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A NOTE ON COVERING SPACES

By

Y. KURATA and M. KATO

Chevalley proved in his book the following proposition, which is effectively used in showing the uniqueness of simply connected covering space:

**Proposition.** Let $\mathfrak{B}$ be a simply connected space. Let $\mathfrak{B}$ be a space, and assume that $(\overline{\mathfrak{B}}, f)$ is a covering space of $\mathfrak{B}$. Let $\varphi$ be a continuous mapping of $\mathfrak{W}$ into $\mathfrak{B}$. Then there exists a continuous mapping $\tilde{\varphi}$ of $\mathfrak{W}$ into $\overline{\mathfrak{B}}$ such that $\varphi = f \cdot \tilde{\varphi}$. If $w_0$ is a point of $\mathfrak{W}$ and if $\tilde{p}_0$ is any point of $\overline{\mathfrak{B}}$ such that $f(\tilde{p}_0) = p_0 = \varphi(w_0)$, $\tilde{\varphi}$ may be constructed as to map $w_0$ upon $\tilde{p}_0$, and is then uniquely determined.

He remarked there that this can be deduced from his principle of monodromy in the case where the space $\mathfrak{B}$ is assumed to be normal. In the present note, we shall however propose a modified principle of monodromy so that the above proposition can be deduced from it without the assumption of normality. We have namely

**Theorem.** Let $\mathfrak{B}$ be a simply connected space. Assume that we have assigned to every $p \in \mathfrak{B}$ a non-empty set $E_p$. Let $\mathfrak{U}$ be an open basis of $\mathfrak{B}$ consisting of connected sets. Assume furthermore that we have assigned to every triple $(U; p, q)$, where $U \in \mathfrak{U}$ and $p, q \in U$, a mapping $\varphi_{p,q}^{(U)}$ of $E_p$ into $E_q$ in such a way that the following conditions are satisfied:

1) If $U \in \mathfrak{U}$ and $p, q, r \in U$, we have
   \[ \varphi_{p,r}^{(U)} = \varphi_{q,r}^{(U)} \cdot \varphi_{p}^{(\prod_{U})} ; \]

2) If $U, V \in \mathfrak{U}$ and $U \supset V$, we have
   \[ \varphi_{p,q}^{(U)} = \varphi_{p,q}^{(V)} \] for all $p, q \in V$;

3) Each $\varphi_{p,q}^{(U)}$ is the identity mapping.

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1) This note is written on the basis of a suggestion of Prof. Azumaya. The authors express here much thanks to him.

2) C. CHEVALLEY, Theory of Lie groups I, Princeton (1946).

3) Loc. cit., p. 50.


Then there exists a mapping \( \psi \) which associates with every \( p \in B \) an element \( \psi(p) \in E_p \) in such a way that \( \psi(q) = \varphi^{(U)}_{p,q} (\psi(p)) \) where \( U \in \mathfrak{W} \) and \( p, q \in U \). Moreover, if \( p \in B \) and \( e'_p \in E_p \) are given, then the mapping \( \psi \) is uniquely selected in such a way that \( \psi(p) = e'_p \).

The proof is more or less same as that of Chapter II, § VII, Theorem 2 in Chevalley's book. But we shall give it here for completeness.

Let \( \overline{B} \) be the union of all sets \( \{p\} \times E_p \), for \( p \in B \). For every \( U \in \mathfrak{W} \) and every point \( (p, e_p) \in \overline{B} \), where \( p \in U \), we define \( \overline{U}(p, e_p) \) as the set of all points \((q, \varphi^{(U)}_{p,q}(e_p))\), for \( q \in U \). Then one can see that 1) \( \overline{U}(p, e_p) \cap (q, e_q) \) if and only if \( \overline{U}(p, e_p) = \overline{U}(q, e_q) \), 2) if \( \overline{U}(p, e_p) \in (p, e'_p) \) then \( e_p = e'_p \), 3) if \( U \supset V \) and \( p \in V \) then \( \overline{U}(p, e_p) \supset \overline{V}(p, e_p) \); observe that each \( \varphi^{(U)}_{p,q} \) is in fact a one-to-one mapping of \( E_p \) onto \( E_q \). It follows from this easily that we can define a topology on \( \overline{B} \) by taking the totality of \( \overline{U}(p, e_p)'s \) as an open basis of \( \overline{B} \).

