LINEAR TOPOLOGIES ON SEMI-ORDERED LINEAR SPACES

By

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Normed semi-order linear spaces are considered first by L. KANTOROVITCH.(1) In this paper we shall consider linear topologies on semi-ordered linear spaces.

Let R be a linear space. A manifold $V \subset R$ is called a *vicinity*, if for any $a \in R$ we can find $\varepsilon > 0$ such that $\xi a \in V$ for $|\xi| \leqq \varepsilon$. A collection of vicinities $\mathfrak{V}$ is said to be a *linear topology* on R, if

1) $U \subset V \in \mathfrak{V}$ implies $U \in \mathfrak{V}$,
2) $U, V \in \mathfrak{V}$ implies $UV \in \mathfrak{V}$,
3) $V \in \mathfrak{V}$ implies $\xi V \in \mathfrak{V}$ for every real number ξ,
4) for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{V}$ such that $\xi U \subset V$ for $|\xi| \leqq 1$,
5) for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{V}$ such that $U \times U \subset V$,

adopting the notations:

$$\xi U = \{\xi x : x \in U\}, \quad U \times V = \{x + y : x \in U, y \in V\}.$$

A subset $\mathfrak{B} \subset \mathfrak{V}$ is called a *basis* of a linear topology $\mathfrak{V}$, if for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{B}$ and $\varepsilon > 0$ such that $\varepsilon U \subset V$.

Let R be now a semi-ordered linear space and universally continuous, that is, for any system of positive elements $a_\lambda \in R \; (\lambda \in \Lambda)$ there exists $\bigcap_{\lambda \in \Lambda} a_\lambda$. In this paper we shall consider only such linear topologies $\mathfrak{V}$ on R that $\mathfrak{V}$ have a basis composed only of vicinities V subject to the conditions:

6) $a \in V, \; |x| \leqq a$ implies $x \in V$,
7) $0 \leqq a_\lambda \in V \; (\lambda \in \Lambda), \; a_\lambda \uparrow_{\lambda \in \Lambda} a$ implies $a \in V$.

Here $a_\lambda \uparrow_{\lambda \in \Lambda} a$ means that for any two $\lambda_1, \lambda_2 \in \Lambda$ we can find $\lambda \in \Lambda$ such that

$$a_\lambda \geqq a_{\lambda_1} \cup a_{\lambda_2}, \quad \text{and } a = \bigcup_{\lambda \in \Lambda} a_\lambda .$$

For such a linear topology, we shall prove as a principal result that the manifold $\{x : a \leqq x \leqq b\}$ is complete as a uniform space in WEIL's

(1) L. KANTOROVITCH: Lineare halbgeordnete Räume, Math. Sbornik, 2 (44), (1937), 121-168.

sense.[2]

For a vicinity V subject to the conditions 6), 7), putting

$$\|x\|_V = \inf_{\xi x \in V} \frac{1}{|\xi|},$$

we obtain a *pseudo-norm* on R. A manifold $A \subset R$ is said to be *topologically bounded*, by a linear topology $\mathfrak{V}$, if $\sup_{x \in A} \|x\|_V < +\infty$ for every such vicinity $V \in \mathfrak{V}$. A linear topology $\mathfrak{V}$ on R is said to be *monotone complete*, if for any topologically bounded system $0 \leqq a_\lambda \in R$ $(\lambda \in \Lambda)$ such that $a_\lambda \uparrow_{\lambda \in \Lambda}$, we can find $a \in R$ for which $a_\lambda \uparrow_{\lambda \in \Lambda} a$. With this definition, we can prove that if a linear topology $\mathfrak{V}$ is monotone complete, then R is complete by $\mathfrak{V}$ in WEIL's sense. This result may be considered as a generalization of the famous RIESZ-FISCHER's theorem about L_p-spaces.

A vicinity V is said to be *convex*, if $V \times V \subset 2V$. A linear topology $\mathfrak{V}$ is said to be *convex*, if $\mathfrak{V}$ has a basis composed only of convex vicinities. There exists a linear topology $\mathfrak{V}$ on R of which the totality of convex vicinities subject to the conditions 6), 7) is a basis. This linear topology $\mathfrak{V}$ is called the *strong topology* of R. A linear topology $\mathfrak{V}$ is said to be *sequential*, if $\mathfrak{V}$ has a basis composed of at most countable vicinities. We shall prove that if a linear topology $\mathfrak{V}$ is sequential, convex, complete, and $\prod_{V \in \mathfrak{V}} V = \{0\}$, then $\mathfrak{V}$ is the strong topology of R.

Let R be now reflexive and $\bar{R}$ its conjugate space.[3] The so-called weak linear topology of R by $\bar{R}$ is not a linear topology in our sense. However there exists the weakest linear topology $\mathfrak{W}$ among our linear topologies by which every $\bar{a} \in \bar{R}$ is topologically continuous. This linear topology $\mathfrak{W}$ is called the *absolute weak topology* of R, as the system of vicinities $\{x : \bar{a}(|x|) \leqq 1\}$ for all positive $\bar{a} \in \bar{R}$ is a basis of $\mathfrak{W}$. We can prove that the absolute weak topology $\mathfrak{W}$ of R is weaker than the strong topology $\mathfrak{S}$ of R, i.e., $\mathfrak{W} \subset \mathfrak{S}$, but $\mathfrak{W}$ is equivalent to $\mathfrak{S}$, i.e., a manifold $A \subset R$ is topologically bounded by $\mathfrak{W}$, if and only if A is so by $\mathfrak{S}$.

A pseudo-norm $\|x\|$ on R is said to be *reflexive*, if for

$$\bar{A} = \{\bar{x} : \sup_{\|x\| \leqq 1} |\bar{x}(x)| \leqq 1\},$$

we have $\|x\| = \sup_{\bar{x} \in \bar{A}} |\bar{x}(x)|$. A linear topology $\mathfrak{V}$ on R is said to be *reflexive*, if $\mathfrak{V}$ has a basis $\mathfrak{B}$ such that the pseudo-norm $\|x\|_V$ is reflexive

2) A. WEIL: Sur les espaces à structure uniforme, Actual. Sci. et Industr. Paris, (1938).

3) H. NAKANO: Modulared semi-ordered linear spaces, Tokyo Math. Book Series I (1950), §22. This book will be denoted by MSLS in this paper.

for every $V \in \mathfrak{V}$. The absolute weak topology of R is reflexive. We shall prove that if the strong topology of R is sequential, then it is reflexive. This result is a generalization of the theorem: if there is a complete norm on R, then there exists a complete reflexive norm on R.

We shall make use of notations in MSLS and the following notations:

$$A^+ = \{x^+ : x \in A\}, \quad A^- = \{x^- : x \in A\}, \quad |A| = \{|x| : x \in A\},$$
$$A^\smile B = \{x^\smile y : x \in A, \ y \in B\}, \quad A_\frown B = \{x_\frown y : x \in A, y \in B\}.$$
$$A \times B = \{x + y : x \in A, \ y \in B\}$$

for manifolds A, B of R.

§ 1. Linear topologies

Let R be a universally continuous semi-ordered linear space. A set of positive elements V is said to be a *positive vicinity*, if

1) for any $a \geqq 0$ we can find $\varepsilon > 0$ such that $\varepsilon a \in V$,

2) $0 \leqq b \leqq a \in V$ implies $b \in V$,

3) $V \ni a_\lambda \uparrow_{\lambda \in \Lambda} a$ implies $a \in V$.

A positive vicinity V is said to be *convex*, if $x, y \in V$, $\lambda + \mu = 1$, $\lambda, \mu \geqq 0$ implies $\lambda x + \mu y \in V$.

