A NOTE ON COVERING SPACES[1)]

By

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CHEVALLEY proved in his book[2)] the following proposition,[3)] which is effectively used in showing the uniqueness of simply connected covering space[4)]:

Proposition. *Let* $\mathfrak{W}$ *be a simply connected space. Let* $\mathfrak{V}$ *be a space, and assume that* $(\tilde{\mathfrak{V}}, f)$ *is a covering space of* $\mathfrak{V}$. *Let* φ *be a continuous mapping of* $\mathfrak{W}$ *into* $\mathfrak{V}$. *Then there exists a continuous mapping* $\tilde{\varphi}$ *of* $\mathfrak{W}$ *into* $\tilde{\mathfrak{V}}$ *such that* $\varphi = f \cdot \tilde{\varphi}$. *If* w_0 *is a point of* $\mathfrak{W}$ *and if* $\tilde{p}_0$ *is any point of* $\tilde{\mathfrak{V}}$ *such that* $f(\tilde{p}_0) = p_0 = \varphi(w_0)$, $\tilde{\varphi}$ *may be constructed as to map* w_0 *upon* $\tilde{p}_0$ *and is then uniquely determined.*

He remarked there that this can be deduced from his principle of monodromy[5)] in the case where the space $\mathfrak{W}$ is assumed to be normal. In the present note, we shall however propose a modified principle of monodromy so that the above proposition can be deduced from it without the assumption of normality. We have namely

Theorem. *Let* $\mathfrak{V}$ *be a simply connected space. Assume that we have assigned to every* $p \in \mathfrak{V}$ *a non empty set* E_p. *Let* $\mathfrak{U}$ *be an open basis of* $\mathfrak{V}$ *consisting of connected sets. Assume furthermore that we have assigned to every triple* $(U; p, q)$, *where* $U \in \mathfrak{U}$ *and* $p, q \in U$, *a mapping* $\varphi^{(U)}_{p,q}$ *of* E_p *into* E_q *in such a way that the following conditions are satisfied:*

1) *If* $U \in \mathfrak{U}$ *and* $p, q, r \in U$, *we have*

$$\varphi^{(U)}_{p,r} = \varphi^{(U)}_{q,r} \cdot \varphi^{(U)}_{p,q};$$

2) *If* $U, V \in \mathfrak{U}$ *and* $U \supset V$, *we have*

$$\varphi^{(U)}_{p,q} = \varphi^{(V)}_{p,q} \qquad \text{for all } p, q \in V;$$

3) *Each* $\varphi^{(U)}_{p,p}$ *is the identity mapping.*

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1) This note is written on the basis of a suggestion of Prof. AZUMAYA. The authors express here much thanks to him.

2) C. CHEVALLEY, *Theory of Lie groups* I, Princeton (1946).

3) Loc. cit., p. 50.

4) Loc. cit., p. 51.

5) Loc. cit., p. 51.

Then there exists a mapping ψ *which associates with every* $p \in \mathfrak{B}$ *an element* $\psi(p) \in E_p$ *in such a way that* $\psi(q) = \varphi^{(U)}_{p,q}(\psi(p))$ *where* $U \in \mathfrak{U}$ *and* $p, q \in U$. *Moreover, if* $p_0 \in \mathfrak{B}$ *and* $e^0_{p_0} \in E_{p_0}$ *are given, then the mapping* ψ *is uniquely selected in such a way that* $\psi(p_0) = e^0_{p_0}$.

The proof is more or less same as that of Chapter II, §VII, Theorem 2 in CHEVALLEY's book. But we shall give it here for completeness.

Let $\widetilde{\mathfrak{B}}$ be the union of all sets $\{p\} \times E_p$, for $p \in \mathfrak{B}$. For every $U \in \mathfrak{U}$ and every point $(p, e_p) \in \widetilde{\mathfrak{B}}$, where $p \in U$, we define $\widetilde{U}(p, e_p)$ as the set of all points $(q, \varphi^{(U)}_{p,q}(e_p))$, for $q \in U$. Then one can see that 1) $\widetilde{U}(p, e_p) \ni (q, e_q)$ if and only if $\widetilde{U}(p, e_p) = \widetilde{U}(q, e_q)$, 2) if $\widetilde{U}(p, e_p) \in (p, e'_p)$ then $e_p = e'_p$, 3) if $U \supset V$ and $p \in V$ then $\widetilde{U}(p, e_p) \supset \widetilde{V}(p, e_p)$; observe that each $\varphi^{(U)}_{p,q}$ is in fact a one-to-one mapping of E_p onto E_q. It follows from this easily that we can define a topology on $\widetilde{\mathfrak{B}}$ by taking the totality of $\widetilde{U}(p, e_p)$'s as an open basis of $\widetilde{\mathfrak{B}}$.

