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A GENERALIZATION OF THE FRENET'S FORMULAE

By

Shimpei YANO

§ 1. Introduction. P. Diénes considered the mixed tensors of rank \( \omega \) \( A_{j_{\nu}}^{f_{\nu}} \) \((x, \dot{x}, \ddot{x}, \ldots)\), whose componentes are functions of a line-element, and applies them to many theorems in his papers\(^{(1)}\). The purpose of the present paper is to show that the Frenet's formulae may take the same form as those found in the Riemannian geometry. At first let us give a brief explanation of notations and contractions according to P. Diénes. The absolute derivatives of a contravariant vector \( A^{i} \) or of a covariant vector \( B_{i} \) along a curve, be expressed by the following forms

\[
\frac{\delta A^{i}}{\delta t} = \frac{dA^{i}}{dt} + f_{i}^{\rho}A^{\rho}, \quad \frac{\delta B_{i}}{\delta t} = \frac{\dot{d}B_{i}}{dt} + f_{i}^{\rho}B_{\rho},
\]

where the connection parameters \( f_{i}^{\rho} \) and \( f_{\rho}^{i} \) are functions of the parameter of the curve and have no relation among themselves. Then the absolute derivative of the tensor \( A_{j_{\nu}}^{f_{\nu}} \) along the curve may by expressed in the form

\[
\frac{\delta A_{j_{\nu}}^{f_{\nu}}}{\delta t} = \frac{dA_{j_{\nu}}^{f_{\nu}}}{dt} + \sum_{i=1}^{\omega} A_{j_{\nu}i}^{f_{\nu}} f_{i}^{\rho} f_{\rho}^{j_{\nu}},
\]

where only one of the two functions \( f_{i}^{\rho} \) and \( f_{\rho}^{j_{\nu}} \) in every term is equal to 1.

The space in which absolute derivatives of tensors are of the form (1) is called a monadromic space. In the monodromic space, the absolute differentiation of the sum or the product of two tensors \( A \) and \( B \) follows to the ordinarily established formal rules,

\[
\frac{\delta (A + B)}{\delta t} = \frac{\delta A}{\delta t} + \frac{\delta B}{\delta t}, \quad \frac{\delta AB}{\delta t} = \frac{\delta A}{\delta t} B + A \frac{\delta B}{\delta t}.
\]

There are two kinds of metric tensors, one of which is the covariant tensor \( g_{ij} \) and the other the contravariant \( g^{ij} \). These tensors are related to each other as \( g_{ij}g^{\rho k} = E_{j}^{k} \), where \( E_{j}^{k} = 0 \) if \( k \neq j \) and \( = 1 \) if \( k = j \). If \( A_{i}^{j} \) are the components of a tensor \( A \), the length of the tensor may be defined and written as

\[
g_{\mu \nu}g^{\alpha \beta}A_{\alpha}^{\mu}A_{\beta}^{\nu} = \left| -g_{\mu \nu}g^{\alpha \beta}A_{\alpha}^{\mu}A_{\beta}^{\nu} \right| = \left| AA \right| = \left| A \right|^2.
\]
Similarly, if \( A_j \) and \( B_j \) are components of two tensors \( A \) and \( B \), whose lengths are \(|A|\) and \(|B|\) respectively, \( \theta \) being the angle between these tensors, we have

\[
|A||B| \cos \theta = -AB.
\]

A prominent property of this space is that there is the following relation between an absolute differentiation and a contraction:

\[
\frac{\delta A^\alpha_j}{\delta t} = \frac{\delta A^\alpha_j}{\delta t} - \{\delta A^\alpha_j\} \cdot \frac{\delta}{\delta t} + A^\alpha_j \cdot \frac{\delta}{\delta t} + C^\alpha_eta,
\]

where \( C^\alpha_\beta = f^\alpha_\beta + f^\alpha_\beta \), and we notice that \( \frac{\delta E^\alpha_k}{\delta t} = C^\alpha_\beta \), therefore the \( C^\alpha_\beta \) is a tensor.

