# EXPONENTS OF MODULARED SEMI-ORDERED LINEAR SPACES

By

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The *modulared semi-ordered linear space* $R$ is a universally continuous semi-ordered linear space where a functional $m(a)$ $(a \in R)$ is defined such as the following properties are satisfied:

1) $0 \leqq m(a) \leqq +\infty$ for every $a \in R$;

2) if $m(\xi a)=0$ for every positive number $\xi>0$, then we have $a=0$;

3) for any element $a \in R$ we can find a positive number $\xi$ such that $m(\xi a)<+\infty$;

4) $m(\xi a)$ is a convex function of $\xi>0$, that is, $\alpha, \beta>0$ implies

$$m\left(\frac{\alpha+\beta}{2} a\right) \leqq \frac{1}{2}\left\{m(\alpha a)+m(\beta a)\right\}$$

for every $a \in R$;

5) $|a| \leqq |b|$ implies $m(a) \leqq m(b)$;

6) $a \frown b=0$ implies $m(a+b)=m(a)+m(b)$;

7) if $0 \leqq a_\lambda \downarrow_{\lambda \in \Lambda} a$, (that is, $a_\lambda$ $(\lambda \in \Lambda)$ are positive, and for any $\lambda_1 \lambda_2 \in \Lambda$ we can find $\lambda \in \Lambda$ such that $a_{\lambda_1} a_{\lambda_2} \leqq a_\lambda$, and $a=\bigcup_{\lambda \in \Lambda} a_\lambda$), then we have

$$m(a)=\sup_{\lambda \in \Lambda} m(a_\lambda).$$

This functional $m(a)$ $(a \in R)$ is called a *modular* on this modulared semi-ordered linear space $R$. The so-called $L_p$-space $(p \geqq 1)$, namely a set of measurable functions $a(t)$ $(0 \leqq t \leqq 1)$ such that

$$\int_0^1 |a(t)|^p dt<+\infty,$$

is obviously a modulared semi-ordered linear space, putting its modular as

$$m(a)=\int_0^1 |a(t)^p| dt.$$

$L_p$-space is but an example of the modulared semi-ordered linear space. However, this is an essentially important example. Because, the modular $m(\xi a)$ is a convex function of $\xi>0$ and the concept of convexity is a direct generalization of that of power, that is, $\xi^p$ for $p \geqq 1$. In other

words, the theory of modulared semi-ordered linear space, which was considered first by NAKANO [7], may be regarded as a direct generalization of the theory of $L_p$-space, in spite of existence of many examples which are not $L_p$-type.

The purpose of this paper is to classify modulars by power functions. For this purpose, we will define two exponents of a modular and discuss properties of these exponents and characterize some properties of the modular by them.

The concept of exponents of modulars is not firstly discussed here. In fact, in particular cases, where the modulars are of unique spectra, they were defined by NAKANO [7]. The exponents defined in this paper can be considered as a generalization of them.

In general cases, where the modular spectra are not always power functions, we must consider derivatives of modulars for the sake of validity of exponents, concerning especially about uniform convexity and uniform eveness. Therefore, §1 and §2 are devoted to argument of the derivatives of modulars and their spectra. In §3 we will define two sorts of exponents of elements and discuss the relations between properties of elements and their exponents. The exponents of a space are defined in §4 and the problem of determining exponents of a space by that of elements of the space will be discussed there. In §5, conjugateness relations of exponents are established. Relations between a modular and its norms are regarded as the first problem which should be solved in the theory of modulared semi-ordered linear space. As the preparations, we will enumerate in §6 some theorems about norms.

In the following three sections, we will consider problems concerning about special types of modulars, namely, bounded modulars, modulars of unique spectra and constant modulars.

Some of results of this paper may be obtained also in modulared linear spaces without semi-ordering. But, it is our cherished opinion that the semi-ordered linear space is most suitable for the development of modular theory, and the opinion has been testified enough in the book: NAKANO [7], to which we owe the terminology and properties of modulars used in this paper. (The general theory of modulard linear spaces without semi-odering was developped by NAKANO [11].).

Before proceeding to the details of the work, I should like to express my gratitude to Professor H. NAKANO for suggesting to me the problems originally, and for much helpful advice since.

## §1. Derivatives of modulars

In the sequel, let $R$ be a modulared semi-ordered linear space and $m$ be its modular.

Since $m(\xi a)$ $(a \in R)$ is a convex function of $\xi > 0$, we can define its right-hand derivative as follows:

$$\pi(\xi/a) = \begin{cases} \lim \dfrac{m((\xi+\varepsilon)a) - m(\xi a)}{\varepsilon} & \text{if} \quad m(\xi a) < +\infty, \\ +\infty & \text{if} \quad m(\xi a) = +\infty, \end{cases}$$

and we have

$$m(\alpha a) = \int_0^{\alpha} \pi(\xi/a)\, d\xi,$$

when $m(\alpha a) < +\infty$.

Now, we will enumerate some properties of the *derivative* as the preparations for the following sections.

$\pi(\xi/a)$ is obviously a increasing function of $\xi > 0$:

$$\pi(\xi/a) \geqq \pi(\eta/a) \qquad \text{for} \quad \xi > \eta \geqq 0,\ a \in R.$$

Moreover, we can prove that

(1) $|a| \leqq |b|$ implies $\pi(\xi/a) \leqq \pi(\xi/b)$ for every $\xi > 0$.

Because, since we have generally the following inequality:

$$m(a+c) + m(b+c) \leqq m(a+b+c) + m(c)$$

for $a, b, c \geqq 0$. We need only put here $a = \varepsilon(b+c)$ for a positive number $\varepsilon > 0$ and letting $\varepsilon \to 0$.

From the definition of derivative, it is easy to see that

(2) $\pi(\xi/\alpha a) = \alpha \pi(\alpha\xi/a)$

for any positive number $\alpha$ and $\xi$.

Moreover we have

(3) $$\frac{m(\xi a)}{\xi} \leqq \pi(\xi/a) \leqq \frac{m((\xi+\eta)a)}{\eta}$$

for any positive number $\xi$ and $\eta$. The right-hand inequality is a immediate consequence of the definition, that is,

$$\pi(\xi/a) \leqq \frac{m((\xi+\eta)a) - m(\xi a)}{\eta} \leqq \frac{m((\xi+\eta)a)}{\eta}.$$

If $m(\alpha a)$ is finite, we have

$$m(\alpha a)=\int_0^{\alpha}\pi(\xi/a)\,d\xi\leqq\int_0^{\alpha}\pi(\alpha/a)\,d\xi=\alpha\pi(\alpha/a)\,.$$

If $(m\alpha a)$ is infinite, $\pi(\alpha/a)$ is also infinite by the definition, and so, the left-hand inequality of (3) is obtained.

By this inequality, we have from the definitions of the modular,

(4) $\pi(\xi/a)=0$ $(\xi>0)$ implies $a=0$,

(5) for any $a\in R$ there exists a number $\alpha>0$ such that $\pi(\alpha/a)<+\infty$.

Let $a\frown b=0$. Then, we have

$$m(\xi(a+b))=m(\xi a)+m(\xi b)$$

and

$$m((\xi+\varepsilon)(a+b))=m((\xi+\varepsilon)a)+m((\xi+\varepsilon)b)$$

for any positive number $\xi$ and $\varepsilon$, because

$$(\xi a)\frown(\xi b)=0 \quad\text{and}\quad ((\xi+\varepsilon)a)\frown((\xi+\varepsilon)b)=0\,.$$

Therefore, by subtracting term by term and letting $\varepsilon\to 0$, we obtain the following inequality:

(6) $\pi(\xi/a+b)=\pi(\xi/a)+\pi(\xi/b)$ for every $\xi>0$,

provided that $a\frown b=0$.

Similarly, we can prove that

(7) $\pi(\xi/a\smile b)+\pi(\xi/a\frown b)=\pi(\xi/a)+\pi(\xi/b)$

for every $\xi>0$ and $a,b\geqq 0$,

(8) if $a_\lambda\downarrow_{\lambda\in\Lambda}0$ and $\pi(1/a_\lambda)<+\infty\,(\lambda\in\Lambda)$, then we have

$$\inf_{\lambda\in\Lambda}\ \pi(1/a_\lambda)=0\,.$$

Because, as there exists a positive number $\varepsilon>0$ such that $m((1+\varepsilon)a_\lambda)<+\infty$, we have

$$\inf_{\lambda\in\Lambda}\ m((1+\varepsilon)a_\lambda)=0$$

by Theorem 35.1 of Nakano [7]. Therefore, we have by (3) that

$$0\leqq\inf_{\lambda\in\Lambda}\pi(1/a_\lambda)\leqq\inf_{\lambda\in\Lambda}\frac{1}{\varepsilon}\,m((1+\varepsilon)a_\lambda)=0\,,$$

as required.

Some of the relations stated above are truth not only for modulars, but also for any convex functions with suitable properties. As to investigations about convex functions, we may have to refer Jensen [3], Orlicz-Birnbaum [12] and W. H. Young [17].

*Remark.* To prove the inequality:

$$m(a+c)+m(b+c)\leqq m(a+b+c)+m(c)$$

for $a, b, c\geqq 0$, we must use the spectral theory which is explained roughly in the next section. As the modular spectra $\omega(\xi, s, \mathfrak{p})$ is a convex function of $\xi>0$, it is not but the definition of convexity that

$$\omega(\alpha+\gamma, s, \mathfrak{p})+\omega(\beta+\gamma, s, \mathfrak{p})\leqq\omega(\alpha+\beta+\gamma, s, \mathfrak{p})+\omega(\gamma, s, \mathfrak{p})$$

for positive numbers $\alpha, \beta, \gamma$. Hence, by the integration:

$$m(\xi s)=\int_{[s]}\omega(\xi, s, \mathfrak{p})m(d\mathfrak{p}s)\ ,$$

we can establish the required inequality.

## §2. Derivative spectra

Since the modulared semi-ordered linear space is a universally continuous semi-ordered linear space, we may need explain some notions about the *spectral theory*, stated in NAKANO [7].

For any positive element $p\in R$, a operator $[p]$ is defined as

$$[p]x=\bigcup_{\nu=1}^{\infty}(x\frown\nu p)$$

for any $x\geqq 0$. This operator $[p]$ is called a *projector*. A system of projectors $\mathfrak{p}$ is called an *ideal*, if 1) $\mathfrak{p}\ni[0]$, 2) $\mathfrak{p}\in[p]\leqq[q]$ implies $\mathfrak{p}\ni[q]$, and 3) $\mathfrak{p}\ni[p],[q]$ implies $\mathfrak{p}\ni[p][q]$. An ideal is said to be *maximal*, if there exists no other ideal containing $\mathfrak{p}$.

For a projector $[p]$, we shall denote by $U_{[p]}$ the set of all maximal ideals $\mathfrak{p}\ni[p]$, and naturally $U_{[0]}=0$.

Considering maximal ideals as points, we obtain a topological space $\mathfrak{S}$ with the neibourhood system $U_{[p]}$ for all projectors $[p]$. This topological space $\mathfrak{S}$ is called the *proper space* of $R$. The proper space is a Hausdorff space and $U_{[p]}$ is open and bicompact.

For any $a\in R$ and $\mathfrak{p}\in\mathfrak{S}$, let us write $(a,\mathfrak{p})=+\infty$ if $\mathfrak{p}\in U_{[a^+]}$, $(a,\mathfrak{p})=0$ if $\mathfrak{p}\bar{\in} U_{[a]}$ and $(a,\mathfrak{p})=-\infty$ if $\mathfrak{p}\in U_{[a^-]}$. Then, if $(a,\mathfrak{p})\neq 0$, for any $b\in R$ there exists uniquely $\lambda_0$ such that $-\infty\leqq\lambda_0\leqq+\infty$ and

$$(\lambda a-b,\mathfrak{p})=\begin{cases}(a,\mathfrak{p}) & \text{for}\quad \lambda>_0\lambda\ ,\\ -(a,\mathfrak{p}) & \text{for}\quad \lambda<\lambda_0\ .\end{cases}$$

This number $\lambda_0$ is called the *relative spectrum* of $b$ by $a$ at $\mathfrak{p}$ and denoted by $\left(\frac{b}{a}, \mathfrak{p}\right)$. Considering as a function of $\mathfrak{p}\in U_{[a]}$, $\left(\frac{b}{a}, \mathfrak{p}\right)$ is continuous and almost finite, namely, finite in an open set being dense in $U_{[a]}$.

For the relative spectrum $\left(\frac{b}{a}, \mathfrak{p}\right)$, we have

$$[a]b = \int_{[a]} \left(\frac{b}{a}, \mathfrak{p}\right) a\mathfrak{p}a,$$

where the integral means the limit of the finite sum:

$$\lim_{\varepsilon \to 0} \sum_{\nu=1}^{\kappa} \left(\frac{b}{a}, \mathfrak{p}_\nu\right) [p_\nu]a \qquad (\mathfrak{p}_\nu \in U_{[p_\nu]}),$$

and the partition:

$$[a] = [p_1] + \cdots\cdots + [p_\kappa], \quad [p_\nu][p_\mu] = 0 \ (\nu \neq \mu),$$

is such that

$$\sup_{\mathfrak{p}_1, \mathfrak{p}_2 \in U_{[p_\nu]}} \left| \left(\frac{b}{a}, \mathfrak{p}_1\right) - \left(\frac{b}{a}, \mathfrak{p}_2\right) \right| < \varepsilon \qquad (\nu = 1, 2, \cdots, \kappa)$$

for a positive number $\varepsilon$.

