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ON INTRINSIC THEORIES IN THE MANIFOLD OF SURFACE-ELEMENTS OF HIGHER ORDER

By

Keinosuke TONOWOKA

Introduction. It is well known that the space in which a measure of a hypersurface: \( x^i = x^i(u^1, u^2, \cdots, u^{n-1}) \), \( i = 1, 2, \cdots, n \) is given by the \((n-1)\)-ple integral: \( \int_{(n-1)} F(x^i, \partial x^i/\partial u^a) \, du^1 \cdots du^{n-1} \) is called a Cartan space. As it is shown by Cartan, this space is to be regarded as a manifold of hyperplane-elements \((x^i, \partial x^i/\partial u^a)\). The geometry of Cartan space were discussed by E. Cartan \[1\] and L. Berwald \[6,7\] at large. Thereafter, T. Oikubo \[9\] and the present author \[10,11\] extended this theory to the \((n-1)\)-ple integral of higher order of special forms. Recently, the present author \[12\] have established a geometry of an \((n-1)\)-ple integral of the second order in general form, but the space in which the theories are discussed was regarded as a manifold of hypersurface-elements of the third order. On the other hand the theory of \(K\)-spreads in an \(n\)-dimensional manifold which are concerned with a system of partial differential equations of the second order was studied at first by J. Douglass, and the theory was treated in the manifold of all \(K\)-dimensional surface-elements of order 1. Thereafter A. Kawaguchi and H. Hombr \[5\] studied the theory of \(K\)-spreads of the \(m\)-th order \((m \geq 2)\), and the manifold of all \(K\)-dimensional surface-elements of the \((m-1)\)-th order was based in this case. In this paper we aim to establish the foundation of differential geometries in the manifold of \(K\)-dimensional surface-elements of higher order under the transformation group of the surface-elements which is deduced from the groups of arbitrary transformations of coordinates and parameters, and treat of the geometry of multiple integral of higher order in detail.

The present author wishes to express his heartfelt thanks to Prof. A. Kawaguchi for his kind guidance during the present researches.

§ 1. The manifold \( F_{n}^{(m)} \) and notations. In an \(n\)-dimensional space \( X_n \) with point coordinates \( x^1, x^2, \cdots, x^n \) a \(K\)-dimensional surface is defined analytically by the parametric equations

(1) Numbers in brackets refer to the references at the end of the paper.
$x^i = x^i(u^a)$, $\alpha = 1, 2, \ldots, K$,

where $u^a$ are $K$ essential parameters for the $K$-dimensional surface.

At every point on this $K$-dimensional surface a $K$-dimensional surface element of the $m$-th order can be determined by

$$x^i = x^i(u^a), \quad p_a^{i(1)} = p_a^{i}, \quad p_a^{i(2)} = p_a^{i} = \frac{\partial x^i}{\partial u^a}, \quad p_a^{i} = p_a^{i} = \frac{\partial x^i}{\partial u^a}.$$ 

$$\cdots \cdots \cdots \cdots , p_a^{i(m)} = p_a^{i} = p_a^{i} = \frac{\partial x^i}{\partial u^a} \frac{\partial x^i}{\partial u^a} \cdots \partial u^a.$$ 

Now, adjoining arbitrary system of values $x^i$, $p_a^{i(1)}$, $p_a^{i(m)}$ to every point in $X_n$, we have the $n \left(\begin{array}{c} K+m \\ K \end{array}\right)$-dimensional manifold $F_n^{(m)}$. We shall name the quantity which is transformed according to the tensor law under the transformation groups of coordinates and parameters:

(1.1) \quad $x^i = x^i(u^1, u^2, \ldots, u^n)$,

(1.2) \quad $u^a = u^a(u^i, u^i, \ldots, u^i)$

the intrinsic quantity according to E. Bortolotti.

We can speak of $x$-transformations or $u$-transformations alone, and of $x$-tensors or $u$-tensors accordingly. Tensor will mean, unless otherwise mentioned, a geometrical object which has the proper law of transformation for both sorts of indices.

Throughout this paper we shall use the notations

$$X^i = \frac{\partial x^i}{\partial x^i}, \quad X^i = \frac{\partial x^i}{\partial x^i}, \quad X^i = \frac{\partial x^i}{\partial x^i}, \quad \cdots \cdots ,$$

$$U^a = \frac{\partial u^a}{\partial u^a}, \quad U^a = \frac{\partial u^a}{\partial u^a}, \quad U^a = \frac{\partial u^a}{\partial u^a}, \quad \cdots \cdots .$$

and

$$F_{a} = \frac{\partial F}{\partial u^a}, \quad F_{i} = \frac{\partial F}{\partial u^i}, \quad F_{a}^{i} = \frac{\partial F}{\partial u^a}, \quad F_{a}^{i} = \frac{\partial F}{\partial u^a}. \quad \text{when the indices } a_1, a_2, \ldots, a_s \text{ consist of } l_1, l_2, \ldots, l_s \text{ blocks of the same indices.}$$

Moreover we shall often use the notations for indices in the form

$I_{(a_1 a_2 \cdots a_s J_{a_1 a_2 \cdots a_s})} = I_{(a_1 a_2 \cdots a_s J_{a_1 a_2 \cdots a_s})}$.

§ 2. Transformation laws of the various quantities. Under the transformation (1.1) the surface-elements in $F_n^{(m)}$ has the laws of transformation as the forms:
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\[ x^t = x^t(x^{t'}), \quad p^t_a = X^t_{i^{t}}p^{i^{t}}_a, \]
\[ p^t_{a(2)} = X^t_{i^{t}}p^{i^{t}}_{a(2)} + X^t_{j^{t}k^{t}}p^{j^{t}}_{a, k^{t}}p^{k^{t}}_{a}, \]
\[ \quad \ldots \quad \]
\[ p^t_{a(m)} = X^t_{i^{t}}p^{i^{t}}_{a(m)} + R^t_{a(m)}(x^t, p_{(1)}, p_{(2)}, \ldots, p_{(m-1)}), \]

so that \( dp^t_{a(t)} = \sum_{s=-t}^{t} \frac{\partial p^t_{a(t)}}{\partial p^{s}_{\beta(s)}} dp^{s}_{\beta(s)} \), putting \( dp^{t}_{a(0)} = dx^t \).

It is well known that

\[ \frac{\partial p^t_{a(t)}}{\partial p^{s}_{\beta(s)}} = \frac{t!}{s!}(s-r)! \frac{\partial p^t_{a(t-r)}}{\partial p^{s}_{\beta(s-r)}} \]
\[ = \left( \begin{array}{c} t \\ s \end{array} \right) \delta^{\beta(s)}_{a(t)} \frac{\partial p_{a(t)}}{\partial x^t} \]

\( (t \geq s \geq r) \),

putting \( \delta^{\beta(s)}_{a(t)} = \delta_{a(t)}^{\beta(s)} \ldots \delta_{a(s)}^{\beta(s)} \).

On the other hand, by the transformations (1.2) the partial derivatives \( f_{;a(r)} = \frac{\partial f}{\partial u^{a_{t}a_{t+1}} \ldots \partial u^{a_{r}}} (r = 1, 2, \ldots, m) \) are transformed in the manners

\[ f_{;a} = U^{a}_{a} f_{;a}, \quad f_{;a(2)} = U^{a}_{a_{1}}U^{a}_{a_{2}} f_{;a(2)} + U^{a}_{a_{1}}f_{;a_{1}}, \ldots, \]

and in general

\[ f_{;a^{s}} = U^{a_{1}}_{a} f_{;a^{s}} \]

so that, we have

\[ p_{a^{s}} = \sum_{t=0}^{s} A^{a(s)}_{a(t)}p_{a(t)} \]

\( (s = 0, 1, \ldots, m) \)

and consequently

\[ dp_{a^{s}} = \sum_{t=0}^{s} A^{a(s)}_{a(t)} dp_{a(t)} \]

\( (s = 0, 1, \ldots, m) \),

putting \( A^{a(0)}_{a(0)} = 1, \quad A^{a(0)}_{a(1)} = A^{a(0)}_{a(2)} = \ldots = 0 \).

It is easily seen that the quantities \( A^{a(t)}_{a(t)} \) are polynomials of the derivatives \( U^{a_{1}}_{a_{1}}, \quad U^{a_{1}}_{a_{1}a_{2}}, \ldots, \quad U^{a_{1}}_{a_{1}a_{2} \ldots \ldots a_{s-1}a_{s}}, \) and determined from the recurring formulae

\[ A^{a(t)}_{a(s)} = \left\{ \begin{array}{ll} U^{a(s)}_{a(s)} & t = s, \\ A^{a(t)}_{a(s)} U^{a(t)}_{a(s)} + A^{a(t)}_{a(s)} & s > t > 1, \\ U^{a(t)}_{a_{1}a_{2} \ldots \ldots a_{s}} & t = 1. \end{array} \right. \]
Now, we shall prove the formula

\[(2.5) \quad \left( \begin{array}{c} r \\ t \end{array} \right) A_{\alpha(s)}^{a(r)} = \sum_{u=t}^{s-r+t} \binom{s}{u} A_{\alpha(u)}^{\alpha(r-t)} A_{\alpha(s-u)}^{a(t)} \cdot \]

Let \(\phi\) and \(\psi\) be any two functions of the parameters \(u^a's\), then it follows that

\[(\phi \cdot \psi)_{;a(r)} A_{\alpha(s)}^{a(r)} = \sum_{u=0}^{\theta} \left( \begin{array}{c} s \\ u \end{array} \right) \left( \sum_{v=0}^{u} \phi_{;a(v)} \psi_{;\alpha(r-t)} A_{\alpha(s)}^{a(v)} A_{\alpha(s-u)}^{a(t)} \right) \]

which is known as the generalized Leibniz formula. Therefore, by the transformation (1.2) we have

\[\sum_{r=0}^{s} (\phi \cdot \psi)_{;a(r)} A_{\alpha(s)}^{a(r)} = \sum_{u=0}^{s} \left( \begin{array}{c} s \\ u \end{array} \right) \phi_{;a(v)} \psi_{;\alpha(r-t)} A_{\alpha(s)}^{a(v)} A_{\alpha(s-u)}^{a(t)} \]

or

\[\sum_{r=0}^{s} \sum_{\nu=0}^{r} \left( \begin{array}{c} r \\ \nu \end{array} \right) \phi_{;a(v)} \psi_{;\alpha(r-t)} A_{\alpha(s)}^{a(v)} A_{\alpha(s-u)}^{a(t)} \]

Comparing the coefficients of \(\phi_{;a(v)} \psi_{;\alpha(r-t)}\) on both sides, we have the formula (2.5).

