vartheta\omega^i_j(d) - d\omega^i_j(\Delta) + \omega^i_h(\Delta)\omega^h_j(d) - \omega^i_h(d)\omega^h_j(\Delta).$$

The four terms on the right hand member of the last equation have just the same forms dividing by δt as those giving curvature tensors. Hence (11) offers us

$$(12) \qquad X\omega^i_j = \delta\xi^i_{\cdot j} + {}^*R^i_{jkl}dx^k\xi^l + {}^*B^i_{jkl}dx^k\frac{\delta\xi^l}{dt} - {}^*B^i_{jlk}\xi^l\omega^k + P^i_{jkl}\omega^k\frac{\delta\xi^l}{dt}$$

Next we have from (7), $X\omega^i = \Delta\omega^i - \xi^i_{\cdot h}\omega^h$, which can be written in another form from another aspect. In fact, since $X\,dx'^i = 0$ and $X\,x'^i = 0$, an application of Lie derivative to the both members of $\omega^i = dx'^i + \omega^i_j x'^j$ gives $X\omega^i = X\omega^i_j x'^j$ and hence

$$X\omega^i = \delta\xi^i_{\cdot j}x'^j + {}^*R^i_{hkl}x'^h dx^k\xi^l + {}^*B^i_{hkl}x'^h dx^k\frac{\delta\xi^l}{dt} - {}^*B^i_{hlk}x'^h\xi^l\omega^k + P^i_{hkl}x'^h\omega^k\frac{\delta\xi^l}{dt}$$

which reduces

$$(13) \qquad X\omega^i = \left(\frac{\delta\xi^i}{dt}\right)_{/k}dx^k + K^i_{kl}dx^k\xi^l,$$

because of (4) and the assumption $C^i_{jk}\xi^j = 0$.

Defining a deformed object Ω:

$$\overline{\Omega}(x,x') = \Omega(x,x') + D\Omega(x,x'), \tag{14}$$

we can see that the deformed vector of a covariant derivative of a vector with respect to the original connection is equal to the covariant derivative of the deformed vector with respect to the deformed connection, that is to say,

$$\overline{\delta v^i} = \delta v^i + D\delta v^i = d(v^i + Dv^i) + (\omega^i_j + D\omega^i_j)(v^j + Dv^j), \quad \text{or}$$

$$D\delta v^i - \delta D v^i = D\omega^i_j v^j, \quad X\delta v^i - \delta X v^i = X\omega^i_j v^j. \tag{15}$$

Since $X\omega^i = X\Gamma^i_j dx^j$, hence

$$Xv^i_{/j} - (Xv^i)_{/j} = X\Gamma^i_{kj} v^k - X\Gamma^k_j v^i_{,(k)}, \quad Xv^i_{/(j)} - (Xv^i)_{/(j)} = XC^i_{kj} v^k. \tag{16}$$

Consider an arbitrary vector v^i, then we see from (7) and (15)

$$X\omega^i_j v^j = X\delta v^i - \delta X v^i = \delta\xi^i_{\cdot j} v^j + \Delta\delta v^i - \delta\Delta v^i$$

from which we get (12), too.

On the other hand, we see easily from (1)

$$X\omega^i = X\Gamma^i_j dx^j, \quad X\omega^i_j = (X\Gamma^i_{jk} + X\Gamma^h_k C^i_{jh})\, dx^k + XC^i_{jk}\omega^k. \tag{17}$$

Comparing the last equations with (12) and (13), we know

$$X\Gamma^i_j = \left(\frac{\delta\xi^i}{dt}\right)_{/j} + K^i_{jl}\xi^l, \tag{18}$$

$$\begin{aligned} X\Gamma^i_{jk} &= \dot{\xi}^i_{\cdot j/k} + {}^*R^i_{jkl}\xi^l + {}^*B^i_{jkl}\frac{\delta\xi^l}{dt} - C^i_{jh}X\Gamma^h_k, \\ XC^i_{jk} &= \xi^i_{\cdot j/(k)} - {}^*B^i_{jlk}\xi^l + P^i_{jkl}\frac{\delta\xi^l}{dt}. \end{aligned} \tag{19}$$