With this definition, we see easily that if V is a positive vicinity (convex), then ξV also is a positive vicinity (convex) for $\xi > 0$, and for two positive vicinity U, V (convex), both UV and $U \times V$ are positive vicinities (convex).

A collection $\mathfrak{V}$ of positive vicinities is called a *linear topology*, if

1′) $U \subset V \in \mathfrak{V}$ implies $U \in \mathfrak{V}$,

2′) $U, V \in \mathfrak{V}$ implies $UV \in \mathfrak{V}$,

3′) $V \in \mathfrak{V}$ implies $\xi V \in \mathfrak{V}$ for every $\xi > 0$,

4′) for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{V}$ such that $U \times U \subset V$.

For a linear topology $\mathfrak{V}$ on R, a subset $\mathfrak{B} \subset \mathfrak{V}$ is called a *basis* of $\mathfrak{V}$, if for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{B}$ ane $\alpha > 0$ such that $\alpha U \subset V$. With this definition, we can prove easily

Theorem 1.1 *If a collection of positive vicinities $\mathfrak{B}$ satisfies*

1″) for any $U, V \in \mathfrak{B}$ we can find $W \in \mathfrak{B}$ and $\alpha > 0$ such that $\alpha W \subset UV$,

2″) for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ and $\alpha > 0$ such that $U \times U \subset \alpha V$,

then there exists uniquely a linear topology $\mathfrak{V}$ of which $\mathfrak{B}$ is a basis.

A linear topology $\mathfrak{V}$ is said to be *convex*, if $\mathfrak{V}$ has a basis composed only of convex positive vicinities. A linear topology $\mathfrak{V}$ is said to be *sequential,* if $\mathfrak{V}$ has a basis composed of at most countable positive vicinities. A sequence of positive vicinities V_ν $(\nu=1,2,\cdots)$ is said to be *decreasing,* if

$$V_\nu \supset V_{\nu+1}\times V_{\nu+1} \qquad \text{for every} \quad \nu=1,2,\cdots.$$

If a linear topology $\mathfrak{V}$ is sequential, then we can find obviously by definition a decreasing sequence $V_\nu\in\mathfrak{V}$ $(\nu=1,2,\cdots)$ as a basis of $\mathfrak{V}$. Such a basis is called a *decreasing basis* of $\mathfrak{V}$. If $V_\nu\in\mathfrak{V}$ $(\nu=1,2,\cdots)$ is a decreasing basis of $\mathfrak{V}$, then for any $V\in\mathfrak{V}$ we can find ν such that $V_\nu\subset V$. Because we can find by definition μ and $\varepsilon>0$ such that $\varepsilon V_\mu\subset V$. For such $\varepsilon>0$, we can find $\nu>\mu$ such that $\frac{1}{2^{\nu-\mu}}<\varepsilon$, and then we have

$$V_\nu\subset\frac{1}{2^{\nu-\mu}}V_\mu\subset\varepsilon V_\mu\subset V,$$

because we have $V_\nu\supset 2V_{\nu+1}$ for every $\nu=1,2,\cdots$.

A decreasing basis $V_\nu\in\mathfrak{V}$ $(\nu=1,2,\cdots)$ is said to be *convex,* if every V_ν $(\nu=1,2,\cdots)$ is convex. With this definition, we see at once by definition

Theorem 1.2. *If a linear topology $\mathfrak{V}$ is sequential and convex, then $\mathfrak{V}$ has a convex decreasing basis.*

A linear topology $\mathfrak{V}$ is said to be of *single vicinity* if $\mathfrak{V}$ has a basis composed only of a single positive vicinity. With this definition we have obviously

Theorem 1.3. *If a linear topology $\mathfrak{V}$ is of single vicinity and convex, then there is a convex positive vicinity which is a basis of $\mathfrak{V}$.*

§ 2. Pseudo-norms

A functional $\|x\|$ $(x\in R)$ on R is said to be a *pseudo-norm* on R, if

1) $0\leqq\|x\|<+\infty$ for every $x\in R$,
2) $|x|\leqq|y|$ implies $\|x\|\leqq\|y\|$,
3) $\|\xi x\|=|\xi|\,\|x\|$ for every real number ξ,
4) $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}x$ implies $\|x\|=\sup\limits_{\lambda\in\Lambda}\|x_\lambda\|$.

A pseudo-norm $\|x\|$ $(x\in R)$ is said to be *convex,* if

$$\|x+y\|\leqq\|x\|+\|y\| \qquad \text{for every} \quad x,y\in R.$$

For a pseudo-norm $\|x\|$ $(x\in R)$, putting

$$V=\{x:\ \|x\|\leqq 1,\quad x\geqq 0\},$$

we see easily that V is a positive vicinity. Furthermore, if $\|x\|(x\in R)$ is convex, then this positive vicinity V is convex.

Conversely, for a positive vicinity V, putting

$$(1) \qquad \|x\|_V = \inf_{\xi|x|\in V} \frac{1}{\xi},$$

we obtain a pseudo-norm $\|x\|_V (x\in R)$, which will be called the *pseudo-norm* of V. With this definition, we see easily

$$(2) \qquad V = \{x : \|x\|_V \leqq 1, \ x \geqq 0\}.$$

Furthermore we can prove easily

$$(3) \qquad \|x\|_{\xi V} = \frac{1}{\xi}\|x\|_V \qquad \text{for} \quad \xi > 0,$$

$$(4) \qquad V \subset U \text{ implies } \|x\|_V \geqq \|x\|_U \qquad \text{for every} \quad x\in R,$$

$$(5) \qquad V\times V \subset U \text{ implies } \|x+y\|_U \leqq \operatorname{Max}\{\|x\|_V, \|y\|_V\}.$$

By virtue of Theorem 1.1, we can prove easily

Theorem 2.1. *For a system of pseudo-norms $\|x\|_\lambda$ $(\lambda\in\Lambda)$ on R, if for any $\lambda\in\Lambda$ we can find $\sigma\in\Lambda$ such that*

$$\|x+y\|_\lambda \leqq \|x\|_\sigma + \|y\|_\sigma \qquad \textit{for every} \quad x, y\in R,$$

then there exists uniquely a linear topology $\mathfrak{V}$ on R such that the totality of

$$V_{\lambda_1,\lambda_2,\dots,\lambda_\kappa} = \{x : \|x\|_{\lambda_\nu} \leqq 1 \ (\nu=1,2,\dots,\kappa), \ x\geqq 0\}$$

for every finite number of elements $\lambda_\nu \in \Lambda$ $(\nu=1,2,\dots,\kappa)$ is a basis of $\mathfrak{V}$.

A pseudo-norm $\|x\|$ $(x\in R)$ is said to be *proper*, if $\|x\|=0$ implies $x=0$. A pseudo-norm is called a *norm*, if it is convex and proper.

Theorem 2.2. *For a convex pseudo-norm $\|x\|$ $(x\in R)$ there exists uniquely a normal manifold N of R such that $\|x\|$ $(x\in N)$ is proper in N and $\|x\| = 0$ for every $x\in N^\perp$.*

Proof. Putting $N = \{x : \|x\|=0\}$, we see easily that N is a normal manifold of R. For such N, it is evident that $\|x\|=0$ for every $x\in N$. Conversely, if $\|x\|=0$, then we have naturally $x\in N$, and hence $[N^\perp]x=0$. Thus $\|x\|$ is proper in $N^\perp$. If $\|x\|$ is proper in a normal manifold M and $\|x\|=0$ for every $x\in M^\perp$, then it is evident that $M^\perp=N$.

