STRONGLY $\pi$-REGULAR RINGS

By

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Arens-Kaplansky [1] and Kaplansky [3] investigated, as generalizations of algebraic algebras and rings with minimum condition, following two types of rings: one is a $\pi$-regular ring, that is, a ring in which for every element $a$ there exists an element $x$ and a positive integer $n$ such that $a^nx=a^n$, and the other is a ring in which for every $a$ there exists an $x$ and an $n$ such that $a^{n+1}x=a^n$ — this we shall call a right $\pi$-regular ring. The present note is devoted mainly to study the latter more precisely. Apparently, the two notions of $\pi$-regularity and right $\pi$-regularity are different ones in general. However we can prove, among others, that under the assumption that a ring is of bounded index (of nilpotency) it is $\pi$-regular if and only if it is right $\pi$-regular. Moreover, we shall show, in this case, that we may find, for every $a$, an element $z$ such that $az=za$ and $a^nz=a^n$, where $n$ is the least upper bound of all indices of nilpotency in the ring. This is obviously a stronger result than a theorem of Kaplansky (2) as well as that of Gertshikoff (3), both of which are stated in section 8 of Kaplansky [3].

1. Strong regularity. Let $A$ be a ring. Let $a$ be an element of $A$. $a$ is called regular (in $A$) if there exists an element $x$ of $A$ such that $axa=a$, while $a$ is said to be right (or left) regular if there exists $x$ such that $a^2x=a$ (or $xa^2=a$). Further, we call $a$ strongly regular if it is both right regular and left regular.

Lemma 1. Let $a$ be a strongly regular element of $A$. Then there exists one and only one element $z$ such that $az=za$, $a^2z=za$ and $az^2=za$, and in particular $a$ is regular. For any element $x$ such that $a^2x=a$, $z$ coincides with $ax^2$. Moreover, $z$ commutes with every element which is commutative with $a$.

Proof. Let $x$, $y$ be two elements such that $a^2x=a$, $ya^2=a$. Then

\begin{equation}
ax = ya^2x = ya,
\end{equation}

so that
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(2) \[ ax^2 = yax = ya. \]

From (1) we have also

(3) \[ axa = ya^2 = a = ax. \]

Now put \( z = ax^2 \). It follows then from (1), (2), (3) that \( az = ayax = ax = ya = yaax = za, \) \( a^2 = axa = axa = a, \) \( ax^2 = yaz = yax = za \), as desired.

Suppose next \( z' \) be any element which satisfies the same equalities as \( z \): \( az' = z' a, \) \( a^2 = a, \) \( ax^2 = z' \). Then, by replacing \( x, y \) in (2) by \( z, z' \) respectively, we get \( z = ax^2 = za \), showing the uniqueness of \( z \).

For the proof of the last assertion, let \( c \) be any element such that \( ac = ca \). Then we have first \( zac = zca = zca^2 = za'c = az = caz \), i.e., \( c \) commutes with \( za = az \). It follows from this now \( zc = z'ac = zca = acz = cz = az \), and this completes our proof.

Lemma 2. Let \( a \) be a right regular element of \( A \) and let \( ax = a \). Then, for any positive integer \( n \), we have

\[
(a-ax^n a^n)^r = \begin{cases} a^r - ax^{n-r+1} a^n & r = 1, 2, \ldots, n, \\ 0 & r = n+1 \end{cases}
\]

Proof. Since the assertion is valid for \( r = 1 \), we may proceed by induction on \( r \) (for fixed \( n \)). Suppose \( r \leq n \) and our lemma holds for \( r \):

\[
(a-ax^n a^n)^r = a^r - ax^{n-r+1} a^n.
\]

Right-multiplying by \( a-ax^n a^n \) and using the relation \( a^{n+1} x^n = a \), which follows immediately from \( ax = a \), we have \( (a-ax^n a^n)^{r+1} = a^{r+1} - ax^{n-r} a^n - ax^{n-r} a^n + ax^{n-r+1} a^n x^n a^n = a^{r+1} - ax^{n-r} a^n \). But when \( r < n \) \( a^{r+1} x^n = a^{r+1} x^{n-r} = a^{r+1} x^{n-r} \), while when \( r = n \) \( a^{r+1} x^n = a^{r+1} x^{n+1} a^n = a \). This completes our induction.