Moreover the expressions (19) can be rewritten without difficulty into

$$\begin{aligned} X\Gamma^i_{jk} &= \xi^i_{/j/k} + R^i_{jkl}\xi^l + B^i_{jkl}\frac{\delta\xi^l}{dt}, \\ XC^i_{jk} &= \Delta C^i_{jk} - \xi^i_{\cdot l}C^l_{jk} + C^i_{lk}\xi^l_{\cdot j} + C^i_{jl}\xi^l_{\cdot k}. \end{aligned} \tag{20}$$

D. In order to find Lie derivatives of various curvature tensors, let us consider the deformed curvature tensors constructed with the deformed parameters of connections:

$$\overline{R}^i_{jkl} = R^i_{jkl} + DR^i_{jkl}, \quad \overline{K}^i_{jk} = K^i_{jk} + DK^i_{jk}, \quad {}^*\overline{R}^i_{jkl} = {}^*R^i_{jkl} + D{}^*R^i_{jkl},$$

$${}^*\overline{B}^i_{jkl} = {}^*B^i_{jkl} + D{}^*B^i_{jkl}, \quad \overline{P}^i_{jkl} = P^i_{jkl} + DP^i_{jkl},$$

where we must remember (6) for operation D, that is,

$$
(21)\quad
\begin{aligned}
&XK^i_{jk} = (X\Gamma^i_j)_{|k} - (X\Gamma^i_k)_{|j}, \quad X^*R^i_{jkl} = XR^i_{jkl} + XC^i_{jh}K^h_{kl} + C^i_{jh}XK^h_{kl}, \\
&XR^i_{jkl} = (X\Gamma^i_{jk})_{|l} - (X\Gamma^i_{jl})_{|k} + X\Gamma^h_k\Gamma^i_{jl,(h)} - X\Gamma^h_l\Gamma^i_{jk,(h)}, \\
&X^*B^i_{jkl} = (X\Gamma^i_{jk})_{|(l)} - (XC^i_{jl})_{|k} + X\Gamma^i_{jh}C^h_{kl} + X\Gamma^h_{kl}C^i_{jh} + X\Gamma^h_k C^i_{jl,(h)}, \\
&XP^i_{jkl} = (XC^i_{jk})_{|(l)} - (XC^i_{jl})_{|(k)} + 2XC^i_{jh}C^h_{[kl]},
\end{aligned}
$$

then we obtain the following expressions of LIE derivatives of curvature tensors, in virtue of RICCI and BIANCHI's identities (3) and (4) and the equations (17) and (19);

$$
(22)\quad
\begin{aligned}
&\text{a.}\quad XK^i_{jk} = \Delta K^i_{jk} - \xi^i_{\cdot l}K^l_{jk} + K^i_{lk}\xi^l_{\cdot j} + K^i_{jl}\xi^l_{\cdot k}, \\
&\text{b.}\quad XR^i_{jkl} = \Delta R^i_{jkl} - \xi^i_{\cdot h}R^h_{jkl} + R^i_{hkl}\xi^h_{\cdot j} + R^i_{jhl}\xi^h_{\cdot k} + R^i_{jkh}\xi^h_{\cdot l}, \\
&\text{c.}\quad X^*B^i_{jkl} = \Delta^*B^i_{jkl} - \xi^i_{\cdot h}{}^*B^h_{jkl} + {}^*B^i_{hkl}\xi^h_{\cdot j} + {}^*B^i_{jhl}\xi^h_{\cdot k} + {}^*B^i_{jkh}\xi^h_{\cdot l}, \\
&\text{d.}\quad XP^i_{jkl} = \Delta P^i_{jkl} - \xi^i_{\cdot h}P^h_{jkl} + P^i_{hkl}\xi^h_{\cdot j} + P^i_{jhl}\xi^h_{\cdot k} + P^i_{jkh}\xi^h_{\cdot l}; \\
&\text{e.}\quad X^*R^i_{jkl} = \Delta^*R^i_{jkl} - \xi^i_{\cdot h}{}^*R^h_{jkl} + {}^*R^i_{hkl}\xi^h_{\cdot j} + {}^*R^i_{jhl}\xi^h_{\cdot k} + {}^*R^i_{jkh}\xi^h_{\cdot l}.
\end{aligned}
$$