A element $s$ of $R$ is said to be *simple*, if $m(s) < +\infty$ and $m([p]s) = 0$ implies $]p]s = 0$. For a simple element $s$ and an arbitrary element $a \in R$, the *relative modular spectrum* of $a$ by $s$ at $\mathfrak{p}$ is defined as

$$\omega(a/s, \mathfrak{p}) = \lim_{[p] \to \mathfrak{p}} \frac{m([p]a)}{m([p]s)},$$

where the limit means the upper limit:

$$\inf_{U_{[q]} \ni \mathfrak{p}} \sup_{\mathfrak{p} \ni [p] \leqq [q]} \frac{m([p]a)}{m([p]s)} = \overline{\lim_{[p] \to \mathfrak{p}}} \frac{m([p]a)}{m([p]s)},$$

or the lower limit:

$$\sup_{U_{[q]} \ni \mathfrak{p}} \inf_{\mathfrak{p} \ni [p] \leqq [q]} \frac{m([p]a)}{m([p]s)} = \underline{\lim_{[p] \to \mathfrak{p}}} \frac{m([p]a)}{m([p]s)},$$

and both limits coincide in this case.

Considering $m([p]a)$ as a measure of $U_{[p][a]}$, we have

$$m([p]a) = \int_{[s]} \omega(a/s, \mathfrak{p}) m(d\mathfrak{p}s).$$

Especially, $\omega(\xi s/s, \mathfrak{p})$ is denoted briefly by $\omega(\xi, s, \mathfrak{p})$ and called the *modular spectrum* by $\xi$ and $s$ at $\mathfrak{p}$.

Now, take a element $a \in R$ such that $\pi(1/a) < +\infty$. Then, we have

$$\pi(1/([p] + [q])a) = \pi(1/[p]a) + \pi(1/[q]a)$$

if $[p][q] = 0$, and $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$ implies that

$$\pi(1/[p_\nu]a) \downarrow_{\nu=1}^{\infty} 0.$$

Therefore, by Theorem 36.1 of NAKANO [7], we can define the *derivative spectrum* for a simple element $s \in R$ as follows:

$$\pi(a/s, \mathfrak{p}) = \lim_{[p] \to \mathfrak{p}} \frac{\pi(1/[p]a)}{m([p]s)},$$

and here we have

$$\pi(1/[p]a) = \int_{[p]} \pi(a/s, \mathfrak{p}) m(d\mathfrak{p}s).$$

Considering as a function of $\mathfrak{p} \in U_{[s]}$, $\pi(a/s, \mathfrak{p})$ is continuous. We will call it the derivative spectrum of $a$ by $s$ at $\mathfrak{p} \in U_{[s]}$.

From the definition, we see easily that,

(1) $\pi(a/s, \mathfrak{p}) = 0$ for $\mathfrak{p} \bar{\in} U_{[a]}$,

(2) $\pi([p]a/s, \mathfrak{p}) = \pi(a/[p]s, \mathfrak{p}) = \pi(a/s, \mathfrak{p})$ for $\mathfrak{p} \in U_{[p]}$.

Since $|a| \geqq |b|$ implies that

$$\pi(1/[p]a) \geqq \pi(1/[p]b)$$

for any projector $[p]$ by (1) of §1, we have

(3) $\pi(a/s, \mathfrak{p}) \geqq \pi(b|s, \mathfrak{p})$

if $|a| \geqq |b|$ and $\mathfrak{p} \in U_{[s]}$.

*Theorem* 2.1 *For the derivative of the modular spectrum by* $s, \xi > 0$ *at* $\mathfrak{p}$:

$$\pi(\xi, s, \mathfrak{p}) = \bigcap_{\varepsilon > 0} \frac{\omega(\xi + \varepsilon, s, \mathfrak{p}) - \omega(\xi, s, \mathfrak{p})}{\varepsilon} \qquad (\mathfrak{p} \in U_{[s]}),$$

*we have*

$$\pi(\alpha, s, \mathfrak{p}) = \lim_{[p] \to \mathfrak{p}} \frac{\pi(\alpha/[p]s)}{m([p]s)},$$

*provided that* $\pi(\alpha/s) < +\infty$.

*Proof*. From the assumption, we have

$$\pi(\alpha, s, \mathfrak{p}) = \inf_{\varepsilon > 0} \frac{\omega(\xi + \varepsilon, s, \mathfrak{p}) - \omega(\xi, s, \mathfrak{p})}{\varepsilon}$$

in an open set being dense in $U_{[s]}$, and the right-hand side is monotonically decreasing with respect to $\varepsilon > 0$. Hence B. LEVY's theorem shows that

$$\int_{[p[} \pi(\alpha, s, \mathfrak{p}) m(d\mathfrak{p}s) = \lim \frac{m((\alpha + \varepsilon)[p]s) - m(\alpha[p]s)}{\varepsilon} = \pi(\alpha[p]s)$$

for any projector $[p]$. This means that

$$\pi(\alpha, s, \mathfrak{p}) = \lim_{[p] \to \mathfrak{p}} \frac{\pi(\alpha/[p]s)}{m([p]s)}.$$

for every $\mathfrak{p}\in U_{[s]}$, so that the proof is completed.

By this theorem, we see easily that

$$\pi(\xi s/s,\mathfrak{p})=\lim_{[p]\to\mathfrak{p}}\frac{\pi(1/\xi[p]s)}{m([p]s)}$$

$$=\xi\lim_{[p]\to\mathfrak{p}}\frac{\pi(\xi/[p]s)}{m([p]s)}=\xi\pi(\xi,s,\mathfrak{p}),$$

if $\pi(\xi/s)<+\infty$, that is, we have

(4) $\pi(\xi s/s,\mathfrak{p})=\xi\pi(\xi,s,\mathfrak{p})$ if $\pi(\xi/s)<+\infty$.

Now, we will prove the main theorem in this section:

*Theorem* 2.2. *If, for a finite, simple element $s$ of $R$, $\pi(\xi,s,\mathfrak{p})$ is continuous with respect to $\xi>0$, then we have*

$$\pi(a/s,\mathfrak{p})=\left(\frac{a}{s},\mathfrak{p}\right)\pi\left(\left(\frac{a}{s},\mathfrak{p}\right),s,\mathfrak{p}\right)$$

*in an open set being dense in $U_{[s]}$, provided that $\pi(1/a)<+\infty$.*

*Proof*. We need only prove that, for a number $\varepsilon$ such that $0<\varepsilon<1$, the inequality:

$$(1-\varepsilon)\left(\frac{a}{s},\mathfrak{p}\right)\pi\left((1-\varepsilon)\left(\frac{a}{s},\mathfrak{p}\right),s,\mathfrak{p}\right)\leqq\pi(a/s,\mathfrak{p})$$

$$\leqq(1+\varepsilon)\left(\frac{a}{s},\mathfrak{p}\right)\pi\left((1+\varepsilon)\left(\frac{a}{s},\mathfrak{p}\right),s,\mathfrak{p}\right)$$

is true in an open set being dense in $U_{[s]}$. Because, when the inequality is proved, the continuity of $\pi(\xi,s,\mathfrak{p})$ gives the proof.

When $\mathfrak{p}\in U_{[s]}-U_{[s][a]}$, we have

$$\left(\frac{a}{s},\mathfrak{p}\right)=\pi(a/s,\mathfrak{p})=0,$$

so that the above inequality is obvious.

Generally, we can find a dense subset $A$ of $U_{[s]}$ such that

$$0<\left(\frac{a}{s},\mathfrak{p}\right)<+\infty$$

for every $\mathfrak{p}\in A$. And, for any $\mathfrak{p}_0\in A$ and a number $\varepsilon$ such that $0<\varepsilon<1$, there exists a projector $[p]$ such that

$$(1-\varepsilon)\left(\frac{a}{s},\mathfrak{p}_0\right)\leqq\left(\frac{a}{s},\mathfrak{p}\right)\leqq(1+\varepsilon)\left(\frac{a}{s},\mathfrak{p}_0\right)$$

for every $\mathfrak{p}\in U_{[p]}$, namely,

$$(1-\varepsilon)\left(\frac{a}{s},\mathfrak{p}_0\right)[p]s\leqq[p]a\leqq(1+\varepsilon)\left(\frac{a}{s},\mathfrak{p}_0\right)[p]s$$

for this projector $[p]$. Therefore, the formula [3] shows that

$$\left(\pi(1-\varepsilon)\left(\frac{a}{s}, \mathfrak{p}_0\right)[p]s/s, \mathfrak{p}_0)\right) \leqq \pi([p]a/s, \mathfrak{p}_0)$$
$$\leqq \pi\left((1+\varepsilon)\left(\frac{a}{s}, \mathfrak{p}_0\right)[p]s/s, \mathfrak{p}_0\right)$$

so that we have

$$\pi\left((1-\varepsilon)\left(\frac{a}{s}, \mathfrak{p}_0\right)s/s, \mathfrak{p}_0\right) \leqq \pi(a/s, \mathfrak{p}_0)$$
$$\leqq \pi\left((1+\varepsilon)\left(\frac{a}{s}, \mathfrak{p}_0\right)s/s, \mathfrak{p}_0\right)$$

by the formula (2). As the element $s$ is finite, we have

$$\pi\left((1+\varepsilon)\left(\frac{a}{s}, \mathfrak{p}_0\right)/s\right) < +\infty ,$$

and so, we can obtain by [4] the inequality which was required above. Hence, the proof is established.

## § 3. The exponents of elements

In this section, we will define the exponents of elements and discuss the relations between properties of the elements and their exponents. $R$ is a modulared semi-ordered linear space with $m$ in the sequel.

Let $a$ be an arbitrary element of $R$. Consider a function of a variable $\xi > 0$:

$$\varphi_a(\alpha, \xi) = \frac{\pi(1/\xi a)}{\xi^\alpha} \qquad (\xi > 0) ,$$

where $\alpha$ is a fixed number such that $\alpha \geqq 1$. It is obvious that

$$\xi^\alpha \varphi_a(\alpha, \xi) = \xi^\beta \cdot \varphi_a(\beta, \xi) = \pi(1/\xi a)$$

for any number $\beta$ such that $\beta \geqq 1$.

If the function $\varphi_a(\alpha, \xi)$ is increasing, then $\varphi_a(\beta, \xi)$ is also increasing, whenever $\alpha \geqq \beta \geqq 1$. In fact, for $\xi > \eta > 0$ we have

$$\varphi_a(\beta, \xi) = \xi^{\alpha-\beta} \varphi_a(\alpha, \xi) \geqq \eta^{\alpha-\beta} \cdot \varphi_a(\alpha, \eta) = \varphi_a(\beta, \eta) ,$$

since $\alpha - \beta \geqq 0$. Similarly, if $\varphi_a(\alpha, \xi)$ is a decreasing function of $\xi > 0$, then $\varphi_a(\beta, \xi)$ is also decreasing, whenever $\beta \geqq \alpha$.

Therefore, we can define as follows:

*Definition.* The greatest lower bound of such number $\alpha \geqq 1$ that $\varphi(\alpha, \xi)$ is decreasing as a function of $\xi > 0$ is called *the upper exponent*

*of $a$ by $\pi$*, and denoted by $\chi^{\pi}(a)$. The least upper bound of such number $\alpha \geqq 1$ that $\varphi_a(\alpha, \xi)$ is increasing as a function of $\xi > 0$ is called *the lower exponent of $a$ by $\pi$* and denoted by $\chi_{\pi}(a)$.

From this definition and §1(2), we see easily that

(1) $\chi^{\pi}(a) = \chi^{\pi}(\xi a) \geqq \chi_{\pi}(\xi a) = \chi_{\pi}(a) \geqq 1$.

for every number $\xi > 0$ and $a \in R$. Moreover, it is obvious that $\varphi_a(\chi^{\pi}(a), \xi)$ is decreasing and $\varphi_a(\chi_{\pi}(a), \xi)$ is increasing as a function of $\xi > 0$.

Between a modular and its exponents, there exist following inequalities:

(2) $\chi_{\pi}(a) \cdot m(a) \leqq \pi(1/a) \leqq \chi^{\pi}(a) \cdot m(a)$. $\quad (a \in R)$.

In fact, we can prove the right-hand inequality as follows: the case, when $\pi(1/a)$ and $m(a)$ are finite, is only to be proved. Then

$$m(a) = \int_0^1 \pi(\xi/a) d\xi \geqq \int_0^1 \xi^{\alpha - 1} \cdot \pi(1/a) d\xi = \frac{1}{\alpha} \cdot \pi(1/a)$$

for $\alpha = \chi^{\pi}(a)$, as required. The proof of the left-hand inequality is quite similar.