A system of pseudo-norms $\|x\|_\lambda$ $(\lambda\in\Lambda)$ is said to be *proper*, if $\|x\|_\lambda = 0$ for all $\lambda\in\Lambda$ implies $x=0$. With this definition, we have

Theorem 2.3. *For a system of pseudo-norms $\|x\|_\lambda$ $(\lambda\in\Lambda)$ on R, if for any $\lambda\in\Lambda$ we can find $\sigma\in\Delta$ such that*

$$\|x+y\|_\lambda \leqq \|x\|_\sigma + \|y\|_\sigma \qquad \textit{for every} \quad x, y\in R,$$

then there exists uniquely a normal manifold N of R such that the system $\|x\|_\lambda$ $(\lambda\in\Lambda)$ is proper in N and $\|x\|_\lambda=0$ for every $\lambda\in\Lambda$ and $x\in N^\perp$.

Proof. Putting $M=\{x:\|x\|_\lambda=0$ for all $\lambda\in\Lambda\}$, we see easily that M is a normal manifold of R and $M^\perp$ satisfies our requirement. Furthermore the uniqueness is obvious.

We shall say that R is *separated* by a linear topology $\mathfrak{V}$, or that $\mathfrak{V}$ is *separative* if $\prod_{V\in\mathfrak{V}} V=\{0\}$. With this definition, we see at once

Theorem 2.4. *A linear topology $\mathfrak{V}$ is separative, if and only if for a basis $\mathfrak{B}$ of $\mathfrak{V}$, the system of pseudo-norms $\|x\|_V$ $(V\in\mathfrak{B})$ is proper.*

§ 3. Completeness

Let $\mathfrak{V}$ be a linear topology on R. A system of manifolds A_λ $(\lambda\in\Lambda)$ is said to be a *CAUCHY system* by $\mathfrak{V}$, if $\prod_{\nu=1}^{\kappa} A_{\lambda_\nu}\neq 0$ for every finite number of elements $\lambda_\nu\in\Lambda$ $(\nu=1,2,\cdots,\kappa)$, and for any $V\in\mathfrak{V}$ we can find $\lambda\in\Lambda$ such that

$$|x-y|\in V \qquad \text{for every} \quad x,y\in A_\lambda .$$

A CAUCHY system A_λ $(\lambda\in\Lambda)$ is said to be *convergent* to a *limit* $a\in R$, if for any $V\in\mathfrak{V}$ we can find $\lambda\in\Lambda$ such that

$$|x-a|\in V \qquad \text{for every} \quad x\in A_\lambda .$$

If $\mathfrak{V}$ is separative, then we see easily that the limit of a CAUCHY system is uniquely determined, if it is convergent.

We see easily by definition that for a basis $\mathfrak{B}$ of $\mathfrak{V}$, a system of manifolds A_λ $(\lambda\in\Lambda)$ is a CAUCHY system by $\mathfrak{V}$, if and only if $\prod_{\lambda=1}^{\kappa} A_{\lambda_\nu}\neq 0$ for every finite number of elements $\lambda_\nu\in\Lambda$ $(\nu=1,2,\cdots,\kappa)$ and for any $V\in\mathfrak{B}$ and $\varepsilon>0$ we can find $\lambda\in\Lambda$ such that

$$\|x-y\|_V\leqq\varepsilon \qquad \text{for every} \quad x,y\in A_\lambda .$$

Furthermore we see that a CAUCHY system A_λ $(\lambda\in\Lambda)$ is convergent to a limit $a\in R$, if and only if for any $V\in\mathfrak{B}$ and $\varepsilon>0$ we can find $\lambda\in\Lambda$ such that

$$\|x-a\|_V\leqq\varepsilon \qquad \text{fer every} \quad x\in A_\lambda .$$

By virtue of the formula §2 (5), we can prove easily

Theorem 3.1. *For two CAUCHY system A_λ and B_λ $(\lambda\in\Lambda)$, all $A_\lambda\smile B_\lambda$, $A_\lambda\frown B_\lambda$, and $A_\lambda\times B_\lambda$ $(\lambda\in\Lambda)$ are CAUCHY systems, furthermore, if A_λ and*

B_λ ($\lambda \in \Lambda$) are convergent respectively to limits a and b, then $A_\lambda \smile B_\lambda$, $A_\lambda \frown B_\lambda$, and $A_\lambda \times B_\lambda$ ($\lambda \in \Lambda$) are convergent to $a \smile b$, $a \frown b$, and $a+b$ respectively.

We see further easily

Theorem 3.2. *For a* CAUCHY *system A_λ ($\lambda \in \Lambda$), all A_λ^+, A_λ^-, $|A_\lambda|$, αA_λ, and $[N]A_\lambda$ ($\lambda \in \Lambda$) are* CAUCHY *systems for every real number α and projection operator $[N]$. If a* CAUCHY *system A_λ ($\lambda \in \Lambda$) is convergent to a limit a, then A_λ^+, A_λ^-, $|A_\lambda|$, αA_λ, and $[N]A_\lambda$ ($\lambda \in \Lambda$) are convergent to a^+, a^-, $|a|$, αa, and $[N]a$ respectively.*

A manifold A of R is said to be *complete* by a linear topology $\mathfrak{B}$, if every CAUCHY system $A_\lambda \subset A$ ($\lambda \in \Lambda$) is convergent to a limit $a \in A$. With this definition we have

Theorem 3.3. *For every positive element $a \in R$, $\{x : |x| \leqq a\}$ is complete by $\mathfrak{B}$.*

Proof. We shall consider firstly the case where $\mathfrak{B}$ is sequential and separative. Let $V_\nu \in \mathfrak{B}$ ($\nu = 1, 2, \cdots$) be a decreasing basis of $\mathfrak{B}$. We set

$$A = \{x : |x| \leqq a\}$$

and assume that $A_\lambda \subset A$ ($\lambda \in \Lambda$) is a CAUCHY system by $\mathfrak{B}$. Then we can find $\lambda_\nu \in \Lambda$ ($\nu = 1, 1, \cdots$) such that

$$\sup_{x, y \in A_{\lambda_\nu}} \|x - y\|_{V_\nu} \leqq \frac{1}{\nu} \qquad (\nu = 1, 2, \cdots).$$

For such $\lambda_\nu \in \Lambda$ ($\nu = 1, 2, \cdots$) we can find

$$a_\mu \in \prod_{\nu=1}^{\mu} A_{\lambda_\nu} \qquad (\mu = 1, 2, \cdots).$$

As $V_{\nu+1} \times V_{\nu+1} \subset V_\nu$, we conclude by the formula §2 (5)

$$\left\| \left(\sum_{\nu=\mu}^{\sigma} |a_{\nu+1} - a_\nu| \right) \right\|_{V_{\mu-1}} \leqq \max_{\mu \leqq \nu \leqq \sigma} \|a_{\nu+1} - a_\nu\|_{V_\nu} \leqq \frac{1}{\mu}.$$

On the other hand we have

$$\bigcup_{\nu=\mu}^{\sigma} a_\nu - a_\mu = \bigcup_{\nu=\mu}^{\sigma} (a_\nu - a_\mu) \leqq \sum_{\nu=\mu}^{\sigma} |a_{\nu+1} - a_\nu|,$$

and hence $\left\| \bigcup_{\nu=\mu}^{\sigma} a_\nu - a_\mu \right\|_{V_{\mu-1}} \leqq \frac{1}{\mu}$. This relation yields by 4) in §2

$$\left\| \bigcup_{\nu=\mu}^{\infty} a_\nu - a_\mu \right\|_{V_{\mu-1}} \leqq \frac{1}{\mu} \qquad (\mu = 2, 3, \cdots).$$

We obtain likewise

$$\left\| a_\mu - \bigcap_{\nu=\mu}^{\infty} a_\nu \right\|_{V_{\mu-1}} \leqq \frac{1}{\mu} \qquad (\mu = 2, 3, \cdots).$$