To find the expressions (22) it must be used

for b. $$\left(\frac{\delta\xi^i}{dt}\right)_{|j|k} - \left(\frac{\delta\xi^i}{dt}\right)_{|k|j} = {}^*R^i_{hjk}\frac{\delta\xi^h}{dt} - K^h_{jk}\left(\frac{\delta\xi^i}{dt}\right)_{|(h)},$$

and $$K^i_{jk|l} + K^i_{kl|j} + K^i_{lj|k} = 0,$$

for a. $$\xi^i_{|j|k|l} - \xi^i_{|j|l|k} = R^i_{hkl}\xi^h_{|j} - R^h_{jkl}\xi^i_{|h} - K^h_{kl}\xi^i_{|j,(h)},$$

and $$R^i_{jkl|h} + R^i_{jlh|k} + R^i_{jhk|l} + B^i_{jkm}K^m_{lh} + B^i_{jlm}K^m_{hk} + B^i_{jhm}K^m_{kl} - B^i_{jkh|l} + B^i_{jlh|k} + R^i_{jkl,(h)} = 0,$$

for c. $$\xi^i_{\cdot j|k|(l)} - \xi^i_{\cdot j|(l)|k} = {}^*B^i_{hkl}\xi^h_{\cdot j} - {}^*B^h_{jkl}\xi^i_{\cdot h} - C^h_{kl}\xi^i_{\cdot j|h},$$

$$\left(\frac{\delta\xi^i}{dt}\right)_{|k|(l)} - \left(\frac{\delta\xi^i}{dt}\right)_{|(l)|k} = {}^*B^i_{hkl}\frac{\delta\xi^h}{dt} - C^h_{kl}\left(\frac{\delta\xi^i}{dt}\right)_{|h},$$

and $$\begin{aligned}
&{}^*B^i_{jlh|k} - {}^*B^i_{jkh|l} + {}^*R^i_{jkl|(h)} + P^i_{jhm}K^m_{kl} + {}^*R^i_{jkm}C^m_{lh} - {}^*R^i_{jlm}C^m_{kh} = 0, \\
&{}^*B^i_{jkh|(l)} - {}^*B^i_{jkl|(h)} + P^i_{jhl|k} + 2{}^*B^i_{jkm}C^m_{[hl]} + {}^*B^i_{jmh}C^m_{kl} - {}^*B^i_{jml}C^m_{kh} = 0,
\end{aligned}$$

for d. $$\xi^i_{\cdot j|(k)|(l)} - \xi^i_{\cdot j|(l)|(k)} = P^i_{hkl}\xi^h_{\cdot j} - P^h_{jkl}\xi^i_{\cdot h} - 2C^h_{[kl]}\xi^i_{\cdot j|(h)},$$

and $$P^i_{jkl|(h)} + P^i_{jlh|(k)} + P^i_{jhk|(l)} + 2P^i_{jkm}C^m_{[lh]} + 2P^i_{jlm}C^m_{[hk]} + 2P^i_{jhm}C^m_{[kl]} = 0,$$

and e. can be verified immediately.

§ 3. **Infinitesimal motion.** Consider an infinitesimal transformation $\bar{x}^i = x^i + \xi^i(x)\,\delta t$ in the space with connections (ω^i, ω^i_j), then we have a deformed space with connections $(\omega^i + D\omega^i,\ \omega^i_j + D\omega^i_j)$.