\[\text{§ 2. Isotropic space.} \]
Let us consider an arbitrarily given continuous, differentiable field of a normalized orthogonal \( n \)-nuple \( A^\alpha \) along a curve. Since the absolute derivative of a vector at any point is also a vector at the point, there must be the functions \( P^{\alpha \beta} \) which satisfy the relations

\[
\frac{\delta A^\alpha}{\delta t} = P^{\alpha \beta} A^\beta \quad (i, k = 1, 2, \ldots, n).
\]

Applying the relation (2) twice we have

\[
\left| \frac{\delta (A^\alpha A^\beta g_{\alpha \beta})}{\delta t} \right|_{(i, i, 1, 2, \ldots, n)} = \left| \frac{\delta (A^\alpha A^\beta g_{\alpha \beta})}{\delta t} \right|_{(i, i, 1, 2, \ldots, n)} + A^\alpha A^\beta g_{\alpha \beta} C^\alpha_\beta.
\]

On the other hand, (3) gives

\[
\left| \frac{\delta (A^\alpha A^\beta g_{\alpha \beta})}{\delta t} \right|_{(i, i, 1, 2, \ldots, n)} = (A^\alpha A^\beta g_{\alpha \beta} + A^\alpha A^\beta g_{\alpha \beta} + A^\alpha A^\beta g_{\alpha \beta} + A^\alpha A^\beta g_{\alpha \beta} + A^\alpha A^\beta g_{\alpha \beta} + A^\alpha A^\beta g_{\alpha \beta}).
\]

Comparing the two above relations, we get

\[
P^{\alpha \beta} + P^{\alpha \beta} + A^\alpha A^\beta \frac{\delta g_{\alpha \beta}}{\delta t} = A^\alpha A^\beta (g_{\alpha \beta} C^\alpha_\beta + g_{\alpha \beta} C^\alpha_\beta).
\]

Particularly, if there is the relation

\[
\frac{\delta g_{\alpha \beta}}{\delta t} = g_{\alpha \beta} C^\alpha_\beta + g_{\alpha \beta} C^\alpha_\beta
\]

at any point of space, (4) may be replaced by the more simple relation
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(6) \[ P^{ij} + P^{ji} = 0 . \]

Similarly, for a normalized orthogonal covariant \( n \)-nuple \( \tilde{A}_a \), there are some functions \( P_{ij} \) along the curve, satisfying the following equations:

(7) \[ \frac{\partial \tilde{A}_a}{\partial t} = P_{ik} \tilde{A}_a \, , \]

(8) \[ \tilde{A}_a \tilde{A}_b \left( g^a \rho C^b_{\rho} + g^b \rho C^a_{\rho} \right) = P_{ij} + P_{ji} + \tilde{A}_a \tilde{A}_b \frac{\partial g^{a \rho}}{\partial t} \, , \]

particularly, if we have the relation

(9) \[ \frac{\partial g^{a \rho}}{\partial t} = g^a \rho C^{b}_{\rho} + g^b \rho C^a_{\rho} \]

at any point of the space, the equation (8) may be replaced by the relation

(10) \[ P_{ij} + P_{ji} = 0 . \]

More generally, consider a normalized orthogonal system of tensors \( \tilde{A}_a^{j \nu} \) along the curve, then any tensor of the same kind may be expressed by a linear form of the tensors \( \tilde{A}_a^{j \nu} \). Then there exist some functions \( P^{a \beta} \) on the curve, which satisfy the equation

(11) \[ \frac{\partial \tilde{A}_a^{j \nu}}{\partial t} = P^{a \beta} \tilde{A}_a^{j \nu} \, , \quad (a, \beta = 1, 2, \ldots, \omega) \]

and we get the following equation:

(12) \[ P^{a \beta} + P^{a \beta} + A^{a \beta} \tilde{A}_a^{j \nu} \left[ \sum_{\rho=1}^{\omega} \left( \frac{\partial g_{\mu \nu \rho}}{\partial t} g_{\nu \mu \rho}^{i_{\mu} j_{\nu} \nu_{\rho}} + \frac{\partial g^{i_{\mu} j_{\nu} \nu_{\rho}}}{\partial t} g_{\nu \mu \rho}^{i_{\mu} j_{\nu} \nu_{\rho}} \right) \right] \]