Consequently we have by the formula §2 (5)

$$\left\| \bigcup_{\nu=\mu}^{\infty} a_\nu - \bigcap_{\nu=\mu}^{\infty} a_\nu \right\|_{V_{\mu-2}} \leqq \frac{1}{\mu} \qquad (\mu=3, 4, \cdots).$$

Thus, putting $l_\mu = \bigcup_{\nu=\mu}^{\infty} a_\nu - \bigcap_{\nu=\mu}^{\infty} a_\nu$, $l = \bigcap_{\mu=1}^{\infty} l_\mu$, we obtain $\| l \|_{V_{\mu-2}} \leqq \frac{1}{\mu}$ for everv $\mu = 3, 4, \cdots$. As $\|x\|_{V_1} \leqq \|x\|_{V_2} \leqq \cdots$ by §2 (4), we conclude hence $\|l\|_{V_\mu} = 0$ for every $\mu = 1, 2, \cdots$, and hecce $l = 0$, as $\mathfrak{B}$ is separative by assumption. Therefore there exists $a \in R$ such that $\lim_{\nu\to\infty} a_\nu = a$, and naturally $a \in A$. Furthermore we have

$$\|a - a_\mu\|_{V_{\mu-2}} \leqq \frac{1}{\mu} \qquad \text{for every } \mu=3, 4, \cdots,$$

because $\bigcup_{\nu=\mu}^{\infty} a_\nu \geqq a \geqq \bigcap_{\nu=\mu}^{\infty} a_\nu$. This relation shows that A_λ $(\lambda \in \Lambda)$ is convergent to a by $\mathfrak{B}$.

Now we consider the general case. Let $A_\lambda \subset A (\lambda \in \Lambda)$ be an arbitrary CAUCHY system by $\mathfrak{B}$ and $V_\nu \in \mathfrak{B}$ $(\nu = 1, 2, \cdots)$ an arbitrary decreasing sequence. By virtue of Theorem 2.3, we can find a normal manifold $N_{V_1, V_2, \dots}$ of R such that the system $\|x\|_{V_\nu}$ $(\nu=1, 2, \cdots)$ is proper in $N_{V_1, V_2, \dots}$ and $\|x\|_{V_\nu} = 0$ for every $x \in N^{\perp}_{V_1, V_2, \dots}$ and $\nu = 1, 2, \cdots$. Recalling Theorem 2.1, we can find then a linear topology $\mathfrak{B}_{V_1, V_2, \dots}$ on $N_{V_1, V_2, \dots}$ such that $[N_{V_1, V_2, \dots}] V_\nu$ $(\nu = 1, 2, \cdots)$ is a basis of $\mathfrak{B}_{V_1, V_2, \dots}$. This linear topology $\mathfrak{B}_{V_1, V_2, \dots}$ is obviously sequential and separative by Theorem 2.4. Furthermore, as $[N_{V_1, V_2, \dots}] A_\lambda$ $(\lambda \in \Lambda)$ is a CAUCHY system by $\mathfrak{B}_{V_1, V_2, \dots}$, there exists uniquely a limit $a \in [N_{V_1, V_2, \dots}] A$ of $[N_{V_1, V_2, \dots}] A_\lambda$ $(\lambda \in \Lambda)$, as proved just above.

Corresponding to every decreasing sequence $V_\nu \in \mathfrak{B}$ $(\nu = 1, 2, \cdots)$, we obtain thus uniquely a normal manifold $N_{V_1, V_2, \dots}$ and a limit $a_{V_1, V_2, \dots} \in [N_{V_1, V_2, \dots}] A$ of $[N_{V_1, V_2, \dots}] A_\lambda$ $(\lambda \in \Lambda)$. We see further by Theorem 3.2 that for every two decreasing sequences V_ν and $U_\nu \in \mathfrak{B}$ $(\nu = 1, 2, \cdots)$, we have

$$[N_{V_1, V_2, \dots}] [N_{U_1, U_2, \dots}] a_{V_1, V_2, \dots} = [N_{V_1, V_2, \dots}] [N_{U_1, U_2, \dots}] a_{U_1, U_2, \dots}.$$

Therefore we can find $a \in A$ such that

$$[N_{V_1, V_2, \dots}] a = a_{V_1, V_2, \dots}$$

for every decreasing sequence $V_\nu \in \mathfrak{B}$ $(\nu = 1, 2, \cdots)$. Such $a \in A$ is a limit of A_λ $(\lambda \in \Lambda)$. Because, for any $V \in \mathfrak{B}$ we can find a decreasing sequence $V_\nu \in \mathfrak{B}$ $(\nu = 1, 2, \cdots)$ such that $V \supset V_1 \times V_1$, and $\lambda \in \Lambda$ such that

$$\sup_{x \in [N_{V_1, V_2, \dots}] A_\lambda} \|x - a_{V_1, V_2, \dots}\|_{V_1} \leqq 1,$$

and hence $\sup_{x \in A_\lambda} \|[N_{V_1, V_2, \dots}](x-a)\|_{V_1} \leqq 1$. As

$$\|[N^{\perp}_{V_1, V_2, \dots}](x-a)\|_{V_1} = 0 ,$$

we obtain by §2 (5)

$$\sup_{x \in A_\lambda} \|x-a\|_V \leqq 1 ,$$

that is, $|x-a| \in V$ for every $x \in A_\lambda$. Therefore A is complete by $\mathfrak{V}$.

Theorem 3.4. *$\{x : a \leqq x \leqq b\}$ is complete by every linear topology $\mathfrak{V}$ for every two elements $a \leqq b$.*

Proof. Putting $A = \{x : |x| \leqq |a| + |b|\}$, $B = \{x : a \leqq x \leqq b\}$, we have obviously $B \subset A$ and A is complete by $\mathfrak{V}$ on account of Theorem 3.3. For a Cauchy system $A_\lambda \subset B\, (\lambda \in \Lambda)$ there exists hence a limit $c \in A$ of $A_\lambda\, (\lambda \in \Lambda)$, and then we obtain by Theorem 3.1 that $(c \smile a) \frown b$ is a limit of

$$(A_\lambda \smile a) \frown b = A_\lambda \qquad (\lambda \in \Lambda) ,$$

and it is evident that $(c \smile a) \frown b \in B$. Therefore B is complete by $\mathfrak{V}$.

§ 4. Topologically bounded manifolds

A manifold A of R is said to be *topologically bound* by a linear topology $\mathfrak{V}$, if

$$\sup_{x \in A} \|x\|_V < +\infty \qquad \text{for every } V \in \mathfrak{V} .$$

With this definition, it is obvious by the formula §2 (4) that a manifold A is topologically bounded by a linear topology $\mathfrak{V}$, if and only if for a basis $\mathfrak{B}$ of $\mathfrak{V}$ we have

$$\sup_{x \in A} \|x\|_V < +\infty \qquad \text{for every } V \in \mathfrak{B} .$$

We can prove easily by definition

Theorem 4.1. *If a manifold A is topologically bounded by a linear topology $\mathfrak{V}$, then all A^+, A^-, $|A|$, αA, $[N]A$ are topologically bound by $\mathfrak{V}$ for every real number α and projection operator $[N]$. If both manifolds A and B are topologically bounded by $\mathfrak{V}$, then all $A \smile B$, $A \frown B$, and $A \times B$ are topologically bounded by $\mathfrak{V}$.*

A manifold A of R is said to be *order bound* or *merely bounded,* if we can find a positive element $a \in R$ such that $|x| \leqq a$ for every $x \in A$. Every bounded manifold is obviously topologically bounded by every linear topology.