Now \( A \) is called a ring of bounded index if indices of nilpotency of all nilpotent elements of \( A \) are bounded; and, in this case, the least upper bound of all indices of nilpotency is called the index of \( A \). (Cf. Jacobson [2], Kaplansky [3].) We can now prove the fundamental

Theorem 1. Let \( A \) be a ring of bounded index. Then every right regular element of \( A \) is (left whence) strongly regular.

Proof. Let \( n \) be the index of \( A \). Let \( A \) be any right regular element of \( A \): \( ax = a. \) Then, since \( (a-ax^n a^n)^{n+1} = 0 \) by Lemma 2, we must have \( (a-ax^n a^n)^n = 0 \). On the other hand, \( (a-ax^n a^n)^n = a^n - ax^n a^n \) by the same lemma, and we obtain \( a^n - ax^n a^n = 0 \). Apply furthermore Lemma 2 to \( n+1 \) instead of \( n \). Then \( (a-ax^n a^n+1)^{n+1} = a^{n+1} - axa^{n+1} = a (a^n - axa^n) \)
= 0, and so \((a - ax^n + a^{n+1})^n = 0\). But \((a - ax^n + a^{n+1})^n = a^n - ax^n a^{n+1}\) again by Lemma 2. Hence it follows \(a^n = ax^n a^{n+1}\). Right-multiply now by \(ax^n\) and make use of the relation \(a^{n+1}x^n = a\). Then we find finally \(a = ax^n a^n\), which shows the left regularity of \(a\).

In connection with the preceding theorem, we want to add the following theorem, although we shall not need it later:

**Theorem 2.** Let \(a\) be a right regular element of \(A\). Then \(a\) is strongly regular if and only if \(r(a^2) = r(a)\), where \(r(\cdot)\) denotes the set of all right annihilators.

**Proof.** The “only if” part is easy to see. So we have only to prove the “if” part. The right regularity of \(a\) implies \(a' A = aA\). The mapping \(u \rightarrow au (u \in aA)\) gives therefore an operator-homomorphism of the right ideal \(aA\) onto itself. Moreover, this is an isomorphism because the kernel is zero by the assumption \(r(a') = r(a)\). Let \(\varphi\) be the inverse mapping of it; \(\varphi\) is also an operator-isomorphism of \(aA\) onto itself. Since \(a \in (a^2 A =) aA\) we have in particular \(qa^2 = a\). From this it follows \((qa)a^2 = q(a^2) = qa^2 = a\), showing the left regularity of \(a\).

**Remark.** Von Neumann called \(A\) a regular ring if every element of \(A\) is regular, while Arens-Kaplansky [1] defined \(A\) to be a strongly regular ring when every element is right regular. However, it was shown in above paper that if \(A\) is strongly regular then every element of \(A\) is indeed strongly regular; this follows also from our Theorem 1 directly, since a strongly regular ring \(A\) has evidently no non-zero nilpotent element. This fact justifies our definition of strong regularity for elements.

2. Strong \(\pi\)-regularity. Let us call an element \(a\) of \(A\) \(\pi\)-regular, right \(\pi\)-regular, or left \(\pi\)-regular if a suitable power of \(a\) is regular, right regular, or left regular respectively. Furthermore we call \(a\) a strongly \(\pi\)-regular if it is both right \(\pi\)-regular and left \(\pi\)-regular. Now it can readily be seen that a power \(a^n\) of \(a\) is right (or left) regular if and only if there exists an element \(x\) such that \(a^{n+1} x = a^n\) (or \(xa^{n+1} = a^n\)). On the other hand, we have

**Lemma 3.** Let \(x, y\) satisfy \(a^{n+1} x = a^n, ya^{m+1} = a^m\) for some \(n, m\). Then they satisfy \(a^{m+1} x = a^m, ya^{n+1} = a^n\) too.

**Proof.** When \(m \geq n\) \(a^{n+1} x = a^m\) follows immediately from \(a^{n+1} x = a^n\). Suppose now \(m < n\). Then \(a^m = ya^{m+1}\) implies \(a^m (= y^m a^{m+2} = \ldots) = y^{n-m} a^n\), and so we obtain \(a^{m+1} x = y^{n-m} a^{n+1} x = y^{n-m} a^n = a^m\). Similarly, we can verify the validity of \(ya^{n+1} = a^n\).
Now we prove

**Theorem 3.** Let $a$ be a strongly $\pi$-regular element of $A$. Suppose that $a^n$ is right regular. Then $a^n$ is in fact strongly regular, and moreover there exists an element $z$ such that $az = za$ and $a^{n+1}z = a^n$.

