A GENERALIZATION OF THE THEOREM OF ORLICZ AND BIRNBAUM

By

Ichiro AMEMIYA

ORLICZ and BIRNBAUM have proved in [1] that an ORLICZ-space L_Φ is finite if and only if the function Φ satisfies the following condition for some $\gamma > 0$:

$$\Phi(2t) \leqslant \gamma\Phi(t) \qquad \text{for every } t \geqslant t_0$$

This fact can be generalized for arbitrary monotone complete modulars[1] on non-discrete spaces, and the modular $m(x)$ satisfying the corresponding condition is said to be *semi-upper bounded*,[2] that is, by definition, if for some $\varepsilon, \gamma > 0$ we have

$$(1) \qquad m(2x) \leqslant \gamma m(x) \qquad \text{for every } x \text{ such that } m(x) \geqslant \varepsilon.$$

(Here always we can take ε arbitrarily small varying γ).

It is obvious that in this case m is also *uniformly finite*.[3]

In discrete spaces, a finite modular is not necessarily be semi-upper bounded nor uniformly finite, and we shall give the necessary and sufficient conditions for them.

In the sequel, R is a modulared semi-ordered linear space with the modular m which we suppose to be monotone complete throughout the paper.

1. A set U of R will be said to involve the modular m, if there exists $\varepsilon > 0$ for which $m(x) \leqslant \varepsilon$ implies $x \in U$. Such a set U must belong to every filtre which is order-convergent to 0, since m is order-continuous.

---

1) For the definition of the modular see H. NAKANO [2].

2) The upper bonded modular was defined in [2]. The "conjugate" of "semi-upper bounded" is: for some $\varepsilon, \alpha, \gamma$ such that $\varepsilon > 0$ and $1 < \alpha < \gamma$, we have $m(\alpha x) \geqslant \gamma m(x)$ for every x such that $m(x) \geqslant \varepsilon$.

3) m is said to be uniformly finite if for every $\gamma > 0$ we have

$$\sup_{\|x\| \leqslant \gamma} m(x) < +\infty,$$

where $\|x\|$ is the modular norm which is defined as $\|x\| \leqslant 1$ is equivalent to $m(x) \leqslant 1$.

Conversely, as the fact playing the fundamental roll in this paper, *for every order-closed set U which belongs to every filtre order-convergent to* 0, *there exist mutually orthogonal normal manifolds $N_i(i=0,1,\cdots n)$ such that $R=N_0+N_1+\cdots+N_n$, N_0 is finite-dimensional and the set $N_0+N_1{\frown}U+\cdots+N_n{\frown}U$ involves m.*

The last statement is equivalent to say that $N_i{\frown}U$ involves m in the space N_i for every $i=1,2,\cdots n$, so what is to prove is the existence of a normal manifold N in which $N{\frown}U$ involves m and which belongs to an arbitrarily given non-atomic maximal dual ideal of the Boolean algebra of all normal manifolds. Let this given ideal be $\mathfrak{p}$ and suppose there exists no normal manifold in $\mathfrak{p}$ satisfying the condition. Now we construct consecutively an orthogonal sequence of elements $x_\nu \in R$ $(\nu=1,2,\cdots)$ such that $m(x_\nu)\leqslant \frac{1}{2^\nu}$, $x_\nu \bar{\in} U$, and $[x_1, x_2,\cdots x_\nu]R \bar{\in} \mathfrak{p}$[4] for every $\nu=1,2,\cdots$. If $x_1, x_2,\cdots x_\nu$ had been taken, then we can find x in $(1-[x_1, x_2,\cdots x_\nu])R$ such that $m(x)\leqslant \frac{1}{2_{\nu+1}}$ and $x \bar{\in} U$, so we can put $x_{\nu+1}=[N]x$[4] where N is not in $\mathfrak{p}$ and sufficiently large to keep $[N]x$ outside of U. The existence of such a sequence x_ν is, however, a contradiction since the monotone completeness of m implies the convergence of the series $\sum_{\nu=1}^{\infty} x_\nu$ and so x_ν is order-convergent to 0. Thus the proof was completed.

2. If there exists another modular m_1 on R which is finite, then the set $\{x: m_1(x)\leqslant 1\}$ satisfies the condition of U in 1. Therefore making use of the same notations, there exist $N_i(i=0,1,\cdots n)$ and since for every x in $N_1{\frown}U+N_2{\frown}U+\cdots+N_n{\frown}U$ we have obviously $m_1(x)\leqslant n$ and for some $\gamma>0$, a set $N_0{\frown}\{x: m_1(x)\leqslant\gamma\}$ involves m in N_0, we see that a set $\{x: m_1(x)\leqslant\alpha\}$ involves m for some $\alpha>0$.

