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**タイトル**
LINEAR TOPOLOGIES ON SEMI-ORDERED LINEAR SPACES

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Normed semi-order linear spaces are considered first by L. Kantorovitch. In this paper we shall consider linear topologies on semi-ordered linear spaces.

Let $R$ be a linear space. A manifold $V \subset R$ is called a vicinity, if for any $a \in R$ we can find $\varepsilon > 0$ such that $\xi a \in V$ for $|\xi| \leq \varepsilon$. A collection of vicinities $\mathfrak{B}$ is said to be a linear topology on $R$, if

1) $U \subset V \in \mathfrak{B}$ implies $U \in \mathfrak{B}$,
2) $U, V \in \mathfrak{B}$ implies $UV \in \mathfrak{B}$,
3) $V \in \mathfrak{B}$ implies $\xi V \in \mathfrak{B}$ for every real number $\xi$,
4) for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ such that $\xi U \subset V$ for $|\xi| \leq 1$,
5) for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ such that $U \times U \subset V$,

adopting the notations:

$$\xi U = \{\xi x : x \in U\}, \quad U \times V = \{x+y : x \in U, y \in V\}.$$ 

A subset $\mathfrak{B} \subset \mathfrak{B}$ is called a basis of a linear topology $\mathfrak{B}$, if for any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ and $\varepsilon > 0$ such that $\varepsilon U \subset V$.

Let $R$ be now a semi-ordered linear space and universally continuous, that is, for any system of positive elements $a_{\lambda} \in R (\lambda \in \Lambda)$ there exists $\cap a_{\lambda}$. In this paper we shall consider only such linear topologies $\mathfrak{B}$ on $R$ that $\mathfrak{B}$ have a basis composed only of vicinities $V$ subject to the conditions:

6) $a \in V, |x| \leq a$ implies $x \in V$,
7) $0 \leq a_{\lambda} \in V (\lambda \in \Lambda), a_{\lambda} \uparrow_{\lambda \in \Lambda} a$ implies $a \in V$.

Here $a_{\lambda} \uparrow_{\lambda \in \Lambda} a$ means that for any two $\lambda, \lambda_{2} \in \Lambda$ we can find $\lambda \in \Lambda$ such that

$$a_{\lambda} \geq a_{\lambda_{1}}, \cup a_{\lambda_{2}}, \text{ and } a = \cup_{\lambda \in \Lambda} a_{\lambda}.$$ 

For such a linear topology, we shall prove as a principal result that the manifold $\{x : a \leq x \leq b\}$ is complete as a uniform space in Weil’s
sense.\(^{(2)}\)

For a vicinity \( V \) subject to the conditions 6), 7), putting

\[
\|x\|_V = \inf_{\xi \in \mathcal{V}} \frac{1}{\xi},
\]

we obtain a pseudo-norm on \( R \). A manifold \( A \subset R \) is said to be topologically bounded, by a linear topology \( \mathcal{B} \), if \( \sup_{x \in A} \|x\|_V < +\infty \) for every such vicinity \( V \in \mathcal{B} \). A linear topology \( \mathcal{B} \) on \( R \) is said to be monotone complete, if for any topologically bounded system \( 0 \leq a_\lambda \in R (\lambda \in \Lambda) \) such that \( a_\lambda \downarrow \lambda \in \Lambda \), we can find \( a \in R \) for which \( a_\lambda \downarrow \lambda \in \Lambda \). With this definition, we can prove that if a linear topology \( \mathcal{B} \) is monotone complete, then \( R \) is complete by \( \mathcal{B} \) in \( \text{Weil's sense} \). This result may be considered as a generalization of the famous Riesz-Fischer's theorem about \( L_p \)-spaces.

A vicinity \( V \) is said to be convex, if \( V \times V \subset 2V \). A linear topology \( \mathcal{B} \) is said to be convex, if \( \mathcal{B} \) has a basis composed only of convex vicinities. There exists a linear topology \( \mathcal{B} \) on \( R \) of which the totality of convex vicinities subject to the conditions 6), 7) is a basis. This linear topology \( \mathcal{B} \) is called the strong topology of \( R \). A linear topology \( \mathcal{B} \) is said to be sequential, if \( \mathcal{B} \) has a basis composed of at most countable vicinities. We shall prove that if a linear topology \( \mathcal{B} \) is sequential, convex, complete, and \( \prod_{\mathcal{B}} V = \{0\} \), then \( \mathcal{B} \) is the strong topology of \( R \).

Let \( R \) be now reflexive and \( \overline{R} \) its conjugate space.\(^{(3)}\) The so-called weak linear topology of \( R \) by \( \overline{R} \) is not a linear topology in our sense. However there exists the weakest linear topology \( \mathfrak{B} \) among our linear topologies by which every \( \overline{a} \in \overline{R} \) is topologically continuous. This linear topology \( \mathfrak{B} \) is called the absolute weak topology of \( R \), as the system of vicinities \( \{x : \overline{a}(|x|) \leq 1\} \) for all positive \( \overline{a} \in \overline{R} \) is a basis of \( \mathfrak{B} \). We can prove that the absolute weak topology \( \mathfrak{B} \) of \( R \) is weaker than the strong topology \( \mathcal{E} \) of \( R \), i.e., \( \mathfrak{B} \subset \mathcal{E} \), but \( \mathfrak{B} \) is equivalent to \( \mathcal{E} \), i.e., a manifold \( A \subset R \) is topologically bounded by \( \mathfrak{B} \), if and only if \( A \) is so by \( \mathcal{E} \).

A pseudo-norm \( \|x\| \) on \( R \) is said to be reflexive, if for

\[
\overline{A} = \{\overline{a} : \sup_{|x| \leq 1} |\overline{a}(x)| \leq 1\},
\]

we have \( \|x\| = \sup_{x \in \overline{A}} |\overline{a}(x)| \). A linear topology \( \mathcal{B} \) on \( R \) is said to be reflexive, if \( \mathfrak{B} \) has a basis \( \mathfrak{B} \) such that the pseudo-norm \( \|x\|_V \) is reflexive


\(^{(3)}\) H. \text{Nakano} : Modulated semi-ordered linear spaces, Tokyo Math. Book Series I (1950), §22. This book will be denoted by MSLS in this paper.
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for every $V \in \mathcal{B}$. The absolute weak topology of $R$ is reflexive. We shall prove that if the strong topology of $R$ is sequential, then it is reflexive. This result is a generalization of the theorem: if there is a complete norm on $R$, then there exists a complete reflexive norm on $R$.

We shall make use of notations in MSLS and the following notations:

$$A^+ = \{x^+ : x \in A\}, \quad A^- = \{x^- : x \in A\}, \quad |A| = \{|x| : x \in A\},$$

$$A \searrow B = \{x \searrow y : x \in A, y \in B\}, \quad A \nearrow B = \{x \nearrow y : x \in A, y \in B\},$$

$$A \times B = \{x + y : x \in A, y \in B\}.$$

for manifolds $A, B$ of $R$.

§ 1. Linear topologies

Let $R$ be a universally continuous semi-ordered linear space. A set of positive elements $V$ is said to be a positive vicinity, if

1) for any $a \geq 0$ we can find $\epsilon > 0$ such that $\epsilon a \in V$,
2) $0 \leq b \leq a \in V$ implies $b \in V$,
3) $V \ni a \uparrow \lambda \alpha a$ implies $a \in V$.

A positive vicinity $V$ is said to be convex, if $x, y \in V$, $\lambda + \mu = 1$, $\lambda, \mu \geq 0$ implies $\lambda x + \mu y \in V$.

With this definition, we see easily that if $V$ is a positive vicinity (convex), then $\xi V$ also is a positive vicinity (convex) for $\xi > 0$, and for two positive vicinity $U, V$ (convex), both $UV$ and $U \times V$ are positive vicinities (convex).