Let us consider an operator \(P^{a(l)}\) applied to any quantity of the manifold \(F^{(m)}_n\), that is

\[P^{a(l)}(L) = \sum_{t=l}^{m} \left( \begin{array}{c} t \\ l \end{array} \right) L_{;\alpha(s)}^{\gamma(s)} dp_{\beta(t-l)}^{i} \]

then we have

**Theorem 1.** Under the transformation group (1.1) and (1.2) the operator \(P^{a(l)}\) has the law of transformation:

\[(2.6) \quad P^{a(l)}(L) = \sum_{u=t}^{m} A_{\alpha(u)}^{a(s)} P^{a(u)}(L) \cdot \]

Proof. If we effect the \(x\)-transformations alone, it follows that

\[P^{a(l)}(L) = \sum_{i=1}^{m} \left( \begin{array}{c} t \\ l \end{array} \right) \sum_{s=t}^{\theta} L_{;\alpha(s)}^{\gamma(s)} \frac{\delta p_{\beta}^{i}(s)}{\delta p_{\beta}^{i}(t-l)} dp_{\beta}^{i}(t-l) \]

\[= \sum_{i=1}^{m} \left( \begin{array}{c} t \\ l \end{array} \right) \sum_{s=t}^{\theta} L_{;\alpha(s)}^{\gamma(s)} \delta_{\alpha(s)}^{a(l)} \frac{\delta p_{\beta}^{i}(s)}{\delta p_{\beta}^{i}(t-l)} dp_{\beta}^{i}(t-l) \]

\[= \sum_{i=1}^{m} \left( \begin{array}{c} s \\ l \end{array} \right) L_{;\alpha(s)}^{\gamma(s)} \sum_{t=0}^{\theta} \frac{\delta p_{\beta}^{i}(t-l)}{\delta p_{\beta}^{i}(s-l)} dp_{\beta}^{i}(t-l) \]

\[= \sum_{i=1}^{m} \left( \begin{array}{c} s \\ l \end{array} \right) L_{;\alpha(s)}^{\gamma(s)} \frac{\delta p_{\beta}^{i}(s-l)}{\delta p_{\beta}^{i}(t-l)} \]
By effecting the $u$-transformations we have

$$P^{a(l)}(L) = \sum_{t=l}^{m} \left( \begin{array}{l} t \\ l \end{array} \right) \sum_{s=0}^{m} L_i \frac{\partial p^{i'}}{\partial p^{i'}_{a(l)}} dp^{i'}_{a(l)\beta(t-l)}$$

$$= \sum_{s=0}^{m} \sum_{t=0}^{m} \left( \begin{array}{l} s \\ u \end{array} \right) L_i \frac{\partial p^{i'}}{\partial p^{i'}_{a(l)}} dp^{i'}_{a(l)\beta(t-l)}$$

Theorem 2. If $T^A$ be an intrinsic quantity of $F^{(m)}$ whose transformation law under the transformations (1.1) and (1.2) is $T^A = \mathcal{A}(T^A)'$, then $P^{a(l)}(T^A)$ obeys the transformation law

$$(2.7) \quad P^{a(l)}(T^A) = \sum_{s=0}^{m} A^{a(l)}_{\alpha(s)\beta} P^{a(s)}(T^A).$$

Theorem 3. If $w^i$ be any vector of $F^{(m)}$ and $L$ be a scalar of $F^{(m)}$, the quantity

$$(2.8) \quad D_j(L)w^j = \sum_{r=0}^{m} \sum_{s=0}^{m} L_i \frac{\partial w^i}{\partial w^i_{a(s)}} \frac{\partial w^i}{\partial w^i_{a(s)}}$$

is invariant under the transformations (1.1) and (1.2), where $\partial / \partial u^s$ denotes the total differentiation with respect to $u^s$, that is

$$w^j_{/\beta} = w^j_{/\beta} + \sum_{s=0}^{m} w^j_{/a(s)} \frac{\partial w^i}{\partial w^i_{a(s)}}.$$
By effecting the \( u \)-transformations we have
\[
\sum_{s=0}^{m} L_{i}^{a(s)} w^{t'}_{a(s)} = \sum_{s=0}^{m} \sum_{r=s}^{m} L_{i}^{b(r)} A_{a(r)}^{a(s)} w^{t'}_{a(s)} = \sum_{s=0}^{m} \sum_{r=s}^{m} L_{i}^{b(r)} A_{a(r)}^{a(s)} w^{t'}_{a(s)} = \sum_{s=0}^{m} \sum_{r=s}^{m} L_{i}^{b(r)} A_{a(r)}^{a(s)} w^{t'}_{a(s)}
\]

**Theorem 4.** Let \( T^{A} \) be an intrinsic quantity of \( F^{(m)} \) whose transformations law under the transformations (1.1) and (1.2) is \( T^{A} = T^{A}_{A}, T^{A}' \), and \( v^{f} \) be any vector of \( F^{(m)} \), then the quantities
\[
D_{j}^{a(u)}(T^{A}) v^{f} = \sum_{r=0}^{m} \left( \begin{array}{c} r \\ u \end{array} \right) T_{i}^{A \beta(r)} v^{f}_{\beta(r)}
\]
are transformed by the transformations (1.1) and (1.2) in the manners
\[
D_{j}^{a(u)}(T^{A}) v^{f} = \sum_{r=0}^{m} \left( \begin{array}{c} r \\ u \end{array} \right) T_{i}^{A \beta(r)} v^{f}_{\beta(r)}
\]

Proof. When we put \( L = T^{A} \) and \( u^{j} = \phi v^{j} \) into (2.8), \( \phi \) being any scalar of \( F^{(m)} \), one obtains the intrinsic quantity
\[
D_{j}(T^{A}) \phi v^{j} = \sum_{r=0}^{m} \sum_{u=0}^{r} \left( \begin{array}{c} r \\ u \end{array} \right) \phi_{\beta(r)} v^{j}_{\beta(r)}
\]
so that we can conclude (2.9) by virtue of (2.2).

§ 3. Intrinsic operators and intrinsic Pfaffian form. Let \( f(x^{i}, p_{a(1)}^{i}, \ldots, p_{a(m)}^{i}) \) be any scalar of \( F^{(m)} \), then it is seen from theorem 2 and the above mentioned theorem that the quantities
\[
(3.1a) \quad \sum_{i=1}^{m} f_{a(i)} P_{j}^{a(i)}(T^{A}),
\]
\[
(3.1b) \quad \sum_{i=1}^{m} f_{a(i)} D_{j}^{a(j)}(T^{A}) v^{j}
\]
are tensors of the same kind with \( T^{A} \).

Suppose now that we have the quantity \( G_{\beta}^{a} \) whose transformation law under the transformations (1.1) and (1.2) is the same as that of coefficient of the affine connection of \( u \)-tensor, that is
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(3.2) \[ U^a_{\beta \tau} = G^a_{\beta \tau} U^a_{\beta \tau} - G^a_{\beta \tau} U^a_{\beta \tau}, \]
then we can derive from \( f_{/a(1)}, f_{/a(2)}, \ldots, f_{/a(t)} \) the intrinsic quantities \( f_{\beta(s)} \) \((s = 1, 2, \ldots, t)\) in the following way.

First of all we see that \( f_{\beta(1)} = f_{/a} \delta^a_{\beta} \) is an intrinsic quantity. Assume that there are the quantities \( 'K^a_{\beta(s)} \) \((l = 1, 2, \ldots, s-1)\) such that \( f_{\beta(s-1)} = \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)} \) is intrinsic, namely

(3.3) \[ \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)} = \left( \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)} \right) U^a_{\beta(s-1)}. \]

Differentiating the above equation with respect to \( u^\beta \) and symmetrizing with respect to the indices \( \beta_1, \beta_2, \ldots, \beta_s \) one gets

(3.4) \[ \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)} + \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)/\beta_1} \]
\[ = \left( \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)} + \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)/\beta_1} \right) U^a_{\beta(s-1)} \]
\[ + (s-1) \left( \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)} \right) U^a_{\beta_1} \ldots U^a_{\beta_{s-2}} U^a_{\beta_{s-1}}. \]

Eliminating \( U^a_{\beta_{s-1}} \) from the above equation and (3.2) putting \( \alpha' = \beta_{s-1}, \beta = \beta_{s-1}, \tau = \beta_s \), we have under the consideration of (3.3)

\[ \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)} + \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)/\beta_1} \]
\[ - (s-1) \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)/\beta_1} \]
\[ = \left( \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)} + \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)/\beta_1} \right) U^a_{\beta(s-1)} \]
\[ + (s-1) \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)/\beta_1} \]
\[ + (s-1) \sum_{l=1}^{s-1} f_{/a(l)}' K^a_{\beta(s-1)/\beta_1} \]
\[ \times U^a_{\beta(s-1)} U^a_{\beta_{s-2}} \ldots U^a_{\beta_0}. \]

Therefore, if we put

(3.5) \[ \delta^a_{\beta(s)} K^a_{\beta(s-1)} + 'K^a_{\beta(s-1)/\beta_1} = (s-1) 'K^a_{\beta(s-1)/\beta_1} \]
\[ \times G^a_{\beta(s-1)/\beta_1}, \]
the quantity \( f_{\beta(s)} = \sum_{l=1}^{s} f_{/a(l)}' K^a_{\beta(s)} \) is also an intrinsic quantity. Thus we can see that there exist the intrinsic quantities \( f_{\beta(s)} = \sum_{l=1}^{s} f_{/a(l)}' K^a_{\beta(s)} \) \((s = 1, 2, \ldots, m)\) whose coefficients are determined from the recurring formula (3.5) putting \( 'K^a_{\beta(1)} = \delta^a_{\beta_1}. \)

It is easily seen from (3.5) that

(3.6) \[ 'K^a_{\beta(s)} = \delta^a_{\beta(s)}, \]
Moreover we can find the quantities $K_{T(t)}^{eta(s)}(1 \leq s \leq t \leq m)$ such that the relations

$\sum_{s=l}^{t} K_{T(t)}^{eta(s)} = \delta_{r(t)}^{\alpha(l)} \quad (1 \leq l \leq t \leq m)$

hold. Specially, we have from (3.5), (3.6) and the above relations

$K_{T(t)}^{eta(s)} = \delta_{r(t)}^{\alpha(l)}, \quad \sum_{s=1}^{t} K_{T(t)}^{eta(s)} = \delta_{r(t)}^{\alpha(l)}$.

Therefore, from (3.1a) and (3.1b) we have the following theorems:

**Theorem 5.** The operators $\mathfrak{P}^{\beta(s)}$ defined by

$\mathfrak{P}^{\beta(s)} = \sum_{r=1}^{m} K_{T(t)}^{eta(s)} P_{r(t)}^{(t)} \quad (s = 1, 2, \ldots, m)$

are intrinsic operators, that is

$\mathfrak{P}^{\beta(s)}(L) = U_{\beta(s)}^{\beta(s)} \mathfrak{P}^{\beta(s)}(L)$.

And if $T^{A}$ be an intrinsic quantity of $F_{n}^{(m)}$ whose transformation law is $T^{A} = \mathfrak{P}^{A} T^{A}$, then $\mathfrak{P}^{\beta(s)}(T^{A})$ are also intrinsic quantities whose transformation laws are

$\mathfrak{P}^{\beta(s)}(T^{A}) = U_{\beta(s)}^{\beta(s)} \mathfrak{P}^{\beta(s)}(T^{A})$.

**Theorem 6.** Let $T^{A}$ be an intrinsic quantity of $F_{n}^{(m)}$ and $v^{i}$ be any vector, then

$\mathfrak{D}^{\beta(s)}(T^{A}) v^{i} = \sum_{r=l}^{m} K_{a(r)}^{\beta(s)} D_{i}^{a(r)}(T^{A}) v^{i} \quad (l = 1, 2, \ldots, m)$

are also intrinsic quantities.

Moreover we have

**Theorem 7.** If the PFAFFian form $\sum_{r=0}^{m} P_{\gamma(s)}^{a(r)} dp_{a(r)}^{i}$ defined in the manifold $F_{n}^{(m)}$ has the tensor character with respect to the index $J$, and $v^{i}$ be any vector of $F_{n}^{(m)}$,

$P_{\gamma(s)}^{a(m)} dv^{i} + \sum_{r=0}^{m} P_{\gamma(s)}^{a(m)} dp_{a(r)}^{i}$

has also the tensor character with respect to the index $J$ and $a_{(m)}$. [5].