If m is itself finite, a modular $m_1(x)=m(\gamma x)$ is also finite for every $\gamma>0$, and hence for some ε and $\alpha>0$, $m(x)\leqslant\varepsilon$ implies $m(2x)\leqslant\alpha$.

3. Now we can prove the following

Theorem. *Every monotone complete finite modular on a space without discrete part*[5] *is semi-upper bounded and uniformly finite, and in the case there exists no complete element, the modular is upper bounded in some normal manifold different from* $\{0\}$.

4) For a set $A\subset R$, $[A]R$ means the least normal manifold containing A and $[A]$ is its projection.

5) That is to say, there exist no atomic element $\neq 0$ in R.

Let $\varepsilon, \gamma > 0$ be such that $m(x) \leqslant 2\varepsilon$ implies $m(2x) \leqslant \varepsilon\gamma$, then every element x such that $m(x) \geqslant \varepsilon$ can be decomposed orthogonally as $x = x_1 + x_2 + \cdots + x_n$ such that $\varepsilon \leqslant m(x_i) \leqslant 2\varepsilon$ $(i = 1, 2, \cdots n)$ and hence we have $m(2x) \leqslant \gamma m(x)$ summing up the inequalities $m(2x_i) \leqslant \varepsilon\gamma \leqslant \gamma m(x_i)$. Therefore m is semi-upper bounded and fortiori uniformly finite.

Now we fix $\gamma > 0$, and for every normal manifold N, let $\varepsilon(N)$ be the least number of $\varepsilon \geqslant 0$ such that $m(x) \geqslant \varepsilon$ and $x \in N$ imply $m(2x) \leqslant \gamma m(x)$. We suppose γ is sufficiently large so that $\varepsilon(R)$ exists. For every orthogonal system of normal manifolds N_λ $(\lambda \in \Lambda)$, we have $\varepsilon(N_\lambda) = 0$ except a countable number of λ, because $\sum_{\lambda \in \Lambda} \varepsilon(N_\lambda) > \varepsilon(R)$ is a contradiction as easily be seen. Therefore if m is not upper bounded in every N, then we have $\varepsilon(N_\lambda) > 0$ for every $\lambda \in \Lambda$, that is, there exists no orthogonal decomposition of R into more than countable number of factors, and this is equivalent to the existence of complete element in our case. Thus the theorem was proved.

If m is moreover constant and the space has no complete element, then m is upper bounded. This particular case of the theorem was proved in [2] (Theorem. 55.10).

In general m is not upper bounded even for simple finite Orlicz-spaces.

4. If the modular norm is continuous (in non-discrete case this is equivalent to say that m is finite), then every infinite orthogonal sequence $\{x_\nu\}$ for which $\{\|x_\nu\|\}$ has a non-zero lower bound is not order-bounded, as the same is true for every continuous norm. Here we shall show that the modular norm, in case it is continuous satisfies a stronger condition than above, that is, *for every* $\varepsilon > 0$ *there exists an integer* n *such that the norm of the sum of* n *mutually orthogonal elements having their norm more than* ε *is always* $\geqslant 1$. To see that the existence of n as above for some $\varepsilon < 1$ is sufficient, because then for ε^m where m is an arbitrary integer we can take n^m in place of n. If the condition is satisfied in both normal manifolds N and M, then we can see easily that the same is also true in $N + M$. Since the sphere $\left\{x : \|x\| \leqslant \frac{1}{2}\right\}$ satisfies the condition of U in 1, and since the finite-dimensional case is trivial, we may restrict ourselves to the case that the sphere involves m, that is, for some $\delta > 0$, $m(x) \leqslant \delta$ implies $\|x\| \leqslant \frac{1}{2}$. Then the sum of n mutually orthogonal elements whose norm are all

more than $\frac{1}{2}$ has its modular more than $n\delta$, so in case $n > \frac{1}{\delta}$, more than 1, that is, its norm is $\geqslant 1$.

5. In the sequel we shall deal with discrete spaces and for this we prefer the concrete treatment as follows: let Λ be an abstract set and for every $\lambda \in \Lambda$, φ_λ be a convex function satisfying the conditions of the modular on real number, R is the totality of real function $x \equiv \{\xi_\lambda\}$ on Λ such that

$$m(\alpha x) = \sum_{\lambda \in \Lambda} \varphi_\lambda(\alpha \xi_\lambda) < +\infty \qquad \text{for some } \alpha > 0 .$$

The modular m on R is defined by the equation above, and we can see easily that m is monotone complete.