From (1) and §1 (2), we see easily that

(3) $\chi_{\pi}(a) \dfrac{m(\xi a)}{\xi} \leqq \pi(\xi/a) \leqq \chi^{\pi}(a) \dfrac{m(\xi a)}{\xi}$

for every number $\xi > 0$ and $a \in R$.

*Remark.* If the modular is of $L_p$-type at a element $a \in R$, that is,

$$m(\xi a) = \xi^p \cdot m(a) \qquad (\xi > 0),$$

then, we can easily obtain the equality in the inequality (3), and $\chi^{\pi}(a) = \chi_{\pi}(a) = p$. Now, we can prove the converse, namely, if, for a simple element $a$, $m(\xi a)$ is a differentiable function of $\xi > 0$, then

$$\pi(\xi/a) = p \frac{m(\xi a)}{\xi} \qquad (\xi > 0)$$

implies that

$$m(\xi a) = \xi^p m(a)$$

for every $\xi > 0$. Because, we have from the assumption

$$\pi(\xi/\eta a) = p \frac{m(\xi \eta a)}{\xi}$$

for any $\xi, \eta > 0$, that is,

$$\frac{\pi(\xi/\eta a)}{m(\xi \eta a)} = p \cdot \frac{1}{\xi},$$

thus the integration gives for $\xi \geqq 1$,

$$\log m(\xi\eta a) - \log m(\eta a) = p \cdot \log \xi .$$

Hence, we have

$$m(\xi\eta a) = \xi^p \cdot m(\eta a) .$$

As $\xi \geqq 1$ and $\eta > 0$ are arbitrary, we have

$$m(\xi a) = \xi^p \cdot m(a)$$

for every $\xi > 0$.

If the upper exponent of $a$ by $\pi$ is finite, then the element $a$ is a finite element, that is,

$$m(\xi a) < +\infty \qquad (\xi > 0) .$$

In fact, if there exists a number $\xi$ such that $m(\xi a) = +\infty$, then we have $\pi(1/\xi a) = +\infty$, so that the function $\varphi_a(\alpha, \xi)$ can not be decreasing for any number $\alpha \geqq 1$. This contradicts to the assumption that $\chi^\pi(a)$ is finite.

*Theorem* 3.2. *Let* $\chi^\pi(a)$ *be finite. Then the function of* $\xi > 0$:

$$\frac{m(\xi a)}{\xi^\alpha}$$

*is decreasing for* $\alpha = \chi^\pi(a)$, *and is increasing for* $\alpha = \chi_\pi(a)$.

*Proof*. By Theorem 3.1, $m(\xi a)$ is a finite, convex function of $\xi > 0$. Hence, the inequality (3) in this section is equivalent to our theorem, because the right-hand derivative of $m\xi a/\xi^\alpha$:

$$\frac{d}{d\xi}\left[\frac{m(\xi a)}{\xi^\alpha}\right] = \frac{1}{\xi^\alpha}\left[\pi(\xi/a) - \alpha\frac{m(\xi a)}{\xi}\right]$$

is always positive for $\alpha = \chi_\pi(a)$, and is always negative for $\alpha = \chi^\pi(a)$.

Apart from this theorem, we can define new exponents as follows: The greatest lower bound of such number $\alpha \geqq 1$ that $m(\xi a)/\xi^\alpha$ is decreasing is called *the upper exponent af a by m* and denoted by $\chi^m(a)$. The least upper bound of such number $\alpha \geqq 1$ that $m(\xi a)/\xi^\alpha$ is increasing is called *the lower exponent of a by m* and denoted by $\chi_m(a)$.

Then it is easy to see that

$$\chi^\pi(a) \geqq \chi^m(a) = \chi^m(\xi a) \geqq \chi_m(\xi a) = \chi_m(a) \geqq \chi_\pi(a)$$

for every $\xi > 0$ and $a \in R$.

A element $a \in R$ for which there exist numbers $\alpha, \gamma > 1$ such that

$$m(\alpha\xi a) \leqq \gamma m(\xi a)$$

for every $\xi > 0$, is called *upper bounded*. For a upper bounded element $a$, it is obvious that

$$0 < m(\xi a) < +\infty ,$$

namely, $0 < \pi(\xi/a) < +\infty$ for every $\xi > 0$.

Upper boundedness of the convex function was considered first by Cooper [2] and next by Birkill [1]. Few years later, Mulholland [6], similarly as Orlicz-Birnbaum [12], considered it as a property of more special functions, namely, *sub-multiplicative* functions $f(\xi)$ which satisfy the following relation:

$$f(\xi\eta) \leqq f(\xi) \cdot f(\eta)$$

for every $\xi$ and $\eta$. The converse inequality:

$$f(\xi\eta) \geqq f(\xi) \cdot f(\eta)$$

for every $\xi$ and $\eta$, was called *super-multiplicativity* of the function by Mulholland [6]. Corresponding to this, we can define *lower boundedness* of elements as follows:

If there exist numbers $\gamma > \alpha > 1$ for which

$$m(\alpha\xi a) \geqq \gamma m(\xi a)$$

holds for every $\xi > 0$, then the element $a \in R$ is called *lower bounded*.

For a lower bounded element $a \in R$ we have

$$\lim_{\xi \to 0} \frac{m(\xi a)}{\xi} = 0 , \qquad \lim_{\xi \to \infty} \frac{m(\xi a)}{\xi} = +\infty ,$$

that is, $\lim_{\xi \to 0} \pi(\xi/a) = 0$ and $\lim_{\xi \to +\infty} \pi(\xi/a) = +\infty$.

Now, we will make clear the relation between exponents and boundedness properties.

*Theorem* 3.3. *In order that a element $a \in R$ be upper bounded, it is necessary and sufficient that $\chi^m(a) < +\infty$.*

*Proof.* Let $a \in R$ be upper bounded, that is, there exists a number $\gamma \geqq 2$ such that

$$m(2\xi a) \leqq \gamma \cdot m(\xi a) \qquad (\xi > 0) .$$

Since we have

$$m(2\xi a) - m(\xi a) \leqq (\gamma - 1) m(\xi a) ,$$

and the convexity of $m(\xi a)$ implies that

$$\pi(1/\xi a) \leqq m(2\xi a) - m(\xi a) ,$$

we obtain

$$\pi(1/\xi a) \leqq (\gamma - 1) \cdot m(\xi a) \qquad (\xi > 0) ,$$

which shows that the function of $\xi > 0$:

$$\frac{m(\xi a)}{\xi^{\gamma - 1}}$$

is decreasing, namely

$$\chi^m(a) \leqq \gamma - 1 .$$

Conversely, if $\chi^m(a) = p$ is finite, then we have

$$m(2\xi a) \leqq 2^p \cdot m(\xi a)$$

for every $\xi > 0$, as required, because $2^p \geqq 2$ .

This theorem is an improvement of a lemma of Cooper [2]-Burkill [1], which asserts that, if $a \in R$ is upper bounded, then we can find numbers $p, \gamma > 0$ such that

$$\frac{m(\xi a)}{\xi^p} \leqq \gamma \frac{m(\eta a)}{\eta^p}$$

for every $\xi > \eta > 0$, that is, the function of $\xi > 0$:

$$\frac{m(\xi a)}{\xi^p}$$

is *quasi-decreasing* by the terminology of Mulholland [5].

Between quasi-decreasingness, and usual decreasingness, there is a wide difference. In fact, a function $f(\xi)$ is quasi-decreasing if and only if there exists a decreasing function $g(\xi)$ such that

$$k \cdot g(\xi) \leqq f(\xi) \leqq g(\xi) \qquad (\xi > 0) ,$$

for a fixed number $k$. To see this, we need only put

$$g(\xi) = \inf_{0 \leqq \eta \leqq \xi} f(\eta) .$$

*Theorem* 3.4. *In order that a element $a \in R$ be lower bounded, it is necessary and sufficient that $\chi_m(a)$ is strictly greater than one.*

*Proof*. When $\chi_m(a)$ is strictly greater than one, we have

$$m(\alpha \xi a) \geqq \alpha^q m(\xi a)$$

for any number $\alpha > 1$ and $\xi > 0$. Therefore, $a$ is lower bounded.

Conversely, if $a$ is lower bounded, then there exist numbers $\alpha, \gamma$

such that $0<\gamma<\alpha<1$ for which we have

$$m(\alpha\xi a)\leqq\gamma\cdot m(\xi a)$$

for any $\xi>0$. Because, we can find numbers $\beta,\delta$ such that $1<\beta<\delta$ for which we have

$$m(\beta\xi a)\geqq\delta\cdot m(\xi a)$$

for any $\xi>0$, so that we have

$$m\left(\frac{1}{\beta}\xi a\right)\leqq\frac{1}{\delta}m(\xi a)\qquad(\xi>0),$$

where $1>\frac{1}{\beta}>\frac{1}{\delta}$. Hence, we have, for such $\alpha$ and $\gamma$,

$$\frac{m(\xi a)-m(\alpha\xi a)}{1-\alpha}\geqq\frac{1-\gamma}{1-\alpha}m(\xi a)$$

and

$$m(\xi a)-m(\alpha\xi a)\leqq(1-\alpha)\pi(\xi/a).$$

Therefore, we obtain

$$\frac{1-\gamma}{1-\alpha}m(\xi a)\leqq\pi(\xi/a)$$

for any $\xi>0$, which implies that

$$\chi_m(a)\geqq\frac{1-\gamma}{1-\alpha}>1.$$

We have defined above two sorts of exponents. Now, we will consider the problem when these exponents coincide. In order to have the equalties:

$$\chi^m(a)=\chi^\pi(a),\quad \chi_m(a)=\chi_\pi(a),$$

it is necessary and sufficient that

$$\varphi_a(\chi^m(a),\xi),\quad\text{or}\quad\varphi_a(\chi_m(a),\xi)$$

is increasing or decreasing respectively as a function of $\xi>0$. And, for this, it is necessary and sufficient that the right-hand derivative of the function of $\xi>0$, $\varphi_a(\alpha,\xi)$ is negative for $\alpha=\chi^m(a)$ and is positive for $\alpha=\chi_m(a)$, provided that $\pi(\xi/a)$ is continuous as a function of $\xi>0$.

But, this is not interesting. As a more simple criterion, we have the following:

*Theorem* 3.5. *If the function of* $\xi>0$:

$$\frac{\pi(1/\xi a)}{m(\xi a)}$$

*is decreasing, then we have* $\chi^{\pi}(a)=\chi^{m}(a)$, *and if the function is increasing, then we have* $\chi_{\pi}(a)=\chi_{m}(a)$.

*Praof.* For $\xi>\eta>0$, since

$$m(\xi a)/\xi^{\chi^{m}(a)}$$

is decreasing with respect to $\xi>0$, we have, for $p=\chi^{m}(a)$,

$$\frac{\pi(1/\xi a)}{\xi^{p}}=\frac{\pi(1/\xi a)}{m(\xi a)}\cdot\frac{m(\xi a)}{\xi^{p}}$$
$$\leqq\frac{\pi(1/\eta a)}{m(\eta a)}\cdot\frac{m(\eta a)}{\eta^{p}}=\frac{\pi(1/\eta a)}{\eta^{p}},$$

and this means that $\chi^{m}(a)=\chi^{\pi}(a)$. Similarly, the later half will be proved.

When $m(\xi a)$ is differentiable as a function of $\xi>0$, there exists a number $p$ such that

$$m(\xi a)=\xi^{p}m(a) \qquad (\xi>0),$$

when and only when we have

$$\frac{\pi(1/\xi a)}{m(\xi a)}=p$$

for every $\xi>0$, as we have stated in the previous remark.

The example which shows the difference between $\chi^{\pi}(a), \chi_{\pi}(a)$ and $\chi^{m}(a)$, $\chi_{m}(a)$ can be easily obtained. Moreover, the following theorem, which is a generalization of the theorem of Pappus in Euclidean space, can be obtained only for $\chi^{\pi}(a)$, $\chi_{\pi}(a)$.

*Theorem* 3.6. *If* $\chi^{\pi}(a)\leqq 2$, *then we have*

$$\omega(\alpha+\beta/a)+\omega(\alpha-\beta/a)\leqq 2\{\omega\alpha/a)+\omega(\beta/a)\}$$

*for* $\alpha>\beta>0$. *If* $\chi_{\pi}(a)\geqq 2$, *then we have*

$$\omega(\alpha+\beta/a)+\omega(\alpha-\beta/a)\geqq 2\{\omega(\alpha/a)+\omega(\beta/a)\}$$

*for* $\alpha>\beta>0$, *where*

$$\omega(\xi/a)=m\left(\xi\frac{a}{|||a|||}\right)$$

*for the second norm* $|||a|||$ $(a\in R)$, *which will be explained in* §6.

*Proof.* Let us denote the right-hand derivative of $\omega(\xi/a)$ by $\tau(\xi/a)$, which evidently has the same properties as $\pi(\xi/a)$. If $\chi^{\pi}(a)\leqq 2$, then

we have

$$\omega(\alpha+\beta/a)+\omega(\alpha-\beta/a)-2\omega(\alpha/a)=\left[\int_0^{\alpha+\beta}+\int_0^{\alpha-\beta}-2\int_0^{\alpha}\right]\tau(\xi/a)d\xi$$
$$=\left[\int_{\alpha}^{\alpha+\beta}-\int_{\alpha-\beta}^{\alpha}\right]\tau(\xi/a)d\xi=\int_0^{\beta}\left[\tau(\xi+\alpha/a)-\tau(\xi+\alpha-\beta/a)\right]d\xi$$
$$\leqq\int_0^{\beta}\tau(\beta/a)d\xi=\beta\tau(\beta/a)=\tau(1/\beta a)\leqq 2\omega(\beta/a)\,,$$

so that the first half of this theorem is proved. The rest can be proved similarly.