Suppose now that we have the quantity $\Gamma_{J}^{J}$ whose transformation law under the transformations (1.1) and (1.2) is

$\Gamma_{J}^{J} = \Gamma_{J}^{J} X_{J}^{k} U_{J}^{k} - X_{J}^{k} U_{J}^{k}$,

then we have

**Theorem 8.** If $\omega_{\beta(s)}^{i}(d) = \sum_{r=0}^{m} P_{\beta(s)}^{i(r)} dp_{a(r)}^{i}$ be an intrinsic PFAFFian
form, then

\[
\omega^t_{\beta(s+1)}(d) = \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) dp_a^{k(r)}(r) + \sum_{r=0}^{s} (P^t_{\beta(s+1)}(d)) dp_a^{k(r)}
\]

\[+ \Gamma^t_{j(s+1)}P^t_{\beta(s+1)}(d) - sG_{(s+1)dp_a^{k(r)}}\]

is also an intrinsic PFaffian form.

Proof. It has been proved in the work [5] of A. Kawaguchi and H. Hombu that this theorem is true under the \(x\)-transformations alone. We shall now prove it under the \(u\)-transformations (\(x\) fixed). Since

\[
P^t_{\beta(s+1)}(d) = U^t_{\beta(s+1)}(d) \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}}
\]

we have

\[
\sum_{r=0}^{s} P^t_{\beta(s+1)}(d) dp_a^{k(r)} = U^t_{\beta(s+1)}(d) \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) dp_a^{k(r)} + \sum_{r=0}^{s} \left( \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \right) dp_a^{k(r)}
\]

\[
= U^t_{\beta(s+1)}(d) \sum_{r=0}^{s} \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \left( \sum_{r=0}^{s} \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \left( \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \right) dp_a^{k(r)}
\]

\[
+ U^{t}_{\beta(s+1)}(d) \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) dp_a^{k(r)}
\]

\[
+ U^{t}_{\beta(s+1)}(d) \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \left( \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \right) dp_a^{k(r)}
\]

\[
\Gamma^t_{\beta(s+1)}P^t_{\beta(s+1)}(d) dp_a^{k(r)} = U^{t}_{\beta(s+1)}(d) \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \left( \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \right) dp_a^{k(r)}
\]

and

\[
\sum_{r=0}^{s} sG^t_{\beta(s+1)}(d) P^t_{\beta(s+1)}(d) dp_a^{k(r)}
\]

\[
= s \left( G^{t}_{\beta(s+1)}(d) \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right)
\]

\[
+ U^t_{\beta(s+1)}(d) \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \left( \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \right) dp_a^{k(r)}
\]

\[
+ U^t_{\beta(s+1)}(d) \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \left( \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \right) \left( \sum_{r=0}^{s} \sum_{r=0}^{s} \left( \sum_{r=0}^{s} P^t_{\beta(s+1)}(d) \frac{\partial p_a^{t(r)}}{\partial p_a^{k(r)}} \right) \right) dp_a^{k(r)}
\]
\[
\begin{align*}
&\times \sum_{r=0}^{s} \sum_{t=r}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} \frac{\partial p_{\beta(s+1)}^{a(r)}}{\partial p_{\beta(s+1)}^{a(r)}} dp_{\alpha(r)}^{k} \\
&= sU_{\beta(s+1)}^{\beta(s+1)} \sum_{t=0}^{s} G_{\beta(s+1)}^{\beta(s+1)} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} dp_{\alpha(r)}^{k} \\
&\quad + U_{\beta(s+1)}^{\beta(s+1)} \sum_{t=0}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} dp_{\alpha(r)}^{k} .
\end{align*}
\]

On the other hand we have
\[
\begin{align*}
&U_{\beta(s+1)}^{\beta(s+1)} \sum_{r=0}^{s} \sum_{t=r}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} \left( \frac{\partial p_{\beta(s+1)}^{a(r)}}{\partial p_{\beta(s+1)}^{a(r)}} \right) dp_{\alpha(r)}^{k} \\
&\quad + A^{a(r)}_{\beta(s+1)} \sum_{t=0}^{s} \sum_{t=r}^{s} A^{a(r)}_{\beta(s+1)} \gamma_{j}^{(t)} \\
&\quad = U_{\beta(s+1)}^{\beta(s+1)} \sum_{r=0}^{s} \sum_{t=r}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} \left( A^{a(r)}_{\beta(s+1)} \sum_{t=0}^{s} \sum_{t=r}^{s} A^{a(r)}_{\beta(s+1)} \gamma_{j}^{(t)} \right) dp_{\alpha(r)}^{k} \\
&\quad + A^{a(r)}_{\beta(s+1)} \sum_{t=0}^{s} \sum_{t=r}^{s} A^{a(r)}_{\beta(s+1)} \gamma_{j}^{(t)} \\
&\quad = U_{\beta(s+1)}^{\beta(s+1)} \sum_{r=0}^{s} \sum_{t=r}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} \\
&\quad + U_{\beta(s+1)}^{\beta(s+1)} \sum_{r=0}^{s} \sum_{t=r}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} \\
&\quad = U_{\beta(s+1)}^{\beta(s+1)} \sum_{r=0}^{s} \sum_{t=r}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} \\
&\quad = U_{\beta(s+1)}^{\beta(s+1)} \sum_{r=0}^{s} \sum_{t=r}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} \\
&\quad = U_{\beta(s+1)}^{\beta(s+1)} \sum_{r=0}^{s} \sum_{t=r}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} \\
&\quad = \omega_{\beta(s+1)}^{\beta(s+1)} (a(r)) \sum_{r=0}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} dp_{\alpha(r)}^{k} ,
\end{align*}
\]

consequently, we have the intrinsic Pfaffian form
\[
\omega_{\beta(s+1)}^{\beta(s+1)} (a(r)) = \sum_{r=0}^{s} P_{\beta(s+1)}^{a(r)} \gamma_{j}^{(t)} dp_{\alpha(r)}^{k} ,
\]

putting
\[
\begin{align*}
P_{\beta(s+1)}^{a(r)} &= P_{\beta(s+1)}^{a(r)} \\
(3.11) P_{\beta(s+1)}^{a(r)} &= P_{\beta(s+1)}^{a(r)} \\
&\quad + \Gamma_{(s+1)}^{(s+1)} P_{\beta(s+1)}^{a(r)} - sG_{(s+1)}^{(s+1)} P_{\beta(s+1)}^{a(r)} .
\end{align*}
\]

§ 4. \(K\)-ple integral of the \(m\)-th order. We shall now proceed to discuss the geometry of the \(K\)-ple integral of the \(m\)-th order:
$\int_{(X^m)} F(x^i, p_{a(1)}^i, \ldots, p_{a(m)}^i) \, du^i \ldots du^k \quad (m > 1)$

by using of the results obtained in the preceding paragraphs, where the function $F(x^i, p_{a(1)}^i, \ldots, p_{a(m)}^i)$ is differentiable to sufficient order with respect to its arguments.

If the integral (4.1) be regarded as defining a measure of $K$-dimensional surface in an $n$-dimensional manifold, it is adequate to suppose that the integral (4.1) is invariant under any parameter transformations. In order this it is necessary and sufficient that the function $F$ is transformed under the parameter transformations (1.2) in the manner

$$F(x^i, p_{a(1)}^i, \ldots, p_{a(m)}^i) = \Delta F(x^i, p_{a^\prime(1)}^i, \ldots, p_{a^\prime(m)}^i),$$

where

$$\Delta = \left| U^a_{a_{\prime}}, \ldots \right|, \ldots$$

From this one has the well known relations [5]

$$(4.3a) \quad \sum_{s=1}^{m} \varepsilon_{\beta(\ldots s-1)}^s F^a_{\beta(\ldots s-1)} = \delta_r^a F,$$

$$(4.3b) \quad \sum_{s=t}^{m} \left( \begin{array}{c} s \\ t \end{array} \right) p_{\beta(s-t)}^t F_{;i}^{\alpha(t)\beta(s-t)} = 0 \quad (m \geq t > 1).$$

When $t = m$, (4.3b) becomes

$$F_{;i}^t a(m) p_t^i = 0 \quad (r = 1, 2, \ldots, K).$$

Differentiating with respect to $p_{a(m)}^i$ we have

$$F_{;i}^t a(m) \lambda(m) p_t^i = 0 \quad (r = 1, 2, \ldots, K),$$

so that

$$(4.4) \quad F_{;i_1 i_2 \ldots i_n} \mu_{i_1 i_2 \ldots i_n} = 0 \quad (r = 1, 2, \ldots, K),$$

$$(4.5) \quad F_{;i_1 i_2 \ldots i_n} \lambda_{i_1 i_2 \ldots i_n} = 0 \quad (r = 1, 2, \ldots, K).$$

On the other hand it is evident that

$$(4.6) \quad \varepsilon_{i_1 i_2 \ldots i_n} \rho^{(a(n)(\lambda(n))} = 0 \quad (r = 1, 2, \ldots, K),$$

where

$\varepsilon_{i_1 i_2 \ldots i_n} = n! \delta_{i_1}^{1} \cdots \delta_{i_n}^{n}.$

We can see from (4.4), (4.5) and (4.6) that there is one system of the quantities $\rho^{(a(n)(\lambda(n))}$ such that

$$F_{;i_1 i_2 \ldots i_n} \mu_{i_1 i_2 \ldots i_n} \rho^{(a(n)(\lambda(n))} = \varepsilon_{i_1 i_2 \ldots i_n} \varepsilon_{j_1 j_2 \ldots j_n} \times p_{i}^{n-k+1} \cdots p_{i}^{n+1} \cdots p_{k}^{n} \rho^{(a(n)(\lambda(n))}, \ldots$$
where \((a_1, a_2, \ldots, a_N)\) and \((\lambda_1, \lambda_2, \ldots, \lambda_N)\) represent \((a_1 \cdots a_m \beta_1 \cdots \beta_m \cdots r_1 \cdots r_m)\) and \((\lambda_1 \cdots \lambda_m \mu_1 \cdots \mu_m \cdots \nu_1 \cdots \nu_m)\) respectively, and consequently \(N = m(n-K)\).

It is easily seen from the definition that under the transformations (1.1) and (1.2) the quantity \(\rho^{a(N)\lambda(N)} = \rho^{(a_1 \cdots a_N) (\lambda_1 \cdots \lambda_N)}\) obeys the transformation law

\[
\rho^{a'(N)\lambda'(N)} = D^{x} \rho^{a(N)\lambda(N)} U_{a(N)^{\prime}}^{\alpha(N)} U_{\lambda(N)^{\prime}}^{\lambda(N)}
\]

or

\[
\rho^{a'(N)\lambda'(N)} = D^{y} \sum_{a(N), \lambda(N)} \rho^{a(N), \lambda(N)} U_{a(N)^{\prime}}^{a'(N)} U_{\lambda(N)^{\prime}}^{\lambda'(N)}
\]

where \(\sum\) denotes the summation for the all different combinations of \(N\) indices \(a_1, a_2, \ldots, a_N\) and the all different combinations of \(N\) indices \(\lambda_1, \lambda_2, \ldots, \lambda_N\), and \(\sum'\) denotes \(\rho^{a(N)\lambda(N)} = \frac{N!}{l_1! l_2! \cdots l_t!} U_{a(N)^{\prime}}^{a(N)}\), where \(a_1, a_2, \ldots, a_N\) consist of \(l_1, l_2, \ldots, l_t\) blocks of the same indices.

Consequently, the \(\left(\frac{K+N-1}{N}\right)\) -rowed determinant \(\rho = |\rho^{a(N)\lambda(N)}|\) is transformed by the transformations (1.1) and (1.2) as follows:

\[
\rho' = \Delta^x D^y |\sum' U_{a(N)}^{a(N)}|\rho,
\]

where \(x = \left(\frac{K+N-1}{N}\right) (n-K-2), y = 2 \left(\frac{K+N-1}{N}\right)\) and \(|U_{a(N)}^{a(N')}|\) represents the \(\left(\frac{K+N-1}{N}\right)\) -rowed determinant.