A collection $\mathcal{B}$ of positive vicinities is called a linear topology, if

1') $U \subset V \in \mathcal{B}$ implies $U \in \mathcal{B}$,
2') $U, V \in \mathcal{B}$ implies $UV \in \mathcal{B}$,
3') $V \in \mathcal{B}$ implies $\xi V \in \mathcal{B}$ for every $\xi > 0$,
4') for any $V \in \mathcal{B}$ we can find $U \in \mathcal{B}$ such that $U \times U \subset V$.

For a linear topology $\mathcal{B}$ on $R$, a subset $\mathcal{B} \subset \mathcal{B}$ is called a basis of $\mathcal{B}$, if for any $V \in \mathcal{B}$ we can find $U \in \mathcal{B}$ and $a > 0$ such that $aU \subset V$. With this definition, we can prove easily

Theorem 1.1 If a collection of positive vicinities $\mathcal{B}$ satisfies

1'') for any $U, V \in \mathcal{B}$ we can find $W \in \mathcal{B}$ and $a > 0$ such that $aW \subset UV$,
2'') for any $V \in \mathcal{B}$ we can find $U \in \mathcal{B}$ and $a > 0$ such that $U \times U \subset aV$,

then there exists uniquely a linear topology $\mathcal{B}$ of which $\mathcal{B}$ is a basis.

A linear topology $\mathcal{B}$ is said to be convex, if $\mathcal{B}$ has a basis composed
only of convex positive vicinities. A linear topology $\mathfrak{B}$ is said to be *sequential*, if $\mathfrak{B}$ has a basis composed of at most countable positive vicinities. A sequence of positive vicinities $V_\nu (\nu = 1, 2, \cdots)$ is said to be *decreasing*, if

$$V_\nu \supset V_{\nu+1} \times V_{\nu+1} \quad \text{for every } \nu = 1, 2, \cdots .$$

If a linear topology $\mathfrak{B}$ is sequential, then we can find obviously by definition a decreasing sequence $V_\nu \in \mathfrak{B} \ (\nu = 1, 2, \cdots)$ as a basis of $\mathfrak{B}$. Such a basis is called a *decreasing basis* of $\mathfrak{B}$. If $V_\nu \in \mathfrak{B} \ (\nu = 1, 2, \cdots)$ is a decreasing basis of $\mathfrak{B}$, then for any $V \in \mathfrak{B}$ we can find $\nu$ such that $V_\nu \subset V$. Because we can find by definition $\mu$ and $\varepsilon > 0$ such that $\varepsilon V_\mu \subset V$. For such $\varepsilon > 0$, we can find $\nu > \mu$ such that $\frac{1}{2^{\nu - \mu}} < \varepsilon$, and then we have

$$V_\nu \subset \frac{1}{2^{\nu - \mu}} V_\mu \subset \varepsilon V_\mu \subset V,$$

because we have $V_\nu \supset 2V_{\nu+1}$ for every $\nu = 1, 2, \cdots$.

A decreasing basis $V_\nu \in \mathfrak{B} \ (\nu = 1, 2, \cdots)$ is said to be *convex*, if every $V_\nu \ (\nu = 1, 2, \cdots)$ is convex. With this definition, we see at once by definition

*Theorem 1.2.* *If* a linear topology $\mathfrak{B}$ is sequential and convex, *then* $\mathfrak{B}$ *has a convex decreasing basis*.

A linear topology $\mathfrak{B}$ is said to be of *single vicinity* if $\mathfrak{B}$ has a basis composed only of a single positive vicinity. With this definition we have obviously

*Theorem 1.3.* *If* a linear topology $\mathfrak{B}$ is of single vicinity and convex, *then there is a convex positive vicinity which is a basis of $\mathfrak{B}$."

**§ 2. Pseudo-norms**

A functional $\|x\| \ (x \in R)$ on $R$ is said to be a *pseudo-norm* on $R$, if

1) $0 \leq \|x\| < + \infty \quad \text{for every } x \in R ,$

2) $|x| \leq |y| \quad \text{implies } \|x\| \leq \|y\| ,$

3) $\|\xi x\| = |\xi| \|x\| \quad \text{for every real number } \xi ,$

4) $0 \leq x_\lambda \ (\lambda \in A) \quad \text{implies } \|x\| = \sup_{\lambda \in A} \|x_\lambda\| .$

A pseudo-norm $\|x\| \ (x \in R)$ is said to be *convex*, if

$$\|x + y\| \leq \|x\| + \|y\| \quad \text{for every } x, y \in R .$$

For a pseudo-norm $\|x\| \ (x \in R)$, putting

$$V = \{x : \|x\| \leq 1 , \ x \geq 0\} ,$$
we see easily that $V$ is a positive vicinity. Furthermore, if $\|x\|(x \in R)$ is convex, then this positive vicinity $V$ is convex.

Conversely, for a positive vicinity $V$, putting

\begin{equation}
\|x\|_V = \inf_{\xi \in \mathbb{R}} \frac{1}{\xi}
\end{equation}

we obtain a pseudo-norm $\|x\|_V (x \in R)$, which will be called the pseudo-norm of $V$. With this definition, we see easily

\begin{equation}
V = \{x : \|x\|_V \leq 1, x \geq 0\}.
\end{equation}

Furthermore we can prove easily

\begin{enumerate}
\item \[\|x+y\|_\xi = \frac{1}{\xi} \|x\|_V \quad \text{for } \xi > 0,\]
\item \[V \subset U \text{ implies } \|x\|_V \geq \|x\|_U \quad \text{for every } x \in R,\]
\item \[V \times V \subset U \text{ implies } \|x+y\|_U \leq \max \{\|x\|_V, \|y\|_V\}.\]
\end{enumerate}

By virtue of Theorem 1.1, we can prove easily

\begin{figure}
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\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure}

\textbf{Theorem 2.1.} For a system of pseudo-norms $\|x\|_\lambda (\lambda \in \Lambda)$ on $R$, if for any $\lambda \in \Lambda$ we can find $\sigma \in \Lambda$ such that

\[\|x+y\|_\lambda \leq \|x\|_\sigma + \|y\|_\sigma \quad \text{for every } x, y \in R,\]

then there exists uniquely a linear topology $\mathcal{B}$ on $R$ such that the totality of

\[V_{\lambda_1, \lambda_2, \ldots, \lambda_k} = \{x : \|x\|_\lambda \leq 1 (\lambda = 1, 2, \ldots, k), x \geq 0\}\]

for every finite number of elements $\lambda \in \Lambda (\lambda = 1, 2, \ldots, k)$ is a basis of $\mathcal{B}$.

A pseudo-norm $\|x\| (x \in R)$ is said to be proper, if $\|x\| = 0$ implies $x = 0$. A pseudo-norm is called a norm, if it is convex and proper.

\textbf{Theorem 2.2.} For a convex pseudo-norm $\|x\| (x \in R)$ there exists uniquely a normal manifold $N$ of $R$ such that $\|x\| (x \in N)$ is proper in $N$ and $\|x\| = 0$ for every $x \in N^\perp$.

\textbf{Proof.} Putting $N = \{x : \|x\| = 0\}$, we see easily that $N$ is a normal manifold of $R$. For such $N$, it is evident that $\|x\| = 0$ for every $x \in N$. Conversely, if $\|x\| = 0$, then we have naturally $x \in N$, and hence $[N^\perp] x = 0$. Thus $\|x\|$ is proper in $N^\perp$. If $\|x\|$ is proper in a normal manifold $M$ and $\|x\| = 0$ for every $x \in M^\perp$, then it is evident that $M^\perp = N$.