Now we shall prove that *m is semi-upper bounded if and only if m is finite and for some $\varepsilon, \gamma > 0$, $\varphi_\lambda(\xi) \geqslant \varepsilon$ implies $\varphi_\lambda(2\xi) \leqslant \gamma \varphi_\lambda(\xi)$ for every $\lambda \in \Lambda$.*

The necessity is obvious. To prove the sufficiency, we proceed as in the proof of the foregoing theorem. For the same $\varepsilon, \gamma > 0$, though we can not, in general, decompose x into x_i such that $\varepsilon \leqslant m(x_i) \leqslant 2\varepsilon$, the following decomposition is possible:

$$x = x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_m + z ,$$

where $\varepsilon \leqslant m(x_i) \leqslant 2\varepsilon$, $y_j (j=1, 2, \cdots m)$ is atomic and $m(y_j) > 2\varepsilon$ and $m(z) \leqslant \varepsilon$. Then summing up the inequalities $m(2x_i) \leqslant \gamma m(x_i)$, $m(2y_j) \leqslant \gamma m(y_j)$ and $m(2z) \leqslant \varepsilon\gamma \leqslant \gamma m(x)$ we have $m(2x) \leqslant 2\gamma m(x)$.

6. Concerning the uniformly finiteness of modulars on discrete spaces, we shall prove that *m is uniformly finite if and only if m is finite and*

$$\sup_{\lambda \in \Lambda} \varphi_\lambda(\xi \alpha_\lambda) < +\infty \qquad \textit{for every } \xi > 0 ,$$

where α_λ is the number such that $\varphi_\lambda(\alpha_\lambda) = 1$.

What is to prove is only the sufficiency. If m is finite, then the modular norm is continuous and so the result obtained in 4 is applicable. For every element x such that $\|x\| \leqslant \gamma$, we can decompose it orthogonally as

$$x = x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_m + z ,$$

where $\frac{1}{2} \leqslant \|x_i\| \leqslant 1 \, (i=1, 2, \cdots n)$, $y_j (j=1, 2, \cdots m)$ are atomic and $\|y_j\| > 1$, and finally $||z||<\frac{1}{2}$. In this case the number $n+m$ has an upper bound which depends only to γ. If $y\equiv\{\eta_\lambda\}$ is atomic and $||y||\leqslant\gamma$, then $\eta_\lambda=0$ except for one λ, say λ_0, and we have $\eta_{\lambda_0}\leqslant\gamma\alpha_{\lambda_0}$, or $m(y)\leqslant\varphi_{\lambda_0}(\gamma\alpha_{\lambda_0})$.

Therefore we have

$$m(x)\leqslant n+m\sup_{\lambda\in\Lambda}\varphi_\lambda(\gamma\alpha_\lambda)+1\,,$$

and this completes the proof.

7. In both case 5 and 6, the conditions for φ_λ are not superfluous and this can be seen by the following fact which displays the essential difference between discrete space and non-discrete one.

For another system $\psi_\lambda(\lambda\in\Lambda)$, if we have for every λ

$$\varphi_\lambda(\xi)=\psi_\lambda(\xi) \quad \text{for every } \xi \text{ such that } \varphi_\lambda(\xi) \text{ or } \psi_\lambda(\xi)\leqslant 1\,,$$

then we obtain the same space R and in general a different modular $m_1(x)=\sum_{\lambda\in\Lambda}\psi_\lambda(\xi_\lambda)$ which coinsides with $m(x)$ in case $m(x)\leqslant 1$, and so their modular norms are the same.

If m is finite and all ψ_λ are finite, then m_1 is also finite. Thus we can vary "larger part of values" of a given finite modular rather arbitrarily without the loss of the finiteness.

Contrarily in non-discrete spaces m is determined by its values on the set $\{x: m(x)\leqslant\varepsilon\}$ for whatever small $\varepsilon>0$.

(September, 1954)

References

[1] W. ORLICZ and Z. BIRNBAUM: Uber die Veraīgemeinerung des Begriffes der Zueinander konjugierten Potenzen, Studia Math., III (1931) p. 1–67.

[2] H. NAKANO: Modulared semi-ordered linear spaces, Tokyo Math. Book Ser., vol. 1, 1950, Maruzen.