In the same manner it should be obtained that

\[
\rho = \Delta^{-x} D^{-y} |U_{a(N)}^{a(N)}|\rho',
\]

so that \(|U_{a(N)}^{a(N')}|^2 |U_{a(N)}^{a(N')}|^2 = 1\). Hence we can conclude that \(|U_{a(N)}^{a(N')}|\) is a power of \(\Delta\) multiplying a suitable constant, since \(|U_{a(N)}^{a(N')}|\) is a homogeneous function of \(U_{a(N)}^{a(N')}\). Under some consideration we see that

\[
|U_{a(N)}^{a(N')}| = \Delta^{\left(\frac{K+N-1}{N-1}\right)}
\]

and consequently we have

\[
\rho' = \Delta^r D^y \rho,
\]

putting \(r = \left(\frac{K+N-1}{N}\right) (n-K-2) - 2 \left(\frac{K+N-1}{N-1}\right)\).

When we put
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\[ P^{a(N)\lambda(N)} = F^{1^{-\frac{'' N}{B^{-}}}}\rho^{rac{-0}{y}}\rho^{\alpha(N)\lambda(N)} \]

and denote by \( P_{a(N)\lambda(N)} \) the inverse of \( P^{a(N)\lambda(N)} \), that is \( P^{a(N)\lambda(N)}P_{a(N)\mu(N)} = \delta_{\mu(N)}^{l(N)} \), assuming that the \((K+N-1)\)-rowed determinant \( P = |P^{a(N)\lambda(N)}| \) does not vanish, these are transformed by the transformations (1.1) and (1.2) in the manners

\[
\begin{align*}
P^{a'(N)\lambda(N)} &= P^{a(N)\lambda(N)}U_{a(N)}^{\alpha'(N)}U_{\lambda(N)}^{l(N)}, \\
P_{a(N)\lambda(N)} &= P_{a(0)\mu(N)}U_{a(N)}^{\alpha(N)}U_{\lambda(N)}^{l(N)},
\end{align*}
\]

§ 5. Determination of the quantities \( G^a_{\beta} \) and \( \Gamma^i_j \). We shall now attempt to determine the quantity \( G^a_{\beta} \) as mentioned in § 3 in use of \( P^{a(N)\lambda(N)} \) and \( P_{a(N)\mu(N)} \). If we put

\[
Q^{\lambda(N)}_{\mu(N)\nu} = P^{\lambda(N)\alpha(N)}(P_{\mu(N)\alpha(N)}U_{\alpha(N)}^{\lambda(N)}),
\]

under the transformation (1.1) and (1.2) it follows that

\[
Q^{\lambda'(N)}_{\mu'(N)\nu'} = P^{\lambda'(N)\alpha'(N)}(P_{\mu'(N)\alpha'(N)}U_{\alpha'(N)}^{\lambda'(N)})U_{\nu'}^{\lambda'(N)},
\]

\[
+ P^{\lambda'(N)\alpha'(N)}(U_{\mu'}^{\alpha'(N)}P_{\mu'(N)\alpha'(N)}U_{\alpha'(N)}^{\lambda'(N)}),
\]

Putting \( \lambda'_1 = \mu'_1, \lambda'_2 = \mu'_2, \ldots, \lambda'_{N-1} = \mu'_{N-1}, \lambda'_N = \lambda, \mu'_N = \mu' \) and contracting over the indices \( \mu'_1, \mu'_2, \ldots, \mu'_{N-1} \) we have

\[
Q^{\lambda'(N)}_{\mu'(N)\nu'} = Q^{\lambda'(N)}_{\mu'(N)\nu'},
\]

\[
+ K^0_{\mu'\sigma}U_{\sigma}^{\mu'},
\]

\[
+ pU_{\mu'}^{\lambda'}U_{\nu'}^{\lambda'} + q\delta_{\mu'}^{\lambda'}\omega_{\nu'}, \log \Delta,
\]

where we put

\[
Q^{\lambda'(N-1)}_{\mu'(N-1)\nu'},
\]

\[
K^\sigma_{\mu'\nu'} = NP^{(\mu'_1, \ldots, \mu'_{N-1} \sigma)}(a'_1, \ldots, a'_{N-1} \lambda'),
\]

\[
p = \frac{(K+2) \cdots (K+N)}{N!} \quad \text{and} \quad q = \frac{(N-1)(K+2) \cdots (K+N-1)}{N!} (N \neq 2),
\]

\[
q = \frac{1}{2} \quad (N = 2).
\]

Moreover from (5.1) we have

\[
Q_{\mu'} = Q_{\mu'}U_{\mu'}^{\lambda'}U_{\nu'}^{\lambda'} + (p+q)\omega_{\nu'}, \log \Delta,
\]

\[
Q_{\mu'} = Q_{\mu'}U_{\mu'}^{\lambda'}U_{\nu'}^{\lambda'} + K^\sigma_{\mu'\nu'}U_{\sigma'}^{\mu'}U_{\sigma'}^{\nu'} + (p+q)\omega_{\nu'}, \log \Delta.
\]
putting \( Q_{\mu'} = Q_{\mu'}^{\lambda'} \).

By eliminating \( \partial_{\nu} \log \Delta \) from (5.1) and (5.2) we get

\[
(5.3) \quad (p+q) Q_{\mu'}^{l', \nu'} - q \delta_{\lambda'}^{l'} Q_{\nu'} = \left[(p+q) Q_{\mu}^{l', \nu} - q \delta_{\mu}^{l'} Q_{\nu} \right] U_{\lambda}^{l'} U_{\nu}^{l'} + \left[ (p+q) \delta_{\rho}^{l'} K_{\mu}^{\alpha'} + p (p+q) \delta_{\mu'}^{\lambda'} \delta_{\rho'}^{\nu'} - q \delta_{\rho}^{l'} K_{\nu}^{\lambda'} \right] U_{\alpha}^{l'} U_{\rho}^{l'}.
\]

Now we put

\[
(p+q) K_{\mu'}^{l', \nu'} \delta_{\rho'}^{l'} + p (p+q) \delta_{\mu'}^{\lambda'} \delta_{\rho'}^{\nu'} - q \delta_{\rho}^{l'} K_{\nu}^{\lambda'} = N_{\lambda'}^{l', \nu'} \delta_{\rho'}^{l'}
\]
and assume that the \( K^{\omega} \)-rowed determinant \( |N_{\mu'}^{l', \nu'} \delta_{\rho'}^{l'}| \) is different from zero, then we can obtain the quantities \( n_{\lambda'}^{l', \nu'} \delta_{\rho'}^{l'} \) such that

\[
N_{\mu'}^{l', \nu'} \delta_{\rho'}^{l'} n_{\lambda'}^{l', \nu'} \delta_{\rho'}^{l'} = \delta_{\rho}^{l'} \delta_{\rho}^{l'} \delta_{\rho}^{l'} \delta_{\rho}^{l'}.
\]

Since \( K_{\mu}^{\alpha'} \) has the tensor character with respect to its indices under the transformations (1.1) and (1.2), the quantities \( N_{\mu}^{l', \nu'} \delta_{\rho'}^{l'} \) and \( n_{\lambda}^{l', \nu'} \delta_{\rho'}^{l'} \) are tensors. Hence, if we put

\[
n_{\lambda}^{l', \nu'} = (p+q) Q_{\mu'}^{l', \nu'} - q \delta_{\mu'}^{l'} Q_{\nu'} = G_{\lambda}^{l', \nu'},
\]
it is easily seen from (5.3) that the quantity \( G_{\lambda}^{l', \nu'} \) obeys the law of transformation:

\[
G_{\lambda}^{l', \nu'} = G_{\lambda}^{l', \nu'} U_{\nu}^{l'} U_{\mu}^{l'} + U_{\nu}^{l'} U_{\mu}^{l'}.
\]

Thus obtained \( G_{\lambda}^{l', \nu'} \) will play an important rôle in our theories.

Let us next consider the Euler vector which is concerned with the first variation of the integral (4.1):

\[
E_{i} (F) = \sum_{r=0}^{m} (-1)^{r} (F;_{i}^{a(r)})_{(a(r))},
\]

It is seen from the first variation of (4.1) that the Euler vector \( E_{i} \) is transformed by the parameter transformations as follows:

\[
(5.5) \quad E_{i} (F') = \Delta E_{i} (F).
\]

If in (5.4) \( F \) be substituted by \( F^{*} = F \phi \), \( \phi \) being any function of \( u \)'s, we have

\[
E_{i} (F \phi) = \sum_{r=0}^{m} E_{i}^{a(r)} \phi_{a(r)},
\]

where \( E_{i}^{a(r)} (r = 0, 1, \ldots, m) \) are vectors of the form

\[
E_{i}^{a(r)} = \sum_{s=r}^{m} (-1)^{s} \left( \begin{array}{c}s \\ r \end{array} \right) (F;_{i}^{a(r)} \delta^{a(r)})_{(a(r))} \quad (r = 0, 1, \ldots, m),
\]

and called the Synge vectors. If we effect the parameter transformations, it follows that
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\[ E_\epsilon (F^\gamma \phi^\gamma) = \sum_{r=0}^{s} E_\epsilon^{a_{r}(s)} \phi_{a_{r}(s)} = \sum_{r=0}^{s} \sum_{s=0}^{r} E_\epsilon^{a_{r}(s)} A_{a_{r}(s)}^{a_{r}(s)} \phi_{a_{r}(s)} \]

On the other hand by (5.5) we have

\[ E_\epsilon (F^\gamma \phi^\gamma) = \Delta E_\epsilon (F^\gamma \phi^\gamma) = \Delta \sum_{r=0}^{s} E_\epsilon^{a_{r}(s)} A_{a_{r}(s)}^{a_{r}(s)} \phi_{a_{r}(s)} \]

Hence, the SYNGE vectors are transformed by the parameter transformations in the manners

\[ E_\epsilon^{a_{r}(s)} = \Delta^{-1} \sum_{r=0}^{s} E_\epsilon^{a_{r}(s)} A_{a_{r}(s)}^{a_{r}(s)} \]

so that we can derive from the SYNGE vectors a system of the intrinsic vectors \( \mathfrak{G}_{a_{r}(s)} \) \( l = 0, 1, \ldots, m \) as similar manner as theorem 5, that is

\[ \mathfrak{G}_{a_{r}(s)} = - \frac{1}{F} \sum_{r=0}^{s} E_\epsilon^{a_{r}(s)} K_{a_{r}(s)}^{a_{r}(s)} \]

where \( v^\gamma \) is a vector of \( F_{(m)} \).