*Remark.* For instance, the next example shows the difference between $\chi^{\pi}(a)$ and $\chi_m(a)$. Let $m(\xi)$ be a modular on the line defined as follows:

$$m(\xi)=\begin{cases}0 & \text{for } 0\leqq\xi\leqq 1\,,\\ \xi^2-\dfrac{1}{\xi} & \text{for } \xi>1\,,\end{cases}$$

then, considering about the element 1, we have

$$\chi_m(1)=2 \qquad \text{ane} \qquad \chi_{\pi}(1)=1\,.$$

By definition, this modular is not simple, that is, $m(a)=0$ does not always imply $a=0$. Therefore $\chi^m(1)=\chi^{\pi}(1)=+\infty$.

## § 4. The exponents of manifolds

The exponents defined in the previous section depend on each element. We will define in this section exponents which depend only on the modular, and discuss the problem determining them by that of elements.

*Definition.* For a subset $M$ of $R$,

$$\chi^m(M)=\sup_{x\in M}\chi^m(x)$$

is called *the upper exponent of $M$ by $m$*, and

$$\chi_m(M)=\inf_{x\in M}\chi_m(x)$$

is called *the lower exponent of $M$ by $m$*. Similarly, we can define *the upper and lower exponents of $M$ by $\pi$*, and they are denoted by $\chi^{\pi}(M)$ and $\chi_{\pi}(M)$ respectively. If $M=R$, they are briefly denoted by $\chi^m$, $\chi_m$ and $\chi^{\pi}$, $\chi_{\pi}$.

It is evident that

$$1\leqq\chi_{\pi}(M)\leqq\chi_m(M)\leqq\chi^m(M)\leqq\chi^{\pi}(M)\leqq+\infty$$

for every subset $M$ of $R$.

Theorem 3.3 shows that, if $\pi(1/\xi a)/m(\xi a)$ is a decreasing function of $\xi>0$ for every $a\in M$, then we have $\chi_m(M)=\chi_\pi(M)$, and if the function of $\xi>0$ is increasing for every $a\in R$, then we have $\chi_m(M)=\chi_\pi(M)$.

At first, we will treat the exponents by $m$.

*Theorem* 4.1. *If, for a simple element $s$ of $R$,*

$$(*)\quad \omega(\xi, s, \mathfrak{p})/\xi^\alpha$$

*is increasing as a function of $\xi>0$ in an open set being dense in $U_{[s]}$, then we have*

$$\chi_m([s]R)\geqq\alpha\,.$$

*If the function (*) is decreasing with respect to $\xi>0$ in an open set being dense in $U_{[s]}$, then we have*

$$\chi^m([s]R)\leqq\alpha\,.$$

*Proof*. Let $a$ be an arbitrary element and $\xi>\eta>0$. We can assume that $\pi(\xi/a)<+\infty$ without loss of genelarity. Then, from the assumption, we have

$$\frac{\omega(\xi,s,\mathfrak{p})}{\xi^\alpha}\geqq\frac{\omega(\eta,s,\mathfrak{p})}{\eta^\alpha}$$

in an open set being dense in $U_{[s]}$. On the other hand, we have

$$\omega(\xi a/s,\mathfrak{p})=\omega\left(\xi\left(\frac{a}{s},\mathfrak{p}\right),s,\mathfrak{p}\right)$$

in an open set being dense in $U_{[s]}$, and that

$$\xi\left(\frac{a}{s},\mathfrak{p}\right)\geqq\eta\left(\frac{a}{s},\mathfrak{p}\right)$$

is obvious. Therefore, we have

$$\frac{\omega(\xi a/s,\mathfrak{p})}{\xi^\alpha}=\frac{\omega\left(\xi\left(\frac{a}{s},\mathfrak{p}\right),s,\mathfrak{p}\right)}{\xi^\alpha}$$

$$\geqq\frac{\omega\left(\eta\left(\frac{a}{s},\mathfrak{p}\right),s,\mathfrak{p}\right)}{\eta^\alpha}=\frac{\omega(\eta a/s,\mathfrak{p})}{\eta^\alpha}$$

in an open set being dhnse in $U_{[s[}$, so that the integration gives

$$\frac{m(\xi[s]a)}{\xi^\alpha}\geqq\frac{m(\eta[s]a)}{\eta^\alpha}$$

that is, $\chi_m([s]a) \geqq a$ for every $a \in R$, and this means that

$$\chi_m([s]R) \geqq \alpha .$$

The later half of this theorem can be proved similarly.

*Theorem* 4.2. *If, for a simple element s of R, we have*

$$\chi_m([p]s) \geqq \alpha$$

*for every projector* $[p]$, *then we have*

$$\chi_m([s]R) \geqq \alpha .$$

*If, for a simple element s of R, we have*

$$\chi^m([p]s) \leqq \alpha$$

*for every projector* $[p]$, *then we have*

$$\chi^m([s]R) \leqq \alpha .$$

*Proof.* For $\xi > \eta > 0$, we have

$$\frac{\omega(\xi, s, \mathfrak{p})}{\xi^\alpha} = \lim_{[p] \to \mathfrak{p}} \frac{m(\xi[p]s)}{\xi^\alpha \cdot m([p]s)}$$

$$\geqq \lim_{[p] \to \mathfrak{p}} \frac{m(\eta[p]s)}{\eta^\alpha \cdot m[p]s)} = \frac{\omega(\eta, s, \mathfrak{p})}{\eta^\alpha} .$$

Hence, the previous theorem gives the proof. The later half can be proved similarly.

*Theorem* 4.3. *If there exists a complete system of elements* $a_\lambda (\lambda \in \Lambda)$ *such that*

$$\chi_m([a_\lambda]R) \geqq \alpha \qquad (\lambda \in \Lambda) ,$$

*then we have*

$$\chi_m(S) \geqq \alpha$$

*for the set of simple elements* $S$. *If there exists a complete system of elements* $a_\lambda (\lambda \in \Lambda)$ *such that*

$$\chi^m([a_\lambda]R) \leqq \alpha \qquad (\lambda \in \Lambda) ,$$

*then we have*

$$\chi^m \leqq \alpha .$$

*Proof.* We need only prove the first half. For any $s \in S$, we have by the assumption,

$$\chi_m([p][a_\lambda]s) \geqq \alpha \qquad (\lambda \in \Lambda)$$

for every projector $[p]$. Therefore, similarly as the proof of the previous theorem, we can know that

$$\frac{\omega(\xi, s, \mathfrak{p})}{\xi^{\alpha}} = \frac{\omega(\xi, [a_\lambda]s, \mathfrak{p})}{\xi^{\alpha}} \qquad (\mathfrak{p} \in U_{[a_\lambda][s]})$$

is an increasing function of $\xi > 0$. Therefore

$$\frac{\omega(\xi, s, \mathfrak{p})}{\xi^{\alpha}}$$

is an increasing function of $\xi > 0$ for every $s \in S$, because

$$U_{[s]} = \Big(\sum_{\lambda \in \Lambda} U_{[a_\lambda][s]}\Big)^{-}$$

by the assumption. Hence, Theorem 4.1 gives

$$\chi_m(S) \geqq \alpha ,$$

since $\chi_m(s) \geqq \alpha$ for every $s \in S$.

*Theorem* 4.4. *If there exists a complete system of simple elements* $a_\lambda$ $(\lambda \in \Lambda)$ *such that*

$$\chi_m([p]a_\lambda) \geqq \alpha \qquad (\lambda \in \Lambda)$$

*for any projector* $[p]$, *then we have*

$$\chi_m(S) \geqq \alpha .$$

*If there exists a complete system of elements* $a_\lambda$ $(\lambda \in \Lambda)$ *such that*

$$\chi^m([p]a_\lambda) \leqq \alpha \qquad (\lambda \in \Lambda)$$

*for any projector* $[p]$, *then we have*

$$\chi^m \leqq \alpha .$$

*Proof*. We need only prove the first half. From the assumption we have

$$\chi_m([a_\lambda]R) \geqq \alpha \qquad (\lambda \in \Lambda)$$

by Theorem 4.2, ans. so the previous theorem gives

$$\chi_m(S) \geqq \alpha ,$$

as required.

In the rest of this section, let the appearing $\pi(\xi/a)$ be all continuous as a functions of $\xi > 0$. Therefore, if $\chi^\pi$ is finite, then we have

$$\pi(a/s, \mathfrak{p}) = \Big(\frac{a}{s}, \mathfrak{p}\Big) \cdot \pi\Big(\Big(\frac{a}{s}, \mathfrak{p}\Big), s, \mathfrak{p}\Big)$$

in an open set being dense in $U_{[s]}$, by Theorem 2.2. Therefore, by the similar argument as the above theorems, we have the following theorems:

*Theorem* 4.5. *If, for a simple element $s$ of $R$,*

$$(\#) \qquad \pi(\xi, s, \mathfrak{p})/\xi^{\alpha-1}$$

*is a decreasing function of $\xi>0$ in an open set being dense in $U_{[s]}$, then we have*

$$\chi_\pi([s]R) \geqq \alpha .$$

*If the function $(\#)$ is increasing with respect to $\xi>0$ in an open set being dense in $U_{[s]}$, then we have*

$$\chi^\pi([s]R) \leqq \alpha .$$

*Theorem* 4.6. *If, for a simple element $s$ of $R$, we have*

$$\chi_\pi([p]s) \geqq \alpha$$

*for every projector $[p]$, then we have*

$$\chi_\pi([s]R) \geqq \alpha .$$

*If, for a simple element $s$ of $R$, we have*

$$\chi^\pi([p]s) \leqq \alpha$$

*for every projector $[p]$, then we have*

$$\chi^\pi([s]R) \leqq \alpha .$$

*Theorem* 4.7. *If there exists a complete system of elements $a_\lambda (\lambda \in \Delta)$ such that*

$$\chi_\pi([a_\lambda]R) \geqq \alpha ,$$

*then we have*

$$\chi_\pi(S) \geqq \alpha$$

*for the set of all simple elements $S$. If there exists a complete system of elements $a_\lambda (\lambda \in \Lambda)$ such that*

$$\chi^\pi([a_\lambda]R) \leqq \alpha ,$$

*then we have*

$$\chi^\pi \leqq \alpha .$$

*Theorem* 4.8. *If there exists a complete system of simple elements $a_\lambda (\lambda \in \Lambda)$ such that*

$$\chi_\pi([p]a_\lambda) \geqq \alpha$$

*for every projector* $[p]$, *then we have*

$$\chi_{\pi}(S) \geqq \alpha .$$

*If there exists a complete system of elements* $a_{\lambda}(\lambda \in \Lambda)$ *such that*

$$\chi_{\pi}([p]a_{\lambda}) \leqq \alpha$$

*for every projector* $[p]$, *then we have*

$$\chi^{\pi} \leqq \alpha ,$$

The results obtained in this section will play important rôles in §7 and §8.

## §5. On conjugate modulars

In this section we will consider relations between exponents of a modular and that of the modular of its conjugate space. Let $R$ be a modulared semi-ordered linear space. The *modular conjugate space* $\bar{R}^m$ of $R$ is composed of such linear functionals $\bar{a}$ on $R$ that

$$a_{\lambda} \downarrow_{\lambda \in \Lambda} 0 \quad \text{implies} \quad \inf_{\lambda \in \Lambda} |\bar{a}(a_{\lambda})| = 0 ,$$

and

$$\sup_{m(x) \leqq 1} |\bar{a}(x)| < +\infty .$$

It is known that (See NAKANO (7)), putting

$$\bar{m}(\bar{a}) = \sup_{x \in R} \{\bar{a}(x) - m(x)\} ,$$

$\bar{R}^m$ is a modulared semi-ordered linear space whose modular is complete. If we define a modular in $L_p$-space as

$$m(a) = \frac{1}{p} \int_0^1 |a(t)|^p \, dt ,$$

then, by easy calculations we have

$$\bar{m}(\bar{a}) = \frac{1}{p'} \int_0^1 |\bar{a}(t)|^{p'} \, dt ,$$

where $\bar{R}^m$ coincides with $L_{p'}$-space, $\frac{1}{p} + \frac{1}{p'} = 1$.

Let us denote the exponents of $\bar{R}^m$ by $\chi^{\bar{m}}$, $\chi_{\bar{m}}$ and $\chi^{\bar{\pi}}$, $\chi_{\bar{\pi}}$ etc.

