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北海道大学の研究集積：HUSCAP
ON THE INTRINSIC DERIVATIVE IN THE NON-HOLONOMIC EWSURFACE

By

Yoshie KATSURADA

Introduction. H. V. CRAIG has shown that $M$-th order intrinsic derivative of an absolute tensor can be expressed in a form of contraction of extensors [1]. The principal purpose of the present paper is its generalization, that is, to find the structure of $M$-th order intrinsic derivative of a contravariant vector in the space called the non-holonomic exsurface which is defined by the $n$ independent Pfaffian equations

\begin{equation}
\sum_{a=0}^{M} \lambda_{a}^{a'}(x, x', \cdots, x^{(M)}) dx^{(a)t} = 0 \quad (a' = 1, \cdots, n)
\end{equation}

in a space $K^{(M)}$ of line-elements of $M$-th order. A non-holonomic space [2] and an exsurface [3] introduced by CRAIG are both special ones of this space.

In the present paper we use certain of ideas, notations and results given in the previous paper [4] without explanation. The present author wishes to express to Prof. A. KAWAGUCHI her sincere thanks for his criticisms.

§1. The non-holonomic exsurface and the intrinsic derivative. Let us consider the Pfaffian equations defined by (0.1), then under "non-holonomic exsurface" which is denoted by $N'_n$ we understand a set of the fields of $n$ mutually independent excovariant extensors $\lambda_{a}'$ associated with each expoint $x^{(a)t}$ of the space $K_n^{(M)}$.

We now define the displacement $ds^{a'}$ of an ideal point in $N'_n$ corresponding to the displacement $dx^{(a)t}$ of an expoint $x^{(a)t}$ in $K_n^{(M)}$ as follows:

\begin{equation}
(1.1) \quad ds^{a'} = \sum_{a=0}^{M} \lambda_{a}^{a'}(x, x', \cdots, x^{(M)}) dx^{(a)t}.
\end{equation}

Then for two infinitesimal displacements $d_1$ and $d_2$, there exists the relation

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(1) Numbers in brackets refer to the references at the end of the paper.

(2) Numbers in parentheses denote times of differentiation with respect to $t$. 
\[ d_\sigma s^a - d_\xi s^a = \omega_{\xi \epsilon \beta j}^a d_\sigma x^{(\alpha)\epsilon} d_\xi x^{(\beta)j}, \]

where

\[(1.2) \quad \omega_{\xi \epsilon \beta j}^a = \frac{\partial \lambda_{a}^{a_{\epsilon}^\prime}}{\partial x^{(\beta)j}} - \frac{\partial \lambda_{\beta j}^{a_{\epsilon}^\prime}}{\partial x^{(\alpha)i}}. \]

Especially, if \( \omega_{\xi \epsilon \beta j}^a \equiv 0 \), we have the relation

\[ \frac{\partial \lambda_{a}^{a_{\epsilon}^\prime}}{\partial x^{(\beta)j}} - \frac{\partial \lambda_{\beta j}^{a_{\epsilon}^\prime}}{\partial x^{(\alpha)i}} \equiv 0 \]

which shows complete integrability of the system of partial differential equations \( \frac{\partial s^a}{\partial x^{(\alpha)i}} = \lambda_{a}^{a_{\alpha}^\prime}(x, x', \cdots, x^{(M)}) \) whose solutions

\[(1.3) \quad s^a = f^a(x, x', \cdots, x^{(M)}, s_0^a) \]

may be considered as the equations of exsurface in an \( n \)-dimensional space. Consequently we can see the following theorem:

**Theorem 1.1.** If \( \omega_{\xi \epsilon \beta j}^a \equiv 0 \) and \( s^a \) are assumed to be rectangular cartesian coordinates of a point in an \( n \)-dimensional Euclidean space \( E \), then the non-holonomic exsurface becomes the exsurface defined by CRAIG ([3] p. 792).