The \( n \left( \begin{array}{lll} K+ & m & -1 \\ m & & \end{array} \right) \)-rowed matrix \( (F_{j;\gamma i}^{a_{r}(s)}) \)
has the rank \( \left( \frac{K+m-1}{m} \right) (n-K) \) at most because of \( F; \tau_{(m)}^{r}, \beta_{(m)}^{(m)} p^{r}_{i} = 0 \) for \( r = 1, 2, \ldots, K \). Suppose that the matrix \( (F; \tau_{(m)}^{r}, \beta_{(m)}^{(m)}) \) is of rank \( \left( \frac{K+m-1}{m} \right) (n-K) \), then by virtue of (5.6) it is seen that we can find the quantities \( G^{k}_{a(m)T(m)} \) such that

\[
(5.8) \quad G^{k}_{a(m)T(m)} F; \tau_{(m)}^{r}, \beta_{(m)}^{(m)} = F'(\delta_{i}^{k} - p_{a}^{k} \mathfrak{E}_{i}^{a}) \delta_{a(m)}^{\beta(m)}.
\]

We see that thus obtained \( G^{k}_{a(m)T(m)} \) obey the transformation law of the form

\[
(5.9) \quad G^{k}_{a(m)T(m)} = G^{k'}_{a(m)T'(m)} X_{i}^{k} X_{j}^{j} U_{a(m)}^{\alpha(m)} U_{T'(m)}^{\tau(m)} + R^{a} P^{i}.
\]

We may write all system of the solutions of (5.8) in the forms

\[
(5.10) \quad G^{k}_{a(m)T(m)} = g^{k}_{a(m)T(m)} + \varphi^{k}_{a(m)T(m)} t_{a}^{k} + \psi^{k}_{a(m)T(m)} t_{a}^{k},
\]

where \( g^{k}_{a(m)T(m)} \) and \( \varphi^{k}_{a(m)T(m)} \) are quantities of \( F_{n}^{(m)} \) and \( \psi^{k}_{a(m)T(m)} \) are any quantities. Accordingly, it is known that the quantity \( G^{k}_{a(m)T(m)} T^{j}_{j} \) is intrinsic and is the same for all system of the solutions of (5.8), when \( T^{j}_{j} \) is an intrinsic quantity satisfying the relations \( p^{j}_{i} T^{j}_{j} = 0 \) for \( r = 1, 2, \ldots, K \), that is to say, the intrinsic quantity \( G^{k}_{a(m)T(m)} T^{j}_{j} \) is uniquely determined by the equations (5.8).

If (5.7) be multiplied by \( G^{k}_{a(m)T(m)} \) and summed for \( i_{1}, i_{2}, \ldots, i_{m} \) and \( j \), we have the intrinsic quantity

\[
m (\delta_{i}^{k} - p_{a}^{k} \mathfrak{E}_{i}^{a}) (K+1) \cdots (K+m-1) G^{k}_{a(m)T(m)} (F; \tau_{(m)}^{r}, \beta_{(m)}^{(m)} - 1) \frac{1}{F} \frac{1}{2} \frac{m(m-1)}{2} F^{(m), \omega_{1}, \omega_{2}} \mathcal{B}_{(m)} G^{a_{(m)}T_{(m)}^{a_{(m)}}} v^{i}.
\]

Putting \( \beta_{i} = a_{1}, \beta_{2} = a_{2}, \ldots, \beta_{m-1} = a_{m-1} \) and contracting over these indices, we get

\[
(\delta_{i}^{k} - p_{a}^{k} \mathfrak{E}_{i}^{a}) (K+1) \cdots (K+m-1) G^{k}_{a(m)T(m)} (F; \tau_{(m)}^{r}, a_{(m-1)}^{a_{(m-1)}}) \frac{1}{F} \frac{1}{2} \frac{m(m-1)}{2} F^{(m), \omega_{1}, \omega_{2}} \mathcal{B}_{(m)} G^{a_{(m)}T_{(m)}^{a_{(m)}}} v^{i}.
\]

Accordingly, if we put

\[
\Lambda^{k}_{t_{a}} = \frac{(m-1)!}{(K+1) \cdots (K+m-1)} \frac{1}{F} \frac{1}{2} \frac{m(m-1)}{2} F^{(m), \omega_{1}, \omega_{2}} \mathcal{B}_{(m)} G^{a_{(m)}T_{(m)}^{a_{(m)}}} v^{i}.
\]

\( (m>2) \).
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\[ \Lambda_{a}^{k} = \frac{1}{K+1} \frac{1}{F} G_{a_{1}a_{2}a_{3}}^{(2)} F_{i}^{j_{1}} a_{1}^{i} + (F_{i}^{j_{1}} a_{1}^{i}) \partial_{i} + 2 \delta_{i}^{j_{1}} F_{i}^{j_{1}} \gamma_{j}^{i} a_{1}^{i} - 2 \delta_{i}^{j_{1}} F_{i}^{j_{1}} \gamma_{j}^{i} \]  

\[ (m = 2), \]

we have the intrinsic quantity

\[ (5.11) \quad \Delta_{a} v^{k} = (\delta^{k}_{i} - \partial_{i}^{k}) v^{k}_{/a} + \Lambda_{a}^{k} v^{i} \]

which is the same for all systems of the solutions of (5.8). If \( w_{i} \) be any covariant vector of \( F_{i}^{(m)} \) such that \( w_{i} p^{i}_{\tau} = 0 (\tau = 1, 2, \ldots, K) \), one obtains from (5.11) the intrinsic derivative of \( w_{i} \):

\[ \Delta_{a} w = w_{/a} - \Lambda_{a}^{k} w^{k} \]

by means of \( \Delta_{a} (v^{i} w_{i}) = (v^{i} w_{i})_{/a} \). Accordingly, it is easily seen that the quantity

\[ (5.12) \quad \Delta_{a} F_{i}^{a_{1}a_{2}a_{3}}^{(m)} = \frac{1}{F} \left( F_{i}^{a_{1}a_{2}a_{3}} - \Lambda_{a_{1}}^{k} F_{i}^{a_{2}a_{3}} + m G_{a_{1}a_{2}a_{3}}^{(m)} \right) \]

is intrinsic. Moreover, we see from (5.10) and the definition of \( \Lambda_{a}^{k} \) that \( \Lambda_{a_{1}}^{k} F_{i}^{a_{1}a_{2}a_{3}}^{(m)} \) does not depend on \( \delta_{i}^{a_{1}} \), so that (5.12) can be written in the form

\[ (5.13) \quad \Delta_{a} F_{i}^{a_{1}a_{2}a_{3}}^{(m)} = H_{a_{1}}^{a_{2}a_{3}} b_{(m+1)} p_{b_{(m+1)}}^{i} + P_{a_{1}a_{2}a_{3}}^{(m)} (x^{i}, p_{b_{(1)}}^{j}, \ldots, p_{b_{(m)}}^{j}). \]

When \( m > 2 \), we have from (4.3b) an identity

\[ \sum_{s = m-1}^{m} \left( \frac{s}{m-1} \right) p_{a_{1}a_{2}a_{3}}^{(m)} F_{i}^{a_{1}a_{2}a_{3}}^{(m-1)} = 0. \]

Differentiating this with respect to \( p_{a_{1}a_{2}a_{3}}^{(m)} \) one obtains

\[ F_{i}^{a_{1}a_{2}a_{3}}^{(m)} b_{(m-1)} p_{a_{1}a_{2}a_{3}}^{(m-1)} p_{a_{1}a_{2}a_{3}}^{(m)} = -m F_{i}^{a_{1}a_{2}a_{3}}^{(m-1)} b_{(m-1)} p_{a_{1}a_{2}a_{3}}^{(m-1)} \]

and consequently it follows that

\[ p_{a_{1}a_{2}a_{3}}^{(m)} \Lambda_{a_{1}}^{k} F_{i}^{a_{1}a_{2}a_{3}}^{(m)} = p_{a_{1}a_{2}a_{3}}^{(m)} (F_{i}^{a_{1}a_{2}a_{3}}^{(m)} - \Lambda_{a_{1}}^{k} F_{i}^{a_{2}a_{3}}^{(m)}) \]

\[ = -F_{i}^{a_{1}a_{2}a_{3}}^{(m)} p_{a_{1}a_{2}a_{3}}^{(m)} + \frac{(m-1)!}{(K+1) \cdots (K+m-1)} \frac{1}{F} G_{a_{1}a_{2}a_{3}}^{(m)} \left( F_{i}^{a_{1}a_{2}a_{3}}^{(m)} - \Lambda_{a_{1}}^{k} F_{i}^{a_{2}a_{3}}^{(m)} \right) \]

\[ = -F_{i}^{a_{1}a_{2}a_{3}}^{(m)} p_{a_{1}a_{2}a_{3}}^{(m)} + \frac{m!}{(K+1) \cdots (K+m-1)} \frac{1}{F} G_{a_{1}a_{2}a_{3}}^{(m)} \left( F_{i}^{a_{1}a_{2}a_{3}}^{(m)} - \Lambda_{a_{1}}^{k} F_{i}^{a_{2}a_{3}}^{(m)} \right) \]

\[ = -F_{i}^{a_{1}a_{2}a_{3}}^{(m)} p_{a_{1}a_{2}a_{3}}^{(m)} + \frac{m!}{(K+1) \cdots (K+m-1)} (\delta_{i}^{a_{1}} - \partial_{i}^{a_{1}}) \delta_{a_{2}a_{3}}^{(m-1)} b_{(m-1)} p_{a_{2}a_{3}}^{(m)} \]

\[ = -F_{i}^{a_{1}a_{2}a_{3}}^{(m)} p_{a_{1}a_{2}a_{3}}^{(m)} + F_{i}^{a_{1}a_{2}a_{3}}^{(m)} p_{a_{1}a_{2}a_{3}}^{(m)} = 0. \]

When \( m = 2 \), it follows that
\[ p_i \Delta_a F_i = p_i \left( F_i + A^a_{\alpha i} F_{a i} \right) \]
\[ = -F_i p_{\alpha i} - \frac{1}{K+1} \frac{1}{F} G^k_{\beta} \left( F_i + A^a_{\alpha i} F_{a i} \right) \delta^i_{\beta} \]
\[ = -\delta^i_{\beta} F_i \delta^i_{\beta} \]

On the other hand, differentiating the identity

\[ F_i = p_i + 2F_i p_{\beta i} = \delta^i_{\beta} F_i \]

with respect to \( p_i \), we have

\[ F_i + 2F_i p_{\beta i} = \delta^i_{\beta} F_i \]

and consequently

\[ p_i \Delta_a F_i = -F_i p_{\alpha i} - \frac{2}{K+1} \frac{1}{F} G^k_{\beta} \delta^i_{\beta} \]

Hence, we have from (5.13) the relations

\[ (5.14) \quad p_i P_{\alpha i} = 0 \quad (r = 1, 2, \ldots, K) \]

Suppose now that the rank of the \( (K+m) \times (m+1) \) -rowed matrix

\( H_{\alpha i}^{a(m)} \) is \( (n-K) \times (m+1) \), then we can find the quantities

\( H_{\alpha i}^{a(m)} \) such that

\[ (5.15) \quad H_{\alpha i}^{a(m)} = \delta_{i}^{j} \delta_{j}^{k} \delta_{k}^{\beta} \]

We may write all systems of the solutions of (5.15) in the form

\[ (5.16) \quad H_{\alpha i}^{a(m)} = h_{\alpha i}^{a(m)} + \varphi_{\alpha i}^{a(m)} \]

where \( h_{\alpha i}^{a(m)} \) and \( \varphi_{\alpha i}^{a(m)} \) are quantities of \( F_{\alpha i}^{(m)} \), and \( \varphi_{\alpha i}^{a(m)} \) are any quantities.

If (5.13) be multiplied by \( H_{\alpha i}^{a(m)} \) and contracted over the indices \( i, a, \alpha_i, \ldots, \alpha_m \), it is obtained the intrinsic quantity

\[ (5.17) \quad (\delta_{i}^{j} - p_{k}^{\beta} \delta_{k}^{j}) P_{\alpha i}^{a(m)} \]

and by (5.14) it is seen that the quantity \( H_{\alpha i}^{a(m)} P_{\alpha i}^{a(m)} \) is the same for all solutions of (5.15).

