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ON TRANSCENDENTAL POINTS IN PROPER SPACES OF DISCRETE SEMI-ORDERED LINEAR SPACES

By

Hidegorô NAKANO

A cardinal number $f$ is said to be singular, if
1) $f >$ the countable density $\aleph_0$,
2) $f > c$ implies $f > 2^c$,
3) for any system of cardinal numbers $c_\lambda < f (\lambda \in \Lambda)$ with a density $< f$ we have $\sum_{\lambda \in \Lambda} c_\lambda < f$.

The existence of singular cardinal numbers is not known yet. It will be extraordinarily great, if exists.

A cardinal number $c$ is said to be regular, if there is no singular cardinal number $\leq c$. The countable density $\aleph_0$ is naturally regular. If a cardinal number $c$ is regular, then $2^c$ also is regular. For a system of regular cardinal numbers $c_\lambda (\lambda \in \Lambda)$, if the density of $\Lambda$ is regular, then $\sum_{\lambda \in \Lambda} c_\lambda$ also is regular.

Let $S$ be a set and $R$ the totality of real functions on $S$. $R$ is then obviously a discrete semi-ordered linear space. The purpose of this paper is to prove: If the density of $S$ is regular, then for any positive linear functional $\Phi$ on $R$, we can find a finite number of elements $s_\nu \in S$ and positive numbers $a_\nu (\nu = 1, 2, \ldots, k)$ such that

$$\Phi (\varphi) = \sum_{\nu=1}^{k} a_\nu \varphi (s_\nu)$$

for every $\varphi \in R$.

Let $R$ be now an arbitrary linear space. We have defined a strongest convex linear topology on $R$, of which the totality of convex vicinities in $R$ is a basis. By virtue of the fact stated just above, we see easily that if the density of $R$ is regular, then $R$ is regular (reflexive) by the strongest convex linear topology.

---

(3) c. f. 2).
§ 1. Transcendental ideals of sets.

Let \( R \) be a set. A collection \( p \) of subsets from \( R \) is said to be an ideal, if

1) \( O \in p \),
2) \( \exists \cup Y \in p \) implies \( X \in p \),
3) \( X, Y \in p \) implies \( XY \in p \).

An ideal \( p \) is said to be maximal, if there is no other ideal including \( p \). For a maximal ideal \( p \), we see easily that for any set \( X \in p \) we can find \( Y \in p \) such that \( XY = O \).

Theorem 1.1. Let \( p \) be a maximal ideal, and \( \Lambda \) a set with a density \( c \). If \( X_{\lambda} \in p (\lambda \in \Lambda) \) implies \( \prod_{\lambda} X_{\lambda} \in p \), then for a set \( \Gamma \) with the density \( 2^c \) we also have that \( X_{\gamma} \in p (\gamma \in \Gamma) \) implies \( \prod_{\gamma} X_{\gamma} \in p \).

Proof. The collection of systems \( (\epsilon_{\lambda})_{\lambda \in \Lambda} \) for \( \epsilon_{\lambda} = 0, 1 \) has by definition the density \( 2^c \). Let \( A(\epsilon_{\lambda})_{\lambda \in \Lambda} \) for all \( (\epsilon_{\lambda})_{\lambda \in \Lambda} \) be a partition of \( R \), that is,

\[
R = \sum_{(\epsilon_{\lambda})_{\lambda \in \Lambda}} A(\epsilon_{\lambda})_{\lambda \in \Lambda},
\]

\[
A(\epsilon_{\lambda})_{\lambda \in \Lambda} A(\delta_{\lambda})_{\lambda \in \Lambda} = O \quad \text{for } (\epsilon_{\lambda})_{\lambda \in \Lambda} \neq (\delta_{\lambda})_{\lambda \in \Lambda}.
\]

For every finite number of elements \( \lambda_{\nu} \in \Lambda (\nu = 1, 2, \cdots, \kappa) \), putting

\[
Y_{\delta_{\lambda_{1}}, \delta_{\lambda_{2}}, \cdots, \delta_{\lambda_{\kappa}}} = \sum_{\epsilon_{\lambda_{\nu}} = 0, 1 (\nu = 1, 2, \cdots, \kappa)} A(\epsilon_{\lambda})_{\lambda \in \Lambda}
\]

for \( \delta_{\lambda_{\nu}} = 0, 1 (\nu = 1, 2, \cdots, \kappa) \), we have obviously