**Proof.** That $a^n$ is strongly regular is an immediate consequence of Lemma 3. Now from Lemma 1 it follows that there exists an element $z$ such that $a^n z = a^n$ and $z$ commutes with every element which is commutative with $a^n$; however the latter condition implies, since $a$ is commutative with $a^n$, that $az = za$. Denoting $a^{n+1}z$ again by $z$, $z$ is evidently the desired element.

**Corollary.** Strongly $\pi$-regular element is $\pi$-regular.

Now we define the **index** of a strongly $\pi$-regular element $a$ as the least integer $n$ such that $a^n$ is right regular. By Lemma 3, the index $n$ is characterized also as the least integer such that $a^n$ is left regular. It is to be noted further that every nilpotent element is strongly $\pi$-regular and its index of nilpotency coincides with the index in the sense defined above, as can be seen quite easily. Furthermore we have

**Lemma 4.** Let $a$ be a strongly $\pi$-regular element of index $n$, and $z$ an element such that $az = za$ and $a^{n+1}z = a^n$ (as in Theorem 3). Then $a-a^z$ is a nilpotent element of index $n$.

**Proof.** Since $az = za$ we have the following binomial expansion:

$$(a-a^z)^n = a^n - \binom{n}{1}a^{n+1}z + \binom{n}{2}a^{n+2}z^2 - \cdots + (-1)^na^nz^n.$$  

But $a^n = a^{n+1}z$ implies $a^n = a^n z^2 - \cdots = a^{n+1}z$. Hence we get

$$(a-a^z)^n = a^n - \binom{n}{1}a^n + \binom{n}{2}a^n - \cdots + (-1)^na^n = (a-a)a^n = 0.$$  

On the other hand, $(a-a^z)^{n-1}$ is, again by a binomial expansion, say, expressible in a form $a^{n-1} - a^n x$ with some $x$; but this is certainly not zero because $a$ is of index $n$. Thus, the index of $a-a^z$ is exactly $n$.

We now obtain from Theorems 1, 3 and Lemma 4 immediately the following

**Theorem 4.** Let $A$ be a ring of bounded index (of nilpotency). Then every right $\pi$-regular element of $A$ is strongly $\pi$-regular and its index does not exceed the index of $A$.

Above results show us in fact the appropriateness of our definition of index for strongly $\pi$-regular elements. This is strengthened further by the following
Remark. Suppose that $A$ is (not necessarily finite dimensional) algebra over a field $K$. Let $a$ be an algebraic element of $A$, and $\mu(\lambda)$ the minimum polynomial of $a$ (without constant term). Jacobson [2] defined the index of $a$ as the largest integer $r$ such that $\lambda^r$ divides $\mu(\lambda)$. Now we want to show that $a$ is then strongly $\pi$-regular and (the Jacobson index) $r$ coincides with the index in our sense. For the proof, we may assume, since $\lambda^r$ divides exactly $\mu(\lambda)$, that $\mu(\lambda)$ is of the form $\lambda^r + a_1\lambda^{r+1} + a_2\lambda^{r+2} + \cdots$ (with $a_i$, $a_2$, $\cdots$ in $K$). It follows then $a_i\lambda\mu(\lambda) = a_i\lambda^{r+1} + a_i^2\lambda^{r+2} + \cdots$, and so we have $\mu(\lambda) - a_i\lambda\mu(\lambda) = \lambda^r + (a_2 - a_1)\lambda^{r+2} + \cdots = \lambda^r - \lambda^{r+1}\nu(\lambda)$, where $\nu(\lambda) = (a_2 - a_1)\lambda + \cdots$ is also a polynomial. Since now $\mu(i)$ has a for a root, so does $\mu(\lambda) - a_i\lambda\mu(\lambda)$ too, i.e., we have $\lambda^r = a^{r+1}\nu(\lambda)$, which shows the strong $\pi$-regularity of $a$. Let $n$ be the index of $a$ (as strongly $\pi$-regular element). Then $n \leq r$, and moreover we have from Lemma 3 that $a^n = a^{r+1}\nu(\lambda)$, that is, $a$ is a root of the polynomial $\lambda^n - \lambda^{n+1}\nu(\lambda)$. Since $\mu(\lambda)$ is the minimum polynomial of $a$, the latter must be divisible by $\mu(\lambda)$, and this implies in particular that $n \geq r$, proving our assertion.