If (5.17) be multiplied by \( (\delta_{i}^{j} - p_{k}^{\beta} \delta_{k}^{j}) \) and summed for \( k \), we have, in consequence of (5.6) and (5.16),
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\[
(\delta^j_{\lambda}-p^j_{\lambda}(\xi^j))\left[p^l_{\Gamma(m+1)}+H^l_{\Gamma(m+1)}(x^l, p^l_{\alpha(1)}, \cdots, p^l_{\alpha(m)})\right]
\]
or
\[
(5.18) \quad T^l_{\Gamma(m+1)} = p^l_{\Gamma(m+1)}+H^l_{\Gamma(m+1)}-p^l_{\lambda}(\xi^l),
\]
putting \( H^l_{\Gamma(m+1)} = h^a_{\alpha(m)}x^k_{\Gamma(m+1)}p^a_{(m)} \), \( \chi^l_{\Gamma(m+1)} = \xi^l(p^l_{\Gamma(m+1)}+H^l_{\Gamma(m+1)}) \). It should be observed that \( H^l_{\Gamma(m+1)} \) is a quantity of \( F^{(m)} \).

Let us now observe the following facts. If \( Q^A \) be any quantity of \( F^{(m)} \) which is transformed by arbitrary transformations of parameters in the manner

\[
Q^A' (x^l, p^l_{\alpha(1)}, \cdots, p^l_{\alpha(m)}) = R^A' (Q^A, U^a_{\alpha(1)}, \cdots, U^a_{\alpha(s)}),
\]
\( R^A' \) being a rational function of its arguments, then the relations

\[
Q^A;_{i}^{\alpha(m)}p^l_{\alpha(l)} = 0 \quad (r = 1, 2, \cdots, K)
\]
hold when \( s < m \). On the other hand, since \( A^a_{\alpha(s)} (t \leq s) \) are polynomials of \( U^a_{\alpha(1)}, U^a_{\alpha(2)}, \cdots, U^a_{\alpha(s-t+1)} \), the equation of transformation of the quantity

\[
F^a_{\alpha(r)} \beta(s-r) = 0 \quad (1 \leq r < s \leq m)
\]
appearing in the course of formation of the SYNGE vector contains the derivatives of \( u^\alpha \)'s not more than the \((m-r)\)-th order. And the quantity \( K^a_{\alpha(s)} (1 \leq s \leq t \leq m-1) \) determined from (3.5) and (3.7) has the transformation law:

\[
K^a_{\alpha(t)} = U^a_{\alpha(s)} \sum_{r=s}^{t} A^a_{\alpha(r)} K^a_{\alpha(r)}
\]
in which \( U^a_{\alpha(s-t+1)} \)'s are contained as the highest derivatives of \( u^\tau \).

Let us now define the differential operator \( D_r \) applied to the quantity \( Q^A \) as follows:

\[
D_r Q^A = Q^A;_{i}^{\alpha} + \sum_{s=t}^{m-1} Q^A;_{i}^{\alpha(s)}p^a_{\alpha(s)} - Q^A;_{i}^{\alpha(m)}(H^a_{\alpha(m)} - p^a_{\lambda}(\xi^a))
\]
or
\[
D_r Q^A = Q^A;_{i}^{\alpha} + \sum_{s=t}^{m-1} Q^A;_{i}^{\alpha(s)}p^a_{\alpha(s)} - Q^A;_{i}^{\alpha(m)}H^a_{\alpha(m)}
\]
then it is also a quantity of \( F^{(m)} \) and its transformation law under (1.1) and (1.2) is the same as that of \( Q^A;_{i}^{\alpha} \). Consequently, if the quantities \( K^a_{\alpha(r)} \) and \( \xi^a_{\alpha(r)} \) in which the operator \( D_r \) is applied instead of the operator \( /r \) are denoted by \( K^a_{\alpha(r)} \) and \( \xi^a_{\alpha(r)} \) respectively, these are quantities of \( F^{(m)} \), and the transformation laws of \( K^a_{\alpha(r)} \) and \( \xi^a_{\alpha(r)} \) are the
same as that of $K_{r(t)}^{\beta(s)}$ and $\mathfrak{G}_{l}^{a(r)}$ respectively. Moreover the same relations as (5.6) hold in this case.

We can now find one and only one system of the quantities $\hat{G}_{a(m)\tau(m)}^{kj}$ satisfying the equations

$$\hat{G}_{a(m)\tau(m)}^{kj} F^{\omega_{l}}_{;j} (\beta(m); i) = F^{\tau}_{;j} (\beta(m); i) \hat{G}_{a(m)\tau(m)}^{kj} (\alpha = 1, 2, \ldots, K),$$

since we have supposed that the rank of the matrix $(F^{\tau}_{;j} (\beta(m); i))$ is $(n-K)(K+m-1)$. And it is easily seen that thus obtained $\hat{G}_{a(m)\tau(m)}^{kj}$ is intrinsic and symmetric with respect to the rows $\alpha(m)$ and $\tau(m)$, so that

(5.19) $$\hat{G}_{a(m)\tau(m)}^{kj} \mathfrak{G}_{\alpha(m)\tau(m)}^{kj} = 0 \quad (\alpha = 1, 2, \ldots, K).$$

If we put

$$\hat{A}_{ia}^{k} = \frac{(m-1)}{(K+1)(K+m-1)} \frac{1}{F} \hat{G}_{a(m-1)\tau(m-1)}^{kj} F^{\omega_{l}}_{;j} (\beta(m-1); i),$$

then

$$(\delta^{k}_{l} - p_{\beta}^{l} \mathfrak{G}^{\beta}) v_{l\alpha}^{i} + \hat{A}_{ia}^{k} v^{i}$$

is an intrinsic quantity as it is seen from (5.11).

When we put $v^{i} = p_{\gamma}^{l} v^{\tau}$, by virtue of (5.6) it follows that $\hat{G}_{i}^{\tau} v^{l} = v^{\tau}$, and consequently we have

(5.20) $$\delta^{k}_{l} - p_{\beta}^{k} \hat{G}_{\beta}^{\gamma} v^{i} + \hat{A}_{ia}^{k} v^{i} = v_{l\alpha}^{k} - p_{\beta}^{k} (v_{l\alpha}^{\beta} + \hat{G}_{a\tau}^{\beta} v^{\tau}) + (p_{\beta}^{k} G_{a\tau}^{\beta} \hat{G}_{\beta}^{\gamma} + p_{\beta}^{k} (\hat{G}_{a\tau}^{\beta} + \hat{A}_{ia}^{k})) v^{i}$$

from which we see that

(5.21) $$A_{a} v^{k} = v_{l\alpha}^{k} + \Gamma_{l\alpha}^{k} v^{i}$$

defines an intrinsic derivative of the vector $v^{k}$, where we put

$$\Gamma_{l\alpha}^{k} = \hat{A}_{ia}^{k} + p_{\beta}^{k} (\hat{G}_{a\tau}^{\beta} + \hat{G}_{a\tau}^{\beta} v^{\tau}).$$

From (5.5), (5.19) and (5.20) we have

$$\hat{G}_{k}^{\beta} (v_{l\alpha}^{k} + \Gamma_{l\alpha}^{k} v^{i}) = v_{l\alpha}^{\beta} + G_{a\tau}^{\beta} v^{\tau},$$

Consequently, if one defines the intrinsic derivative along the $K$-dimensional surface $x^{i} = x^{i}(u^{a})$ by (5.21), putting
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$p^i_{\alpha(s)}=\frac{\partial x^i}{\partial u^a}\cdots\frac{\partial x^i}{\partial u^s}$ $(s=0,1,\cdots,m)$, then

$$\Delta_{\alpha}v^\beta\equiv\frac{\partial}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^\alpha}$$

may be regarded as the intrinsic derivative induced on the $K$-dimensional surface. Therefore, the covariant differential of the vector $v^k$ along a $K$-dimensional surface is given by

$$\delta_s v^k = dv^k + \Gamma_{j\tau}^{k}v^j du^\tau$$

and the induced covariant differential is given by

$$\delta_s v^\beta = du^\beta + G_{\alpha\gamma}^\beta v^\alpha du^\gamma$$

when we put $p^i_{\alpha(s)}=\frac{\partial x^i}{\partial u^a}\cdots\frac{\partial x^i}{\partial u^s}$ $(s=0,1,\cdots,m)$.

In order to determine the base connections and the connections in $F_n^{(m)}$ we put

$$\Gamma_{\alpha\tau}^{\beta} = \Lambda_{J\tau}^{\beta} + D_{\alpha} + G_{\alpha\gamma}^{\beta} p_{\beta}^{i}$$

then it is a quantity of $F_n^{(m)}$ and has the same transformation law as that of $I_{\alpha\tau}^{\beta}$ under the transformations (1.1) and (1.2), that is,

$$(5.22) \quad \Gamma_{\alpha\tau}^{\beta} = \hat{\Gamma}_{\alpha\tau}^{\beta}, X_{\beta}^{\prime\prime}U_{\tau}^{T} - X_{\beta}^{\prime\prime}p_{\beta}^{i}U_{\tau}^{T},$$

as it is seen from the tensor character of (5.21).

§ 6. Base connections in $F_n^{(m)}$ and covariant differentials. In order to define the base connections in $F_n^{(m)}$, we shall introduce the intrinsic Pfaffian forms by means of theorem 8.

Since $dx^i$ may be regarded as an intrinsic Pfaffian form,

$$\omega_{\beta(1)}^{i}(d) = P_{\beta(1)}^{a(1)} dp_{a(1)}^{k} + P_{\beta(1)}^{i} dx^k$$

is an intrinsic Pfaffian form, where the coefficients $P_{\beta(1)}^{a(1)}$ and $P_{\beta(1)}^{i}$ are determined from the recurring formulae (3.11) as follows:

$$P_{\beta(1)}^{a(1)} = \delta_{a_{\tau}}^{i}, \quad P_{\beta(1)}^{i} = \hat{\Gamma}_{k\beta}^{i},$$

so that

$$\omega_{\beta(1)}^{i}(d) = dp_{\beta(1)}^{i} + \hat{\Gamma}_{k\beta}^{i} dx^k$$

is an intrinsic Pfaffian form of $F_n^{(m)}$. In general, we obtain the intrinsic Pfaffian forms

$$(6.1) \quad \omega_{\beta(s)}^{i}(d) = \sum_{r=0}^{s} P_{\beta(s)}^{a(r)} dp_{a(r)}^{k} (s=0,1,\cdots,m),$$
where the coefficients $P_{\beta(s)k}^{a(r)}$ are determined from the recurring formulae
\[
P_{\beta(s)k}^{a(r)} = P_{\beta(s-1)k}^{a(r'-1)} a^{r}_{\beta s} + D_{\beta(s-1)}^{a(r)} + \Gamma_{\beta(s-1)k}^{a(r)} - (s-1) G_{\beta(s-1)k}^{a(r)}
\]
\[\text{for } 0 \leq r \leq s \leq m,\]
putting $P_{\beta(0)k}^{a(0)} = \delta_{k}^{a}$.

It is evident from the above recurring formulae that the coefficients of the differential $dp_{\beta(s)}^{k}$ in the Pfaffian form $\omega_{\beta(s)k}^{a}$, say $P_{\beta(s)k}^{a(r)}$, is of the form $\delta_{k}^{a} \delta_{\beta(s)}^{a}$, so that (6.1) becomes
\[
\omega_{\beta(s)k}^{a}(d) = dp_{\beta(s)k}^{a} + \sum_{r=0}^{s-1} P_{\beta(s)k}^{a(r)} dp_{\alpha(r)}^{k} \quad (s=0,1, \cdots, m).
\]

Hence, we can define the base connections in $F_{n}^{(m)}$ by the equations
\[
\omega_{\beta(s)k}^{a}(d) = 0 \quad (s=0,1, \cdots, m).
\]

We shall next introduce a covariant differential of vector of $F_{n}^{(m)}$ by means of theorem 5.