The infinitesimal transformation which lets vanish LIE derivative of the base connection, that is $D\omega^i = 0$, is called *an infinitesimal affine collineation.* The infinitesimal affine collineation for which $D\omega^i_j = 0$ is called *an infinitesimal C'-motion.* The geometrical meaning of an infinitesimal C'-motion is clearly that the original space and the deformed one obey to the same law of displacement, that is, the both spaces have the same

connections.

Moreover, $X\omega^i = X\Gamma^i_j dx^j = 0$ gives $X\Gamma^i_j = 0$ and $X\omega^i_j = 0$ leads us to $X\Gamma^i_{jk} = 0$, $XC^i_{jk} = 0$ in use of (17). Thus we have the

Theorem I. *In order that an infinitesimal transformation be an infinitesimal C'-motion, it is necessary and sufficient that* Lie *derivatives of connection parameters Γ^i_{jk}, C^i_{jk} of the space vanish at the same time.*

And from (15) it can be concluded that if the operator of Lie derivative is commutable, the infinitesimal transformation must be an infinitesimal C'-motion.

Now we shall proceed to consider the integrability conditions of the differential equations $X\omega^i_j = 0$. For convenience's sake, we put the somewhat restrictive condition $\xi^i_{,(j)} = \xi^i_{/(j)} = 0$. To find suitable forms, we shall deform the equation $X\omega^i = 0$ into

$$(23) \qquad X\omega^i = \delta\left(\frac{\delta\xi^i}{dt}\right) + K^i_{jl}\xi^l dx^j - \left(\xi^i_{.j} - 2C^i_{[jl]}\frac{\delta\xi^l}{dt}\right)\omega^j = 0,$$

and, according to the condition $\xi^i_{/(j)} = 0$, the equation $X dx^i = 0$ into

$$(24) \qquad X dx^i = \delta\xi^i - \left(\xi^i_{.j} - C^i_{jl}\frac{\delta\xi^l}{dt}\right)dx^j = 0.$$

Regarding (23), (24) and the equation that (12) is equal to zero as a system of compatible differential equations in ξ^i, $\frac{\delta\xi^i}{dt}$ and $\xi^i_{.j}$, we get its integrability conditions: $(X\omega^i)' = 0$, $(X\omega^i_j)' = 0$ and $(X dx^i)' = 0$ whose tensor forms are

$$(25) \qquad \begin{aligned} &(X\omega^i_j)' + [\omega^i_h X\omega^h_j] + [\omega^h_j X\omega^i_h] = 0, \\ &(X\omega^i)' + [\omega^i_h X\omega^h] = 0, \text{ and } (X dx^i)' + [\omega^i_h X dx^h] = 0. \end{aligned}$$

Calculating the first equation of the above, we have from (17) and (21),

$$(26) \quad \tfrac{1}{2} X^* R^i_{jkl} - X\Gamma^h_k {}^*B^i_{jhl} = 0, \quad X^* B^i_{jkl} + X\Gamma^h_k P^i_{jhl} = 0, \quad XP^i_{jkl} = 0.$$

From the second,

$$(27) \qquad XK^i_{jk} = 0, \quad X\Gamma^i_{jk} + X\Gamma^h_j C^i_{hk} = 0.$$

And the third gives no condition, for it holds identically.