A system of pseudo-norms $\|x\|_\lambda (\lambda \in \Lambda)$ is said to be proper, if $\|x\|_\lambda = 0$ for all $\lambda \in \Lambda$ implies $x = 0$. With this definition, we have

\textbf{Theorem 2.3.} For a system of pseudo-norms $\|x\|_\lambda (\lambda \in \Lambda)$ on $R$, if for any $\lambda \in \Lambda$ we can find $\sigma \in \Lambda$ such that

\[\|x+y\|_\lambda \leq \|x\|_\sigma + \|y\|_\sigma \quad \text{for every } x, y \in R,\]
then there exists uniquely a normal manifold $N$ of $R$ such that the system $||x||_\lambda (\lambda \in \Lambda)$ is proper in $N$ and $||x||_\lambda = 0$ for every $\lambda \in \Lambda$ and $x \in N^\perp$.

Proof. Putting $M = \{ x : ||x||_\lambda = 0 \text{ for all } \lambda \in \Lambda \}$, we see easily that $M$ is a normal manifold of $R$ and $M^\perp$ satisfies our requirement. Furthermore the uniqueness is obvious.

We shall say that $R$ is separated by a linear topology $\mathfrak{B}$, or that $\mathfrak{B}$ is separative if $\prod_{V \in \mathfrak{B}} V = \{ 0 \}$. With this definition, we see at once

Theorem 2.4. A linear topology $\mathfrak{B}$ is separative, if and only if for a basis $\mathfrak{B}$ of $\mathfrak{B}$, the system of pseudo-norms $||x||_V (V \in \mathfrak{B})$ is proper.

§ 3. Completeness

Let $\mathfrak{B}$ be a linear topology on $R$. A system of manifolds $A_\lambda (\lambda \in \Lambda)$ is said to be a CAUCHY system by $\mathfrak{B}$, if $\prod_{\nu=1}^{r_V} A_{\lambda\nu} \neq 0$ for every finite number of elements $\lambda, \nu \in \Lambda (\nu = 1, 2, \cdots, r_V)$, and for any $V \in \mathfrak{B}$ we can find $\lambda \in \Lambda$ such that

$$|x - y| \in V$$

for every $x, y \in A_\lambda$.

A CAUCHY system $A_\lambda (\lambda \in \Lambda)$ is said to be convergent to a limit $a \in R$, if for any $V \in \mathfrak{B}$ we can find $\lambda \in \Lambda$ such that

$$|x - a| \in V$$

for every $x \in A_\lambda$.

If $\mathfrak{B}$ is separative, then we see easily that the limit of a CAUCHY system is uniquely determined, if it is convergent.

We see easily by definition that for a basis $\mathfrak{B}$ of $\mathfrak{B}$, a system of manifolds $A_\lambda (\lambda \in \Lambda)$ is a CAUCHY system by $\mathfrak{B}$, if and only if $\prod_{\nu=1}^{r_V} A_{\lambda\nu} \neq 0$ for every finite number of elements $\lambda, \nu \in \Lambda (\nu = 1, 2, \cdots, r_V)$ and for any $V \in \mathfrak{B}$ and $\varepsilon > 0$ we can find $\lambda \in \Lambda$ such that

$$||x - y||_V \leq \varepsilon$$

for every $x, y \in A_\lambda$.

Furthermore we see that a CAUCHY system $A_\lambda (\lambda \in \Lambda)$ is convergent to a limit $a \in R$, if and only if for any $V \in \mathfrak{B}$ and $\varepsilon > 0$ we can find $\lambda \in \Lambda$ such that

$$||x - a||_V \leq \varepsilon$$

for every $x \in A_\lambda$.

By virtue of the formula §2(5), we can prove easily

Theorem 3.1. For two CAUCHY system $A_\lambda$ and $B_\lambda (\lambda \in \Lambda)$, all $A_\lambda \setminus B_\lambda$, $A_\lambda \cap B_\lambda$, and $A_\lambda \times B_\lambda (\lambda \in \Lambda)$ are CAUCHY systems, furthermore, if $A_\lambda$ and
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$B_{\lambda}(\lambda \in \Lambda)$ are convergent respectively to limits $a$ and $b$, then $A_{\lambda} \cup B_{\lambda}$, $A_{\lambda} \cap B_{\lambda}$, and $A_{\lambda} \times B_{\lambda}(\lambda \in \Lambda)$ are convergent to $a \cup b$, $a \cap b$, and $a+b$ respectively.

We see further easily

Theorem 3.2. For a CAUCHY system $A_{\lambda}(\lambda \in \Lambda)$, all $A_{\lambda}^{+}, A_{\lambda}^{-}, |A_{\lambda}|, aA_{\lambda}$, and $[N]A_{\lambda}(\lambda \in \Lambda)$ are CAUCHY systems for every real number $a$ and projection operator $[N]$. If a CAUCHY system $A_{\lambda}(\lambda \in \Lambda)$ is convergent to a limit $a$, then $A_{\lambda}^{+}, A_{\lambda}^{-}, |A_{\lambda}|, aA_{\lambda}$, and $[N]A_{\lambda}(\lambda \in \Lambda)$ are convergent to $a^{+}, a^{-}, |a|, aa$, and $[N]a$ respectively.

A manifold $A$ of $R$ is said to be complete by a linear topology $\mathcal{B}$, if every CAUCHY system $A_{\lambda} \subset A(\lambda \in \Lambda)$ is convergent to a limit $a \in A$. With this definition we have

Theorem 3.3. For every positive element $a \in R$, $\{x : |x| \leqq a\}$ is complete by $\mathcal{B}$.

Proof. We shall consider firstly the case where $\mathcal{B}$ is sequential and separative. Let $V_{\nu} \in \mathcal{B}(\nu = 1, 2, \cdots)$ be a decreasing basis of $\mathcal{B}$. We set

$A = \{x : |x| \leqq a\}$

and assume that $A_{\lambda} \subset A(\lambda \in \Lambda)$ is a CAUCHY system by $\mathcal{B}$. Then we can find $\lambda_{\nu} \in \Lambda(\nu = 1, 1, \cdots)$ such that

$\sup_{x, y \in A_{\lambda}} \|x - y\|_{\nu} \leqq \frac{1}{\nu}(\nu = 1, 2, \cdots).$

For such $\lambda_{\nu} \in \Lambda(\nu = 1, 2, \cdots)$ we can find

$a_{\mu} \in \prod_{\nu = 1}^{\mu} A_{\lambda, \nu}(\mu = 1, 2, \cdots).$

As $V_{\nu+1} \times V_{\nu+1} \subset V_{\nu}$, we conclude by the formula §2(5)

$\left\| \left( \sum_{\nu = 1}^{\sigma} |a_{\nu+1} - a_{\nu}| \right) \right\|_{V_{\mu - 1}} \leqq \max_{\mu \leqq \nu \leqq \sigma} \|a_{\nu+1} - a_{\nu}\|_{\nu} \leqq \frac{1}{\mu}.$

On the other hand we have

$\sigma_{\nu = 1}^{\sigma} a_{\nu} - a_{\mu} = \sigma_{\nu = 1}^{\sigma} (a_{\nu} - a_{\mu}) \leqq \sigma_{\nu = 1}^{\sigma} |a_{\nu+1} - a_{\nu}|,$

and hence $\left\| \sigma_{\nu = 1}^{\sigma} a_{\nu} - a_{\mu} \right\|_{V_{\mu - 1}} \leqq \frac{1}{\mu}$. This relation yields by 4) in §2

$\left\| \bigcup_{\nu = 1}^{\infty} a_{\nu} - a_{\mu} \right\|_{V_{\mu - 1}} \leqq \frac{1}{\mu}(\mu = 2, 3, \cdots).$

We obtain likewise

$\left\| a_{\mu} - \bigcap_{\nu = 1}^{\infty} a_{\nu} \right\|_{V_{\mu - 1}} \leqq \frac{1}{\mu}(\mu = 2, 3, \cdots).$
Consequently we have by the formula §2 (5)

\[ \left\| \bigcup_{\nu=\mu}^{\infty} a_{\nu} - \bigcap_{\nu=\mu}^{\infty} a_{\nu} \right\|_{V_{\mu-B}} \leq \frac{1}{\mu} \quad (\mu=3, 4, \cdots). \]