\[
R = \sum_{\epsilon_{\lambda_{\nu}} = 0, 1} Y_{\epsilon_{\lambda_{1}}, \epsilon_{\lambda_{2}}, \cdots, \epsilon_{\lambda_{\kappa}}} \quad \text{for every } \lambda_{1}, \lambda_{2}, \cdots, \lambda_{\kappa} \in \Lambda,
\]

and

\[
Y_{\epsilon_{\lambda_{1}}, \epsilon_{\lambda_{2}}, \cdots, \epsilon_{\lambda_{\kappa}}} Y_{\epsilon'_{\lambda_{1}}, \epsilon'_{\lambda_{2}}, \cdots, \epsilon'_{\lambda_{\kappa}}} = O,
\]

if \( \epsilon_{\lambda_{\nu}} = \epsilon'_{\lambda_{\nu}} \) for some \( \nu \). Thus for each finite number of elements \( \lambda_{\nu} \in \Lambda (\nu = 1, 2, \cdots, \kappa) \) we can find uniquely \( \delta_{\lambda_{\nu}} = 0, 1 (\nu = 1, 2, \cdots, \kappa) \) such that

\[
Y_{\delta_{\lambda_{1}}, \delta_{\lambda_{2}}, \cdots, \delta_{\lambda_{\kappa}}} \in p.
\]

As \( Y_{\epsilon_{\lambda_{1}}, \epsilon_{\lambda_{2}}, \cdots, \epsilon_{\lambda_{\kappa}}} \supset Y_{\epsilon'_{\lambda_{1}}, \epsilon'_{\lambda_{2}}, \cdots, \epsilon'_{\lambda_{\kappa}}, \epsilon_{\lambda_{\kappa+1}}} \), we see easily further that there exists uniquely \( (\delta_{\lambda})_{\lambda \in \Lambda} \) such that

\[
Y_{\delta_{\lambda_{1}}, \delta_{\lambda_{2}}, \cdots, \delta_{\lambda_{\kappa}}} \in p \quad \text{for every } \lambda_{1}, \lambda_{2}, \cdots, \lambda_{\kappa} \in \Lambda,
\]

Then, as the totality of systems \( \lambda_{1}, \lambda_{2}, \cdots, \lambda_{\kappa} \in \Lambda \) also has the density \( c \),
we have by assumption
\[ A_{\delta_{\lambda}} \in \prod_{\lambda \in \Lambda} Y_{\delta_{\lambda}, \delta_{\lambda_2}, \ldots, \delta_{\lambda_k}} \in \mathfrak{p}. \]

Therefore, for a set \( \Gamma \) with the density \( 2^c \), if \( \sum A_\tau = R, A_\tau A_\tau' = O \) for \( \tau \neq \tau' \), then there exists uniquely \( \tau \in \Gamma \) such that \( A_\tau \in \mathfrak{p} \). If \( \sum B_\tau = R \), then we can find by the transfinite induction subsets \( A_\tau \subset B_\tau (\tau \in \Gamma) \) such that
\[ \sum_{\tau \in \Gamma} A_\tau = R, \ A_\tau A_\tau' = 0 \quad \text{for} \quad \tau \neq \tau', \]
and hence there exists \( \tau \in \Gamma \) such that \( B_\tau \in \mathfrak{p} \). If \( X_\tau \in \mathfrak{p} (\tau \in \Gamma) \), then we have obviously
\[ R = \sum_{\tau \in \Gamma} (R - X_\tau) + \prod_{\tau \in \Gamma} X_\tau, \quad R - X_\tau \in \mathfrak{p} \quad \text{for every} \quad \tau \in \Gamma, \]
and consequently \( \prod_{\tau \in \Gamma} X_\tau \in \mathfrak{p} \), as proved just above.

Theorem 1.2. Let \( \mathfrak{p} \) be a maximal ideal and \( \Gamma \) a set for which \( X_\tau \in \mathfrak{p} \) \( (\tau \in \Gamma) \) implies \( \prod_{\tau \in \Gamma} X_\tau \in \mathfrak{p} \). For a system of sets \( \Lambda_\tau \ (\tau \in \Gamma) \) of \( \{X_\tau \in \mathfrak{p} (\lambda \in \Lambda_\tau) \} \) implies \( \prod_{\tau \in \Gamma} X_\tau \in \mathfrak{p} \) for every \( \tau \in \Gamma \), then \( X_\lambda \in \mathfrak{p} (\lambda \in \sum_{\gamma \in \Gamma} \Lambda_\gamma) \) implies \( \prod_{\tau \in \Gamma} X_\tau \in \mathfrak{p} \).