The quantities \( v^\alpha \) determined by an excontravariant extensor \( v^\alpha \) of range \( R (R \leq M) \) in \( K_n^{(M)} \) such that

\[ \left( \begin{array}{l} M \\ R \end{array} \right) v^\alpha = \sum_{a=M-R}^{M} \left( \begin{array}{c} a \\ M-R \end{array} \right) \lambda_{a}^{a_{\alpha}^\prime} v^{a-M+R} \]

are called the vector components in \( N_n' \), corresponding to \( v^\alpha \). Similarly, the tensor components in \( N_n' \) corresponding to higher order excontravariant extensors can be defined. If \( v^{(\alpha)\epsilon} \), \( a = 0, 1, \cdots, R \), are the components of the extensor obtained by differentiating the components of a vector \( v^{\alpha} \) in \( K_n^{(M)} \) \( a \) times along a parameterized arc, then the vector components in \( N_n' \) of \( v^{(\alpha)\epsilon} \) are given by

\[ \left( \begin{array}{l} M \\ R \end{array} \right) v^\alpha = \sum_{a=M-R}^{M} \left( \begin{array}{c} a \\ M-R \end{array} \right) \lambda_{a}^{a_{\alpha}^\prime} v^{(a-M+K)\epsilon} \]

we call such the quantities \( v^\alpha_R \) \( R \)-th order intrinsic derivatives of the components \( v^\alpha \) in \( N_n' \) corresponding to \( v^{\epsilon} \), i.e.,

\[ v^\alpha \equiv \lambda_{M^\alpha}^a v^{\alpha} \]

and put

\[ \frac{\partial^R v^\alpha}{\partial t^R} \equiv \frac{\partial}{\partial t^R} v^\alpha. \]

Next we shall discuss such the intrinsic derivative of \( v^\alpha \).
In the case that $R=1$, we have

\begin{equation}
M \frac{\partial v^{a'}}{\partial t} \equiv v^{a'} = \sum_{a=M-1}^{M} \lambda_{a}^{a'} v^{(a-M+1)i} \\
= M \lambda_{M}^{a'} v^{(1)i} + \lambda_{M-1j}^{a} v^{j}'.
\end{equation}

If we consider the reciprocal contravariant vectors $\lambda_{i}^{a'}$ of the covariant vectors $\lambda_{Mi}^{a'}$, which satisfy the equation

$$\lambda_{Mi}^{a'} \lambda_{a'}^{i} = \delta_{a'}^{i} \quad (\delta_{a'}^{i}: \text{Kronecker delta})$$

under the assumption that $|\lambda_{M}^{a'}| \neq 0$. Then multiplying the last result of (1.4) by $\frac{1}{M} \lambda_{i}^{a'}$ and summing with respect to $a'$, we have

$$\frac{\partial v^{a'}}{\partial t} \lambda_{a}^{i} = v^{(1)i} + \frac{1}{M} \lambda_{a}^{i} \lambda_{a'}^{j} v^{k},$$
onputting

\begin{equation}
(1.5)
\Gamma_{i}^{j} = \frac{1}{M} \lambda_{a}^{i} \lambda_{M}^{j} \lambda_{a'}^{k},
\end{equation}

the above equation is rewritten in the form

$$\frac{\partial v^{a'}}{\partial t} \lambda_{i}^{a'} = v^{(1)i} + \Gamma_{i}^{j} v^{k}.$$  

Consequently, $K_{(M)}$ is the space with the affine connection $\Gamma_{i}^{j}$, $\Gamma_{i}^{j}$ being the function of $x^{(a)i}$, $a=0,1,\cdots,M$. For the structure of $\frac{\partial v^{a'}}{\partial t}$ referred to the components of the non-holonomic exsurface, we can state the following theorem.