Thus, putting \( l_{\mu} = \bigcup_{\nu=\mu}^{\infty} a_{\nu} - \bigcap_{\nu=\mu}^{\infty} a_{\nu} \), \( l = \bigcap_{\mu=1}^{\infty} l_{\mu} \), we obtain \( \|l\|_{V_{\mu-0}} \leq \frac{1}{\mu} \) for every \( \rho = 3, 4, \cdots \). By \( \S 2 \) (4), we conclude hence \( \|l\|_{V_{1}} \leq \|l\|_{V_{\mu}} = 0 \) for every \( \rho = 1, 2, \cdots \), and hence \( l = 0 \), as \( \mathcal{B} \) is separative by assumption. Therefore there exists \( a \in R \) such that \( \lim_{\nu \to \infty} a_{\nu} = a \), and naturally \( a \in A \). Furthermore we have

\[ \|a - a_{\mu}\|_{V_{\mu-2}} \leq \frac{1}{\mu} \]

for every \( \mu = 3, 4, \cdots \), because \( \bigcup_{\nu=\mu}^{\infty} a_{\nu} \geq a \geq \bigcap_{\nu=\mu}^{\infty} a_{\nu} \). This relation shows that \( A_{\lambda} (\lambda \in \Lambda) \) is convergent to \( a \) by \( \mathcal{B} \).

Now we consider the general case. Let \( A_{\lambda} \subset A (\lambda \in \Lambda) \) be an arbitrary Cauchy system by \( \mathcal{B} \) and \( V_{\nu} \in \mathcal{B} (\nu=1, 2, \cdots) \) an arbitrary decreasing sequence. By virtue of Theorem 2.3, we can find a normal manifold \( N_{V_{1}, V_{2}, \cdots} \) of \( R \) such that the system \( \|x\|_{V_{\nu}} (\nu=1, 2, \cdots) \) is proper in \( N_{V_{1}, V_{2}, \cdots} \) and \( \|x\|_{V_{\nu}} = 0 \) for every \( x \in N_{V_{1}, V_{2}, \cdots} \) and \( \nu = 1, 2, \cdots \). Recalling Theorem 2.1, we can find then a linear topology \( \mathcal{B}_{V_{1}, V_{2}, \cdots} \) on \( N_{V_{1}, V_{2}, \cdots} \) such that \( [N_{V_{1}, V_{2}, \cdots}] \mathcal{B}_{V_{1}, V_{2}, \cdots} A_{\lambda} (\lambda \in \Lambda) \) is a Cauchy system by \( \mathcal{B}_{V_{1}, V_{2}, \cdots} \), there exists uniquely a limit \( a \in [N_{V_{1}, V_{2}, \cdots}] A_{\lambda} (\lambda \in \Lambda) \), as proved just above.

Corresponding to every decreasing sequence \( V_{\nu} \in \mathcal{B} (\nu=1, 2, \cdots) \), we obtain thus uniquely a normal manifold \( N_{V_{1}, V_{2}, \cdots} \) and a limit \( a_{V_{1}, V_{2}, \cdots} \in [N_{V_{1}, V_{2}, \cdots}] A_{\lambda} (\lambda \in \Lambda) \). We see further by Theorem 3.2 that for every two decreasing sequences \( V_{\nu} \) and \( U_{\nu} \in \mathcal{B} (\nu=1, 2, \cdots) \), we have

\[ [N_{V_{1}, V_{2}, \cdots}] [N_{U_{1}, U_{2}, \cdots}] a_{V_{1}, V_{2}, \cdots} = [N_{V_{1}, U_{2}, \cdots}] a_{V_{1}, U_{2}, \cdots}. \]

Therefore we can find \( a \in A \) such that

\[ [N_{V_{1}, V_{2}, \cdots}] a = a_{V_{1}, V_{2}, \cdots} \]

for every decreasing sequence \( V_{\nu} \in \mathcal{B} (\nu=1, 2, \cdots) \). Such \( a \in A \) is a limit of \( A_{\lambda} (\lambda \in \Lambda) \). Because, for any \( V \in \mathcal{B} \) we can find a decreasing sequence \( V_{\nu} \in \mathcal{B} (\nu=1, 2, \cdots) \) such that \( V \supset V_{1} \times V_{1} \), and \( \lambda \in \Lambda \) such that

\[ \sup_{x \in [N_{V_{1}, V_{2}, \cdots} A_{\lambda}]} \|x - a_{V_{1}, V_{2}, \cdots}\|_{V_{1}} \leq 1, \]
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and hence \( \sup_{x \in A_{\lambda}} \|[N_{V_{1}}, V_{\sim}] x - a\|_{V_{1}} \leq 1 \). As
\[
\|[N_{V_{1}}, V_{\sim}] x - a\|_{V_{1}} = 0,
\]
we obtain by §2(5)
\[
\sup_{x \in A_{\lambda}} \|x - a\|_{V} \leq 1,
\]
that is, \( |x - a| \in V \) for every \( x \in A_{\lambda} \). Therefore \( A \) is complete by \( \mathfrak{B} \).

**Theorem 3.4.** \( \{x : a \leq x \leq b\} \) is complete by every linear topology \( \mathfrak{B} \) for every two elements \( a \leq b \).

**Proof.** Putting \( A = \{x : |x| \leq |a| + |b|\} \), \( B = \{x : a \leq x \leq b\} \), we have obviously \( B \subset A \) and \( A \) is complete by \( \mathfrak{B} \) on account of Theorem 3.3. For a Cauchy system \( A_{\lambda} \subset B (\lambda \in \Lambda) \) there exists hence a limit \( c \in A \) of \( A_{\lambda} (\lambda \in \Lambda) \), and then we obtain by Theorem 3.1 that \( (c \wedge a) \cap b \) is a limit of
\[
(A_{\lambda} \leftarrow a) \wedge b = A_{\lambda} (\lambda \in \Lambda),
\]
and it is evident that \( (c \wedge a) \cap b \in B \). Therefore \( B \) is complete by \( \mathfrak{B} \).

§ 4. Topologically bounded manifolds

A manifold \( A \) of \( \mathbb{R} \) is said to be **topologically bound** by a linear topology \( \mathfrak{B} \), if
\[
\sup_{x \in A} \|x\|_{V} < +\infty \quad \text{for every } V \in \mathfrak{B}.
\]
With this definition, it is obvious by the formula §2(4) that a manifold \( A \) is topologically bounded by a linear topology \( \mathfrak{B} \), if and only if for a basis \( \mathfrak{B} \) of \( \mathfrak{B} \) we have
\[
\sup_{x \in A} \|x\|_{V} < +\infty \quad \text{for every } V \in \mathfrak{B}.
\]

We can prove easily by definition

**Theorem 4.1.** If a manifold \( A \) is topologically bounded by a linear topology \( \mathfrak{B} \), then all \( A^{+}, A^{-}, |A|, aA, [N] A \) are topologically bound by \( \mathfrak{B} \) for every real number \( a \) and projection operator \( [N] \). If both manifolds \( A \) and \( B \) are topologically bounded by \( \mathfrak{B} \), then all \( A^{\wedge} B, A_{\wedge} B, \) and \( A \times B \) are topologically bounded by \( \mathfrak{B} \).

A manifold \( A \) of \( \mathbb{R} \) is said to be **order bound** or merely **bounded**, if we can find a positive element \( a \in \mathbb{R} \) such that \( |x| \leq a \) for every \( x \in A \). Every bounded manifold is obviously topologically bounded by every linear topology.