A linear topology $\mathfrak{V}$ on R is said to be *monotone complete,* if for any topologically bounded manifold of positive elements $a_\lambda\uparrow_{\lambda\in\Lambda}$, we can find $a\in R$ such that $a_\lambda\uparrow_{\lambda\in\Lambda}a$.

Theorem 4.2. *If a linear topology* $\mathfrak{V}$ *on* R *is monotone complete, then* R *is complete by* $\mathfrak{V}$.

Proof. Let A_λ $(\lambda\in\Lambda)$ be a CAUCHY system by $\mathfrak{V}$. We suppose firstly that $\mathfrak{V}$ is separative. As A_λ^+ $(\lambda\in\Lambda)$ also is by Theorem 3.2 a CAUCHY system, corresponding to every $x\geqq 0$, we obtain uniquely by Theorem 3.3 a limit a_x of a CAUCHY system $A_\lambda^+{}_\frown x$ $(\lambda\in\Lambda)$. For this limit a_x, we have obviously by Theorem 3.1 $0\leqq a_x\uparrow_{x\geqq 0}$. Furthermore the system $a_x\,(x\geqq 0)$ is topologically bounded by $\mathfrak{V}$. Because for any $V\in\mathfrak{V}$ we can find by definition $U\in\mathfrak{V}$ such that $U\times U\times U\times U\subset V$, and $\lambda_1\in\Lambda$ such that $\|y-z\|_U\leqq 1$ for every $y,\ z\in A_{\lambda_1}^+$, and hence by § 2 (5) $\sup_{y\in A_{\lambda_1}^+}\|y\|_{U\times U}<+\infty$. For any $x\geqq 0$ we$\|_{U\times U}$ can find by definition, $\lambda_2\in\Lambda$ such that

$$\|a_x-z\|_{U\times U}\leqq 1 \qquad \text{for every} \quad z\in A_{\lambda_2}^+{}_\frown x.$$

For an element $b\in A_{\lambda_1}^+A_{\lambda_2}^+$, we have then by §2 (5)

$$\|a_x\|_V\leqq \operatorname{Max}\{1,\ \|b_\frown x\|_{U\times U}\}\leqq \operatorname{Max}\{1,\ \|b\|_{U\times U}\},$$

and hence $\|a_x\|_V\leqq \operatorname{Max}\{1,\ \sup_{y\in A_{\lambda_1}^+}\|y\|_{U\times U}\}$ for every $x\geqq 0$.

Therefore there exists by assumption $a\in R$ such that $a_x\uparrow_{x\geqq 0}a$. As we have by Theorem 3.1

$$a_{x\frown}y=a_{x\cap y} \qquad \text{for every} \quad x,y\geqq 0,$$

we obtain $a_\frown x=a_x$ for every $x\geqq 0$. For any $V\in\mathfrak{V}$ we can find $U\in\mathfrak{V}$ such that $U\times U\subset V$, and further $\lambda_0\in\Lambda$ such that

$$\sup_{y,z\in A_{\lambda_0}^+}\|y-z\|_U\leqq 1.$$

Thus, for any $y\in A_{\lambda_0}^+$, putting $x=y^\smile a$, we can find $\lambda_1\in\Lambda$ such that

$$\sup_{z\in A_{\lambda_1}^+}\|z_\frown x-a\|_U=\sup_{z\in A_{\lambda_1}^+}\|z_\frown x-a_x\|_U\leqq 1,$$

and for $z\in A_{\lambda_0}^+A_{\lambda_1}^+$ we have

$$\|y-z_\frown x\|_U=\|y_\frown x-z_\frown x\|_U\leqq\|y-z\|_U\leqq 1.$$

Consequently we obtain by §2 (5)

$$\|y-a\|_V\leqq 1 \qquad \text{for every} \quad y\in A_{\lambda_0}^+.$$

Therefore a is a limit of A_λ^+ $(\lambda\in\Lambda)$. We obtain likewise a limit b of A_λ^-

$(\lambda \in \Lambda)$. Thus we see by Theorem 3.1 that $a-b$ is a limit of A_λ $(\lambda \in \Lambda)$.

In general, we can find by Theorem 2.3 a normal manifold N of R, such that the system of pseudo-norms $\|x\|_V$ $(V \in \mathfrak{V})$ is proper in N and $\|x\|_V = 0$ for every $x \in N^\perp$ and $V \in \mathfrak{V}$. Then there exists a limit $a \in N$ of $[N]A_\lambda$ $(\lambda \in \Lambda)$, as proved just above. This limit a also is a limit of A_λ $(\lambda \in \Lambda)$, because for any $V \in \mathfrak{V}$ we can find $U \in \mathfrak{V}$ such that $U \times U \subset V$, and we have by §2 (5) for every $x \in R$

$$\|x-a\|_V \leqq \|[N]x-a\|_U .$$

A linear topology $\mathfrak{V}$ on R is said to be *complete*, if R is complete by $\mathfrak{V}$. We can state then by Theorem 4.2 that every monotone complete linear topology is complete.

Theorem 4.3. *If a linear topology $\mathfrak{V}$ on R is separative, convex, and complete, and a manifold A of R is topologically bounded by $\mathfrak{V}$, then we have for every positive vicinity W*

$$\sup_{x \in A} \|x\|_W < +\infty .$$

Proof. If $\sup_{x \in A} \|x\|_W = +\infty$, then we can find $x_\nu \in A$ $(\nu = 1, 2, \cdots)$ such that $\|x_\nu\|_W \geqq \nu 2^\nu$ for every $\nu = 1, 2, \cdots$. As A is by assumption topologically bounded by $\mathfrak{V}$, we have obviously $\sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \|x_\nu\|_V < +\infty$ for every $V \in \mathfrak{V}$. As $\mathfrak{V}$ is convex and complete by assumption, we can find $a \in R$ such that

$$\lim_{\mu \to \infty} \left\| \sum_{\nu=1}^{\mu} \frac{1}{2^\nu} |x_\nu| - a \right\|_V = 0 \qquad \text{for every } V \in \mathfrak{V} .$$

As $\mathfrak{V}$ is separative by assumption, we conclude easily that $a = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} |x_\nu|$, and hence we have

$$\|a\|_W \geqq \frac{1}{2^\nu} \|x_\nu\|_W \geqq \nu \qquad \text{for every } \nu = 1, 2, \cdots,$$

contradicting $\|a\|_W < +\infty$.

§ 5. Equivalence

A linear topology $\mathfrak{V}$ on R is said to be *equivalent* to a linear topology $\mathfrak{U}$ on R, if $\mathfrak{V}$ has the same topologically bounded manifolds with $\mathfrak{U}$, that is, a manifold A is topologically bounded by $\mathfrak{V}$ if and only if A is so by $\mathfrak{U}$. With this definition, we have obviously

Theorem 5.1. *If a linear topology $\mathfrak{V}$ is monotone complete, then every* *linear topology equivalent to* $\mathfrak{V}$ *is also monotone complete.*

We shall say that a linear topology $\mathfrak{V}$ on R is *stronger* than a linear topology $\mathfrak{U}$ on R, or that $\mathfrak{U}$ is *weaker* than $\mathfrak{V}$, if $\mathfrak{V} \supset \mathfrak{U}$. With this definition we have obviously by Theorem 4.3.

Theorem 5.2. *If a linear topology* $\mathfrak{V}$ *is separative, convex, and complete, then every linear topology stronger than* $\mathfrak{V}$ *is equivalent to* $\mathfrak{V}$.

By virtue of Theorem 1.1, we see easily that there exists uniquely a linear topology $\mathfrak{V}$ of which the totality of convex vicinity in R is a basis. This linear topology $\mathfrak{V}$ is called the *strong topology* of R. With this definition, we have obviously that the strong topology of R is the strongest convex linear topology on R, that is, the strong topology of R is stronger than every other convex linear topology on R.

