A GENERALIZATION OF THE THEOREM OF ORLICZ AND BIRNBAUM

By

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Orlicz and Birnbaum have proved in [1] that an Orlicz-space $L_\Phi$ is finite if and only if the function $\Phi$ satisfies the following condition for some $r > 0$:

$$\Phi(2t) \leq r\Phi(t) \quad \text{for every } t \geq t_0$$

This fact can be generalized for arbitrary monotone complete modulars on non-discrete spaces, and the modular $m(x)$ satisfying the corresponding condition is said to be semi-upper bounded, that is, by definition, if for some $\epsilon, r > 0$ we have

$$m(2x) \leq r m(x) \quad \text{for every } x \text{ such that } m(x) \geq \epsilon.$$  

(Here always we can take $\epsilon$ arbitrarily small varying $r$).

It is obvious that in this case $m$ is also uniformly finite.

In discrete spaces, a finite modular is not necessarily be semi-upper bounded nor uniformly finite, and we shall give the necessary and sufficient conditions for them.

In the sequel, $R$ is a modularized semi-ordered linear space with the modular $m$ which we suppose to be monotone complete throughout the paper.

1. A set $U$ of $R$ will be said to involve the modular $m$, if there exists $\epsilon > 0$ for which $m(x) \leq \epsilon$ implies $x \in U$. Such a set $U$ must belong to every filtre which is order-convergent to 0, since $m$ is order-continuous.

1) For the definition of the modular see H. Nakano [2].

2) The upper bonded modular was defined in [2]. The "conjugate" of "semi-upper bounded" is: for some $\epsilon, \alpha, r$ such that $\epsilon > 0$ and $1 < \alpha < r$, we have $m(ax) \geq r m(x)$ for every $x$ such that $m(x) \geq \epsilon$.

3) $m$ is said to be uniformly finite if for every $r > 0$ we have

$$\sup_{||x|| \leq r} m(x) < +\infty,$$

where $||x||$ is the modular norm which is defined as $||x|| \leq 1$ is equivalent to $m(x) \leq 1$. 

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1) For the definition of the modular see H. Nakano [2].
Conversely, as the fact playing the fundamental roll in this paper, for every order-closed set $U$ which belongs to every filtre order-convergent to 0, there exist mutually orthogonal normal manifolds $N_i(i=0,1,\cdots,n)$ such that $R=N_0+N_1+\cdots+N_n$, $N_0$ is finite-dimensional and the set $N_0+N_1\cap U+\cdots+N_n\cap U$ involves $m$.

The last statement is equivalent to say that $N_i\cap U$ involves $m$ in the space $N_i$ for every $i=1,2,\cdots,n$, so what is to prove is the existence of a normal manifold $N$ in which $N\cap U$ involves $m$ and which belongs to an arbitrarily given non-atomic maximal dual ideal of the Boolean algebra of all normal manifolds. Let this given ideal be $\mathfrak{p}$ and suppose there exists no normal manifold in $\mathfrak{p}$ satisfying the condition. Now we construct consecutively an orthogonal sequence of elements $x_{\nu}\in R (\nu=1,2,\cdots)$ such that $m(x_{\nu})\leq \frac{1}{2^\nu}$, $x_{\nu}\not\in U$, and $[x_1,x_2,\cdots x_{\nu}] R\in \mathfrak{p}$ for every $\nu=1,2,\cdots$. If $x_1,x_2,\cdots x_{\nu}$ had been taken, then we can find $x$ in $(1-[x_1,x_2,\cdots x_{\nu}]) R$ such that $m(x)\leq \frac{1}{2^{\nu+1}}$ and $x\not\in U$, so we can put $x_{\nu+1}=[N]x^\nu$ where $N$ is not in $\mathfrak{p}$ and sufficiently large to keep $[N]x$ outside of $U$. The existence of such a sequence $x_{\nu}$ is, however, a contradiction since the monotone completeness of $m$ implies the convergence of the series $\sum_{\nu=1}^\infty x_{\nu}$ and so $x_{\nu}$ is order-convergent to 0. Thus the proof was completed.

2. If there exists another modular $m_i$ on $R$ which is finite, then the set $\{x: m_i(x)\leq 1\}$ satisfies the condition of $U$ in 1. Therefore making use of the same notations, there exist $N_i(i=0,1,\cdots,n)$ and since for every $x$ in $N_1\cap U+N_2\cap U+\cdots+N_n\cap U$ we have obviously $m_i(x)\leq n$ and for some $r>0$, a set $N_{r}\{x: m_i(x)\leq r\}$ involves $m$ in $N_r$, we see that a set $\{x: m_i(x)\leq a\}$ involves $m$ for some $a>0$.