A linear topology \( \mathfrak{B} \) on \( \mathbb{R} \) is said to be **monotone complete**, if for any
topologically bounded manifold of positive elements $a_{\lambda}^{\uparrow}_{\lambda \in \Lambda}$, we can find $a \in R$ such that $a_{\lambda}^{\uparrow}_{\lambda \in \Lambda} a$.

Theorem 4.2. If a linear topology $\mathfrak{B}$ on $R$ is monotone complete, then $R$ is complete by $\mathfrak{B}$.

Proof. Let $A_{\lambda} (\lambda \in \Lambda)$ be a Cauchy system by $\mathfrak{B}$. We suppose firstly that $\mathfrak{B}$ is separative. As $A_{\lambda}^{+} (\lambda \in \Lambda)$ also is by Theorem 3.2 a Cauchy system, corresponding to every $x \geq 0$, we obtain uniquely by Theorem 3.3 a limit $a_{x}$ of a Cauchy system $A_{\lambda}^{+} x (\lambda \in \Lambda)$. For this limit $a_{x}$, we have obviously by Theorem 3.1 $0 \leq a_{x}^{\uparrow}_{x \geq 0}$. Furthermore the system $a_{x} (x \geq 0)$ is topologically bounded by $\mathfrak{B}$. Because for any $V \in \mathfrak{B}$ we can find by definition $U \in \mathfrak{B}$ such that $U \times U \subset V$, and $\lambda \in \Lambda$ such that $\|y - z\|_{U} \leq 1$ for every $y, z \in A_{\lambda}^{+}$, and hence by §2(5) $\sup_{y \in A_{\lambda}^{+}} \|y\|_{\mathfrak{B}} < +\infty$.

For any $x \geq 0$ we $\|x \wedge y\|$ can find by definition, $\lambda \in \Lambda$ such that

$$\|a_{x} - z\|_{\mathfrak{B}} \leq 1 \quad \text{for every} \quad z \in A_{\lambda}^{+} x .$$

For an element $b \in A_{\lambda}^{+} A_{\lambda}^{+}$, we have then by §2(5)

$$\|a_{x}\|_{\mathfrak{B}} \leq \sup_{\lambda \in \Lambda} \|b_{\lambda} x\|_{\mathfrak{B}} \leq \sup_{\lambda \in \Lambda} \|b\|_{\mathfrak{B}} ,$$

and hence $\|a_{x}\|_{\mathfrak{B}} \leq \sup_{\lambda \in \Lambda} \|y\|_{\mathfrak{B}} \cdot v \leq \sup_{\lambda \in \Lambda} \|y\|_{\mathfrak{B}}$ for every $x \geq 0$.

Therefore there exists by assumption $a \in R$ such that $a_{x}^{\uparrow}_{x \geq 0} a$. As we have by Theorem 3.1

$$a_{x} \wedge y = a_{y \cap y} \quad \text{for every} \quad x, y \geq 0 ,$$

we obtain $a \wedge x = a_{x}$ for every $x \geq 0$. For any $V \in \mathfrak{B}$ we can find $U \in \mathfrak{B}$ such that $U \times U \subset V$, and further $\lambda \in \Lambda$ such that

$$\sup_{y \in A_{\lambda}^{+}} \|y - z\|_{U} \leq 1 .$$

Thus, for any $y \in A_{\lambda}^{+}$, putting $x = y \wedge a$, we can find $\lambda \in \Lambda$ such that

$$\sup_{\lambda \in \Lambda} \|z \wedge x - a_{x}\|_{\mathfrak{B}} = \sup_{\lambda \in \Lambda} \|z \wedge x - a_{x}\|_{\mathfrak{B}} \leq 1 ,$$

and for $z \in A_{\lambda}^{+} A_{\lambda}^{+}$, we have

$$\|y - z \wedge x\|_{U} = \|y \wedge x - z \wedge x\|_{U} \leq \|y - z\|_{U} \leq 1 .$$

Consequently we obtain by §2(5)

$$\|y - a\|_{\mathfrak{B}} \leq 1 \quad \text{for every} \quad y \in A_{\lambda}^{+} .$$

Therefore $a$ is a limit of $A_{\lambda}^{+} (\lambda \in \Lambda)$. We obtain likewise a limit $b$ of $A_{\lambda}^{+}$.
(λ ∈ Δ). Thus we see by Theorem 3.1 that α − β is a limit of \( A_λ (λ ∈ Δ) \).

In general, we can find by Theorem 2.3 a normal manifold \( N \) of \( R \), such that the system of pseudo-norms \( \|x\|_V (V ∈ ℱ) \) is proper in \( N \) and \( \|x\|_V = 0 \) for every \( x ∈ N^⊥ \) and \( V ∈ ℱ \). Then there exists a limit \( α ∈ [N] A_λ (λ ∈ Δ) \), as proved just above. This limit \( α \) also is a limit of \( A_λ (λ ∈ Δ) \), because for any \( V ∈ ℱ \) we can find \( U ∈ ℱ \) such that \( U × U ⊆ V \), and we have by §2 (5) for every \( x ∈ R \)

\[ \|x − α\|_V ≤ ||[N] x − α||_V. \]

A linear topology \( ℱ \) on \( R \) is said to be complete, if \( R \) is complete by \( ℱ \). We can state then by Theorem 4.2 that every monotone complete linear topology is complete.

**Theorem 4.3.** If a linear topology \( ℱ \) on \( R \) is separative, convex, and complete, and a manifold \( A \) of \( R \) is topologically bounded by \( ℱ \), then we have for every positive vicinity \( W \)

\[ \sup_{x ∈ A} \|x\|_W < +∞. \]

**Proof.** If \( \sup_{x ∈ A} \|x\|_W = +∞ \), then we can find \( x_ν ∈ A (ν = 1, 2, \cdots) \) such that \( \|x_ν\|_W ≥ ν 2^ν \) for every \( ν = 1, 2, \cdots \). As \( A \) is by assumption topologically bounded by \( ℱ \), we have obviously \( \sum_{ν=1}^{∞} \frac{1}{2^ν} \|x_ν\|_V < +∞ \) for every \( V ∈ ℱ \).

As \( ℱ \) is convex and complete by assumption, we can find \( α ∈ R \) such that

\[ \lim_{μ → ∞} \left| \sum_{ν=1}^{μ} \frac{1}{2^ν} |x_ν| - α \right|_V = 0 \quad \text{for every } V ∈ ℱ. \]

As \( ℱ \) is separative by assumption, we conclude easily that \( α = \sum_{ν=1}^{∞} \frac{1}{2^ν} |x_ν| \), and hence we have

\[ \|α\|_W ≥ \frac{1}{2^ν} \|x_ν\|_W ≥ ν \quad \text{for every } ν = 1, 2, \cdots, \]

contradicting \( \|α\|_W < +∞ \).

§ 5. Equivalence

A linear topology \( ℱ \) on \( R \) is said to be equivalent to a linear topology \( \Pi \) on \( R \), if \( ℱ \) has the same topologically bounded manifolds with \( \Pi \), that is, a manifold \( A \) is topologically bounded by \( ℱ \) if and only if \( A \) is so by \( \Pi \). With this definition, we have obviously

**Theorem 5.1.** If a linear topology \( ℱ \) is monotone complete, then every
linear topology equivalent to \( \mathfrak{B} \) is also monotone complete.

We shall say that a linear topology \( \mathfrak{B} \) on \( R \) is stronger than a linear topology \( \mathfrak{H} \) on \( R \), or that \( \mathfrak{H} \) is weaker than \( \mathfrak{B} \), if \( \mathfrak{B} \supseteq \mathfrak{H} \). With this definition we have obviously by Theorem 4.3.

**Theorem 5.2.** If a linear topology \( \mathfrak{B} \) is separative, convex, and complete, then every linear topology stronger than \( \mathfrak{B} \) is equivalent to \( \mathfrak{B} \).