Recalling Theorem 5.2, we obtain at once

Theorem 5.3. *If a linear topology* $\mathfrak{V}$ *on* R *is separative, convex, and complete, then* $\mathfrak{V}$ *is equivalent to the strong topology of* R.

Theorem 5.4. *If a linear topology* $\mathfrak{V}$ *on* R *is sequential and equivalent to a linear topology* $\mathfrak{U}$ *on* R, *then* $\mathfrak{V}$ *is stronger than* $\mathfrak{U}$.

Proof. Let $V_\nu \in \mathfrak{V} (\nu=1,2,\cdots)$ be a decreasing basis of $\mathfrak{V}$. If $\mathfrak{V}$ is not stronger than $\mathfrak{U}$, then we can find $U \in \mathfrak{U}$ such that $U \bar{\in} \mathfrak{V}$. For such U, there is a sequence $a_\nu \in R$ $(\nu=1,2,\cdots)$ such that

$$\nu U \bar{\ni} a_\nu \in V_\nu \qquad \text{for every} \quad \nu=1,2,\cdots,$$

and hence we havh by the formulas (2), (3) in §2

$$\|a_\nu\|_{V_\nu} \leqq 1, \quad \|a_\nu\|_U \geqq \nu \qquad \text{for every} \quad \nu=1,2,\cdots.$$

Then $\{a_1, a_2, \cdots\}$ is a bounded by $\mathfrak{V}$ but not by $\mathfrak{U}$, contradicting assumption.

On account of this Theorem 5.4, we conclude by Theorem 5.3

Theorem 5.5. *If a linear topology* $\mathfrak{V}$ *on* R *is sequential, separative, convex, and complete, then* $\mathfrak{V}$ *is the strong topology of* R.

§6. Continuous linear topologies

A pseudo-norm $\|x\|$ on R is said to be *continuous*, if $R \ni x_\nu \downarrow_{\nu=1}^\infty 0$ implies $\lim_{\nu\to\infty} \|x_\nu\| = 0$. A linear topology $\mathfrak{V}$ on R is said to be *continuous*, if the preudo-norm $\|x\|_V$ is continuous for every $V \in \mathfrak{V}$. With this definition, we see at once by the formulas (3), (4) in §2 that $\mathfrak{V}$ is continuous if and only if for a basis $\mathfrak{B}$ of $\mathfrak{V}$, the pseudo-norm $\|x\|_V$ is continuous for every $V \in \mathfrak{B}$.

Theorem 6.1. *If a linear topology* $\mathfrak{V}$ *on* R *is sequential, separative and* *continuous, then* R *is superuniversally continuous, that is, for any system of positive elements* $a_\lambda \in R$ $(\lambda \in \Lambda)$ *we can find countable* $\lambda_\nu \in \Lambda$ $(\nu=1, 2, \cdots)$ *such that* $\bigcap_{\nu=1}^{\infty} a_{\lambda_\nu} = \bigcap_{\lambda \in \Lambda} a_\lambda$.

Proof. Let $V_\nu \in \mathfrak{V}$ $(\nu=1, 2, \cdots)$ be a decreasing basis of $\mathfrak{V}$. $0 \leqq x_\lambda \downarrow_{\lambda \in \Lambda}$ implies then

$$\inf_{\lambda \in \Lambda} \{\sup_{x_\sigma \leqq x_\lambda} \|x_\lambda - x_\sigma\|_{V_\nu}\} = 0 \qquad \text{for every} \quad \nu=1, 2, \cdots .$$

Because, if $0 \leqq x_\lambda \downarrow_{\lambda \in \Lambda}$ and

$$\inf_{\lambda \in \Lambda} \{\sup_{x_\sigma \leqq x_\lambda} \|x_\lambda - x_\sigma\|_{V_\nu}\} \geqq \varepsilon > 0$$

for some ν, then we can find $\lambda_\mu \in \Lambda (\mu=1, 2, \cdots)$ such that

$$x_{\lambda_1} \geqq x_{\lambda_2} \geqq \cdots, \quad \|x_{\lambda_\mu} - x_{\lambda_{\mu+1}}\|_{V_\nu} \geqq \varepsilon \qquad (\mu=1, 2, \cdots).$$

Then, putting $x_0 = \bigcap_{\mu=1}^{\infty} x_{\lambda_\mu}$, we have $x_{\lambda_\mu} - x_0 \downarrow_{\mu=1}^{\infty} 0$, but

$$\|x_{\lambda_\mu} - x_0\|_{V_\nu} \geqq \|x_{\lambda_\mu} - x_{\lambda_{\mu+1}}\|_{V_\nu} \geqq \varepsilon$$

for every $\mu=1, 2, \cdots$, contradicting the assumption that $\mathfrak{V}$ is continuous.

Therefore for $0 \leqq x_\lambda \downarrow_{\lambda \in \Lambda}$ we can find $\lambda_\nu \in \Lambda (\nu=1, 2, \cdots)$ such that $x_{\lambda_\nu} \downarrow_{\nu=1}^{\infty}$ and

$$\sup_{x_\sigma \leqq x_{\lambda_\nu}} \|x_{\lambda_\nu} - x_\sigma\|_{V_\nu} \leqq \frac{1}{2^\nu} \qquad \text{for every} \quad \nu=1, 2, \cdots.$$

Then, putting $x_0 = \bigcap_{\nu=1}^{\infty} x_{\lambda_\nu}$, we have for every $\sigma \in \Lambda$

$$\|x_{\lambda_\nu} - x_0 \frown x_\sigma\|_{V_\nu} \leqq \frac{1}{2^\nu} \qquad (\nu=1, 2, \cdots),$$

because $x_{\lambda_\nu} - x_{\lambda_\mu} \frown x_\sigma \uparrow_{\mu=1}^{\infty} x_{\lambda_\nu} - x_0 \frown x_\sigma$, $\|x_{\lambda_\nu} - x_{\lambda_\mu} \frown x_\sigma\|_{V_\nu} \leqq \frac{1}{2^\nu}$ for $\mu \geqq \nu$. Thus we obtain naturally for every $\sigma \in \Lambda$

$$\|x_0 - x_0 \frown x_\sigma\|_{V_\nu} \leqq \frac{1}{2^\nu} \qquad (\nu=1, 2, \cdots).$$

As $\mathfrak{V}$ is separative by assumption, we obtain hence $x_0 - x_0 \frown x_\sigma = 0$, and consequently $x_0 \leqq x_\sigma$ for every $\sigma \in \Lambda$. Therefore $x_\lambda \downarrow_{\lambda \in \Lambda} x_0$.

Theorem 6.2. *If a linear topology* $\mathfrak{V}$ *on* R *is continuous, then* $a_\lambda \downarrow_{\lambda \in \Lambda} 0$ *implies* $\inf_{\lambda \in \Lambda} \|a_\lambda\|_V = 0$ *for every* $V \in \mathfrak{V}$.