(26) and (27) are moreover not independent of each others, that is, $X\omega^i = 0$ follows from $X\omega^i_j = 0$. (26) and (27) are therefore reduced to $X^* R^i_{jkl} = 0$, $X^* B^i_{jkl} = 0$ and $XP^i_{jkl} = 0$. But these are still not independent. On the reason of $XR^i_{hkl}x'^h = XK^i_{kl}$ and (21), $X^* R^i_{jkl} = 0$ is reduced to $XR^i_{jkl} = 0$ and $XK^i_{jk} = 0$ is neglegible. From (16) it follows that

$$XC^i_{jl/k} - (XC^i_{jl})_{/k} = X\Gamma^i_{hk}C^h_{jl} - X\Gamma^h_{jk}C^i_{hl} - X\Gamma^h_{lk}C^i_{jh} - X\Gamma^h_k C^i_{jl,(h)}$$

and then $XC^i_{jl|k}=0$. And also since $X^*B^i_{jkl}=XB^i_{jkl}-XC^i_{jl|k}$, then $X^*B^i_{jkl}=0$ is neglegible[3]. Therefore, instead of (26) and (27), the integrability conditions of $X\omega^i_j=0$ and (23) and (24) can be stated by

$$(28)\qquad XR^i_{jkl}=0\,,\quad XP^i_{jkl}=0\,.$$

Followingly, we have from (16)

$$(29)\quad XR^i_{jkl|h}=0\,,\quad XP^i_{jkl|h}=0\,,\quad XR^i_{jkl|(h)}=0\,,\quad XP^i_{jkl|(h)}=0\,.$$

Repeating this process, we can see that LIE derivatives of all curvature tensors and their covariant derivatives of two kinds must be all equal to zero. Hence we have the

Theorem II. *In order that this space admits an infinitesimal C'-motion, it is necessary and sufficient that there exists a positive integer N such that the first N sets of the equations (28), (29), ····· be compatible in ξ^i, $\frac{\delta\xi^i}{dt}$ and $\xi^i_{.j}$ of which all solutions satisfy the $(N+1)$-st set of equations.*

Under any infinitesimal affine collineation, the law of displacement of P, F' in the deformed space is the same as that of the original, but that of a natural reference R_i is $dR_i=(\omega^j_i+D\omega^j_i)\,R_j$, where $X\omega^i_j=XC^i_{jk}\omega^k$. Hence an infinitesimal affine collineation which satisfies the equation $XC^i_{jk}=0$ becomes an infinitesimal C'-motion.

For the admittability of an infinitesimal affine collineation, since $XB^i_{jkl}=(X\Gamma^i_{jk})_{,(l)}$, the equations (23), (24) and the following relation must be compatible; $X\omega^i_j=XC^i_{jk}\omega^k$. Then the integrability conditions are expressible evidently from the above stated results by

$$\tfrac{1}{2}X^*R^i_{jkl}-X\Gamma^{h*}_{k}B^i_{jhl}=\tfrac{1}{2}XC^i_{jh}K^h_{kl}\,,\quad X^*B^i_{jkl}+X\Gamma^h_kP^i_{jhl}=-XC^i_{jl|k}\,,$$
$$XP^i_{jkl}=XP^i_{jkl}\ \text{and}\ XK^i_{jk}=0\,,\quad X\Gamma^i_{jk}+X\Gamma^h_jC^i_{hk}=0\,.$$

It can bee seen that the independent conditions of the last are only

$$(30)\qquad XK^i_{jk}=0^{(4)}.$$

From (4) and (16) it follows that

$$(31)\qquad XK^i_{jk|l}=0\,,\ \text{and}\ XK^i_{jk,(l)}=XR^i_{ljk}=0\,.$$

Repeating this process, we can see that LIE derivatives of tensor K^i_{jk} and its covariant derivatives by x^i and its partial derivatives by x'^i must be all equal to zero. Hence we have the

(3) $XB^i_{jkl}=(X\Gamma^i_{jk})_{,(l)}$ can be verified. Although this must require some difficult calculation, lemma II in §4 gives to this easily proof.

(4) It can be seen from (4) and lemma II in §4 that $XK^i_{jk}=0$ is equivalent to $XR^i_{jkl}=0$.