*Theorem* 5.1. *Under the above definitions, we have*

$$\frac{1}{\chi^m} + \frac{1}{\chi_{\bar{m}}} = \frac{1}{\chi_m} + \frac{1}{\chi^{\bar{m}}} = 1 .$$

*Proof.* If, for a number $\alpha>1$,

$$m(\xi a)/\xi^{\alpha}$$

is increasing for any $a\in R$ as a function of $\xi>0$, then for such a number $\beta$ that $(\alpha-1)(\beta-1)=1$, the function of $\xi>0$:

$$\bar{m}(\xi\bar{a})/\xi^{\beta}$$

is decreasing for any $\bar{a}\in\bar{R}^m$. Because, we have for $\xi\geqq 1$,

$$\bar{m}(\bar{a})=\sup_{x\in R}\{\bar{a}(x)-m(x)\}\geqq\sup_{x\in R}\left\{\bar{a}(x)-\frac{m(\xi x)}{\xi^{\alpha}}\right\}$$

$$=\frac{1}{\xi^{\alpha}}\cdot\sup_{x\in R}\{\xi^{\alpha-1}\bar{a}(\xi x)-m(\xi x)\}=\frac{1}{\xi^{\alpha}}\cdot\bar{m}(\xi^{\alpha-1}\bar{a})\,.$$

Therefore we have, for such a number $\eta$ that $\xi^{\alpha}=\eta^{\beta}=\xi\eta$,

$$\eta^{\beta}\cdot\bar{m}(\bar{a})\geqq\bar{m}(\eta\bar{a})\,,$$

that is, $\bar{m}(\xi\bar{a})/\xi^{\beta}$ is decreasing for any $\bar{a}\in\bar{R}^m$ with respect to $\xi>0$.

Conversely, if for a number $\alpha$, the function of $\xi>0$:

$$\bar{m}(\xi\bar{a})/\xi^{\alpha}$$

is decreasing for any $\bar{a}\in\bar{R}^m$, then

$$m(\xi a)/\xi^{\beta}$$

is increasing for any $a\in R$ with respect to $\xi>0$, where $(\alpha-1)(\beta-1)=1$.

Similarly, we can prove the dual properties, from which the proof is established.

A sequence of elements $a_\nu\,(\nu=1,2,\cdots)$ is said to be *modular bounded*, if we have

$$\sup_{\nu=1,2,\cdots}m(\xi a_\nu)<+\infty$$

for any number $\xi>0$. Then we have:

*Theorem* 5.2. *If $m$ is complete and $1<\chi_m\leqq\chi^m<+\infty$, then every modular bounded sequence $a_\nu\,(\nu=1,2,\cdots)$ contains a subsequence $a_{\nu_\mu}\,(\mu=1,2,\cdots)$ and there exists $a\in R$ for which we have*

$$\lim_{\mu\to\infty}\bar{a}(a_{\nu_\mu})=\bar{a}(a)$$

*for any $\bar{a}\in\bar{R}^m$.*

*Proof.* At first, we can prove that the sequence $a_\nu\,(\nu=1,2,\cdots)$ is *equi-continuous* (See NAKANO [7], §27), namely, for any $\bar{a}_\nu\downarrow_{\nu=1}^{\infty}0$, $\bar{a}_\nu\in\bar{R}$ and $\varepsilon>0$, there exists $\nu_0$ for which we have

$$\bar{a}_{\nu_0}(|a_\nu|) \leqq \varepsilon \qquad (\nu=1,2,\cdots).$$

Since we have

$$\xi\bar{a}_\mu(|a_\nu|) \leqq \bar{m}(\xi\bar{a}_\mu)+m(a_\nu),$$

and $\chi_m>1$ implies $\chi^{\bar{m}}<+\infty$ by the previous theorem, there exists $\nu_0$ for which we have

$$\xi\bar{a}_{\nu_0}(|a_\nu|) \leqq \xi\varepsilon+m(a_\nu) \qquad (\nu=1,2,\cdots),$$

so that we have

$$\bar{a}_{\nu_0}(|a_\nu|) \leqq \varepsilon \qquad (\nu=1,2,\cdots).$$

On the other hand, the completeness of $m$ implies that $R$ is reflexive and $\bar{R}=\bar{R}^m$. Therefore, Theorem 27.5 of NAKANO [7] gives a subsequence $a_{\nu_\mu}(\mu=1,2,\cdots)$and $a\in R$ such that

$$\lim_{\mu\to\infty}\bar{a}(a_{\nu_\mu}) = \bar{a}(a)$$

for any $\bar{a}\in\bar{R}^m$.

*Remark.* This theorem is a generalization of one given by ORLICZ[15], which was stated for functions $f(x)$ such that

$$\int_0^1 M[\alpha f(x)]dx < +\infty$$

for a suitable number $\alpha>0$, where $M[\xi]$ is a continuous convex function satisfying some conditions. The set of all such functions was denoted by $L^M$.

Next, we will prove a theorem concerning about the exponents $\chi^\pi$ and $\chi_\pi$.

*Theorem* 5.3. *For any element $a\in R$ such that $\chi^\pi(a)<+\infty$, there exists a linear functional $\bar{a}\in\bar{R}^m$ such that*

$$\frac{1}{\chi^\pi(a)}+\frac{1}{\chi_{\bar{\pi}}(\bar{a})}=1.$$

*Proof.* Since the element $a$ is simple and domestic, puttihg

$$\bar{a}(x) = \int_{[a]}\left(\frac{x}{a},\mathfrak{p}\right)m(d\mathfrak{p}a),$$

we have $0\leqq\bar{a}\in\bar{R}^m$, and

$$\bar{m}(\xi\bar{a}[p]) = \int_{[p]}\bar{\omega}(\xi,a,\mathfrak{p})m(d\mathfrak{p}a),$$

by Theorem 38.11 of NAKANO [7], where $\bar{\omega}(\xi,a,\mathfrak{p})$ is the conjugate modular spectrum, namely,

$$\bar{\omega}(\xi,a,\mathfrak{p})=\bigcup_{x\in D}\left\{\xi\left(\frac{x}{a},\mathfrak{p}\right)-\omega(x/a,\mathfrak{p})\right\},$$

denoting by $D$ the set of all demestic elements of $R$. (See NAKANO [7], §37). On the other hand, since the modular spectrum $\omega(\xi,a,\mathfrak{p})$ is a finite, increasing, convex function of $\xi>0$ for each $\mathfrak{p}\in U_{[a]}$, the function

$$\pi(\xi,a,\mathfrak{p})=\bigcap_{\varepsilon>0}\frac{1}{\varepsilon}\left\{\omega(\xi+\varepsilon,a,\mathfrak{p})-\omega(\xi,a,\mathfrak{p})\right\}$$

is finite and continuous with respect to $\mathfrak{p}\in U_{[a]}$. Therefore, we have

$$\bar{\omega}(\pi(\alpha,a,\mathfrak{p}),a,\mathfrak{p})=\int_0^\alpha \xi d_\xi \pi(\xi,a,\mathfrak{p})$$

for any $\alpha>0$ and $\mathfrak{p}\in U_{[a]}$, by Theorem 37.8 of NAKANO [7]. Hence, putting $\eta=\pi(\xi,a,\mathfrak{p})$, we have

$$D_\eta\bar{\omega}(\eta,a,\mathfrak{p})=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{\bar{\omega}(\pi(\xi+\varepsilon,a,\mathfrak{p}),a,\mathfrak{p})-\bar{\omega}(\pi(\xi,a,\mathfrak{p}),a,\mathfrak{p}\right\}=\xi,$$

so that the function $D_\eta\bar{\omega}(\eta,a,\mathfrak{p})$ is a inverse function of $\pi(\xi,a,\mathfrak{p})$. Therefore, since

$$\pi(\xi/a)=\int_{[a]}(\pi\xi,a,\mathfrak{p})\,m(d\mathfrak{p}a)$$

and

$$\bar{\pi}(\xi/\bar{a})=\inf_{\varepsilon>0}\frac{\bar{m}((\xi+\varepsilon)\bar{a})-\bar{m}(\xi\bar{a})}{\varepsilon}=\int_{[a]}D_\xi\bar{\omega}(\xi,a,\mathfrak{p})m(d\mathfrak{p}a),$$

we have

$$\frac{1}{\chi^\pi(a)}+\frac{1}{\chi_{\bar{\pi}}(\bar{a})}=1,$$

as required.

Let $1<\chi_\pi\leqq\chi^\pi<+\infty$. Then, since the modular $m$ is simple and finite, the set of functionals:

$$\bar{a}(x)=\int_{[a]}\left(\frac{x}{a},\mathfrak{p}\right)m(d\mathfrak{p}a)$$

is complete in $\bar{R}^m$, because it is easily seen that

$$C_{\bar{a}}=U_{[a]}.$$

Hence, Theorem 5.3 and 4.8 shows that

$$\chi_{\bar{\pi}}\geqq\chi^\pi/\chi^\pi-1>1,$$

provided that all $\bar{\pi}(\xi/\bar{a})$ are continuous with respect to $\xi>0$.

Now, let $1<\chi_\pi\leqq\chi^\pi<+\infty$ and the modular $m$ be monotone complete,

and all $\bar{\pi}(\xi/\bar{a})$ $(\bar{a}\in\bar{R}^m)$ are continuous with respect to $\xi>0$. Then, $R$ is *conjugately similar* by Theorem 61.5 of NAKANO, [7], because, in this case, $m$ and its conjugate modular $\bar{m}$ are both normal and monotone complete. Namely, between $R$ and its modular conjugate space $\bar{R}^m$, there is a one-to-one correspondence $R\ni a\to a^{\bar{R}}\in R$ such that

$$m(a)+\bar{m}(a^{\bar{R}})=a^{\bar{R}}(a)\,,$$

where

$$a^{\bar{R}}(x)=\int_{[a]}\pi(1,a,\mathfrak{p})\left(\frac{x}{a}\,,\mathfrak{p}\right)m(d\mathfrak{p}a),$$

by Theorem 39.1 of NAKANO [7], where $\pi(\xi,a,\mathfrak{p})$ denotes the right-hand derivative of the modular spectrum $\omega(\xi,a,\mathfrak{p})$. In other words, we have the following theorem which is a generalization of that obtained by ORLICZ [15]:

*Theorem* 5.4. *If $m$ is monotone complete, $1<\chi_\pi\leqq\chi^\pi<+\infty$ and all $\bar{\pi}(\xi/\bar{a})$ $(\bar{a}\in\bar{R}^m)$ are continuous with respect to $\xi>0$, then, for any $\bar{a}\in\bar{R}^m$, we can find a element $a\in R$ such that*

$$\bar{a}(x)=\int_{[a]}\pi(1,a,\mathfrak{p})\left(\frac{x}{a}\,,\mathfrak{p}\right)m(a\mathfrak{p}a)$$

*for every $x\in R$.*

*Remark.* When $R$ is conjugately similar and $m$ is its modular, then we have

$$\pi(\xi/a)=(\xi a)^{\bar{R}}(a)\qquad(a\in R)$$

for almost all $\xi>0$. Because, we can easily see that

$$\lim_{\varepsilon\to 0}((\xi+\varepsilon)a)^{\bar{R}}(a)=(\xi a)^{\bar{R}}(a)$$

and

$$\lim_{\varepsilon\to 0}((\xi+\varepsilon)a)^{\bar{R}}(a)=\pi(\xi/a)$$

for almost all $\xi>0$.

## §6. On norms by modulars

In the modulared semi-ordered linear space, there defined two sorts of norms, that is, *the first norm*:

$$\|a\|=\inf_{\xi>0}\frac{1+m(\xi a)}{\xi}\,,$$

and *the second norm*:

$$\|a\| = \inf_{m(\xi a)\leqq 1} \frac{1}{|\xi|},$$

and we have

$$\|a\| \leqq \|a\| \leqq 2\|a\|$$

for every $a \in R$. These norms were introduced generally by NAKANO [7], and the first norm was defined in a classical form by ORLICZ [14]. Forms stated above were obtained by I. AMEMIYA, who has given the following theorem in a slightly different form:

*Theorem* 6.1. *We have*

$$\|a\|^{\chi_m} \geqq m(a) \quad \textit{and} \quad \|a\|^{\chi^m} \leqq m(a),$$

*if* $\|a\| \leqq 1$, *and*

$$\|a\|^{\chi^m} \geqq m(a) \quad \textit{and} \quad \|a\|^{\chi_m} \leqq m(a),$$

*if* $\|a\| \geqq 1$. *Moreover, we have*

$$\|a+b\|^{\chi_m} \leqq \|a\|^{\chi_m} + \|b\|^{\chi_m},$$
$$\|a+b\|^{\chi^m} \geqq \|a\|^{\chi^m} + \|b\|^{\chi^m},$$

*provided that* $|a| \frown |b| = 0$.

*Proof*. If $\|a\| \leqq 1$, then, putting $\chi^m = \alpha$, we have

$$m(a)/\|a\|^{\alpha} \geqq m\left(\frac{a}{\|a\|}\right),$$

so that $m(a) \geqq \|a\|^{\alpha}$. Similarly, we can prove that $\|a\| \geqq 1$ implies $\|a\|^{\chi^m} \geqq m(a)$. Since we have

$$\left\|\frac{a}{\|a+b\|}\right\|, \left\|\frac{b}{\|a+b\|}\right\| \leqq 1,$$

we can conclude that

$$\frac{\|a\|^{\alpha} + \|b\|^{\alpha}}{\|a+b\|^{\alpha}} \leqq m\left(\frac{a}{\|a+b\|}\right) + m\left(\frac{b}{\|a+b\|}\right) = m\left(\frac{a+b}{\|a+b\|}\right) = 1,$$

so that we have $\|a\|^{\alpha} + \|b\|^{\alpha} \geqq \|a+b\|^{\alpha}$. The other formulae can be proved quite similarly.