\[ = A^{a \beta} \tilde{A}_a^{j \nu} \left[ \sum_{\rho=1}^{\omega} \left( C_{\mu \nu \rho} + C_{\nu \mu \rho} \right) g_{\nu \mu \rho}^{i_{\mu} j_{\nu} \nu_{\rho}} \right] + \left( C^{i_{\mu} j_{\nu} \nu_{\rho}} + C^{j_{\nu} \nu_{\rho}} \right) g_{\nu \mu \rho}^{i_{\mu} j_{\nu} \nu_{\rho}} \, , \]

where

\[ C^{i_{\mu} j_{\nu} \nu_{\rho}} = g^{i_{\mu} \rho} C^{j_{\nu} \rho} \, , \quad C_{\mu j} = g_{\mu \rho} C^{\rho j} . \]

If the relations (5) and (9) are satisfied simultaneously in the space, then the relation (12) is simply expressed as in (6) and (10) by the form

(10)’ \[ P^{a \beta} + P^{a \beta} = 0 . \]

The relations (6) and (8) give some dependence among \( f_{\rho}^{i} \), \( f^{i}_{\rho} \) and \( g_{ij} \) or \( g^{ij} \). We call therefore the space in which (6) and (8) are satisfied, an isotropic space. We shall examine in detail the isotropic space.
From (5) we have
\[ \frac{\partial g^{ij}g_{jk}}{\partial t} (j, \rho) = \frac{\partial g^{ij}}{\partial t} g_{jk} + C_{k} + g^{i\rho}g_{k\sigma}C^{\sigma} \]
onumber
on the other hand from (2) the following equation is obtained:
\[ \frac{\partial g^{ij}g_{jk}}{\partial t} (j, \rho) = \frac{\partial E_{k}^{i}}{\partial t} + g^{i\rho}g_{k\sigma}C^{\sigma}. \]
Comparing the above two equations we have
\[ (5)' \quad \frac{\partial g^{ij}}{\partial t} = 0. \]
Conversely, the relation (5) can be deduced from (5)'. Similarly it will be seen that the relation (9) is equivalent to the form
\[ (9)' \quad \frac{\partial g_{\alpha\beta}}{\partial t} = 0. \]

In the isotropic space we get the relations \( \frac{\partial g^{\alpha\beta}}{\partial t} = \frac{\partial g_{\alpha\beta}}{\partial t} = 0, \) which are equivalent to the equation \( g^{\alpha\beta}C_{\beta}^{\alpha} + g_{\alpha\beta}C_{\beta}^{\alpha} = 0, \) but do not necessitate the condition \( C_{\beta}^{\alpha} = 0, \) namely \( f_{\beta}^{\alpha} + f_{\alpha}^{\beta} = 0 \) as in the Riemannian geometry.

Since these relations are satisfied both in the Finsler space and the Cartan space, the two spaces are isotropic spaces.

§ 3. The Frenet's formulae and the curvature. Consider an arbitrarily given continuous differentiable tensor field \( B(1), \) along a curve and linearly independent tensors \( B \) defined by the following equations:
\[ \frac{\partial B_{j}^{i'}}{\partial t} = B_{j}^{i'}, \ldots, \frac{\partial B_{j}^{i''}}{\partial t} = B_{j}^{i''}. \]
By the Schmidt's method as expounded in the Riemannian geometry, we have then a normalized orthogonal system of tensor fields \( A(1) \) along a curve
\[ \mathcal{A} = \begin{pmatrix} (1,1), \ldots (1,i-1) B \\ \vdots \\ (i,1), \ldots (i,i-1), B \end{pmatrix} \sqrt{D_{i-1}D_{i}}, \quad \text{where} \quad D_{i} = \begin{pmatrix} (1,1), \ldots (1,i) \\ \vdots \\ (i,1), \ldots (i,i) \end{pmatrix}, \quad (i,j) = |B B|. \]
Since \( A \) are homogeneous linear forms of \( B, B, \ldots, B, \) we have
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\[ \frac{\delta A}{\delta t} = a_1 B + a_2 B + \cdots + a_l B + \frac{D_{t-1}}{\sqrt{D_{t-1}D_t}} B. \]

But \( B \) is a homogeneous linear form of \( A, A, \ldots, A \). Hence

\[ \frac{\delta A}{\delta t} = \sum_{f=1}^{\ell+1} P^{i} A \] i.e. \( P^{\ell, \ell+k} = 0 \), when \( k > 1 \).