Proof. We have obviously by assumption that \( X_\tau \in \mathfrak{p} (\lambda \in \sum_{\gamma \in \Gamma} \Lambda_\gamma) \) implies \( \prod_{\tau \in \Gamma} X_\lambda \in \mathfrak{p} \) for every \( \tau \in \Gamma \), and hence
\[ \prod_{\tau \in \Gamma} X_\lambda = \prod_{\tau \in \Gamma} (\prod_{\lambda \in \Lambda_\gamma} X_\lambda) \in \mathfrak{p}. \]

A maximal ideal \( \mathfrak{p} \) is said to be transcendental, if \( X_\nu \in \mathfrak{p} (\nu = 1, 2, \ldots) \) implies \( \prod_{\nu = 1}^\infty X_\nu \in \mathfrak{p} \). With this definition, we conclude immediately by Theorems 1.1 and 1.2.

Theorem 1.3. If the density of \( R \) is regular, then for every transcendental maximal ideal \( \mathfrak{p} \), if we have \( \prod X \in \mathfrak{p} \), and hence \( \prod X \) is composed only of a single element.

§ 2. Transcendental points of discrete semi-ordered linear spaces

Let \( R \) be a discrete semi-ordered linear space\(^{(4)} \) and \( a_\lambda \in R (\lambda \in \Lambda) \) a basis of \( R \), i.e., \( a_\lambda \wedge a_\rho = 0 \) for \( \lambda \neq \rho \) and for each positive element \( x \in R \) we can find uniquely a system of real numbers \( \xi_\lambda \geq 0 (\lambda \in \Lambda) \) such that

\(^{(4)}\) e. f. 1.)
For a positive element $x = \bigcup_{\lambda \in \Lambda} \xi_{\lambda} a_{\lambda}$, putting
\[ \Lambda_x = \{ \lambda : \xi_\lambda \neq 0 \} , \]
we see easily:
\[ [a] x = \bigcup_{\lambda \in \Lambda_a} \xi_{\lambda} a_{\lambda} \]
for every positive element $x = \bigcup_{\lambda \in \Lambda} \xi_{\lambda} a_{\lambda}$; $\Lambda_a \subseteq \Lambda_b$ if and only if $[a] \leq [b]$; $\Lambda_{a \cup b} = \Lambda_a + \Lambda_b$, $\Lambda_{a \cap b} = \Lambda_a \Lambda_b$.

Thus every projector $[a]$ may be represented by the set $\Lambda_a$. Therefore every point of the proper space of $R$ may be considered as a maximal ideal of subsets from $\Lambda$. Furthermore, for a maximal ideal $p$ of subsets from $\Lambda$, if $p \ni \Lambda_a$ for a positive element $a \in R$, then $p$ is a point of the proper space of $R$ and $p \in U_{[a]}$.

A point $p$ of the proper space of $R$ is said to be transcendental\(^{(5)}\), if $p \in U_{[a_{v\lambda}]} (v = 1, 2, \cdots)$ implies $p \in U_{[b]} \subseteq \prod_{\nu=1}^\infty U_{[a_{v\lambda}]}$ for some $b \in R$. With this definition, it is evident that a point $p$ is transcendental if and only if $p$ is a transcendental maximal ideal.

For a positive element $a \in R$, the density of $\Lambda_a$ is called the dimension of $a$.

**Theorem 2.1.** If the dimension of every positive element of $R$ is regular, then the proper space of $R$ has no transcendental point up to isolated points.

**Proof.** For a transcendental point $p$ of the proper space of $R$, we can find obviously a positive element $a \in R$ such that $p \in U_{[a]}$. Then we can consider $p$ as a transcendental maximal ideal of subsets from $\Lambda_a$. Therefore we can find by Theorem 1.3 $\lambda \in \Lambda$ such that $U_{[a_{v\lambda}]}$ is composed only of the single point $p$, and hence $p$ is an isolated point.

From this Theorem 2.1 we conclude immediately

**Theorem 2.2.** If the density of $R$ is regular, then the proper space of $R$ has no transcendental point up to isolated points.