**Theorem 1.2.** The relation

$$\frac{\partial v^{a'}}{\partial t} = \frac{dv^{a'}}{dt} + \Gamma_{i}^{a'} v^{i}.$$ 

holds good, where

$$\Gamma_{i}^{a'} = \Gamma_{i}^{a'} \lambda_{M}^{a'} \lambda_{b}^{k} - \sum_{a=0}^{M} \frac{\partial \lambda_{M}^{a'} \lambda_{b}^{k}}{\partial x^{(a)i}} x^{(a+1)i} \lambda_{b}^{k},$$

$$= \frac{1}{M} \lambda_{M-1}^{a'} \lambda_{b}^{k} - \lambda_{M}^{(a')i} \lambda_{b}^{k}.$$ 

**Proof.** Differentiating the expression $v^{i} = \lambda_{a}^{i} v^{a'}$ deduced from $v^{a'} = \lambda_{M}^{a'} v^{i}$ with respect to $t$, we have
$v^{(1)i} = \lambda_{a}^{i}, v^{(1)+1} y^{a}$,

let replace $v^{(1)i}$ in the last result of (1.4) with the right-hand member of the above equation, then it follows that

$$\frac{d v^{a}}{dt} = \lambda_{a}^{i}, y^{a} + \frac{\partial v^{a}}{\partial x'^{(a)}} \lambda_{a}^{i}, v + \lambda_{a}^{i} \frac{\partial v^{a}}{\partial x'^{(a)}} y^{a} + \frac{1}{M} \lambda_{a}^{i} \lambda_{a}^{j} y^{a} v^{b}$$

$$= v^{(1)} + \left(\frac{1}{M} \lambda_{a}^{i} \lambda_{a}^{j} y^{a} v + \lambda_{a}^{i} \frac{\partial v^{a}}{\partial x'^{(a)}} y^{a} \right) v^{b}$$

$$= v^{(1)} + \Gamma_{b}^{a} v^{b},$$

on making use of $\frac{\partial v^{a}}{\partial x'^{(a)}} y^{a} = -\lambda_{a}^{i}, \frac{\partial v^{a}}{\partial x'^{(a)}} y^{a} .

**Theorem 1.3.** There exists the relation

$$(1.6) \quad M \frac{d v^{a}}{dt} = \sum_{\beta = M-1}^{M} \left(\begin{array}{c} M \beta' \end{array}\right) N_{\beta b}^{a} v^{b(\beta' - M+1)},$$

where

$$N_{\beta b}^{a} = \sum_{\beta = M-1}^{M} \left(\begin{array}{c} M \beta' \end{array}\right) \lambda_{a}^{i} \lambda_{a}^{j} y^{a} v^{b(\beta' - M+1)} .$$

**Proof.** Calculating the right-hand member of (1.6), it follows that

$$\sum_{\beta = M-1}^{M} \left(\begin{array}{c} M \beta' \end{array}\right) N_{\beta b}^{a} v^{b(\beta' - M+1)} = N_{\beta b}^{a} v^{b(\beta' - M+1)} + MN_{\beta b}^{a} v^{b(\beta' - M+1)}$$

$$= \left\{ \lambda_{a}^{i} \lambda_{a}^{j} y^{a} v + \lambda_{a}^{i} \frac{\partial v^{a}}{\partial x'^{(a)}} y^{a} v^{b(1)} \right\}$$

$$= \left\{ v^{(1)} + \left(\lambda_{a}^{i} \lambda_{a}^{j} y^{a} v + \frac{1}{M} \lambda_{a}^{i} \lambda_{a}^{j} y^{a} v^{b(1)} \right) \right\}$$

$$= M \left\{ v^{(1)} + \left(\frac{1}{M} \lambda_{a}^{i} \lambda_{a}^{j} y^{a} v + \lambda_{a}^{i} \lambda_{a}^{j} y^{a} v^{b(1)} \right) \right\}$$

$$= M \left\{ v^{(1)} + \left(\lambda_{a}^{i} \lambda_{a}^{j} y^{a} v + \lambda_{a}^{i} \lambda_{a}^{j} y^{a} v^{b(1)} \right) \right\}$$

$$= M v^{(1)} + \Gamma_{b}^{a} v^{b} = M \frac{d v^{a}}{dt} .$$

In general, for the structure of the higher order intrinsic derivative of $v^{a}$, we have the