Let ϖ be the mapping of $\widetilde{\mathfrak{B}}$ onto $\mathfrak{B}$ which is defined by $\varpi(p, e_p) = p$. Then each $\widetilde{U}(p, e_p)$ is clearly mapped by ϖ topologically onto U and so is connected. Since $\varpi^{-1}(U) = \bigcup_{e_p \in E_p} \widetilde{U}(p, e_p)$ and $\widetilde{U}(p, e_p)$'s are mutually disjoint, it follows that U is evenly covered by $\widetilde{\mathfrak{B}}$ with respect to ϖ. Let $\widetilde{\mathfrak{B}}_0$ be the component of $(p_0, e^0_{p_0})$ in $\widetilde{\mathfrak{B}}$, and let ϖ_0 be the contraction of ϖ to $\widetilde{\mathfrak{B}}$. Then $(\widetilde{\mathfrak{B}}_0, \varpi_0)$ is a covering space of $\mathfrak{B}$. Since $\mathfrak{B}$ is simply connected, ϖ_0 is a homeomorphism. Therefore, we can define a mapping ψ by

$$\varpi_0^{-1}(p) = (p, \psi(p)) \qquad \text{for } p \in \mathfrak{B}.$$

Let $U \in \mathfrak{U}$ and $p, q \in U$. Then the connected set $\widetilde{U}(p, \psi(p))$ is contained in $\widetilde{\mathfrak{B}}_0$ and hence we have $(q, \varphi^{(U)}_{p,q}(\psi(p))) \in \widetilde{\mathfrak{B}}_0$, i.e., $\psi(q) = \varphi^{(U)}_{p,q}(\psi(p))$.

It remains only to prove the uniqueness of ψ. Let ψ' be any mapping satisfying the conditions 1), 2), 3) and $\psi'(p_0) = e^0_{p_0}$. Let A be the set of points p such that $\psi'(p) = \psi(p)$. Assume that U is a set in $\mathfrak{U}$ such that $U \frown A \neq \emptyset$. Let p be any point of U and q a point of $U \frown A$. Then we have $\varphi^{(U)}_{p,q}(\psi'(p)) = \psi'(q) = \psi(q) = \varphi^{(U)}_{p,q}(\psi(p))$. Since $\varphi^{(U)}_{p,q}$ is one-to-one, it follows $\psi'(p) = \psi(p)$, which implies $U \subset A$. Therefore A is open and closed, whence $A = \mathfrak{B}$, because A is not empty and $\mathfrak{B}$ is connected.

Now we show that it is possible to derive the above mentioned proposition from our theorem. Let $\mathfrak{U}$ be an open basis of $\mathfrak{B}$ which consists of evenly covered sets and $\mathfrak{U}'$ the totality of open connected sets W in $\mathfrak{W}$ such that $\varphi(W) \subset U$ for some $U \in \mathfrak{U}$, which forms obviously an open basis of $\mathfrak{W}$. Since U is evenly covered and $\varphi(W)$ is connected in this case, it follows that $\varphi(W)$ is also evenly covered, that is, every component $\widetilde{W}_\lambda$ of $f^{-1}(\varphi(W))$ mapped by f topologically onto $\varphi(W)$. Now we set $E_w = f^{-1}(\varphi(w))$, for every point $w \in \mathfrak{W}$. Take two points w_1, w_2 in W. Then each $\widetilde{W}_\lambda \frown E_{w_i}$ consists of only one point $\tilde{p}_\lambda^{(i)}$ and we have $\bigcup_\lambda \{\tilde{p}_\lambda^{(i)}\} = E_{w_i}$, for $i = 1, 2$. We define a mapping $\varphi_{w_1, w_2}^{(W)}$ by

$$\varphi_{w_1, w_2}^{(W)}(\tilde{p}_\lambda^{(1)}) = \tilde{p}_\lambda^{(2)}.$$

Then $\varphi_{w_1, w_2}^{(W)}$'s satisfy obviously the conditions 1), 2) and 3) in the above theorem, and our assertion follows immediately.

We note further that Chapter II, §VII, Theorem 3 in CHEVALLEY's book can also be deduced directly from our theorem in the similar way as in his proof.