By virtue of Theorem 1.1, we see easily that there exists uniquely a linear topology \( \mathfrak{B} \) of which the totality of convex vicinity in \( R \) is a basis. This linear topology \( \mathfrak{B} \) is called the strong topology of \( R \). With this definition, we have obviously that the strong topology of \( R \) is the strongest convex linear topology on \( R \), that is, the strong topology of \( R \) is stronger than every other convex linear topology on \( R \).

Recalling Theorem 5.2, we obtain at once

**Theorem 5.3.** If a linear topology \( \mathfrak{B} \) on \( R \) is separative, convex, and complete, then \( \mathfrak{B} \) is equivalent to the strong topology of \( R \).

**Theorem 5.4.** If a linear topology \( \mathfrak{B} \) on \( R \) is sequential and equivalent to a linear topology \( \mathfrak{H} \) on \( R \), then \( \mathfrak{B} \) is stronger than \( \mathfrak{H} \).

**Proof.** Let \( V_{\nu} \in \mathfrak{B} \) \((\nu=1, 2, \ldots)\) be a decreasing basis of \( \mathfrak{B} \). If \( \mathfrak{B} \) is not stronger than \( \mathfrak{H} \), then we can find \( U \in \mathfrak{H} \) such that \( U \in \mathfrak{B} \). For such \( U \), there is a sequence \( a_{\nu} \in R \) \((\nu=1, 2, \ldots)\) such that

\[
\nu U \ni a_{\nu} \in V_{\nu} \quad \text{for every } \nu=1, 2, \ldots,
\]

and hence we have by the formulas (2), (3) in §2

\[
\|a_{\nu}\|_{V_{\nu}} \leq 1, \quad \|a_{\nu}\|_{V} \geq \nu \quad \text{for every } \nu=1, 2, \ldots.
\]

Then \( \{a_{1}, a_{2}, \ldots\} \) is a boundedly \( \mathfrak{B} \) but not by \( \mathfrak{H} \); contradicting assumption.

On account of this Theorem 5.4, we conclude by Theorem 5.3.

**Theorem 5.5.** If a linear topology \( \mathfrak{B} \) on \( R \) is sequential, separative, convex, and complete, then \( \mathfrak{B} \) is the strong topology of \( R \).

§6. Continuous linear topologies

A pseudo-norm \( \|x\| \) on \( R \) is said to be continuous, if \( R \ni x_{\nu} \uparrow x, 0 \) implies \( \lim_{\nu \to \infty} \|x_{\nu}\| = 0 \). A linear topology \( \mathfrak{B} \) on \( R \) is said to be continuous, if the pseudo-norm \( \|x\|_{\nu} \) is continuous for every \( V \in \mathfrak{B} \). With this definition, we see at once by the formulas (3), (4) in §2 that \( \mathfrak{B} \) is continuous if and only if for a basis \( \mathfrak{B} \) of \( \mathfrak{B} \), the pseudo-norm \( \|x\|_{\nu} \) is continuous for every \( V \in \mathfrak{B} \).

**Theorem 6.1.** If a linear topology \( \mathfrak{B} \) on \( R \) is sequential, separative and
continuous, then \( R \) is superuniversally continuous, that is, for any system of positive elements \( a_\lambda \in R (\lambda \in \Lambda) \) we can find countable \( \lambda_\nu \in \Lambda (\nu=1,2,\cdots) \) such that
\[
\bigcap_{\nu=1}^\infty a_{\lambda_\nu} = \bigcap_{\lambda \in \Lambda} a_\lambda.
\]

**Proof.** Let \( V_\nu \in \mathfrak{B} (\nu=1,2,\cdots) \) be a decreasing basis of \( \mathfrak{B} \). If \( 0 \leq x_\lambda \downarrow \lambda \in \Lambda \) implies then
\[
\inf_{\lambda \in \Lambda} \{ \sup_{\sigma \leq \lambda} ||x_\lambda - x_\sigma||_{V_\nu} \} = 0 \quad \text{for every } \nu=1,2,\cdots.
\]

Because, if \( 0 \leq x_\lambda \downarrow \lambda \in \Lambda \) and
\[
\inf_{\lambda \in \Lambda} \{ \sup_{\sigma \leq \lambda} ||x_\lambda - x_\sigma||_{V_\nu} \} \geq \varepsilon > 0
\]
for some \( \nu \), then we can find \( \lambda_\mu \in \Lambda (\mu=1,2,\cdots) \) such that
\[
x_{\lambda_1} \geq x_{\lambda_2} \geq \cdots, \quad ||x_{\lambda_{\mu+1}} - x_{\lambda_\mu}||_{V_\nu} \geq \varepsilon \quad (\mu=1,2,\cdots).
\]
Then, putting \( x_0 = \bigcap_{\mu=1}^\infty x_{\lambda_\mu} \), we have \( x_{\lambda_{\mu+1}} - x_0 \downarrow \mu \geq 0 \), but
\[
||x_{\lambda_{\mu+1}} - x_0||_{V_\nu} \geq ||x_{\lambda_{\mu+1}} - x_{\lambda_{\mu+1}}||_{V_\nu} \geq \varepsilon
\]
for every \( \mu=1,2,\cdots \), contradicting the assumption that \( \mathfrak{B} \) is continuous.

Therefore for \( 0 \leq x_\lambda \downarrow \lambda \in \Lambda \) we can find \( \lambda_\nu \in \Lambda (\nu=1,2,\cdots) \) such that \( x_{\lambda_\nu} \downarrow \nu = 0 \) and
\[
\sup_{\sigma \leq \lambda} ||x_{\lambda_{\nu}} - x_\sigma||_{V_\nu} \leq \frac{1}{2^\nu} \quad \text{for every } \nu=1,2,\cdots.
\]
Then, putting \( x_0 = \bigcap_{\nu=1}^\infty x_{\lambda_\nu} \), we have for every \( \sigma \in \Lambda \)
\[
||v_{\lambda_{\nu}} - x_0 ||_{V_\nu} \leq \frac{1}{2^\nu} \quad (\nu=1,2,\cdots),
\]
because \( x_{\lambda_{\nu}} - x_{\lambda_\mu} \cap x_\sigma \cup_{\mu=1}^\infty x_{\lambda_{\nu}} - x_0 \cap x_\sigma, \quad ||x_{\lambda_{\nu}} - x_{\lambda_\mu} \cap x_\sigma ||_{V_\nu} \leq \frac{1}{2^\nu} \) for \( \mu \geq \nu \).

Thus we obtain naturally for every \( \sigma \in \Lambda \)
\[
||x_0 - x_0 \cap x_\sigma||_{V_\nu} \leq \frac{1}{2^\nu} \quad (\nu=1,2,\cdots).
\]
As \( \mathfrak{B} \) is separative by assumption, we obtain hence \( x_0 - x_0 \cap x_\sigma = 0 \), and consequently \( x_0 \leq x_\sigma \) for every \( \sigma \in \Lambda \). Therefore \( x_0 \downarrow \lambda \in \Lambda \).

**Theorem 6.2.** If a linear topology \( \mathfrak{B} \) on \( R \) is continuous, then \( a_\lambda \downarrow \lambda \in \Lambda 0 \) implies
\[
\inf_{\lambda \in \Lambda} ||a_\lambda||_{V} = 0 \quad \text{for every } V \in \mathfrak{B}.
\]

**Proof.** For any \( V \in \mathfrak{B} \) we can find a decreasing sequence \( V_\nu \in \mathfrak{B} (\nu=1,2,\cdots) \) such that \( V_1 \times V_1 \subset V \). For such \( V_\nu \in \mathfrak{B} (\nu=1,2,\cdots) \), we can
find by Theorem 2.3 a normal manifold $N$ of $R$ such that the system of pseudo-norms $\|x\|_{r_{\nu}} (\nu =1, 2, \cdots)$ is proper in $N$ and $\|x\|_{r_{\nu}} = 0$ for every $x \in N^{\perp}$ and $\nu =1, 2, \cdots$. Then the linear topology on $N$, of which $\{x : \|x\|_{r_{\nu}} \leq 1, 0 \leq x \in N\} (\nu =1, 2, \cdots)$ is a basis, is obviously sequential, separative, and continuous. Thus $N$ is superuniversally continuous by Theorem 6.1. Therefore, if $R \ni a_{\lambda} \downarrow \lambda e \Lambda 0$, then we can find $\lambda_{\mu} \in \Lambda (\mu =1, 2, \cdots)$ such that $\|N\|a_{\lambda_{\mu}} \downarrow_{yAl}^{\infty} 0$, and hence $\lim_{\mu \rightarrow \infty} \|\alpha_{\lambda_{\mu}}\|_{V_{1}} = 0$, because $\mathfrak{B}$ is continuous by assumption.