It follows by (10) and (11) that in an isotropic space we have also

\( P^{\ell+k} = P^{\ell, \ell+k} = 0 \) and \( P^{\omega} = 0 \), when \( k > 1 \).

On the other hand

\[ P^{\ell+1} = \frac{\delta A}{\delta t} \left| \frac{D_{t-1}}{\sqrt{D_{t-1}D_t}} \right| = \frac{\delta A}{\delta t} \left| B \right| \frac{D_{t-1}}{\sqrt{D_{t-1}D_t}} = \frac{\sqrt{D_{t-1}D_{t+1}}}{D_t}. \]

Let \( \theta \) be the extremal angle between a tensor \( \frac{\delta A}{\delta t} \) and a plane \( P \) which is determined by the tensor \( B, B, \ldots, B \). P. Diénes used the relation\(^{(1)}\)

\[ \cos \theta = \frac{\sum_{\rho=1}^{\ell} D^{(\ell)} B B}{\sqrt{\sum_{\rho=1}^{\ell} D^{(\ell)} B B}} = \frac{(1, 1), \cdots, (1, k-1), (1, k+1), \cdots, (1, \omega-1)}{(1, 1), \cdots, (1, k-1), (1, k+1), \cdots, (1, \omega-1)} \]

Denote by \( \alpha \) the angle between \( F \) and \( P \) \((C : t || t')\). The limit of the quotient \( \frac{\alpha}{\Delta t} \) or that of \( \frac{\sin \alpha}{\Delta t} \), as \( \Delta t \to 0 \), is called the \( \ell \)th curvature of the tensor field. He had following relations\(^{(1)}\):

\[ \frac{1}{r_t} = \lim_{\Delta t \to 0} \frac{\alpha}{\Delta t} = \frac{\sqrt{D_{t-1}D_{t+1}}}{D_t}. \]

Hence we have the Frenet's formulae, as in the Riemannian geometry:

\[ \frac{\delta A^{(\ell)}}{\delta t} = \frac{1}{r_t} A^{(\ell)} \frac{A^{(\ell)}}{\delta t} - \frac{1}{r_{t-1}} A^{(\ell-1)} \frac{A^{(\ell-1)}}{\delta t} \] \( (\ell = 2, \cdots, \omega-1) \),

\[ \frac{\delta A^{(\omega)}}{\delta t} = \frac{1}{r_1} A^{(\omega)} \frac{A^{(\omega)}}{\delta t}, \quad \frac{\delta A^{(\omega-1)}}{\delta t} = - \frac{1}{r_{\omega-1}} A^{(\omega-1)} \frac{A^{(\omega-1)}}{\delta t}. \]

Consider a parallel polyhedron composed of tensors \( B, B, \ldots, B \), then its volume \( \sqrt{V_t} \) is expressed in the form
\[ \sqrt{V_{e}} = \sqrt{(1,1), \cdots, (1,i)} \]

Since
\[ 0 = \begin{vmatrix} (1,1), \cdots, (1,i), (1,i+1) \\ (i,1), \cdots, (i,i), (i,i+1) \\ (j,1), \cdots, (j,i), (j,i+1) \end{vmatrix} = (j,i+1)V_{e} - \sum_{k=1}^{i} (i,k) D_{k}, \quad j \leq i, \]

\[ V_{e+1} = \begin{vmatrix} (1,1) \cdots \cdots (1,i+1) \\ (i,1) \cdots \cdots (i,i) \cdots (i,i+1) \\ (i+1,1) \cdots (i+1,i) \cdots (i+1,i+1) \end{vmatrix} = (i+1,i+1)V_{e} - \sum_{k=1}^{i} D_{k}(i+1), \]

we have at last
\[ \sqrt{V_{e+1}} = \sqrt{(i+1,i+1)} \sin \theta \sqrt{V_{e}}, \quad r_{e}^{2} = \frac{V_{e-1}V_{e+1}}{V_{e}^{2}}. \]

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