Now we say that a ring $A$ is $\pi$-regular, right $\pi$-regular, left $\pi$-regular, or strongly $\pi$-regular if so is every element of $A$ respectively. (Cf. Kaplansky [3].) Evidently $A$ is strongly $\pi$-regular if and only if it is both right $\pi$-regular and left $\pi$-regular. Moreover, strong $\pi$-regularity of $A$ implies $\pi$-regularity of $A$, according to Corollary of Theorem 3. However, the converse is also true provided $A$ is assumed to be of bounded index. Namely, we have

**Theorem 5.** Under the assumption that $A$ is of bounded index, the following four conditions are equivalent to each other:

i) $A$ is $\pi$-regular,

ii) $A$ is right $\pi$-regular,

iii) $A$ is left $\pi$-regular,

iv) $A$ is strongly $\pi$-regular.

**Proof.** That ii) implies iv) is a direct consequence of Theorem 4. By right-left symmetry, iii) implies also iv). Therefore we have only to prove that ii) follows from i).

Suppose that $A$ is a $\pi$-regular ring of index $n$. Let $a$ be an element of $A$. Then $a^n$ is $\pi$-regular, that is, there exists an integer $n'(\geq 1)$ such that $a^{nn'}$ is regular. Put $r = nn'$. Then $r \geq n$ and there exists an element $x$ such that $a^r x a^r = a^n$. Write $e = a^r x$. Then $e$ is an idempotent and satisfies $eA = a^r A$. Similarly, the $\pi$-regularity of, say, $a^{r+1}$ implies the existence of an integer $s$ and an idempotent $f$ such that $s > r$ and $fA$
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$a^*A$. Since then $a^*A \supset a^*A$, $fA$ is necessarily a direct summand right subideal of $eA$. Hence we can construct, as is well-known, two orthogonal idempotents $f$, and $g$ such that $e=f_1+g$ and $f,A=fA(-a^*A)$. Now take any primitive ideal $P$ of $A$. By Kaplansky [3, Theorem 2.3] the residue class ring $\overline{A}=A/P$ is a (full) matrix ring over a division ring of degree at most $n$. Denote by $\overline{a}$ the residue class of $a$ modulo $P$, and consider the chain of right ideals $\overline{A} \supset \overline{a}A \supset \overline{a}^2A \supset \cdots$. It follows then, since the degree of the simple ring $\overline{A}$ is equal to the composition length for right ideals of $\overline{A}$, that $\overline{a}^n\overline{A}=\overline{a}^{n+1}\overline{A}=\cdots$, and we have in particular $\overline{a}^r\overline{A}=\overline{a}^s\overline{A}$. Write further by $\overline{e}, \overline{f}_1, \overline{g}$ the residue classes of $e, f_1, g$ modulo $P$ respectively. Then $\overline{e}\overline{A}=\overline{a}^r\overline{A}$, $\overline{f}_1\overline{A}=\overline{a}^s\overline{A}$ whence $\overline{e}\overline{A}=\overline{f}_1\overline{A}$. On the other hand, $\overline{e}=\overline{f}_1+\overline{g}$ and $\overline{f}_1, \overline{g}$ are orthogonal idempotents; hence $\overline{e}\overline{A}$ is the direct sum of $\overline{f}_1\overline{A}$ and $\overline{g}\overline{A}$: $\overline{e}\overline{A}=\overline{f}_1\overline{A} + \overline{g}\overline{A}$. This implies evidently that $\overline{g}\overline{A}=0$, i.e., $g=0$ or $g \in P$. This is the case for every primitive ideal $P$, and so $g$ must lie in the intersection of all $P$'s i.e. the (Jacobson) radical of $A$. If we observe however that 0 is the only quasi-regular idempotent, it follows indeed $g=0$, and this shows that $a^rA(-eA=f_1A)=a^*A$ whence $a^rA=a^{r+1}A$. The latter equality implies, since $a^r=a^ra^r$ is in $a^rA$, the right $\pi$-regularity of $a$. Thus, the proof of our theorem is concluded.

Remark. The radical of a $\pi$-regular ring as well as that of a right $\pi$-regular ring is always a nil-ideal, as was shown in Kaplansky [3, section 2] and Arens-Kaplansky [1, Theorem 3.1]; the assumption in the latter that (the right $\pi$-regular ring) $A$ is of bounded index being superfluous for proving our assertion.

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Bibliography