Proof. For any $V \in \mathfrak{V}$ we can find a decreasing sequence $V_\nu \in \mathfrak{V}$ $(\nu=1, 2, \cdots)$ such that $V_1 \times V_1 \subset V$. For such $V_\nu \in \mathfrak{V} (\nu=1, 2, \cdots)$, we can find by Theorem 2.3 a normal manifold N of R such that the system of pseudo-norms $\|x\|_{V_\nu}$ $(\nu=1,2,\cdots)$ is proper in N and $\|x\|_{V_\nu}=0$ for every $x\in N^\perp$ and $\nu=1,2,\cdots$. Then the linear topology on N, of which $\{x: \|x\|_{V_\nu}\leqq 1,\ 0\leqq x\in N\}$ $(\nu=1,2,\cdots)$ is a basis, is obviously sequential, separative, and continuous. Thus N is superuniversally continuous by Theorem 6.1. Therefore, if $R\ni a_\lambda\downarrow_{\lambda\in\Lambda}0$, then we can find $\lambda_\mu\in\Lambda\,(\mu=1,2,\cdots)$ such that

$$[N]a_{\lambda_\nu}\downarrow_{\nu=1}^{\infty}0,$$

and hence $\lim_{\mu\to\infty}\|[N]a_{\lambda_\mu}\|_{V_1}=0$, because $\mathfrak{V}$ is continuous by assumption. As $\|[N^\perp]a_{\lambda_\mu}\|_{V_1}=0$, we obtain hence by §2(5)

$$\|a_{\lambda_\mu}\|_V\leqq\|[N]a_{\lambda_\mu}\|_{V_1}\qquad\text{for every}\quad \mu=1,2,\cdots.$$

Consequently we have $\lim_{\mu\to\infty}\|a_{\lambda_\mu}\|_V=0$. Thus we have naturally

$$\inf_{\lambda\in\Lambda}\|a_\lambda\|_V=0.$$

Theorem 6.3. *If a linear topology $\mathfrak{V}$ on R is sequential, separative, continuous, and complete, then R is regularly complete, that is, for any double sequence $a_{\nu,\mu}\downarrow_{\nu=1}^{\infty}0$ $(\mu=1,2,\cdots)$, we can find ν_μ $(\mu=1,2,\cdots)$ such that $\sum_{\mu=1}^{\infty}a_{\nu_\mu,\mu}$ is convergent.*

Proof. Let $V_\nu\in\mathfrak{V}\,(\nu=1,2,\cdots)$ be a decreasing basis of $\mathfrak{V}$. If $a_{\nu,\mu}\downarrow_{\nu=1}^{\infty}0$ $(\mu=1,2,\cdots)$, then we have

$$\lim_{\nu\to\infty}\|a_{\nu,\mu}\|_{V_\mu}=0\qquad\text{for every}\quad\mu=1,2,\cdots,$$

because $\mathfrak{V}$ is continuous by assumption. Thus we can find ν_μ $(\mu=1,2,\cdots)$ such that $a_{\nu_\mu,\mu}\in V\,(\mu=1,2,\cdots)$. Then we have obviously

$$\sum_{\mu=\sigma}^{\rho}a_{\nu_\mu,\mu}\in V_{\sigma-1}\qquad\text{for}\quad\rho>\sigma.$$

As $\mathfrak{V}$ is complete and separative, we see easily that $\sum_{\mu=1}^{\infty}a_{\nu_\mu,\mu}$ is convergent. Therefore R is regularly complete.

§7. Linear functionals

Let $\mathfrak{V}$ be a linear topology on R. A linear functional φ on R is said to be *topologically bounded* by $\mathfrak{V}$, if $\sup_{x\in A}|\varphi(x)|<+\infty$ for every topologically bounded manifold A.

For any positive element $a\in R$, $\{x: 0\leqq x\leqq a\}$ is obviously topologically bounded by $\mathfrak{V}$. Thus we have

Theorem 7.1. *If a linear functional φ on R is topologically bounded by $\mathfrak{V}$, then φ is bounded, that is,*

$$\sup_{0\leqq x\leqq a}|\varphi(x)|<+\infty \qquad \text{for every} \quad a\geqq 0.$$

Conversely we have

Theorem 7.2. *If a linear topology $\mathfrak{V}$ on R is separative, convex, and complete, then every bounded linear functional φ on R is topologically bounded by $\mathfrak{V}$.*

Proof. Let φ be a positive linear functional on R. If φ is not topologically bounded by $\mathfrak{V}$, then we can find a sequence $a_\nu\geqq 0\ (\nu=1, 2,\cdots)$ such that $\{a_1, a_2,\cdots\}$ is topologically bounded, but

$$\varphi(a_\nu)\geqq \nu 2^\nu \qquad (\nu=1, 2,\cdots).$$

Then we have obviously $\sum_{\nu=1}^{\infty}\frac{1}{2^\nu}\|a_\nu\|_V<+\infty$ for every $V\in\mathfrak{V}$. As $\mathfrak{V}$ is separative, convex, and complete by assumption, we obtain hence that $\sum_{\nu=1}^{\infty}\frac{1}{2^\nu}a_\nu$ is convergent, and putting $a=\sum_{\nu=1}^{\infty}\frac{1}{2^\nu}a_\nu$, we have that $\varphi(a)\geqq\varphi\left(\frac{1}{2^\nu}a_\nu\right)\geqq\nu$ for every $\nu=1, 2,\cdots$, contradicting $\varphi(a)<+\infty$.

A linear functional φ on R is said to be *topologically continuous* by a linear topology $\mathfrak{V}$, if we can find $V\in\mathfrak{V}$ such that

$$|\varphi(x)|\leqq\|x\|_V \qquad \text{for every} \quad x\in R.$$

With this definition, we see at once by the formulas (3), (4) in §2 that a linear functional φ on R is topologically continuous by $\mathfrak{V}$, if and only if for a basis $\mathfrak{B}$ of $\mathfrak{V}$ we can find $V\in\mathfrak{B}$ and $\alpha>0$ such that

$$|\varphi(x)|\leqq\alpha\|x\|_V \qquad \text{for every} \quad x\in R.$$

If a linear functional φ on R is topologically continuous by $\mathfrak{V}$, then φ is obviously by definition topologically bounded by $\mathfrak{V}$.

If a linear functional φ on R is *universally continuous*, that is, if $x_\lambda\downarrow_{\lambda\in\Lambda}0$ implies $\inf_{\lambda\in\Lambda}|\varphi(x_\lambda)|=0$, then, putting

$$V=\{x:\sup_{|y|\leqq x}|\varphi(y)|\leqq 1,\ x\geqq 0\},$$

we see easily that V is a convex positive vicinity. Thus we have

Theorem 7.3. *If a linear functional φ on R is universally continuous, then φ is topologically continuous by the strong topology of R.*

Recalling Theorem 6.2, we obtain immediately

Theorem 7.4. *If a linear topology $\mathfrak{B}$ on R is continuous, then every topologically continuous linear functional on R is universally continuous.*

If a convex pseudo-norm $\|x\|$ on R is not continuous, then we can find a linear functional φ on R such that

$$\sup_{\|x\|\leqq 1} |\varphi(x)| < +\infty ,$$

but there is a sequence $a_\nu \downarrow_{\nu=1}^{\infty} 0$ for which we have $\lim_{\nu\to\infty} \varphi(a_\nu) > 0$. (c.f. MSLS Theorem 31.10). Therefore we have

Theorem 7.5. *For a convex linear topology $\mathfrak{B}$ on R, if every topologically continuous linear functional on R is continuous, then $\mathfrak{B}$ is continuous.*

§ 8. Reflexive linear topologies

Let R be a reflexive semi-ordered linear space and $\bar{R}$ the conjugate space of R. For any positive $\bar{a} \in \bar{R}$, putting

(1) $$V_{\bar{a}} = \{x : \bar{a}(x) \leqq 1, \ x \geqq 0\} ,$$

we obtain obviously a convex positive vicinity $V_{\bar{a}}$. For this $V_{\bar{a}}$ we have obviously

(2) $$\|x\|_{V_{\bar{a}}} = \bar{a}(|x|) \qquad \text{for every} \quad x \in R ,$$

because $\|x\|_{V_{\bar{a}}} = \inf_{\xi |x| \in V_{\bar{a}}} \frac{1}{\xi} = \inf_{\bar{a}(\xi |x|) \leqq 1} \frac{1}{\xi} = \bar{a}(|x|)$.