The later half of this theorem means that:
If the upper exponent $\chi^m$ is finite, then

$$a \frown b = 0, \quad \|a\| = \|b\| = 1, \quad \|\xi a + \eta b\| = 1, \quad \xi, \eta \geqq 0$$

implies $\xi^{\chi^m} + \eta^{\chi^m} \leqq 1$. If the lower exponent $\chi_m$ is strictly greater than one, then the above condition implies $\xi^{\chi_m} + \eta^{\chi_m} \geqq 1$.

Therefore we have:

*Theorem* 6.2. *If* $\chi^m < +\infty$, *then the second norm is uniformly monotone, and, if* $\chi_m > 1$, *then the second norm is uniformly flat.*

For the first norm, corresponding to Theorem 6.1, we have:

*Theorem* 6.3. *We have*

$$\|a\| \leqq p^{\frac{1}{p}} p'^{\frac{1}{p'}} |||a||| \qquad (a \in R),$$

*if* $p \leqq 2$, *and*

$$\|a\| \leqq q^{\frac{1}{q}} q'^{\frac{1}{q'}} |||a||| \qquad (a \in R),$$

*if* $q \leqq 2$, *and always we have*

$$\text{Min.}\left\{p^{\frac{1}{p}} p'^{\frac{1}{p'}},\ q^{\frac{1}{q}} q'^{\frac{1}{q'}}\right\} |||a||| \leqq \|a\| \leqq 2|||a|||,$$

*provided that*

$$p = \chi^m,\ q = \chi_m,\ p' = \chi_{\bar{m}},\ q' = \chi^{\bar{m}}$$

*are finite.*

***Proof.*** We can assume that $|||a||| = 1$ without loss of generality. By the definition of the first norm, we see easily that

$$\|a\| = \inf_{\epsilon > 0} \frac{1+m(\xi a)}{\xi} \leqq \inf_{\epsilon \geqq 1} \frac{1+m(\xi a)}{\xi} \leqq \inf_{\epsilon \geqq 1} \frac{1+\xi^p}{\xi} = p^{\frac{1}{p}} p'^{\frac{1}{p'}},$$

if $p \leqq p'$ namely $p \leqq 1$ by Theorem 4.1, and

$$\|a\| \leqq \inf_{0 \leqq \epsilon \leqq 1} \frac{1+m(\xi a)}{\xi} \leqq \inf_{0 \leqq \epsilon \leqq 1} \frac{1+\xi^q}{\xi} = q^{\frac{1}{q}} q'^{\frac{1}{q'}},$$

if $q \geqq q'$, namely $q \geqq 2$.

Since we have

$$\|a\| \leqq 1 + m(a),$$

and $|||a||| \leqq 1$ implies $m(a) \leqq 1$, it is obvious that $\|a\| \leqq 2$.

On the other hand, we have

$$\begin{aligned}
\|a\| &= \text{Min.}\left\{\inf_{\epsilon \geqq 1} \frac{1+m(\xi a)}{\xi},\ \inf_{0 \leqq \epsilon \leqq 1} \frac{1+m(\xi a)}{\xi}\right\} \\
&\geqq \text{Min.}\left\{\inf_{\epsilon \geqq 1} \frac{1+\xi^q}{\xi},\ \inf_{0 \leqq \epsilon \leqq 1} \frac{1+\xi^p}{\xi}\right\} \\
&\geqq \text{Min.}\left\{\inf_{\epsilon > 0} \frac{1+\xi^q}{\xi},\ \inf_{\epsilon > 0} \frac{1+\xi^p}{\xi}\right\} \\
&= \text{Min.}\left\{q^{\frac{1}{q}} q'^{\frac{1}{q'}},\ p^{\frac{1}{p}} p'^{\frac{1}{p'}}\right\},
\end{aligned}$$

as required.

From this theorem, when $\chi^m = \chi_m = p$, we have

$$\|a\| = p^{\frac{1}{p}} p'^{\frac{1}{p'}} \|\|a\| \qquad \text{if} \quad 1 < p < +\infty ,$$

and

$$\|a\| = \|\|a\| \qquad \text{if} \quad p = 1 \text{ or } +\infty .$$

In the spaces $L_p\,(p \geqq 1)$, if we put

$$m(a) = \frac{1}{p}\int_0^1 |a(t)|^p dt ,$$

then we have $\chi^m = \chi_m = p$, and hence, the above qualities are satisfied, where we can see easily that

$$\|\|a\| = \left(\frac{1}{p}\int_0^1 |a(t)|^p dt\right)^{\frac{1}{p}} .$$

Therefore, in this case, we have

$$(*) \qquad \|\|a+b\|\|^p = \|\|a\|\|^p + \|\|b\|\|^p$$

whenever $|a| \frown |b| = 0$.

Generally, *in any modulared semi-ordered linear space, having at least two linearly independent elements, when the second norm satisfies the above condition* $(*)$ *for a exponent* $\alpha$, *then we have*

$$\chi_m \leqq \alpha \leqq \chi^m .$$

Because, when $|a| \frown |b| = 0$, puttihg

$$\|\|a\| = \xi^{-\alpha'} , \quad \|\|b\| = \xi^{-\alpha'}$$

for $\alpha' = \dfrac{\alpha}{\alpha - 1}$, we have

$$\|\|\xi a\|\|^\alpha + \|\|\eta b\|\|^\alpha = \|\|\xi a + \eta b\|\|^\alpha \geqq (\|\|\xi a\|\|^p + \|\|\eta b\|\|^p)^{\frac{\alpha}{p}}$$

for $p = \chi^m$. Therefore we have

$$(\|\|a\| + \|\|b\|\|)^{\frac{p}{\alpha}} \geqq \|\|a\|\|^{\frac{p}{\alpha}} + \|\|b\|\|^{\frac{p}{\alpha}} ,$$

which implies that $p \geqq \alpha$. Similar argument shows that $\chi_m \leqq \alpha$, which establishes the above propositions.

Hence, if $a, b \geqq 0$ implies

$$\|\|a+b\|\| = \|\|a\|\| + \|\|b\|\| ,$$

then we have $\chi_m = 1$. And, if $|a| \frown |b| = 0$ implies

$$\|\|a+b\|\| = \text{Max.}\,\{\|\|a\|\|, \|\|b\|\|\} ,$$

then we have $\chi^m = +\infty$.

In these cases, we can not exchange $\chi_m$ and $\chi^m$ each other. As the example, we need only define on the plane as follows:

$$m((x,y)) = \begin{cases} +\infty & \text{if } y>1, \\ |x| & \text{if } 0 \leqq y \leqq 1. \end{cases}$$

Then we see easily that

$$\chi^m((0,1)) = \chi_m((0,1)) = +\infty,$$
$$\chi^m((0,1)) = \chi_m((0,1)) = 1,$$
$$\chi^m((1,1)) = +\infty, \quad \chi_m((1,1)) = 1,$$

and $\chi^m = +\infty$, $\chi_m = 1$, where

$$\|(x,y)\| = |x| + |y|,$$
$$|||(x,y)||| = \text{Max.}\{|x|, |y|\}.$$

For the first norm, we have always

$$\|a\| \leqq 1 + m(a) \qquad (a \in R).$$

Here the eauality does not always hold. For instance, if $\chi^m = \chi_m = 1$, then the left-hand side is strictly smaller than the right-hand side. Concerning about this, we have the following theorem:

*Theorem* 6.4. *If there is a number $\alpha > 0$ which satisfies the relation:*

$$(\sharp) \qquad \|\alpha a\| = 1 + m(\alpha a),$$

*then we have $\chi^m(a) > 1$. Conversely, if $\chi_m(a) > 1$ and $\pi(\xi/a)$ is continuous as a function of $\xi > 0$, then we can find a number $\alpha > 0$ for which the fomula ($\sharp$) is satisfied.*

*Proof*. The relation ($\sharp$) implies that

$$\frac{1 + m(\alpha a)}{\alpha} \leqq \frac{1 + m(\xi a)}{\xi}$$

for every $\xi > 0$ by the definition of the first norm. Therefore, we have, for any number $\varepsilon > 0$,

$$\alpha(1 + m((\alpha + \varepsilon)a)) \geqq (\alpha + \varepsilon)(1 + m(\alpha a)),$$

that is,

$$\alpha(m((\alpha + \varepsilon)a) - m(\alpha a)) \geqq \varepsilon(1 + m(\alpha a)).$$

Hence, we have, letting $\varepsilon$ tends to 0,

$$\pi(\alpha/a) \geqq \frac{1 + m(\alpha a)}{\alpha}.$$

Since

$$\pi(1/\alpha a) \leqq \chi^m(a) \cdot m(\alpha a),$$

we have

$$(\chi^m(a) - 1)\, m(\alpha a) \geqq 1,$$

which means that $\chi^m(a) > 1$.

Next, we will prove the later half of this theorem. As there is a number $\xi > 0$ such that $\pi(\xi/a) < +\infty$, we have

$$\lim_{\xi \to 0} \pi(\xi/a) = 0.$$

Therefore, there is a number $\xi_1 > 0$ such that

$$\pi(1/\xi_1 a) < 1 + m(\xi_1 a).$$

If $\pi(1/\xi a) < 1 + m(\xi a)$ for every $\xi > 0$, since

$$\chi_m(a) \cdot m(\xi a) \leqq \pi(1/\xi a),$$

we have

$$(\chi_m(a) - 1) \cdot m(\xi a) < 1.$$

As the convexity of $m(\xi a)$ implies that

$$\lim_{\xi \to \infty} m(\xi a) = +\infty,$$

we have $\chi_m(a) = 1$ from the above inequality. Hence, if $\chi_m(a) > 1$, then we have a number $\xi_2 > 0$ such that

$$\pi(1/\xi_2 a) \geqq 1 + m(\xi_2 a).$$

Therefore, the continuity of $\pi(\xi/a)$ implies the existence of such a number $\alpha > 0$ which satisfies the formula (♯).

*Remark.* In this theorem, we cannot replace $\chi^m(a)$ and $\chi_m(a)$ bv $\chi_m(a)$ and $\chi^m(a)$ respectively. For example, we need only consider the following modulars on the line:

$$m(\xi) = e^{\xi} - 1 \qquad (\xi > 0,$$

and

$$m(\xi) = \begin{cases} \xi^2 & \left(0 \leqq \xi \leqq \frac{1}{2}\right), \\ \xi - \frac{1}{2} & \left(\xi > \frac{1}{2}\right). \end{cases}$$

In the norm theory, the fundamental inequality is that of HÖLDER. But, in the theory of modulars, the inequality:

$$(*) \qquad |\bar{a}(a)| \leqq m(a)+\bar{m}(\bar{a}) \qquad (a\in R,\ \bar{a}\in \bar{R}^m)$$

plays the essential rôle in stead of the HÖLDER's inequality. As (*) is stronger than HÖLDER's inequality, we can state HÖLDER's inequality by words of modulars using (*). That is, under the same assumption as Theorem 6.3, we have

$$|\bar{a}(a)| \leqq \begin{cases} p^{\frac{1}{p}} p'^{\frac{1}{p'}} m(a)^{\frac{1}{p}} \bar{m}(\bar{a})^{\frac{1}{p'}} & \text{if } p\cdot m(a)\leqq p'\bar{m}(\bar{a}), \\ q^{\frac{1}{q}} q'^{\frac{1}{q'}} m(a)^{\frac{1}{q}} \bar{m}(\bar{a})^{\frac{1}{q'}} & \text{if } q'\bar{m}(\bar{a})\leqq q m(a)\,. \end{cases}$$

For the proof, we need only apply the operation of exponents to the formula:

$$|\bar{a}(a)| \leqq m(\xi a)+\bar{m}\left(\frac{1}{\xi}a\right) \qquad (\xi>0)\,,$$

and calculate the minimum value of the right-hand side.

## §7. On bounded modulars

A modular $m$ on $R$ is called *upper bounded*, if there exist numbers $\alpha, \gamma>1$ such that

$$m(\alpha x)\leqq \gamma\cdot m(x)$$

for every element $x\in R$, and $m$ is called *lower bounded*, if there exist numbers $\gamma>\alpha>1$ such that

$$m(\alpha x)\geqq \gamma m(x)$$

for every $x\in R$.

If $m$ is upper bounded, then the conjagate modular $\bar{m}$ of $m$ is lower bounded, and if $m$ is lower bounded, then $\bar{m}$ is upper bounded. An upper bounded modular is uniformly simple and uniformly finite, and the lower bounded modular is uniformly monotone and uniformly increasing.

If a modular is upper or lower bounded, then every element is upper or lower bounded respectively. But the connverse is not always true. Practically, there is a non-upper bounded modular which has upper bounded elements. In order that a modular be upper or lower bounded, it is necessary that the appearing factors $\alpha$ and $\gamma$ are uniquely determined for every element. Thus the properties of boundedness

of modulars are, so to speak, "uniform" properties.

