A NOTE ON COVERING SPACES[1]

By

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CHEVALLEY proved in his book[2] the following proposition,[3] which is effectively used in showing the uniqueness of simply connected covering space[4]:

Proposition. *Let $\mathfrak{W}$ be a simply connected space. Let $\mathfrak{V}$ be a space, and assume that $(\tilde{\mathfrak{V}}, f)$ is a covering space of $\mathfrak{V}$. Let φ be a continuous mapping of $\mathfrak{W}$ into $\mathfrak{V}$. Then there exists a continuous mapping $\tilde{\varphi}$ of $\mathfrak{W}$ into $\tilde{\mathfrak{V}}$ such that $\varphi = f \cdot \tilde{\varphi}$. If w_0 is a point of $\mathfrak{W}$ and if $\tilde{p}_0$ is any point of $\tilde{\mathfrak{V}}$ such that $f(\tilde{p}_0) = p_0 = \varphi(w_0)$, $\tilde{\varphi}$ may be constructed as to map w_0 upon $\tilde{p}_0$ and is then uniquely determined.*

He remarked there that this can be deduced from his principle of monodromy[5] in the case where the space $\mathfrak{W}$ is assumed to be normal. In the present note, we shall however propose a modified principle of monodromy so that the above proposition can be deduced from it without the assumption of normality. We have namely

Theorem. *Let $\mathfrak{V}$ be a simply connected space. Assume that we have assigned to every $p \in \mathfrak{V}$ a non empty set E_p. Let $\mathfrak{U}$ be an open basis of $\mathfrak{V}$ consisting of connected sets. Assume furthermore that we have assigned to every triple $(U; p, q)$, where $U \in \mathfrak{U}$ and $p, q \in U$, a mapping $\varphi_{p,q}^{(U)}$ of E_p into E_q in such a way that the following conditions are satisfied:*

1) *If $U \in \mathfrak{U}$ and $p, q, r \in U$, we have*

$$\varphi_{p,r}^{(U)} = \varphi_{q,r}^{(U)} \cdot \varphi_{p,q}^{(U)};$$

2) *If $U, V \in \mathfrak{U}$ and $U \supset V$, we have*

$$\varphi_{p,q}^{(U)} = \varphi_{p,q}^{(V)} \qquad \text{for all } p, q \in V;$$

3) *Each $\varphi_{p,p}^{(U)}$ is the identity mapping.*

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2) C. CHEVALLEY, *Theory of Lie groups* I, Princeton (1946).

3) Loc. cit., p. 50.

4) Loc. cit., p. 51.

5) Loc. cit., p. 51.

Then there exists a mapping ψ which associates with every $p\in\mathfrak{B}$ an element $\psi(p)\in E_p$ in such a way that $\psi(q)=\varphi_{p,q}^{(U)}(\psi(p))$ where $U\in\mathfrak{U}$ and $p, q\in U$. Moreover, if $p_0\in\mathfrak{B}$ and $e_{p_0}^0\in E_{p_0}$ are given, then the mapping ψ is uniquely selected in such a way that $\psi(p_0)=e_{p_0}^0$.

The proof is more or less same as that of Chapter II, §VII, Theorem 2 in CHEVALLEY's book. But we shall give it here for completeness.

Let $\widetilde{\mathfrak{B}}$ be the union of all sets $\{p\}\times E_p$, for $p\in\mathfrak{B}$. For every $U\in\mathfrak{U}$ and every point $(p, e_p)\in\widetilde{\mathfrak{B}}$, where $p\in U$, we define $\widetilde{U}(p, e_p)$ as the set of all points $(q, \varphi_{p,q}^{(U)}(e_p))$, for $q\in U$. Then one can see that 1) $\widetilde{U}(p, e_p)\ni(q, e_q)$ if and only if $\widetilde{U}(p, e_p)=\widetilde{U}(q, e_q)$, 2) if $\widetilde{U}(p, e_p)\in(p, e_p')$ then $e_p=e_p'$, 3) if $U\supset V$ and $p\in V$ then $\widetilde{U}(p, e_p)\supset\widetilde{V}(p, e_p)$; observe that each $\varphi_{p,q}^{(U)}$ is in fact a one-to-one mapping of E_p onto E_q. It follows from this easily that we can define a topology on $\widetilde{\mathfrak{B}}$ by taking the totality of $\widetilde{U}(p, e_p)$'s as an open basis of $\widetilde{\mathfrak{B}}$.