If $m$ is itself finite, a modular $m_i(x)=m(rx)$ is also finite for every $r>0$, and hence for some $\varepsilon$ and $a>0$, $m(x)\leq \varepsilon$ implies $m(2x)\leq a$.

3. Now we can prove the following

**Theorem.** Every monotone complete finite modular on a space without discrete part\(^5\) is semi-upper bounded and uniformly finite, and in the case there exists no complete element, the modular is upper bounded in some normal manifold different from $\{0\}$.

4) For a set $A\subset R$, $[A] R$ means the least normal manifold containing $A$ and $[A]$ is its projection.

5) That is to say, there exist no atomic element $\neq 0$ in $R$.
Let $\varepsilon, \gamma > 0$ be such that $m(x) \leq 2\varepsilon$ implies $m(2x) \leq \varepsilon \gamma$, then every element $x$ such that $m(x) > \varepsilon$ can be decomposed orthogonally as $x = x_1 + x_2 + \cdots + x_n$ such that $\varepsilon \leq m(x_i) \leq 2\varepsilon (i = 1, 2, \cdots, n)$ and hence we have $m(2x) \leq \gamma m(x)$ summing up the inequalities $m(2x_i) \leq \gamma \varepsilon \leq \gamma m(x_i)$. Therefore $m$ is semi-upper bounded and fortiori uniformly finite.

Now we fix $\gamma > 0$, and for every normal manifold $N$, let $\varepsilon (N)$ be the least number of $\varepsilon \geq 0$ such that $m(x) \geq \varepsilon$ and $x \in N$ imply $m(2x) \leq \gamma m(x)$. We suppose $\gamma$ is sufficiently large so that $\varepsilon (R)$ exists. For every orthogonal system of normal manifolds $N_\lambda (\lambda \in \Lambda)$, we have $\varepsilon (N_\lambda) > 0$ except a countable number of $\lambda$, because $\sum_{\lambda \in \Lambda} \varepsilon (N_\lambda) > \varepsilon (R)$ is a contradiction as easily be seen. Therefore if $m$ is not upper bounded in every $N$, then we have $\varepsilon (N_\lambda) > 0$ for every $\lambda \in \Lambda$, that is, there exists no orthogonal decomposition of $R$ into more than countable number of factors, and this is equivalent to the existence of complete element in our case. Thus the theorem was proved.

If $m$ is moreover constant and the space has no complete element, then $m$ is upper bounded. This particular case of the theorem was proved in [2] (Theorem. 55.10).

In general $m$ is not upper bounded even for simple finite ORLICZ-spaces.

4. If the modular norm is continuous (in non-discrete case this is equivalent to say that $m$ is finite), then every infinite orthogonal sequence $\{x_\nu\}$ for which $\{|x_\nu|\}$ has a non-zero lower bound is not order-bounded, as the same is true for every continuous norm. Here we shall show that the modular norm, in case it is continuous satisfies a stronger condition than above, that is, for every $\varepsilon > 0$ there exists an integer $n$ such that the norm of the sum of $n$ mutually orthogonal elements having their norm more than $\varepsilon$ is always $\geq 1$. To see that the existence of $n$ as above for some $\varepsilon < 1$ is sufficient, because then for $\varepsilon^n$ where $m$ is an arbitrary integer we can take $n^n$ in place of $n$. If the condition is satisfied in both normal manifolds $N$ and $M$, then we can see easily that the same is also true in $N + M$. Since the sphere $\{x : ||x|| \leq \frac{1}{2}\}$ satisfies the condition of $U$ in 1, and since the finite-dimensional case is trivial, we may restrict ourselves to the case that the sphere involves $m$, that is, for some $\delta > 0$, $m(x) \leq \delta$ implies $||x|| \leq \frac{1}{2}$. Then the sum of $n$ mutually orthogonal elements whose norm are all
more than 1/2 has its modular more than nδ, so in case n > 1/δ, more than 1, that is, its norm is \( \geq 1 \).