Theorem III. *In order that this space admits an infinitesimal affine collineation, it is necessary and sufficient that there exists a positive integer N, such that the first N sets of the equations* (30), (31), $\cdots\cdots$ *be compatible in* ξ^i, $\frac{\delta\xi^i}{dt}$ *and* $\xi^i_{\cdot j}$ *of which all solutions satisfy the* $(N+1)$*-st set of the equations.*

§ 4. Infinitesimal motion in the space with arc length $s = \int [A_i x''^i + B]^{\frac{1}{p}} dt$. As the arc length should remain unaltered by a transformation of parameter t, A_i must be a vector and homogeneous of degree p-2 in the x'^i and B is homogeneous of degree p in the x'^i and is transformed as $\bar{B}(\bar{x}\bar{x}') = B(xx') - A_i \frac{\partial x^i}{\partial \bar{x}^a} \frac{\partial \bar{x}^a}{\partial x^j \partial x^k} x'^j x'^k$ under a point transformation.

By means of these A_i and B, Prof. A. Kawaguchi defines Γ^i and C^i_{jk} as follows ;

$$2\Gamma^l = (2A_{ik}x'^k - B_{(i)})\, G^{il}\,,$$

$$C^i_{jk} = \frac{G^{li}}{p-3}\left\{A_{l(j)(k)} + A_{k(l)(j)} + (p-3)A_{j(l)(k)}\right\} + \frac{G^{li}_{\cdot\cdot(k)}}{(p-3)^2}\left\{A_{l(j)} + (p-2)\,A_{j(l)}\right\} \qquad \text{for} \quad p \neq 3\,,$$

$$= G^{li}_{\cdot\cdot(k)}\,\{A_{l(j)} + A_{j(l)}\} \qquad \text{for} \quad p = 3\,,$$

where $\quad G_{ij} = 2A_{i(j)} - A_{j(i)}\,,\quad G_{ik}G^{il} = \delta^l_k\,,\quad A_{ik} = \frac{\partial A_i}{\partial x^k}\,,\quad B_{(i)} = \frac{\partial B}{\partial x'^{(i)}}\,.$

We can easily verify that these Γ^i and C^i_{jk} satisfy our conditions [I], [II]. We shall hence consider the space with the connections ω^i, ω^i_j defined as (1).

In this space the infinitesimal transformation which satisfies the equation $XF = 0$, where $F = A_i x''^i + B$, does not change an arc length. Next, we shall show that the infinitesimal transformation which satisfies $XF = 0$ becomes an infinitesimal C'-motion defined in §3. In order to show this, we must verify the following two lemmas.

Lemma I. *If $\xi^i(x)$ is a function of x^i only, then $Xx^{(a)i} = 0$,*

where $\quad x^{(a)i} = \frac{d^a x^i}{dt^a}\,, \qquad a = 1, 2, \cdots\cdots, m\,.$

Proof. Definition (6) gives $Xx^{(a)i} = (\bar{x}^i)^{(a)} - \overline{(x^{(a)i})}$. But we have on the other hand

$$(\bar{x}^i)^{(a)} = (x^i + \xi^i(x)\,\delta t)^{(a)} = x^{(a)i} + \xi^{i(a)}\,\delta t\,,$$

$$(\overline{x^{(\alpha)i}}) = \frac{\delta \overline{x}^{(\alpha-1)i}}{\delta x^{(\beta-1)j}} x^{(\beta)j} = \frac{\partial (x^{(\alpha-1)i} + \xi^{i(\alpha-1)}\delta t)}{\partial x^{(\beta-1)j}} x^{(\beta)j} = (\delta^{\alpha}_{\beta}\delta^{i}_{j} + \xi^{i(\alpha-1)}_{,(\beta-1)j}dt)\, x^{(\beta)j}$$
$$= x^{(\alpha)i} + \xi^{i(\alpha)}\,\delta t\,,$$

consequently we have $Xx^{(\alpha)i} = 0$.