Let \( \overline{\omega} \) be the mapping of \( \overline{B} \) onto \( B \) which is defined by \( \overline{\omega}(p, e_p) = p \). Then each \( \overline{U}(p, e_p) \) is clearly mapped by \( \overline{\omega} \) topologically onto \( U \) and so is connected. Since \( \overline{\omega}^{-1}(U) = \bigcup \overline{U}(p, e_p) \) and \( \overline{U}(p, e_p)'s \) are mutually disjoint, it follows that \( U \) is evenly covered by \( \overline{B} \) with respect to \( \overline{\omega} \). Let \( \overline{\mathfrak{B}}_0 \) be the component of \((p_0, e^{p_0}_p)\) in \( \overline{B} \), and let \( \overline{\omega}_0 \) be the contraction of \( \overline{\omega} \) to \( \overline{B} \). Then \((\overline{\mathfrak{B}}_0, \overline{\omega}_0)\) is a covering space of \( B \). Since \( B \) is simply connected, \( \overline{\omega}_0 \) is a homeomorphism. Therefore, we can define a mapping \( \psi \) by

\[
\overline{\omega}_0^{-1}(p) = (p, \psi(p)) \quad \text{for} \quad p \in B.
\]

Let \( U \in \mathfrak{W} \) and \( p, q \in U \). Then the connected set \( \overline{U}(p, \psi(p)) \) is contained in \( \overline{\mathfrak{B}}_0 \) and hence we have \((q, \varphi^{(U)}_{p,q}(\psi(p))) \in \overline{\mathfrak{B}}_0 \), i.e., \( \psi(q) = \varphi^{(U)}_{p,q}(\psi(p)) \).

It remains only to prove the uniqueness of \( \psi \). Let \( \psi' \) be any mapping satisfying the conditions 1), 2), 3) and \( \psi'(p_0) = e^{p_0}_p \). Let \( A \) be the set of points \( p \) such that \( \psi'(p) = \psi(p) \). Assume that \( U \) is a set in \( \mathfrak{W} \) such that \( U \cap A \neq \emptyset \). Let \( p \) be any point of \( U \) and \( q \) a point of \( U \cap A \). Then we have \( \varphi^{(U)}_{p,q}(\psi'(p)) = \psi'(q) = \psi(q) = \varphi^{(U)}_{p,q}(\psi(p)) \). Since \( \varphi^{(U)}_{p,q} \) is one-to-one, it follows \( \psi'(p) = \psi(p) \), which implies \( U \subset A \). Therefore \( A \) is open and closed, whence \( A = B \), because \( A \) is not empty and \( B \) is connected.

Now we show that it is possible to derive the above mentioned proposition from our theorem. Let \( \mathfrak{W} \) be an open basis of \( B \) which consists of evenly covered sets and \( \mathfrak{W}' \) the totality of open connected sets \( W \) in \( B \) such that \( \psi(W) \subset U \) for some \( U \in \mathfrak{W} \), which forms obviously an open basis of \( B \). Since \( U \) is evenly covered and \( \psi(W) \) is connected
in this case, it follows that \( \varphi(W) \) is also evenly covered, that is, every component \( \overline{W}_\lambda \) of \( f^{-1}(\varphi(W)) \) mapped by \( f \) topologically onto \( \varphi(W) \). Now we set \( E_w = f^{-1}(\varphi(w)) \), for every point \( w \in \mathbb{R} \). Take two points \( w_1, w_2 \) in \( W \). Then each \( \overline{W}_\lambda \sim E_{w_i} \) consists of only one point \( \tilde{p}_\lambda^{(i)} \) and we have \( \bigcup_\lambda \{ \tilde{p}_\lambda^{(i)} \} = E_{w_i} \), for \( i = 1, 2 \). We define a mapping \( \varphi_{w_1, w_2}^{(W)} \) by

\[
\varphi_{w_1, w_2}^{(W)} (\tilde{p}_\lambda^{(i)}) = \tilde{p}_\lambda^{(2)}.
\]

Then \( \varphi_{w_1, w_2}^{(W)} \)'s satisfy obviously the conditions 1), 2) and 3) in the above theorem, and our assertion follows immediately.

We note further that Chapter II, §VII, Theorem 3 in Chevalley's book can also be deduced directly from our theorem in the similar way as in his proof.