Now, we are going to compute the degree of the uniformity by its exponents.

*Theorem* 7.1. *A modular $m$ is upper bounded, if and only if $\chi^m$ is finite, and then we have*

$$\text{Min.}\{1, \xi^{\chi^m}\} \leqq \omega(\xi/a) \leqq \text{Max.}\{1, \xi^{\chi^m}\}.$$

*If, moreover, $\chi^\pi$ is finite, then we have*

$$\omega(\alpha/a) \geqq \alpha^{\chi^\pi}/[(\chi^\pi - 1)\beta^{\chi^\pi} + 1],$$

*provided that* $0 \leqq \alpha \leqq \beta < 1$ *ann* $0 \neq a \in R$.

*Proof*. If $m$ is upper bounded, there exists a number $\gamma \geqq 2$ such that

$$m(2x) \leqq \gamma m(x)$$

for every $x \in R$. By the same argument as Theorem 3.2, we have

$$\chi^m = \sup_{x \in R} \chi^m(x) \leqq \gamma - 1,$$

that is, $\chi^m$ is finite. Conversely, if $\chi^m = p$ is finite, then we have

$$m(2x) \leqq 2^p m(x)$$

for every $x \in R$, that is, $m$ is upper bounded, because $2^p \geqq 2$.

When $\chi^m = p$ is finite, it is obvious by the definition that

$$m(\xi a) \geqq \xi^p m(a) \qquad \text{if} \quad 0 \leqq \xi \leqq 1,$$

and

$$m(\xi a) \leqq \xi^p m(a) \qquad \text{if} \quad \xi \geqq 1.$$

Since $m$ is upper bounded, we have

$$\omega(\xi/a) = m(\xi a)$$

if $m(a) = 1$. Therefore we have

$$\text{Min.}\{1, \xi^{\chi^m}\} \leqq \omega(\xi/a) \leqq \text{Max.}\{1, \xi^{\chi^m}\}.$$

Let $\chi^\pi = p$ is finite and $0 < \beta < 1$, $m(a) = 1$. Then, for $0 < \alpha \leqq \beta$, we have

$$\omega(\alpha/a) = \int_0^\alpha \pi(\xi/a)\, d\xi$$

$$\geqq \int_0^\alpha \frac{\pi(\beta/a)}{\beta^{p-1}} \cdot \xi^{p-1} d\xi = \frac{\pi(\beta a/)}{\beta^{p-1}} \cdot \frac{\alpha^p}{p}.$$

On the other hand, we can see easily that

$$1 = \int_0^1 \pi(\xi/a)\,d\xi = \left(\int_0^\beta + \int_\beta^1\right)\pi(\xi/a)\,d\xi$$
$$\leqq \beta\pi(\beta/a) + \int_\beta^1 \frac{\pi(\beta/a)}{\beta^p}\cdot\xi^{p-1}d\xi = \frac{(p-1)\beta^p+1}{p\cdot\beta^{p-1}}\cdot\pi(\beta/a)\,.$$

Therefore, combining with the above formula, we have

$$\omega(\alpha/a) \geqq \frac{\alpha^p}{(p-1)\beta^p+1}\,,$$

as required.

*Theorem* 7.2. *A modular $m$ is lower bounded, if and only if $\chi_m$ is strictly greater than one. Then we have*

$$\omega(\xi/a) \leqq \xi^{\chi_m} \qquad \textit{if} \quad 0 \leqq \xi \leqq 1\,,$$

*and*

$$\omega(\xi/a) \geqq \xi^{\chi_m} \qquad \textit{if} \quad \xi \geqq 1\,.$$

*If, moreover, $\chi_\pi$ is strictly greater than one, then we have*

$$\omega(\xi/a) \leqq \xi^{\chi_\pi}/(1-\xi^{\chi_\pi}) \qquad \textit{if} \quad 0 \leqq \xi \leqq 1\,.$$

*Proof*. If $m$ is lower bounded, then the conjugate modular is upper bounded. Therefore $\chi^{\bar{m}}$ is finite by the previous theorem, so that $\chi_m$ is strictly greater than one, because

$$(\chi_m - 1)(\chi^{\bar{m}} - 1) = 1\,,$$

by Theorem 5.1.

We can prove the first half similarly as the previous theorem.

When $q = \chi_\pi > 1$, $0 \leqq \alpha \leqq 1$ and $0 \neq a \in R$, we have

$$\omega(\alpha/a) = \int_0^\alpha \tau(\xi/a)\,d\xi$$
$$\leqq \int_0^\alpha \frac{\tau(\alpha/a)}{\alpha^{q-1}}\cdot\xi^{q-1}d\xi = \frac{\tau(\alpha/a)}{q\cdot\alpha^{q-1}}\cdot\alpha^q\,.$$

On the other hand, we have

$$1 \geqq \omega(1/a) = \int_0^1 \tau(\xi/a)d\xi \geqq \int_\alpha^1 \tau(\xi/a)\,d\xi$$
$$\geqq \frac{\tau(\alpha/a)}{\alpha^{q-1}}\int_\alpha^1 \xi^{q-1}d\xi = \tau(\alpha/a)\cdot\frac{1-\alpha^q}{q\cdot\alpha^{q-1}}\,.$$

Therefore, combining with the above formula, we have

$$\omega(\alpha/a) \leqq \frac{\alpha^q}{1-\alpha^q}\,,$$

as required.

From these theorems, it is not difficult to see that:

*Theorem* 7.3. *For a simple element $s$ of $R$, we have*

$$\xi^{\chi_m} \leqq \omega(\xi, s, \mathfrak{p}) \leqq \xi^{\chi^m} \qquad \text{if} \quad \xi \geqq 1,$$

*and*

$$\xi^{\chi^m} \leqq \omega(\xi, s, \mathfrak{p}) \leqq \xi^{\chi_m} \qquad \text{if} \quad 0 \leqq \xi \leqq 1,$$

*and, moreover we have,*

$$\frac{\xi^{\chi^\pi}}{(\chi^\pi - 1)\xi^{\chi^\pi} + 1} \leqq \omega(\xi, s, \mathfrak{p}) \leqq \frac{\xi^{\chi_\pi}}{1 - \xi^{\chi_\pi}} \qquad \text{if} \quad 0 \leqq \xi \leqq 1,$$

*provided that the appearing exponents are finite,*

We will write here some more results about uniform convexity of modulars. A modular $m$ on $R$ is said to be *uniformly convex*, if for any $\gamma, \varepsilon > 0$ there exists $\delta > 0$ such that

$$0 \leqq \alpha < \beta \leqq \gamma, \qquad \beta - \alpha \geqq \varepsilon$$

implies

$$\frac{\omega(\alpha/a) + \omega(\beta/a)}{2} \geqq \omega\left(\frac{\alpha+\beta}{2}/a\right) + \delta$$

for all $a \in R$. A modular $m$ on $R$ is said to be *uniformly finite*, if

$$\sup_{x \in R} \omega(\xi/x) < +\infty$$

for any number $\xi > 0$. If a modular $m$ is uniformly convex and uniformly finite, then the second norm by $m$ and the first norm by $\overline{m}$ are uniformly convex in Clarkson's sense. A modular $m$ is said to be *uniformly even*, if, for any $\gamma, \varepsilon > 0$ there exists $\delta > 0$ such that

$$\gamma \geqq \alpha \geqq |\beta|, \qquad \alpha - \beta \leqq \delta$$

implies

$$\frac{\omega(\alpha/a) + \omega(\beta/a)}{2} \leqq \omega\left(\frac{\alpha+\beta}{2}/a\right) + (\alpha - \beta)\varepsilon$$

for all $0 \neq a \in R$. This property characterises the conjugate modulars of uniformly convex modulars. These were defined and proved in Nakano [7] §50 and §51.

Now, new sufficient conditions for a modular should be uniformly convex or uniformly even are obtained:

*Theorem* 7.4. *If a modular $m$ is uniformly simple:*

$$\inf_{0 \neq x \in R} \omega(\xi/x) > 0 \qquad (\xi > 0),$$

*and* $\chi_\pi \geqq 2$, *then the modular is uniformly convex.*

*Proof.* By Theorem 3.6, we have

$$\frac{\omega(\alpha/a)+\omega(\beta/a)}{2} \geqq \omega\left(\frac{\alpha+\beta}{2}/a\right)+\omega\left(\frac{\alpha-\beta}{2}/a\right)$$

for $\alpha > \beta > 0$ and $0 \neq a \in R$, and uniform simpleness implies

$$\inf_{0 \neq x \in R} \omega\left(\frac{\alpha-\beta}{2}/a\right) > 0,$$

so that the proof is established.

Since the modular is uniformly simple if $\chi^m < +\infty$, this theorem shows that, a modular is uniformly convex if $2 < \chi_\pi \leqq \chi^m < +\infty$.

*Theorem* 7.5. *If a modular m is uniformly monotone:*

$$\lim_{\xi \to 0} \frac{1}{\xi} \sup_{0 \neq x \in R} \omega(\xi/x) = 0,$$

*and* $\chi^\pi \leqq 2$, *then the modular is uniformly even.*

*Proof.* By Theorem 3.6, we have

$$\frac{\omega(\alpha/a)+\omega(\beta/a)}{2} \leqq \omega\left(\frac{\alpha+\beta}{2}/a\right)+\omega\left(\frac{\alpha-\beta}{2}/a\right)$$

for $\alpha > \beta > 0$ and $0 \neq a \in R$, and uniform monotonity implies the proof.

Since the modular is uniformly monotone if $\chi_m > 1$, this theorem shows that a modular is uniformly even if $1 < \chi_m \leqq \chi^\pi \leqq 2$.

*Remark.* Theorem 7.4 and 7.5 are improvements of theorems given in NAKANO [7] §52 and §53, which assert that a modular is uniformly convex if it is uniformly simple and of convex type, and uniformly even, if it is uniformly monotone and of concave type. A modular $m$ is said to be *of convex type*, if its derivative $\pi(\xi/a)$ is a convex function of $\xi > 0$ and $\lim_{\xi \to 0} \pi(\xi/a) = 0$ for every $a \in R$, and *of concave type* if $\pi(\xi/a)$ is a concave function of $\xi > 0$ for every $a \in R$. It is obvious that, we have $\chi_\pi \geqq 2$ if the modular is of convex type, and $\chi^\pi \leqq 2$ if the modular is of concave type.

## §8. On modulars of unique spectra

Let $R$ be a modulared semi-ordered linear space and $m$ be its modular. Since the modular spectrum $\omega(\xi, s, \mathfrak{p})$ $(\mathfrak{p} \in U_{[s]})$ for a simple element $s$ is a convex function of $\xi > 0$ and $\omega(0, s, \mathfrak{p}) = 0$, we can define

its exponents quite similarly as that were defined in §3. We will denote such exponents as $\chi^m(s,\mathfrak{p})$, $\chi_m(s,\mathfrak{p})$ and $\chi^\pi(s,\mathfrak{p})$, $\chi_\pi(s,\mathfrak{p})$, and call them *spectral exponents by $m$ or by $\pi$* respectively.

Then, we have the following inequalities:

$$\varliminf_{[p]\to\mathfrak{p}} \chi^m([p]s) \geqq \chi^m(s,\mathfrak{p}) \geqq \chi_m(s,\mathfrak{p}) \geqq \varlimsup_{[p]\to\mathfrak{p}} \chi_m([p]s)$$

for every $\mathfrak{p}\in U_{[s]}$. Because, if there exists such a number $\alpha>0$ that

$$\chi^m(s,\mathfrak{p}) > \alpha > \varliminf_{[p]\to\mathfrak{p}} \chi^m([p]s)\ ,$$

the left-hand side inequality implies the existence of a projector $[p]$ such that

$$\chi^m([q]s) > \alpha \qquad (\mathfrak{p}\ni[q]\leqq[p])\ ,$$

and the right-hand side inequality means that, for any projector $[p]$ there exists $[q]\leqq[p]$ such that

$$\alpha > \chi^m([q]s)\ .$$

Hence, we have the contradiction. Another inequality can be proved similarly.

Concerning about the case when the equality holds in the above inequality, we have the following:

*Theorem* 8.1. *If the spectral upper exponent $\chi^m(s,\mathfrak{p})$ is a upper semi-continuous function of $\mathfrak{p}$ at a point $\mathfrak{p}_0\in U_{[s]}$, then we have*

$$\chi^m(s,\mathfrak{p}_0) = \lim_{[p]\to\mathfrak{p}_0} \chi^m([p]s)\ ,$$

*and if the spectral lower exponent $\chi_m(s,\mathfrak{p})$ is a lower iemi-continuous function of $\mathfrak{p}$ at a point $\mathfrak{p}_0\in U_{[s]}$, then we have*

$$\chi_m(s,\mathfrak{p}_0) = \lim_{[p]\to\mathfrak{p}_0} \chi_m([p]s)\ .$$

*Proof*. At first we will prove that

$$\sup_{\mathfrak{p}\in U_{[s]}} \chi^m(s,\mathfrak{p}) = \sup_{[p]\leqq[s]} \chi^m([p]s)\ .$$

Let $\alpha>0$ be an arbitrary number such that

$$\alpha \geqq \sup_{\mathfrak{p}\in U_{[s]}} \chi^m(s,\mathfrak{p})\ .$$

Then the function $\xi>0$:

$$\frac{\omega(\xi,s,\mathfrak{p})}{\xi^\alpha}$$

is decreasing for every $\mathfrak{p}\in U_{[p]}$, so that the integration gives that the function of $\xi>0$;

$$\frac{m(\xi[p]s)}{\xi^{\alpha}}$$

is decreasing for any $[p]\leqq[s]$. This means that

$$\alpha \geqq \sup_{[p]\leqq[s]} \chi^m([p]s) .$$

Hence we have

$$\sup_{\mathfrak{p}\in U_{[s]}} \chi^m(s,\mathfrak{p}) \geqq \sup_{[p]\leqq[s]} \chi^m([p]s) .$$

Another inequality can be proved similarly.