Lemma II. *If $\xi^i(x)$ is a function of x^i only, the operator of* LIE *derivative with respect to the infinitesimal transformation and that of partial derivative with respect to $x^{(\alpha)i}$ are commutative.*

Proof. For example, we take an extensor T'^{pi} of type $(1, 0, m, m)$[5]. Then we have from definition (6), $XT'^{pi} = T'^{pi}_{,(\tau)k}\xi^{k(\tau)} - \xi^{i(p)}_{,(\tau)k}T'^{\tau k}$. Differentiating partially the last equation with respect to $x^{(q)j}$,

$$(XT'^{pi})_{,(q)j} = T'^{pi}_{,(\tau)k,(q)j}\xi^{k(\tau)} + T'^{pi}_{,(\tau)k}\xi^{k(\tau)}_{,(q)j} - \xi^{i(p)}_{,(\tau)k,(q)j}T'^{\tau k} - \xi^{i(p)}_{,(\tau)k}T'^{\tau k}_{,(q)j}\,.$$

Since $T'^{pi}_{,(q)j}$ is transformed as $\overline{T}^{pi}_{,(q)j} = X^{(p)i}_{(a)m}T'^{am}_{,(b)n}\overline{X}^{(b)n}_{(q)j} + T'^{(a)m}X^{(p)i}_{(a)m,(c)l}\overline{X}^{(c)l}_{(q)j}$, then

$$XT'^{pi}_{,(q)j} = T'^{pi}_{,(q)j,(\tau)k}\xi^{k(\tau)} - T'^{\tau k}_{,(q)j}\xi^{i(p)}_{,(\tau)k} + T'^{pi}_{,(\tau)k}\xi^{k(\tau)}_{,(q)j} - \xi^{i(p)}_{,(\tau)k,(q)j}T'^{\tau k}\,.$$

Therefore we find $(XT'^{pi})_{,(q)j} = XT'^{pi}_{,(q)j}$.

Now, since $Xx''^i = 0$ follows from lemma I, it is seen that $XF = XA_i x''^i + XB = 0$ which leads to

$$XA_i = 0\,, \qquad XB = 0\,, \tag{33}$$

because XA_i and XB do not contain x''^i. Since from definition (32) of C^i_{jk}, C^i_{jk} is made of A_i and its derivatives only, we must have $XC^i_{jk} = 0$ from lemma II. On the other hand, since Γ^i is made of A_i and B and their derivatives, we see $X\Gamma^i = 0$, and obviously $X\Gamma^i_{jk} = X\dfrac{\partial^2\Gamma^i}{\partial x'^j\partial x'^k}$ $= \dfrac{\partial^2}{\partial x'^j\partial x'^k}X\Gamma^i = 0$. These results lead to the following, by sake of theorem I,

Theorem IV. *The infinitesimal transformation which preserves an arc length is an infinitesimal C'-motion.*

In §3, an infinitesimal affine collineation $X\omega^i = 0$ gives LIE derivative of connection ω^i_j as $X\omega^i_j = XC^i_{jk}\omega^k$, while in §4, we obtain $XC^i_{jk} = 0$ from $XA_i = 0$ only. Hence we have the

Theorem V. *The infinitesimal affine collineation which satisfies the equation $XA_i = 0$ becomes an infinitesimal C'-motion.*

Since we can easily verify $B = 2\Gamma^i A_i$, it must be $XB = 2X\Gamma^i A_i + 2\Gamma^i XA_i$. Then we obtain the

(5) Lemma II holds good for any quantity Ω.

Theorem VI. *The infinitesimal affine collineation which satisfies the equation $XA_i=0$ preserves an arc length.*

In this space there exists a system of paths defined by the differential equations $x''^i + 2\Gamma^i = 0$. We can easily verify the following

Theorem VII. *The necessary and sufficient condition for that an infinitesimal transformation carries paths of this space into those of the same space and preserves the base connection is that the transformation is an infinitesimal affine collineation of this space.*