As $\|N\|a_{\lambda_{\mu}} \downarrow_{yAl}^{\infty} 0$, we obtain hence by §2 (5)

$1 \leq \mu =1, 2, \cdots$.

Consequently we have $\lim \|a_{\lambda_{\mu}}\|_{r_{1}} = 0$. Thus we have naturally

$\inf \|a_{\lambda}\|_{r} = 0$.

**Theorem 6.3.** If a linear topology $\mathfrak{B}$ on $R$ is sequential, separative, continuous, and complete, then $R$ is regularly complete, that is, for any double sequence $a_{\nu, \mu} \downarrow_{yAl}^{\infty} 0 (\mu =1, 2, \cdots)$, we can find $\nu_{\mu} (\mu =1, 2, \cdots)$ such that $\sum_{\mu=1}^{\infty} a_{\nu_{\mu}, \mu} \in V_{\sigma}$ is convergent.

**Proof.** Let $V_{1} \in \mathfrak{B} (\nu =1, 2, \cdots)$ be a decreasing basis of $\mathfrak{B}$. If $a_{\nu, \mu} \downarrow_{yAl}^{\infty} 0 (\mu =1, 2, \cdots)$, then we have

$\lim_{\nu \rightarrow \infty} \|a_{\nu, \mu}\|_{r_{\mu}} = 0$ for every $\mu =1, 2, \cdots$,

because $\mathfrak{B}$ is continuous by assumption. Thus we can find $\nu_{\mu} (\mu =1, 2, \cdots)$ such that $a_{\nu_{\mu}, \mu} \in V_{\mu}$ is convergent. Then we have obviously

$\sum_{\mu=1}^{\infty} a_{\nu_{\mu}, \mu} \in V_{\sigma-1}$ for $\rho > \sigma$.

As $\mathfrak{B}$ is complete and separative, we see easily that $\sum_{\mu=1}^{\infty} a_{\nu_{\mu}, \mu}$ is convergent. Therefore $R$ is regularly complete.

§ 7. Linear functionals

Let $\mathfrak{B}$ be a linear topology on $R$. A linear functional $\varphi$ on $R$ is said to be *topologically bounded* by $\mathfrak{B}$, if $\sup_{x \in A} |\varphi(x)| < + \infty$ for every topologically bounded manifold $A$.

For any positive element $a \in R$, $\{x : 0 \leq x \leq a\}$ is obviously topologically bounded by $\mathfrak{B}$. Thus we have
Theorem 7.1. If a linear functional \( \varphi \) on \( R \) is topologically bounded by \( \mathcal{B} \), then \( \varphi \) is bounded, that is,
\[
\sup_{0 \leq x \leq a} |\varphi(x)| < +\infty \quad \text{for every } a \geq 0.
\]

Conversely we have

Theorem 7.2. If a linear topology \( \mathcal{B} \) on \( R \) is separative, convex, and complete, then every bounded linear functional \( \varphi \) on \( R \) is topologically bounded by \( \mathcal{B} \).

Proof. Let \( \varphi \) be a positive linear functional on \( R \). If \( \varphi \) is not topologically bounded by \( \mathcal{B} \), then we can find a sequence \( a_{\nu} \geq 0 (\nu = 1, 2, \cdots) \) such that \( \{a_{\nu}, a_{\nu}, \cdots\} \) is topologically bounded, but
\[
\varphi(a_{\nu}) \geq \nu 2^{\nu} \quad (\nu = 1, 2, \cdots).
\]
Then we have obviously \( \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu}} ||a_{\nu}||_{V} < +\infty \) for every \( V \in \mathcal{B} \). As \( \mathcal{B} \) is separative, convex, and complete by assumption, we obtain hence that \( \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu}} a_{\nu} \) is convergent, and putting \( a = \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu}} a_{\nu} \), we have that \( \varphi(a) \geq \varphi \left( \frac{1}{2^{\nu}} a_{\nu} \right) \geq \nu \) for every \( \nu = 1, 2, \cdots \), contradicting \( \varphi(a) < +\infty \).

A linear functional \( \varphi \) on \( R \) is said to be \textit{topologically continuous} by a linear topology \( \mathcal{B} \), if we can find \( V \in \mathcal{B} \) such that
\[
|\varphi(x)| \leq ||x||_{V} \quad \text{for every } x \in R.
\]
With this definition, we see at once by the formulas (3), (4) in §2 that a linear functional \( \varphi \) on \( R \) is topologically continuous by \( \mathcal{B} \), if and only if for a basis \( \mathcal{B} \) of \( \mathcal{B} \) we can find \( V \in \mathcal{B} \) and \( a > 0 \) such that
\[
|\varphi(x)| \leq a ||x||_{V} \quad \text{for every } x \in R.
\]
If a linear functional \( \varphi \) on \( R \) is topologically continuous by \( \mathcal{B} \), then \( \varphi \) is obviously by definition topologically bounded by \( \mathcal{B} \).

If a linear functional \( \varphi \) on \( R \) is \textit{universally continuous}, that is, if \( x_{\lambda} \downarrow 0 \) implies \( \inf_{\lambda} |\varphi(x_{\lambda})| = 0 \), then, putting
\[
V = \{x: \sup_{|y| \leq 1} |\varphi(y)| \leq 1, x \geq 0\},
\]
we see easily that \( V \) is a convex positive vicinity. Thus we have

Theorem 7.3. If a linear functional \( \varphi \) on \( R \) is universally continuous, then \( \varphi \) is topologically continuous by the strong topology of \( R \).

Recalling Theorem 6.2, we obtain immediately
Theorem 7.4. If a linear topology \( \mathfrak{B} \) on \( R \) is continuous, then every topologically continuous linear functional on \( R \) is universally continuous.

If a convex pseudo-norm \( ||x|| \) on \( R \) is not continuous, then we can find a linear functional \( \varphi \) on \( R \) such that
\[
\sup_{|x|\leq 1} |\varphi(x)| < +\infty,
\]
but there is a sequence \( a_{\nu} \downarrow 0 \) for which we have \( \lim_{\nu \to +\infty} \varphi(a_{\nu}) > 0 \). (c.f. MSLS Theorem 31.10). Therefore we have

Theorem 7.5. For a convex linear topology \( \mathfrak{B} \) on \( R \), if every topologically continuous linear functional on \( R \) is continuous, then \( \mathfrak{B} \) is continuous.