5. In the sequel we shall deal with discrete spaces and for this we prefer the concrete treatment as follows: let \( \Lambda \) be an abstract set and for every \( \lambda \in \Lambda, \varphi_{\lambda} \) be a convex function satisfying the conditions of the modular on real number, \( R \) is the totality of real function \( x = \{ \xi_{\lambda} \} \) on \( \Lambda \) such that

\[
m(ax) = \sum_{\lambda \in \Lambda} \varphi_{\lambda}(a \xi_{\lambda}) < +\infty \quad \text{for some } a > 0.
\]

The modular \( m \) on \( R \) is defined by the equation above, and we can see easily that \( m \) is monotone complete.

Now we shall prove that \( m \) is semi-upper bounded if and only if \( m \) is finite and for some \( \varepsilon, r > 0, \varphi_{\lambda}(\xi) \geq \varepsilon \) implies \( \varphi_{\lambda}(2\xi) \leq r\varphi_{\lambda}(\xi) \) for every \( \lambda \in \Lambda \).

The necessity is obvious. To prove the sufficiency, we proceed as in the proof of the foregoing theorem. For the same \( \varepsilon, r > 0 \), though we can not, in general, decompose \( x \) into \( x_{i} \) such that \( \varepsilon \leq m(x_{i}) \leq 2\varepsilon \), the following decomposition is possible:

\[
x = x_{1} + x_{2} + \cdots + x_{n} + y_{1} + y_{2} + \cdots + y_{m} + z,
\]

where \( \varepsilon \leq m(x_{i}) \leq 2\varepsilon \), \( y_{j} (j=1,2,\cdots m) \) is atomic and \( m(y_{j}) \geq 2\varepsilon \) and \( m(z) \leq \varepsilon \). Then summing up the inequalities \( m(2x_{i}) \leq r m(x_{i}), m(2y_{j}) \leq r m(y_{j}) \) and \( m(2z) \leq \varepsilon r \leq r m(x) \) we have \( m(2x) \leq 2r m(x) \).

6. Concerning the uniformly finiteness of modulars on discrete spaces, we shall prove that \( m \) is uniformly finite if and only if \( m \) is finite and

\[
\sup_{\lambda \in \Lambda} \varphi_{\lambda}(\xi a_{\lambda}) < +\infty \quad \text{for every } \xi > 0,
\]

where \( a_{\lambda} \) is the number such that \( \varphi_{\lambda}(a_{\lambda}) = 1 \).

What is to prove is only the sufficiency. If \( m \) is finite, then the modular norm is continuous and so the result obtained in 4 is applicable. For every element \( x \) such that \( \|x\| \leq r \), we can decompose it orthogonally as

\[
x = x_{1} + x_{2} + \cdots + x_{n} + y_{1} + y_{2} + \cdots + y_{m} + z,
\]

where \( \frac{1}{2} \leq \|x_{i}\| \leq 1 (i=1,2,\cdots n), y_{j} (j=1,2,\cdots m) \) are atomic and \( \|y_{j}\| > 1 \),
and finally $||z|| < \frac{1}{2}$. In this case the number $n+m$ has an upper bound which depends only to $r$. If $y \equiv \{\eta_\lambda\}$ is atomic and $||y|| < r$, then $\eta_\lambda = 0$ except for one $\lambda$, say $\lambda_0$, and we have $\eta_\lambda < r a_\lambda$, or $m(y) < \varphi_\lambda(ra_\lambda)$. Therefore we have

$$m(x) \leq n + m \sup_{\lambda \in \Lambda} \varphi_\lambda(ra_\lambda) + 1,$$

and this completes the proof.

7. In both case 5 and 6, the conditions for $\varphi_\lambda$ are not superfluous and this can be seen by the following fact which displays the essential difference between discrete space and non-discrete one.

For another system $\varphi_\lambda(\lambda \in \Lambda)$, if we have for every $\lambda$ $\varphi_\lambda(\xi) = \psi_\lambda(\xi)$, for every $\xi$ such that $\varphi_\lambda(\xi)$ or $\psi_\lambda(\xi) \leq 1$, then we obtain the same space $R$ and in general a different modular $m_i(x) = \sum_{\lambda \in \Lambda} \varphi_\lambda(\xi_\lambda)$ which coinsides with $m(x)$ in case $m(x) \leq 1$, and so their modular norms are the same.

If $m$ is finite and all $\varphi_\lambda$ are finite, then $m_i$ is also finite. Thus we can vary "larger part of values" of a given finite modular rather arbitrarily without the loss of the finiteness.

Contrarily in non-discrete spaces $m$ is determined by its values on the set $\{x : m(x) \leq \varepsilon\}$ for whatever small $\varepsilon > 0$.

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References