Recalling Theorem 1.1, we see easily that there exists uniquely a linear topology $\mathfrak{W}$ on R such teat the system $V_{\bar{a}}$ $(0 \leqq \bar{a} \in \bar{R})$ is a basis of $\mathfrak{W}$. This linear topology $\mathfrak{W}$ is called the *absolute weak topology* of R. With this definition we have

Theorem 8.1. *The absolute weak topology $\mathfrak{W}$ of R is separative, convex, continuous, and monotone complete.*

Proof. It is evident by definition that $\mathfrak{W}$ is separative, convex, and continuous. If a system of positive elements $x_\lambda \uparrow_{\lambda\in\Lambda}$ is topologically bounded by $\mathfrak{W}$, then we have by the formula (2)

$$\sup_{\lambda\in\Lambda} \bar{a}(x_\lambda) = \sup_{\lambda\in\Lambda} \|x_\lambda\|_{V_{\bar{a}}} < +\infty$$

for every positive $\bar{a} \in \bar{R}$. Therefore there exists $a \in R$ such that $x_\lambda \uparrow_{\lambda\in\Lambda} a$. (c.f. MSLS. Theorem 24.4)

Theorem 8.2. *A manifold A of R is topologically bounded by the absolute weak topology $\mathfrak{W}$ if and only if A is weakly bounded, that is,*

$$\sup_{x\in A} |\bar{x}(x)| < +\infty \qquad \textit{for every} \quad \bar{x} \in \bar{R} .$$

Proof. If A is weakly bounded, then we have

$$\sup_{x \in A} \bar{a}(|x|) < +\infty \qquad \text{for} \quad 0 \leqq \bar{a} \in \bar{R}$$

(MSLS. Theorem 24.15). Thus we obtain by (2)

$$\sup_{x \in A} \|x\|_{V_{\bar{a}}} < +\infty \qquad \text{for} \quad 0 \leqq \bar{a} \in \bar{R},$$

and hence A is topologically bounded by $\mathfrak{W}$. Conversely, if A is topologically bounded by $\mathfrak{W}$, then we have by (2)

$$\sup_{x \in A} |\bar{a}(x)| \leqq \sup_{x \in A} |\bar{a}|(|x|) = \sup_{x \in A} \|x\|_{V_{|\bar{a}|}} < +\infty,$$

and hence A is weakly bounded.

Recalling Theorem 5.3, we obtain by Theorem 8.1

Theorem 8.3. *The strong topology of R is separative and equivalent to the absolute weak topology of R.*

A pseudo-norm $\|x\|$ on R is said to be *reflexive*, if for

$$\bar{A} = \{\bar{x} : \sup_{\|x\| \leqq 1} |\bar{x}(x)| \leqq 1\},$$

we have $\|x\| = \sup_{\bar{x} \in \bar{A}} |\bar{x}(x)|$ for every $x \in R$. With this definition, we see at once that every reflexive pseudo-norm is convex.

Let $\bar{\mathfrak{W}}$ be the absolute weak topology of the conjugate space $\bar{R}$. For every topologically bounded manifold $\bar{A}$ of $\bar{R}$ by $\bar{\mathfrak{W}}$, putting

$$V = \{x : |\bar{x}|(x) \leqq 1 \quad \text{for every } \bar{x} \in \bar{A},\ x \geqq 0\},$$

we see easily that V is a positive vicinity in R and the pseudo-norm $\|x\|_V$ is reflexive.

Theorem 8.4. *If a pseudo-norm $\|x\|$ $(x \in R)$ is convex and continuous, then it is reflexive.*

Proof. By virtue of BANACH's extension theorem, for any $a \in R$ we can find a linear functional φ on R such that

$$\varphi(a) = \|a\|, \quad |\varphi(x)| \leqq \|x\| \qquad \text{for every} \quad x \in R.$$

As $\|x\|$ $(x \in R)$ is convex and continuous by assumption, we see by Theorem 6.2 that φ is universally continuous, and hence $\varphi \in \bar{R}$. Furthermore, putting

$$\bar{A} = \{\bar{x} : \sup_{\|x\| \leqq 1} |\bar{x}(x)| \leqq 1\},$$

we have obviously $\varphi \in \bar{A}$, and hence

$$\sup_{\bar{x} \in \bar{A}} |\bar{x}(a)| \geqq \varphi(a) = \|a\|.$$

On the other hand, it is evident that $\|a\| \geqq \sup_{\bar{x}\in\bar{A}} |\bar{x}(a)|$. Thus we conclude $\|a\| = \sup_{\bar{x}\in\bar{A}} |\bar{x}(a)|$ for every $a \in R$, that is, the pseudo-norm $\|x\|$ $(x \in R)$ is reflexive by definition.

A linear topology $\mathfrak{V}$ on R is said to be *reflexive*, if there is a basis $\mathfrak{B}$ of $\mathfrak{V}$ such that $\|x\|_V$ is reflexive for every $V \in \mathfrak{B}$. With this definition, we have obviously by Theorem 8.4

Theorem 8.5. *If a linear topology $\mathfrak{V}$ on R is convex and continuous, then $\mathfrak{V}$ is reflexive.*

Consequently we obtain by Theorem 8.1

Theorem 8.6. *The absolute weak topology of R is reflexive.*

Theorem 8.7. *If the strong topology of R is sequential, then it is reflexive.*

Proof. Let V_ν $(\nu=1, 2, \cdots)$ be the convex decreasing basis of the strong topology of R. Putting

$$\bar{A}_\nu = \{\bar{x} : \sup_{x\in V_\nu} \bar{x}(x) \leqq 1, \quad 0 \leqq \bar{x} \in \bar{R}\},$$

we see easily that every $\bar{A}_\nu$ $(\nu=1, 2, \cdots)$ is topologically bounded by the absolute weak topology $\bar{\mathfrak{W}}$ of $\bar{R}$. Thus, putting

$$U_\nu = \{x : \sup_{\bar{x}\in\bar{A}_\nu} \bar{x}(x) \leqq 1, \quad 0 \leqq x \in R\},$$

we obtain a convex positive vicinity U_ν in R such that $\|x\|_{U_\nu}$ is reflexive. For any positive $\bar{a} \in \bar{R}$, putting

$$V_{\bar{a}} = \{x : \bar{a}(x) \leqq 1, \quad 0 \leqq x \in R\},$$

we obtain a convex vicinity $V_{\bar{a}}$ and hence we can find ν such that $V_{\bar{a}} \supset V_\nu$, because V_ν $(\nu=1, 2, \cdots)$ is a basis of the strong topology of R. For such ν, we have obviously $\bar{a} \in \bar{A}_\nu$, and consequently $U_\nu \subset V_{\bar{a}}$. Therefore the convex linear topology $\mathfrak{V}$, of which U_ν $(\nu=1, 2, \cdots)$ is a basis, is stronger than the absolute weak topology of R. Recalling Theorem 5.2, we see that $\mathfrak{V}$ is monotone complete, and hence $\mathfrak{V}$ coincides by Theorem 7.5 with the strong topology of R. Furthermore $\mathfrak{V}$ is obviously reflexive. Consequently the strong topology of R is reflexive.

If a norm $\|x\|$ on R is complete, that is, if the linear topology $\mathfrak{V}$, of which $\{x : \|x\| \leqq 1, \ 0 \leqq x \in R\}$ is a basis, is complete, then $\mathfrak{V}$ is by Theorem 5.5 the strong topology of R, and hence reflexive by Theorem 8.7. Therefore we obtain.

Theorem 8.8. *If there is a complete norm on R, then there exists a complete reflexive norm on R.*