§ 8. Reflexive linear topologies

Let \( R \) be a reflexive semi-ordered linear space and \( \overline{R} \) the conjugate space of \( R \). For any positive \( \overline{a} \in \overline{R} \), putting
\[
V_{\overline{a}} = \{ x : \overline{a}(x) \leq 1, \ x \geq 0 \},
\]
we obtain obviously a convex positive vicinity \( V_{\overline{a}} \). For this \( V_{\overline{a}} \) we have obviously
\[
||x||_{V_{\overline{a}}} = \overline{a}(|x|)
\]
for every \( x \in R \), because
\[
||x||_{V_{\overline{a}}} = \inf_{\xi \in V_{\overline{a}}} \frac{1}{\xi} = \inf_{\xi \leq 1} \frac{1}{\xi} = \overline{a}(|x|).
\]

Recalling Theorem 1.1, we see easily that there exists uniquely a linear topology \( \mathfrak{W} \) on \( R \) such that the system \( V_{\overline{a}} (0 \leq \overline{a} \in \overline{R}) \) is a basis of \( \mathfrak{W} \). This linear topology \( \mathfrak{W} \) is called the absolute weak topology of \( R \). With this definition we have

Theorem 8.1. The absolute weak topology \( \mathfrak{W} \) of \( R \) is separative, convex, continuous, and monotone complete.

Proof. It is evident by definition that \( \mathfrak{W} \) is separative, convex, and continuous. If a system of positive elements \( x_{\lambda} \uparrow_{\lambda \in \Lambda} \) is topologically bounded by \( \mathfrak{W} \), then we have by the formula (2)
\[
\sup_{\lambda \in \Lambda} \overline{a}(x_{\lambda}) = \sup_{\lambda \in \Lambda} ||x_{\lambda}||_{\overline{a}} < +\infty
\]
for every positive \( \overline{a} \in \overline{R} \). Therefore there exists \( a \in R \) such that \( x_{\lambda} \uparrow_{\lambda \in \Lambda} a \).
(c.f. MSLS. Theorem 24.4)

Theorem 8.2. A manifold \( A \) of \( R \) is topologically bounded by the absolute weak topology \( \mathfrak{W} \) if and only if \( A \) is weakly bounded, that is,
\[
\sup_{x \in A} |\overline{x}(x)| < +\infty \quad \text{for every } \overline{x} \in \overline{R}.
\]
Proof. If $A$ is weakly bounded, then we have
\[ \sup_{x \in A} \overline{a}(|x|) < +\infty \quad \text{for} \quad 0 \leq \overline{a} \in \overline{R} \]
(MSLS. Theorem 24.15). Thus we obtain by (2)
\[ \sup_{x \in A} \|x\|_{\overline{a}} < +\infty \quad \text{for} \quad 0 \leq \overline{a} \in \overline{R} , \]
and hence $A$ is topologically bounded by $\mathfrak{B}$. Conversely, if $A$ is topologically bounded by $\mathfrak{B}$, then we have by (2)
\[ \sup_{x \in A} |\overline{a}(x)| \leq \sup_{x \in A} |\overline{a}|(|x|) = \sup_{x \in A} \|x\|_{\overline{a}} < +\infty , \]
and hence $A$ is weakly bounded.

Recalling Theorem 5.3, we obtain by Theorem 8.1

**Theorem 8.3.** The strong topology of $R$ is separative and equivalent to the absolute weak topology of $R$.

A pseudo-norm $\|x\|$ on $R$ is said to be reflexive, if for
\[ \overline{A} = \{\overline{x} : \sup_{|x| \leq 1} |\overline{x}(x)| \leq 1\} , \]
we have $\|x\| = \sup_{x \in A} |\overline{x}(x)|$ for every $x \in R$. With this definition, we see at once that every reflexive pseudo-norm is convex.

Let $\mathfrak{B}$ be the absolute weak topology of the conjugate space $\overline{R}$. For every topologically bounded manifold $\overline{A}$ of $\overline{R}$ by $\mathfrak{B}$, putting
\[ V = \{x : |\overline{x}|(x) \leq 1 \quad \text{for every} \quad \overline{x} \in \overline{A}, x \geq 0\} , \]
we see easily that $V$ is a positive vicinity in $R$ and the pseudo-norm $\|x\|_V$ is reflexive.

**Theorem 8.4.** If a pseudo-norm $\|x\| (x \in R)$ is convex and continuous, then it is reflexive.

Proof. By virtue of Banach's extension theorem, for any $a \in R$ we can find a linear functional $\varphi$ on $R$ such that
\[ \varphi(a) = \|a\| , \quad |\varphi(x)| \leq \|x\| \quad \text{for every} \quad x \in R . \]
As $\|x\| (x \in R)$ is convex and continuous by assumption, we see by Theorem 6.2 that $\varphi$ is universally continuous, and hence $\varphi \in \overline{R}$. Furthermore, putting
\[ \overline{A} = \{\overline{x} : \sup_{|x| \leq 1} |\overline{x}(x)| \leq 1\} , \]
we have obviously $\varphi \in \overline{A}$, and hence
\[ \sup_{x \in \overline{A}} |\overline{x}(a)| \geq \varphi(a) = \|a\| . \]
On the other hand, it is evident that $||a|| \geq \sup_{x \in A} |\overline{x}(a)|$. Thus we conclude $||a|| = \sup_{x \in A} |\overline{x}(a)|$ for every $a \in R$, that is, the pseudo-norm $||x||$ ($x \in R$) is reflexive by definition.

A linear topology $\mathfrak{B}$ on $R$ is said to be reflexive, if there is a basis $\mathfrak{B}$ of $\mathfrak{B}$ such that $||x||_V$ is reflexive for every $V \in \mathfrak{B}$. With this definition, we have obviously by Theorem 8.4

**Theorem 8.5.** If a linear topology $\mathfrak{B}$ on $R$ is convex and continuous, then $\mathfrak{B}$ is reflexive.

Consequently we obtain by Theorem 8.1

**Theorem 8.6.** The absolute weak topology of $R$ is reflexive.

**Theorem 8.7.** If the strong topology of $R$ is sequential, then it is reflexive.

**Proof.** Let $V_\nu (\nu = 1, 2, \cdots)$ be the convex decreasing basis of the strong topology of $R$. Putting

$$\overline{A}_\nu = \{\overline{x} : \sup_{z \in \overline{V}_\nu} \overline{\alpha}(z) \leq 1, \ 0 \leq \overline{x} \in \overline{R}\},$$

we see easily that every $\overline{A}_\nu (\nu = 1, 2, \cdots)$ is topologically bounded by the absolute weak topology $\mathfrak{B}$ of $\overline{R}$. Thus, putting

$$U_\nu = \{x : \sup_{z \in \overline{A}_\nu} \overline{\alpha}(z) \leq 1, \ 0 \leq x \in R\},$$

we obtain a convex positive vicinity $U_\nu$ in $R$ such that $||x||_{U_\nu}$ is reflexive. For any positive $\overline{a} \in \overline{R}$, putting

$$V_\overline{a} = \{x : \overline{\alpha}(x) \leq 1, \ 0 \leq x \in R\},$$

we obtain a convex vicinity $V_\overline{a}$ and hence we can find $\nu$ such that $V_\overline{a} \supset V_\nu$, because $V_\nu (\nu = 1, 2, \cdots)$ is a basis of the strong topology of $R$. For such $\nu$, we have obviously $\overline{a} \in \overline{A}_\nu$, and consequently $U_\nu \subset V_\overline{a}$. Therefore the convex linear topology $\mathfrak{B}$, of which $U_\nu (\nu = 1, 2, \cdots)$ is a basis, is stronger than the absolute weak topology of $R$. Recalling Theorem 5.2, we see that $\mathfrak{B}$ is monotone complete, and hence $\mathfrak{B}$ coincides by Theorem 7.5 with the strong topology of $R$. Furthermore $\mathfrak{B}$ is obviously reflexive. Consequently the strong topology of $R$ is reflexive.

If a norm $||x||$ on $R$ is complete, that is, if the linear topology $\mathfrak{B}$, of which $\{x : ||x|| \leq 1, \ 0 \leq x \in R\}$ is a basis, is complete, then $\mathfrak{B}$ is by Theorem 5.5 the strong topology of $R$, and hence reflexive by Theorem 8.7. Therefore we obtain

**Theorem 8.8.** If there is a complete norm on $R$, then there exists a complete reflexive norm on $R$.  