We have seen in theorem 5 that the transformation law of the quantity $\mathfrak{P}_{\beta}(L)$ defined by
\[
\mathfrak{P}_{\beta}(L) = \sum_{t \sim \beta \rightarrow 1}^{m} K_{\gamma(t)}^{\beta} \sum_{s \in t}^{m} \left( \begin{array}{l} s \\ t \end{array} \right) \mathcal{L}_{;k}^{(s)}(s-t) dp_{\alpha(s-t)}^{k}
\]
is
\[
\mathfrak{P}_{\beta}(L) = U_{\beta}^{\beta}, \mathfrak{P}_{\beta'}(L),
\]
when $L$ is a quantity of $F_{n}^{(m)}$. If we put $L = \tilde{\mathcal{I}}_{\beta}$ into $\mathfrak{P}_{\beta}(L)$ and contract over the index $\beta$, it follows from (5.22) that the quantity $\mathfrak{P}_{\beta}(\tilde{\mathcal{I}}_{\beta})$ is transformed by the transformations (1.1) and (1.2) in the manner
\[
\mathfrak{P}_{\beta}(\tilde{\mathcal{I}}_{\beta}) = \mathfrak{P}_{\beta'}(\tilde{\mathcal{I}}_{\beta'}) \mathcal{X}_{\beta}^{\beta'} \mathcal{X}_{\beta'}^{\beta} = K \mathcal{X}_{\beta}^{\beta'} \mathcal{X}_{\beta'}^{\beta} dx' dx'',
\]
so that the Pfaffian form
\[
\mathcal{I}_{\beta} = \frac{1}{K} \mathfrak{P}_{\beta}(\tilde{\mathcal{I}}_{\beta})
\]
obeys the transformation law
\[
\mathcal{I}_{\beta} = \mathcal{X}_{\beta}^{\beta'} \mathcal{I}_{\beta'} - \mathcal{X}_{\beta'}^{\beta} \mathcal{X}_{\beta'}^{\beta} dx' dx''.
\]
Consequently, if $v^{i}$ be a vector of $F_{n}^{(m)}$,
\[
\delta v^{i} = dv^{i} + \mathcal{I}_{\beta}^{i} v^{j}
\]
defines a covariant differential of the vector $v^{i}$.

We may write (6.4) in the form
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(6.5) \[ \delta v^t = dv^t + \sum_{s=0}^{m-1} C^t_{j(s)} v^t dp^s_{\beta(s)} , \]

when we put

\[ C^t_{j(s)} = \frac{1}{K} \sum_{s=0}^{m-1} \left( \begin{array}{c} t+s \\ t \end{array} \right) K_{\tau(t)} \gamma_{j(s)} \delta_{\beta(s)} \]

\[ \delta v^t = dv^t + \sum_{s=0}^{m-1} C^t_{j(s)} v^t dp^s_{\beta(s)} \]

On the other hand, we may also introduce another intrinsic differential of vector by means of theorem 7.

From (6.2) we have the intrinsic Pfaffian form

\[ \omega_{\beta(m)}^t = dp^t_{\beta(m)} + \sum_{s=0}^{m} P^t_{\beta(m) a(s)} dp^s_{a(s)} . \]

Applying theorem 7 one obtains the intrinsic quantity

\[ \delta_{t}^s \omega_{\beta(m)}^t \delta v^t + \sum_{s=0}^{m} P^t_{\beta(m) a(s)} dp^s_{a(s)} \]

from which we obtain the intrinsic differential of the vector \( v^t \):

\[ ' \delta v^t = dv^t + \frac{m!}{K(K+1) \cdots (K+m-1)} \sum_{s=0}^{m-1} P^t_{\beta(m) a(s)} dp^s_{a(s)} . \]

\section{7. Covariant derivatives, torsion tensors and curvature tensors.}

When we put

\[ \delta v^t = \sum_{s=0}^{m} (\nabla^t_{j(s)} v^t) \omega^t_{\beta(s)} \]

it is obtained from (6.2) and (6.5) the recurring formulae for the covariant derivatives \( \nabla^t_{j(s)} v^t \) \( (s = 0, 1, \ldots, m) \), that is

\[ \nabla^t_{j(s)} v^t = v^t_{;j(s)} + C^t_{j(s)} v^t \]

\[ \nabla^t_{j(s)} v^t = v^t_{;j(s)} + C^t_{j(s)} v^t - \sum_{s=t+1}^{m} P^t_{j(s) a(s)} \nabla^t_{a(s)} v^t \]

We shall next determine the torsion tensors and the curvature tensors of \( F^t_{(m)} \).

If \( \delta_1 \) and \( \delta_2 \) denote the intrinsic differential operators corresponding to the increments \( d_1 \) and \( d_2 \) respectively, we can find the torsion tensors \( A^t_{a(r) j(s)} \) and \( A^t_{j(s)} \) by means of the equations

\[ \delta_1 (\omega^t_{a(r)} (d_1) - \delta_2 \omega^t_{a(r)} (d_2) = \sum_{p=0}^{r} \sum_{q=0}^{r} A^t_{a(r) j(s)} \omega^t_{\beta(q)} (d_2) \omega^t_{\gamma(q)} (d_1) \]

\[ + \sum_{p=0}^{r} \sum_{q=0}^{r} A^t_{a(r) j(s)} \omega^t_{\beta(q)} (d_2) \omega^t_{\gamma(q)} (d_1) \]

Indeed, we have

\[ \text{[The symbol } [\omega^t_{\beta(q)} (d_2) \omega^t_{\gamma(q)} (d_1)] \text{ means } \omega^t_{\beta(q)} (d_2) \omega^t_{\gamma(q)} (d_1) - \omega^t_{\beta(q)} (d_1) \omega^t_{\gamma(q)} (d_2) \].
\[ A_{\alpha(r)fk}^{l6(r)\tau(q)} = C_{jk}^{iT(q)}\delta_{a(r)}^\beta(r) - C_{kf}^{l6(r)}\delta_{a(r)}^\Gamma(q) - \sum_{s\leftrightarrow q+1}^{m-1} A_{a(r)f}^{i6(r)\omega_l(s)}P_{\omega(s)k}^{lT(q)} \]

\[ (r=m, m-1, \ldots, 0; q=m-1, \ldots, 0) \]

where we put \( P_{\omega_{\backslash}s)k}^{l\tau(q)} = 0 (s \leq q) \) and \( A_{lk}^{i\Gamma(q)} = A_{\alpha(0)lk}^{\mathcal{B}(0)T(q)} \).

Next, by means of the equation

\[
[\delta_3 - \delta_2 \delta_1] v^i = \sum_{t=0}^{m-1} R_{lhk}^{i\alpha(t)\beta(m)} \left[ \omega_{\beta(n\iota)}^{k}(d_{J})(o_{a(t)}^{h}(d_{3}) \right] v^l
\]

we have the recurring formulae for the curvature tensors:

\[
R_{lhk}^{ia(m-1)\beta(m)} = C_{lh;k}^{ia(m-1)\beta(m)}
\]

\[
R_{lhk}^{ia(t)\delta(m)} = C_{lh;k}^{ia(t)\beta(m)} - \sum_{s=t+1}^{m-1} R_{lr'h}^{\Gamma(s)\delta(m)} P_{t(\epsilon)h}^{qa(t)}
\]

\[ (t=m-1, m-2, \ldots, 1, 0) \]

\[
R_{lf}^{i\delta^{(t)}\alpha(\epsilon)} = C_{fh;f}^{i\beta(t)\alpha(\epsilon)} - C_{ff;f}^{la(s)/;(t)} + C_{jf}^{ia(s)}C_{lh}^{f\beta(t)}
\]

\[ - C_{fh}^{i\delta^{(t)}}C_{lf}^{fa(s)} + R_{lf}^{i\alpha(s)\tau_{k}(m)} P_{\Gamma(m)h}^{k.l(\ell)} - R_{lhk}^{i\delta^{(t)T(m)}} P_{T(m)f}^{k\alpha(s)} + \sum_{r=s+1}^{m-1} R_{lgk}^{i\delta(r)\mu(m)} P_{\mu(m)h}^{k\beta(t)}P_{\delta(r)f}^{ga(s)}
\]

\[ (t, s = m-1, m-2, \ldots, 1, 0) \]

\section{Method of A. KAWAGUCHI.}

In the case of one parameter
Prof. A. Kawaguchi [3] has introduced a base connection in the manifold of line-elements of higher order. We shall generalize this method to the manifold of surface-elements of higher order.

By theorem 5 the quantities
\[
\frac{1}{F'} \mathfrak{P}^{T(s)}(F_{;i}^{a(m)}) = \frac{1}{F'} \sum_{t=s}^{m} R_{\beta}^{\gamma(s)}(t) \sum_{l=1}^{m} \left( \begin{array}{l} t \nolimits \end{array} \right) F^{a(m)}_{;i} \beta(l) \mu(t-r) dp_{\mu(t-r)}^{l}
\]
are intrinsic. Putting \( t - l = r \), we have the intrinsic quantities
\[
(8.1) \quad \frac{1}{F'} \binom{m}{s} F^{a(m)}_{;i} \beta(s) \nu(s) \mu(s) dp_{\mu(s)}^{j} + \frac{1}{F'} \sum_{r=0}^{m-s-1} M_{i}^{a(m)\tau(S)\mu(r)} dp_{\mu(r)}^{j}
\]
where
\[
M_{i}^{a(m)\tau(S)\mu(r)} = \sum_{l=s}^{m-r} K_{\beta(l)}^{a(m)\tau(S)\mu(r)} \left( \begin{array}{l} l+r \nolimits \end{array} \right) F^{a(m)}_{;i} \beta(l) \mu(r) .
\]
Hence, we can derive from (8.1) the intrinsic Pfaffian forms
\[
(8.2) \quad \delta p_{\nu(m-s)}^{k} = (\delta_{i}^{k} - p_{a}^{k} \Phi_{J}^{a(m)}) dp_{\nu(m-s)}^{j} + \sum_{t=0}^{m-s-1} N_{T(m-s)f}^{k\beta(t)} dp_{\beta(t)}^{j}
\]
where
\[
N_{T(m-s)f}^{k\beta(t)} = \frac{1}{F} \binom{m+K-1}{s} G_{\nu(m-s)\nu(s)a(m)}^{ik} M^{a(m)\nu(s)\beta(t)} .
\]

Moreover we have the intrinsic quantity
\[
\frac{1}{F} \delta F^{a(m)}_{;i} = \frac{1}{F} (dF^{a(m)}_{;i} - \Gamma_{i}^{j} F^{a(m)}_{;j})
\]
from which one gets the intrinsic Pfaffian form
\[
(8.3) \quad \delta p_{\nu(m)}^{k} = (\delta_{i}^{k} - p_{a}^{k} \Phi_{J}^{a}) dp_{\nu(m)}^{j} + \sum_{t=0}^{m-1} N_{\nu(m)}^{k\beta(t)} dp_{\beta(t)}^{j}
\]
where
\[
N_{\nu(m)}^{k\beta(t)} = \frac{1}{F} G_{\nu(m)\nu(s)}^{k} (F^{a(m)}_{;i} \beta(t) - C_{i}^{\beta(t)} F^{a(m)}_{;i}) .
\]

We may define the base connections in \( F^{a(m)}_{;i} \) by the equations
\[
\partial p_{T(m)}^{i} = 0 \quad (t = 0, 1, \cdots, m)
\]
Using of (6.5), (8.2) and (8.3) we can determine the covariant derivatives \( \nabla^{a(m)}_{\nu} \) under the conditions \( \nabla^{a(m)}_{\nu} \delta p_{T(m)}^{i} = 0 \) (\( i = 1, 2, \cdots, K \)), that is,
§ 9. Method of D. D. Kosambi. D. D. Kosambi has introduced a system of covariant derivatives in his work [2] on the path space of higher order by using of the special method. We shall now generalize this method in our metric space.