Let ϖ be the mapping of $\widetilde{\mathfrak{B}}$ onto $\mathfrak{B}$ which is defined by $\varpi(p, e_p)=p$. Then each $\widetilde{U}(p, e_p)$ is clearly mapped by ϖ topologically onto U and so is connected. Since $\varpi^{-1}(U)=\bigcup_{e_p\in E_p}\widetilde{U}(p, e_p)$ and $\widetilde{U}(p, e_p)$'s are mutually disjoint, it follows that U is evenly covered by $\widetilde{\mathfrak{B}}$ with respect to ϖ. Let $\widetilde{\mathfrak{B}}_0$ be the component of $(p_0, e_{p_0}^0)$ in $\widetilde{\mathfrak{B}}$, and let ϖ_0 be the contraction of ϖ to $\widetilde{\mathfrak{B}}$. Then $(\widetilde{\mathfrak{B}}_0, \varpi_0)$ is a covering space of $\mathfrak{B}$. Since $\mathfrak{B}$ is simply connected, ϖ_0 is a homeomorphism. Therefore, we can define a mapping ψ by

$$\varpi_0^{-1}(p) = (p, \psi(p)) \qquad \text{for } p\in\mathfrak{B}.$$

Let $U\in\mathfrak{U}$ and $p, q\in U$. Then the connected set $\widetilde{U}(p, \psi(p))$ is contained in $\widetilde{\mathfrak{B}}_0$ and hence we have $(q, \varphi_{p,q}^{(U)}(\psi(p)))\in\widetilde{\mathfrak{B}}_0$, i. e., $\psi(q)=\varphi_{p,q}^{(U)}(\psi(p))$.

It remains only to prove the uniqueness of ψ. Let ψ' be any mapping satisfying the conditions 1), 2), 3) and $\psi'(p_0)=e_{p_0}^0$. Let A be the set of points p such that $\psi'(p)=\psi(p)$. Assume that U is a set in $\mathfrak{U}$ such that $U\frown A\neq\emptyset$. Let p be any point of U and q a point of $U\frown A$. Then we have $\varphi_{p,q}^{(U)}(\psi'(p))=\psi'(q)=\psi(q)=\varphi_{p,q}^{(U)}(\psi(p))$. Since $\varphi_{p,q}^{(U)}$ is one-to-one, it follows $\psi'(p)=\psi(p)$, which implies $U\subset A$. Therefore A is open and closed, whence $A=\mathfrak{B}$, because A is not empty and $\mathfrak{B}$ is connected.

Now we show that it is possible to derive the above mentioned proposition from our theorem. Let $\mathfrak{U}$ be an open basis of $\mathfrak{B}$ which consists of evenly covered sets and $\mathfrak{U}'$ the totality of open connected sets W in $\mathfrak{W}$ such that $\varphi(W)\subset U$ for some $U\in\mathfrak{U}$, which forms obviously an open basis of $\mathfrak{W}$. Since U is evenly covered and $\varphi(W)$ is connected in this case, it follows that $\varphi(W)$ is also evenly covered, that is, every component $\widetilde{W}_\lambda$ of $f^{-1}(\varphi(W))$ mapped by f topologically onto $\varphi(W)$. Now we set $E_w = f^{-1}(\varphi(w))$, for every point $w \in \mathfrak{W}$. Take two points w_1, w_2 in W. Then each $\widetilde{W}_\lambda \frown E_{w_i}$ consists of only one point $\tilde{p}_\lambda^{(i)}$ and we have $\bigcup_\lambda \{\tilde{p}_\lambda^{(i)}\} = E_{w_i}$, for $i=1, 2$. We define a mapping $\varphi_{w_1, w_2}^{(W)}$ by

$$\varphi_{w_1, w_2}^{(W)}(\tilde{p}_\lambda^{(1)}) = \tilde{p}_\lambda^{(2)}.$$

Then $\varphi_{w_1, w_2}^{(W)}$'s satisfy obviously the conditions 1), 2) and 3) in the above theorem, and our assertion follows immediately.

We note further that Chapter II, §VII, Theorem 3 in CHEVALLEY's book can also be deduced directly from our theorem in the similar way as in his proof.