§ 3. Universally complete discrete semi-ordered linear spaces

Let a discrete semi-ordered linear space $R$ be universally complete\(^{(6)}\),

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i.e., for every orthogonal system of positive elements $x_r \in R (r \in \Gamma')$ there exists $\bigcup_{r \in \Gamma'} x_r$. $R$ is then obviously totally unbounded\(^{(7)}\), i.e., for an orthogonal sequence of positive elements $x_v \in R (v=1,2,\cdots)$, if $\bigcup_{v=1}^{\infty} x_v$ exists, then we can find a sequence of positive numbers $a_v \uparrow_{v=1}^{\infty} +\infty$ for which $\bigcup_{v=1}^{\infty} a_v x_v$ exists. Therefore every positive linear functional on $R$ is continuous.\(^{(8)}\)

**Theorem 3.1.** If a discrete semi-ordered linear space $R$ is universally complete and the density of $R$ is regular, then for every positive linear functional $\Phi$ on $R$ we can find a finite number of discrete positive elements $a_v \in R (v=1,2,\cdots)$ such that

$$
\Phi(x) = \sum_{v=1}^{\nu} \Phi([a_v]x) \quad \text{for every } x \in R.
$$

**Proof.** Let $\mathcal{G}_{\Phi}$ be the characteristic set\(^{(9)}\) of $\Phi$. If $\mathcal{G}_{\Phi}$ contains infinite points, then we can find a sequence of positive elements $a_v \in R (v=1,2,\cdots)$ such that $[a_v][a_\mu] = 0$ for $\nu \neq \mu$ and $U_{[a_\nu]} \mathcal{G}_{\Phi} = 0$ for every $v = 1,2,\cdots$. Then we have $\Phi(a_v) > 0$ for every $v=1,2,\cdots$, and hence, putting

$$
a = \bigcup_{\nu=1}^{\infty} \frac{\nu}{\Phi(a_\nu)} a_\nu,
$$

we have

$$
\Phi(a) \geq \Phi \left( \frac{\nu}{\Phi(a_\nu)} a_\nu \right) = \nu \quad \text{for every } \nu=1,2,\cdots,
$$

contradicting $\Phi(a) < +\infty$. Thus $\mathcal{G}_{\Phi}$ is composed only of a finite number of points. Furthermore every point of $\mathcal{G}_{\Phi}$ is transcendental. Because, if a point $p \in \mathcal{G}_{\Phi}$ is not transcendental, then we can find by definition a sequence of projectors $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$, such that $U_{[p_\nu]} \supset p$, but $U_{[p_\nu]}$ does not contains any other point of $\mathcal{G}_{\Phi}$ for every $v=1,2,\cdots$. As $p \in \mathcal{G}_{\Phi}$, we can find a positive element $a \in R$ such that $\Phi([p]a) > 0$ for $U_{[p]} \supset p$. Then we have

$$
\Phi([p_\nu]a) = \Phi([p_\nu]a) - \Phi([p_\nu][-p_\nu]a) = \Phi([p_\nu]a),
$$

because $U_{[p_\nu] \cdot [-p_\nu]} \mathcal{G}_{\Phi} = 0$. As $\Phi$ is continuous, we have hence

$$
\Phi([p_\nu]a) = \lim_{\nu \rightarrow \infty} \Phi([p_\nu]a) = 0,
$$

contradicting $\Phi([p_\nu]a) > 0$. Therefore $\mathcal{G}_{\Phi}$ is composed only of a finite number of transcendental points. As the density of $R$ is regular by

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\(^{(8)}\) c. f. 7) Theorem 19.8.

\(^{(9)}\) c. f. 7) §20.
assumption, we see by Theorem 2.2 that $\mathcal{S}_p$ is composed only of a finite number of isolated points. Thus we can find a finite number of discrete positive elements $a_\nu \in R (\nu = 1, 2, \ldots, \kappa)$ such that

$$\Phi(x) = \sum_{\nu=1}^{\kappa} \Phi([a_\nu]x) \quad \text{for every } x \in R.$$  

Recalling a theorem in an earlier paper we obtain immediately by this Theorem 3.1

**Theorem 3.2.** Let $R$ be a universally complete, discrete semi-ordered linear space with a regular density. For a positive linear functional $\Phi$ on $R$, if

$$\text{Min} \{ \Phi(x), \Phi(y) \} = 0 \quad \text{for } x \searrow y = 0,$$

then there exists a positive discrete element $a \in R$ such that

$$\Phi(x) = \Phi([a]x) \quad \text{for every } x \in R. \quad (11)$$

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(10) c. f. 5) Satz 8.

(11) The same problem was considered by E. Hewitt, but he could not succeed to prove. E. Hewitt: Linear functionals on spaces of continuous functions, Fund. Math. 37 (1950), 161-189.