In this equality, since $[s]$ may be replaced by any neighbourhood of $\mathfrak{p}$, taking infimum of both sides, we have

$$\overline{\lim_{[p]\to\mathfrak{p}}}\ \chi^m([p]s) = \chi^m(s,\mathfrak{p}) ,$$

as $\chi^m(s,\mathfrak{p})$ is upper semi-continuous as a function of $\mathfrak{p}$. Hence there exists the limit:

$$\lim_{[p]\to\mathfrak{p}} \chi^m([p]s)$$

and it is equal to $\chi^m(s,\mathfrak{p})$. The later half of this theorem is proved quite similarly.

A modular $m$ is said to be *of unique spectra,* if for any simple element $s_1$ and $s_2$, we have

$$\omega(\xi,s_1,\mathfrak{p}) = \omega(\xi,s_2,\mathfrak{p})$$

for every $\mathfrak{p}\in U_{[s_1][s_2]}$. In this case, there exists a continuous function $\rho(\mathfrak{p})$ on $U_{[s]}$ such that

$$\omega(\xi,s,\mathfrak{p}) = \xi^{\rho(\mathfrak{p})}$$

for all $\mathfrak{p}\in U_{[s]}$ and $\xi>0$. Therefore, it is obvious that

$$\chi^m(s,\mathfrak{p}) = \chi_m(s,\mathfrak{p}) = \rho(\mathfrak{p}) \qquad (\mathfrak{p}\in U_{[s]})$$

for this continuous function $\rho(\mathfrak{p})$. Moreover we have the following theorem:

*Theorem* 8.2. *In order that a modular $m$ should be of unique spectra in $[s]R$ for a simple element $s$, it is necessary and sufficient that*

$$\overline{\lim_{[p]\to\mathfrak{p}}}\ \chi^m([p]s) = \underline{\lim_{[p]\to\mathfrak{p}}}\ \chi_m([p]s) \qquad (\mathfrak{p}\in U_{[s]}) .$$

*Proof*. Since the necessity is almost evident, we need only prove the converse. Putting

$$\rho(\mathfrak{p}) = \overline{\lim_{[p]\to\mathfrak{p}}}\, \chi^m([p]s) = \underline{\lim_{[p]\to\mathfrak{p}}}\, \chi_m([p]s) \qquad (\mathfrak{p}\in U_{[s]}),$$

for any number $\varepsilon>0$ there exists by the definition of the limit a projector $[p_0]$ such that

$$\rho(\mathfrak{p})+\varepsilon \geqq \chi^m([p]s)$$

and

$$\rho(\mathfrak{p})-\varepsilon \leqq \chi_m([p]s)$$

for every $[p]\leqq[p_0]$. Therefore we have

$$\chi^m(s,\mathfrak{p}) = \chi_m(s,\mathfrak{p}) = \rho(\mathfrak{p})$$

for every $\mathfrak{p}\in U_{[s]}$, namely,

$$\omega(\xi,s,\mathfrak{p}) = \xi^{\rho(\mathfrak{p})}$$

for $\mathfrak{p}\in U_{[s]}$, which means that $m$ is of unique spectra in $[s]R$.

The function $\rho(\mathfrak{p})$ can be extended uniquely and continuously to the proper space $\mathfrak{S}$ and the numbers:

$$\rho_u = \sup_{\mathfrak{p}\in\mathfrak{S}} \rho(\mathfrak{p}) \quad \text{and} \quad \rho_l = \inf_{\mathfrak{p}\in\mathfrak{S}} \rho(\mathfrak{p})$$

have been called *the upper and lower exponent of* $m$ respectively. It is obvious that, if $m$ is of unique spectra, we have

$$\chi^\pi = \chi^m = \rho_u \quad \text{and} \quad \chi_\pi = \chi_m = \rho_l .$$

Hence the following theorem is not but a paraphrase of Theorems 54.7–10 of Nakano [7]:

*Theorem* 8.3. *Let $m$ be of unique spectra. Then, in order that $\chi^m$ is finite, it is necessary and sufficient that $m$ is uniformly simple or that $m$ is uniformly finite. In order that $\chi_m$ is strictly greater than one, it is necessary and sufficient that $m$ is uniformly monotone or that $m$ is unifirmly increasing. In order that $m$ is uniformly convex or that $m$ is uniformly even, it is necessary and sufficient that*

$$1<\chi_m \leqq \chi^m < +\infty .$$

Let $p(t)$ be a measurable function on $0\leqq t\leqq 1$ and be $p(t)\geqq 1$. Then the set of measurable functions $a(t)$ such that

$$\int_0^1 |\alpha a(t)|^{p(t)} dt < +\infty$$

for some number $\alpha>0$ is a modulared semi-ordered linear space, taking its modular as

$$m(a)=\int_0^1 \frac{1}{p(t)}|a(t)|^{p(t)}dt\,.$$

This space is denoted by $L_{p(t)}$ and discussed first by NAKANO [9]. The above modular on this space $L_{p(t)}$ is of unique spectra, and

$$\chi^m=\sup_{0\leqq t\leqq 1} p(t) \quad \text{and} \quad \chi_m=\inf_{0\leqq t\leqq 1} p(t)\,.$$

The corresponding sequence space is the set of sequences $\{\xi_\nu\}$, for which

$$\sum_{\nu=1}^{\infty}|\alpha\xi_\nu|^{p_\nu}<+\infty$$

for some number $\alpha>0$. Hence the numbers $p_\nu\geqq 1$ $(\nu=1,2,\cdots)$ determines this space, which is denoted by $l(p_1,p_2\cdots)$. In the sequence space of this kind, a pathological phenomenon occurs, namely, we may have

$$l(p_1,p_2,\cdots)=l(q_1,q_2,\cdots)$$

for different sequences $\{p_\nu\}$ and $\{q_\nu\}$. The problem when these two spaces coincide is considered first by ORLICZ [13] and solved completely by NAKANO [11].

Before concluding this section, we will state a theorem concerning about the relation between a modular and its norms:

*Theorem* 8.4. *If $m$ is of unique spectra, $\chi^m<+\infty$ and*

$$\|[p]a\|=2\,|||[p]a|||$$

*for any $[p]\leqq[a]$, then we have*

$$m(\xi a)=\xi^2 m(a) \qquad (\xi>0)\,.$$

*Proof*. By Theorem 7.3, we have

$$\|a\|\leqq p^{\frac{1}{p}}p'^{\frac{1}{p'}}\,|||a|||$$

if $p=\chi^m(a)\leqq 2$, where $p'=\dfrac{p}{p-1}$. Therefore, if we have

$$\|[p]a\|=2\,|||[p]a|||$$

for a projector $[p]$, then $\chi^m([p]a)\leqq 2$ implies $\chi^m([p]a)=2$, because we have

$$1\leqq p^{\frac{1}{p}}p'^{\frac{1}{p'}}\leqq 2\,.$$

Hence, $\chi^m([p]a)$ must be greater than 2 for any projector $[p]\leqq[a]$.

Similarly, we can conclude that $\chi_m([p]a)$ is smaller than 2 for any $[p]\leqq[a]$. Therefore we have

$$\chi^m([p]a)\geqq 2\geqq\chi_m([p]a)$$

for any $[p]\leqq[a]$. Since $m$ is of unique spectra by the assumption, we have

$$\rho(\mathfrak{p})=\overline{\lim_{[p]\to\mathfrak{p}}}\,\chi^m([p]a)=\underline{\lim_{[p]\to\mathfrak{p}}}\,\chi_m([p]s)=2$$

by Theorem 8.2, because every element is simple. Therefore, as

$$\omega(\xi,a,\mathfrak{p})=\xi^2 \qquad (\xi>0)$$

for all $\mathfrak{p}\in U_{[a]}$, we have

$$m(\xi a)=\xi^2 m(a),$$

as required.

In this theorem, the assumption "$\chi^m<+\infty$" can not be dropped (but may be weakend). As the example, we need only consider the following modular on the line space:

$$m(x)=\begin{cases} x & (0\leqq x\leqq 1), \\ +\infty & (x>1), \end{cases}$$

where we have

$$\chi^m=+\infty, \qquad \chi_m=1.$$

## §9. On constant modulars

Let $R$ be a modulared semi-ordered linear space and $m$ be its modular. A element $c$ of $R$ is said to be *constant*, if it is simple and

$$\frac{m(\xi[p]c)}{m([p]c)}=\frac{m(\xi c)}{m(c)}$$

for every number $\xi>0$ and every projector $[p]$. When, for any element $a$ and $b$, there exists a constant element $c$ such that

$$[c]a, \qquad [c]b\neq 0,$$

then the modular is called *constant modular*.

In the theory of constant modulars, the following theorem which has given by Nakano [7] is fundamental:

*If a modular $m$ is constant, then for any constant elements $0\neq a$, $b\in R$,*

*there exists a number* $\alpha>0$ *such that* $\alpha b$ *is constant and*

$$\frac{m(\xi a)}{m(a)}=\frac{m(\xi\alpha b)}{m(\alpha b)}$$

*for all* $\xi>0$.

From this theorem, it is obvious that, for any constant element $a$ and $b$, we have

$$\chi^m(a)=\chi^m(b) \quad \text{and} \quad \chi_m(a)=\chi_m(b)\,,$$

and moreover,

$$\chi^\pi(a)=\chi^\pi(b) \quad \text{and} \quad \chi_\pi(a)=\chi_\pi(b)\,.$$

Since, if $c\in R$ is a constont element, $[p]c$ is also constant for every projector $[p]$, we have

$$\chi^m([p]c)=\chi^m(c) \quad \text{and} \quad \chi_m([p]c)=\chi_m(c)$$

and

$$\chi^\pi([p]c)=\chi^\pi(c) \quad \text{and} \quad \chi_\pi([p]c)=\chi_\pi(c)$$

for every projector $[p]$. Therefore, by Theorem 4.4, we have the following theorem:

*Theorem* 9.1. *If a modular* $m$ *is constant and not singular, then we have*

$$\chi^m=\chi^m(c) \quad \textit{and} \quad \chi_m=\chi_m(c)$$

*for every constant element* $c\in R$. *Moreover, if* $\pi(\xi/c)$ *is a continuous function of* $\xi>0$, *then we have*

$$\chi^\pi=\chi^\pi(c) \quad \textit{and} \quad \chi_\pi=\chi_\pi(c)\,.$$

Therefore, it is evident that, $m$ is upper bounded when and only when there exists an upper bounded constant element, and $m$ is lower bounded when and only when there exists a lower bounded constant element.

As an example of the modulared semi-ordered linear space with constant modular, the simplest one is the space $L_p\,(p\geqq 1)$. Here, we will introduce somewhat eccentric spaces of this kind.

The function:

$$\varphi(\xi)=e^\xi-\xi-1 \qquad (\xi>0)\,,$$

is a convex, increasing function of $\xi>0$ and $\varphi(0)=0$. Therefore, the space $L_{\exp.}$ of measurable functions $a(t)\,(0\leqq t\leqq 1)$ for which the integral:

$$\int_0^1 \varphi(\alpha|a(t)|)dt$$

is finite for some number $\alpha>0$ is a modulared semi-ordered linear space and its modular:

$$m(a)=\int_0^1 \varphi(|a(t)|)\,dt$$

is complete. Namely, this space is a Banach space. (But, it is difficult to obtain the concrete form of its norms.) As the constant function 1 belong to this space, this modular is constant. Hence we have

$$\chi^m=\chi^m(1), \qquad \chi_m=\chi_m(1)$$

and

$$\chi^\pi=\chi^\pi(1), \qquad \chi_\pi=\chi_\pi(1),$$

by Theorem 9.1. It is easily seen that

$$\chi^m=\chi^\pi=+\infty \quad \text{and} \quad \chi_m=\chi_\pi=2.$$

This modular is not finite and not uniformly convex, and the conjugate space is generated by the convex function:

$$\psi(\xi)=(1+\xi)\log(1+\xi)-\xi.$$

These spaces are so-called Orlicz spaces, which were concidered first by Orlicz [14], and then by Orlicz [15], Zygmund [19], Takahashi [16] Zaanen [18], Krasnoselski [4] and others. And Orlicz spaces may be considered as examples of modulared semi-ordered linear spaces with constant modulars.

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