If \( v^j \) be a vector of \( F_{n}^{(m)} \), the quantity \( \nabla^a_{\dot{i}} v^j = v^j_{;\dot{i}} \) is an intrinsic derivative.

Now we see that when \( m > 1 \),

\[
\nabla^a_{\dot{i}} v^j = v^j_{;\dot{i}} + S_{\dot{i}\beta}^{\dot{\alpha}} v^j_{\dot{\alpha}}
\]

is an intrinsic quantity, because \( \nabla^a_{\dot{i}} v^j \) is a tensor. Moreover we see that \( \nabla^a_{\dot{i}} v^j \) may be written in the form

\[
\nabla^a_{\dot{i}} v^j = v^j_{;\dot{i}} + S_{\dot{i}\beta}^{\dot{\alpha}} v^j_{\dot{\alpha}}
\]

where

\[
S_{\dot{i}\beta}^{\dot{\alpha}} = \frac{m}{K + m - 1} \left( \delta_{\dot{\alpha}}^{\dot{\beta}} - \delta_{\dot{\beta}}^{\dot{\alpha}} \right)
\]

putting \( G_{\beta m} = G_{\beta m}^{l} \).

We shall prove that in general the representations

\[
\nabla^a_{\dot{i}} v^j = v^j_{;\dot{i}} + \sum_{s=r+1}^{m} S_{\dot{i}\beta}^{\dot{\alpha}} v^j_{\dot{\alpha}}
\]

and

\[
\nabla^a_{\dot{i}} v^j = v^j_{;\dot{i}} + \sum_{s=1}^{m} S_{\dot{i}\beta}^{\dot{\alpha}} v^j_{\dot{\alpha}} + \frac{1}{K} \Gamma_{\dot{i}\beta}^{\dot{\alpha}} v^j_{\dot{\alpha}}
\]

is true when we put

\[
\nabla^a_{\dot{i}} v^j = \frac{r + 1}{K + r} \left( \nabla^a_{\dot{i}} \delta_{\dot{\beta}}^{\dot{\alpha}} - \delta_{\dot{\beta}}^{\dot{\alpha}} \nabla^a_{\dot{i}} \delta_{\dot{\alpha}}^{\dot{\beta}} \right) v^j
\]

\[
- \left( 1 - \delta_{\dot{\alpha}}^{\dot{\beta}} \right) \frac{r + 1}{K + r} \left( \nabla^a_{\dot{i}} \delta_{\dot{\beta}}^{\dot{\alpha}} \Gamma_{\dot{i}\beta}^{\dot{\alpha}} \right) v^j \quad (r = 0, 1, \ldots, m - 1).
\]
First of all we have
\[
\mathcal{K}^a_i(m)(D_\tau v^j) - D_\tau(\mathcal{P}_i^a(m)v^j) = v_i^{j(a(m-1)}}v^j_i - H^i_{\beta(m)y^l_a\alpha(m)}\mathcal{P}^a_{\beta(m)y^l_j},
\]
or by (9.1)
\[
(9.3) \quad (9.3) \quad \mathcal{P}_i^a(m)(D_\tau v^j) - D_\tau(\mathcal{P}_i^a(m)v^j) = \delta_{\alpha(m)}^{a(m)}v^j_i + U^l_i_{\beta(m)y^l_a\alpha(m)}\mathcal{P}^a_{\beta(m)y^l_j},
\]
where
\[
(9.4) \quad U^l_i_{\beta(m)y^l_a\alpha(m)} = -S^a_{\alpha(m)}\mathcal{P}_i^a(m-1)\delta_{r^{m}}^{a(m)} - H^l_{\beta(m)y^l_a\alpha(m)}
\]
\[H^l_{\beta(m)y^l_a\alpha(m)}\] being that of § 5.

Let us now assume that the representations of two kinds:
\[
(9.5) \quad \mathcal{P}_i^a(r)v^j = v_i^{j(a(r-1)} + \sum_{s=r+1}^{m}S^a_{i\beta(s)}v^j + \frac{1}{K}(\mathcal{P}_i^a\Gamma_{i\beta}^{j(s)})v^j(i = r, m-1, \ldots, t)
\]
and
\[
(9.6) \quad \mathcal{P}_i^a(r+1)D_\tau v^j - D_\tau(\mathcal{P}_i^a(r+1)v^j)
\]
\[= \delta_{\alpha(r+1)}^{a(r+1)}v^j + \sum_{s=r+1}^{m}U^l_i_{\beta(s)y^l_a\alpha(r+1)}\mathcal{P}^a_{\beta(s)y^l_j},
\]
are true, then, after some calculation we see that the equalities
\[
\mathcal{P}_i^a(r)v^j = v_i^{j(a(r-1)} + \sum_{s=r}^{m}S^a_{i\beta(s)}v^j + \frac{1}{K}(\mathcal{P}_i^a\Gamma_{i\beta}^{j(s)})v^j(i = r, m-1, \ldots, t)
\]
and
\[
\mathcal{P}_i^a(r+1)D_\tau v^j - D_\tau(\mathcal{P}_i^a(r+1)v^j)
\]
\[= \delta_{\alpha(r+1)}^{a(r+1)}v^j + \sum_{s=r+1}^{m}U^l_i_{\beta(s)y^l_a\alpha(r+1)}\mathcal{P}^a_{\beta(s)y^l_j},
\]
hold good, where the coefficients \(S^a_{i\beta(s)}\) and \(U^l_i_{\beta(s)y^l_a\alpha(r)}\) are determined from (9.2), (9.4) and the recurring formulae
\[
S^a_{i\beta(s)} = \frac{r+1}{K+r} \left( \sum_{l=r+1}^{m}S^a_{l\beta(s)}U^l_i_{\beta(s)y^l_a\alpha(s)} - D_\tau S^a_{i\beta(s)} - H^l_{\beta(s)y^l_a\alpha(s)} \right) \quad (r < m),
\]
\[
S^a_{i\beta(s)} = \frac{r+1}{K+r} \left( S^a_{i\beta(s)} + \Gamma_{i\beta}^{j(s)} \delta_{\alpha(s)}^{a(s)} \right) \quad (r = m, m-1, \ldots, t).
\]
\[ - (r - 1) G_{\beta r}^{\alpha r} \partial_{\beta (r-1)}^a \partial_{\beta (r-1)}^b G_{\beta r}^{\alpha r} \]

\[ S_{\beta r}^{(r-1)} \tau = \frac{r+1}{K+r} (S_{\beta r}^{(r-1)} \tau - D_{\tau} S_{\beta r}^{(r-1)} \tau + \sum_{t=r+1}^{s} S_{\beta t}^{(r-1)} \tau \ U_{\beta t}^{(r-1)} \ \omega_{\tau}) \]

\[ (s=r+1, r+2, \ldots, m-1) \]

and

\[ U_{\beta (m-1) r}^{\alpha r} = \sum_{t=r+1}^{m} \delta_{\beta (m-1)}^{(t)} U_{\beta (m-1) r}^{\alpha (t)} - S_{\beta (m-1)}^{(r-1)} \delta_{\beta (m-1)}^{\alpha (r)} \]

\[ U_{\beta (r-1) r}^{\alpha r} = \Gamma_{\beta (r-1) r} S_{\beta (r-1)}^{(r)} - r G_{\beta (r-1) r}^{(r)} \delta_{\beta (r-1)}^{(r)} \delta_{\beta (r-1)}^{(r)} + S_{\beta (r-1)}^{(r)} \delta_{\beta (r-1)}^{(r)} \delta_{\beta (r-1)}^{(r)} \]

\[ U_{\beta (s) r}^{\alpha r} = \sum_{t=r+1}^{s} S_{\beta (t)}^{(s)} U_{\beta (s) r}^{\alpha (r)} - S_{\beta (s)}^{(r-1)} \delta_{\beta (s)}^{\alpha (r)} \]

\[ (s=r+1, \ldots, m-1) \]

Consequently, we have the intrinsic Pfaffian forms of the second kind

\[ \frac{K}{3} \delta_{\beta (s)}^{(r)} = dp_{\beta (s)}^{(r)} - \sum_{r=0}^{s-1} S_{\beta (r)}^{(s)} dp_{\beta (r)}^{(s)} \]

\[ (s=0, 1, \ldots, m) \]

§ 10. Metric tensors and metric connection. If we put

\[ G_{\beta (m-1) r}^{(m-1)} f_{\beta (m-1) r} = G_{\alpha \beta} \]

it is easily seen from (5.6), (5.9) and (5.10) that when \( m > 2 \), \( G_{\alpha \beta} \) is an intrinsic quantity of \( F_{n}^{(m)} \) and is the same for all the solutions \( G_{\alpha (m-1) r}^{(m-1)} f_{\beta (m-1) r} \) of the equations (5.8). When \( m=2 \), we put

\[ \frac{1}{F} G_{\beta (2) r}^{(2)} F_{\beta (2) r}^{(2)} = f \]

and derive the intrinsic vectors \( H_{\tau}^{\alpha} \) from the scalar \( f \) as if we derive the intrinsic vectors \( E_{\tau}^{\alpha} \) from the scalar \( F \). If we put

\[ (\delta_{\tau}^{a} - p_{\tau}^{a} \delta_{\tau}^{a}) (\delta_{\tau}^{b} - p_{\tau}^{b} \delta_{\tau}^{b}) G_{\alpha r}^{(r)} H_{r}^{\alpha} = G_{\alpha \beta} \]

this is an intrinsic quantity of \( F_{n}^{(m)} \) and is the same for all the solutions \( G_{\alpha (2) r}^{(2)} f_{\beta (2) r} \).

Moreover, if we put \( F_{n}^{(2)} G_{n}^{(2)} = g_{\alpha \beta} \) assuming that \( G = |G_{\alpha \beta}| \neq 0 \), then the measure of \( K \)-dimensional surface is given by

\[ \int_{(K)} |g_{\alpha \beta}| \\frac{1}{2} du^{1} \ldots du^{K} \]

Hence, it is adequate to take \( g_{\alpha \beta} \) as the metric tensor on the \( K \)-dim-
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Intrinsic surface, when we put 

\[ p_{a(s)}^{i} = \frac{\partial x^{i}}{\partial u^{a}} \cdots \partial u^{a_{s}} \quad (s = 0, \ldots, m) \]

Now we put

\[ \frac{1}{F} g_{\alpha_{1}\beta_{1}} g_{\alpha_{2}\beta_{2}} \cdots g_{\alpha_{m}\beta_{m}} F_{\alpha_{1}^{(m)}} \beta_{1}^{(m)} + g_{a\beta} \mathfrak{G}_{i}^{\alpha} \mathfrak{G}_{f}^{\beta} + g_{if} = g_{if} \]

and assume that the determinant \(|g_{ij}|\) does not vanish, then \(g_{ij}\) is a tensor of \(F_{n}^{(m)}\) and the relation

\[ g_{ij} p_{\alpha}^{i} p_{\beta}^{j} = g_{\alpha\beta} \]

holds good, so that we may take \(g_{ij}\) as the metric tensor of \(F_{n}^{(m)}\). If \(g^{a\beta}\) and \(g^{if}\) be the inverses of \(g_{a\beta}\) and \(g_{ij}\) respectively, it is easily seen that

\[ g^{ij} g_{a\beta} \mathfrak{G}_{i}^{\beta} = p_{a}^{i}, \quad g^{ij} \mathfrak{G}_{j}^{\alpha} \mathfrak{G}_{i}^{\beta} = g^{a\beta} \]

By the method of Prof. A. KAWAGUCHI [4] we obtain the metric connection:

\[ dv^{i} + \frac{1}{2} (\Gamma_{j}^{i} - g^{ik} \Gamma_{k}^{l} g_{lj} + g^{ik} dg_{jk}) v^{j} = 0 \]

References