**Theorem 1.4.** It follows that

$$(M) \frac{d^{R} v^{a}}{dt^{R}} = \sum_{\beta = M-R}^{M} \left(\begin{array}{c} M \beta' \end{array}\right) N_{\beta b}^{a} v^{b(\beta' - M+R)} ,$$

where
On the Intrinsic Derivative in the Non-Holonomic Exsurface

\[ N_{b'}^{a'} = \sum_{\alpha=\beta}^{M} \left( \frac{\alpha}{\beta} \right) \lambda_{a'}^{j(a-\beta')} \]

Such the quantities \( N_{b'}^{a'} \) are called as intrinsic derivation coefficients of the non-holonomic exsurface.

**Proof.** By virtue of the definition, we can observe

\[ \left( \frac{M}{R} \right) \frac{\partial^{R} v^{a'}}{\partial t^{R}} = \sum_{\alpha}^{M} \left( \frac{\alpha}{M-R} \right) \lambda_{a'}^{j(a-M+R)} v^{(a-M+R)} \]

Differentiating the equation \( v^{j} = \lambda_{b'}^{j} v^{b'} (a-M+R) \) times by LEIBNITZ's rule, the following expression is obtained

\[ v^{j(a-M+R)} = \sum_{\alpha}^{M} \left( \frac{\alpha}{M-R} \right) \lambda_{a'}^{j(a-M+R-\beta)} v^{a'(\beta)} \]

Accordingly, it follows that

\[
\left( \frac{M}{R} \right) \frac{\partial^{R} v^{a'}}{\partial t^{R}} = \sum_{\alpha}^{M} \left( \frac{\alpha}{M-R} \right) \lambda_{a'}^{j(a-M+R)} v^{a'(\beta)}
\]

\[ = \sum_{\beta}^{R} \left( \frac{M-R+\beta}{M-R} \right) \lambda_{a'}^{j(a-M+R-\beta)} v^{a'(\beta)} \]

\[ = \sum_{\beta}^{R} (M-R+\beta) N_{b'}^{a'} v^{b'(\beta-M+K)} \]

(putting: \( \beta' = M-R+\beta \)).

**§ 2. Specialization.** The corresponding statements for the exsurface \( x^{a'} = f^{a'}(x, x', \cdots, x^{(M)}) \) are obtained by replacing \( \lambda_{a'}^{j} \) in the preceding statements with \( \frac{\partial f^{a'}}{\partial x^{(a')i}} \).

Let us consider a non-holonomic space \( \overline{N}_{n} \) which is given by \( n \) independent Pfaffian equations

\[ \lambda_{i}^{a'}(x) \frac{\partial x^{i}}{\partial t^{a'}} = 0 \quad (a' = 1, \ldots, n) \]

corresponding to an \( n \)-dimensional space \( K_{n} \) with an affine connection \( \Gamma_{j}^{i}(x) \) referred to the coordinate system \( x^{i} \), then if we take the quantities \( \lambda_{i}^{a'} T_{a'}^{b'} \) in place of \( \lambda_{a'}^{j} \) in the preceding statements, \( T_{a'}^{b'} \) being the higher order extensive derivative of \( \delta_{j}^{i} \) introduced by CRAIG, that is, \( D^{M} | \delta_{j}^{i}([1] p.338) \), the corresponding results for the higher order intrinsic derivatives of tensors in \( \overline{N}_{n} \) are observed. Also if \( \overline{N}_{n} \) coincides...
with the original $n$-dimensional space $K_n$, then in use of the quantities $T_{a^j}$ corresponding to $\lambda_{a^j}$, the above statements will give the results obtained by CRAIG [1].

(July 1952)

References.


