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MODULARS ON SEMI-ORDERED LINEAR SPACES I

By

Michiyo MIYAKAWA and Hidegorô NAKANO

In an earlier paper [1], one of the authors defined modulars on linear spaces and discussed their properties: a functional $m(x)$ on a linear space $R$ is said to be a modular on $R$, if

1) $m(0) = 0$;
2) $m(-a) = m(a)$ for every $a \in R$;
3) for any $a \in R$ we can find a positive number $\alpha$ such that $m(\alpha a) < +\infty$;
4) $m(\xi a) = 0$ for every positive number $\xi$ implies $a = 0$;
5) $a + \beta = 1$, $a, \beta \geq 0$ implies for every $a, b \in R$ $m(\alpha a + \beta b) \leq \alpha m(a) + \beta m(b)$;
6) $m(a) = \sup_{0 \leq \xi < 1} m(\xi a)$ for every $a \in R$.

For universally continuous semi-ordered linear spaces $R$, modulars were considered with adding conditions: 7) $|a| \leq |b|$ implies $m(a) \leq m(b)$, 8) $|a| < |b| = 0$ implies $m(a + b) = m(a) + m(b)$, and 9) $0 \leq a_{\lambda} \uparrow \lambda \in \Lambda$ implies $m(a) = \sup_{\lambda \in \Lambda} m(a_{\lambda})$. (cf. [2])

In this paper we shall discuss modulars on lattice ordered linear spaces only with adding condition 7).

§1. Modulars on linear spaces

Firstly we shall give a rough sketch of the properties of modulars on linear spaces which are obtained in [1] and [3], and will be used in this paper. Let $m(x)$ ($x \in R$) be a modular on a linear space $R$. A linear functional $\bar{\alpha}(x)$ ($x \in R$) on $R$ is said to be modular bounded, if we can find positive numbers $\alpha, \beta$ such that

$$\alpha \bar{\alpha}(x) \leq \beta + m(x)$$

for every $x \in R$.

The totality of modular bounded linear functionals on $R$ also builds a linear space which will be called the modular associated space of $R$ and denoted by $\bar{R}$. For each $\bar{a} \in \bar{R}$, putting
we obtain a modular $\overline{m}$ on $\overline{R}$, which will be called the \textit{conjugate modular} of $m$. Then we have the reflexive relation:

$$m(a) = \sup_{\overline{x} \in \overline{R}} \{ \overline{\alpha}(a) - \overline{m}(\overline{x}) \} \quad (a \in R)$$

Putting

(1) \quad \left\| x \right\| = \inf_{m(x) \leq 1} \frac{1}{x} \quad (x \in R)

we obtain a norm on $R$, which will be called the \textit{second norm} of $m$. Concerning the second norm, we have

$$m(x) \leq \left\| x \right\| \quad \text{if} \quad \left\| x \right\| \leq 1,$$

$$m(x) \geq \left\| x \right\| \quad \text{if} \quad \left\| x \right\| \geq 1.$$

Putting

$$\| a \| = \sup_{\overline{m}(\overline{x}) \leq 1} | \overline{x}(a) | \quad (a \in R),$$

we also obtain another norm on $R$, which will be called the \textit{first norm} of $m$. Between the first and the second norms there is the relation:

$$\left\| x \right\| \leq \| x \| \leq 2 \left\| x \right\| \quad (x \in R).$$

The first norm also may be defined as

(2) \quad \left\| x \right\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi} \quad (x \in R).

For the first and the second norm of the conjugate modular $\overline{m}$ we have

$$\left\| x \right\| = \sup_{\| x \| \leq 1} | \overline{x}(x) |, \quad \left\| x \right\| = \sup_{\| x \| \leq 1} | \overline{x}(x) |$$

$$\left\| \overline{x} \right\| = \sup_{\| x \| \leq 1} | \overline{x}(x) |, \quad \left\| \overline{x} \right\| = \sup_{\| x \| \leq 1} | \overline{x}(x) | \quad (x \in R, \overline{x} \in \overline{R})).$$

A linear functional $\overline{x}$ on $R$ is modular bounded if and only if $\overline{x}$ is norm bounded, that is,

$$\sup_{m(x) \leq 1} | \overline{x}(x) | < +\infty \quad (x \in R).$$

A sequence $x_{\nu} \in R \ (\nu = 1, 2, \cdots)$ is said to be \textit{modular convergent} to $x \in R$, if

$$\lim_{\nu \to \infty} m(\xi(x_{\nu} - x)) = 0 \quad \text{for every} \quad \xi > 0.$$
With this definition we have that a sequence \( x_\nu \in R \) \((\nu=1,2,\cdots)\) is modular convergent to \( x \in R \) if and only if it is norm convergent, that is,
\[
\lim_{\nu \to \infty} \|x_\nu - x\| = 0.
\]

A modular \( m \) on \( R \) is said to be complete, if
\[
\lim_{\nu,\mu \to \infty} m(\xi (x_\nu - x_\mu)) = 0 \text{ for every } \xi > 0
\]
implies the modular convergence of the sequence \( x_\nu \in R \) \((\nu=1,2,\cdots)\). With this definition, a modular \( m \) on \( R \) is complete if and only if the first or second norm of \( m \) is complete. The conjugate modular \( \overline{m} \) of any modular \( m \) on \( R \) is always complete on \( \overline{R} \).

From the postulate 5) we conclude easily for \( 0 < \varepsilon \leq 1 \)
\[
(3) \quad m(x) \leq m(y) + \frac{\varepsilon}{1+\varepsilon} m((1+\varepsilon)y) + \frac{\varepsilon^2}{1+\varepsilon} m\left(\frac{1+\varepsilon}{\varepsilon^2} (x-y)\right).
\]

§ 2. Monotone modulars

Let \( R \) be a lattice ordered linear space. A modular \( m \) on \( R \) is said to be monotone if \( |x| \leq |y| \) implies \( m(x) \leq m(y) \). With this definition we have obviously by the formulas (1) and (2) in §1 that if a modular \( m \) on \( R \) is monotone, then both the first and the second norm of \( m \) are monotone too, that is, \( |x| \leq |y| \) implies \( \|x\| \leq \|y\| \) and \( \|x\| \leq \|y\| \).

A modular \( m \) on \( R \) is said to be upper semi-continuous, if \( m \) is monotone and \( 0 \leq x_\lambda \uparrow \lambda \in \Lambda \) \( x \) implies
\[
m(x) = \sup_{\lambda \in \Lambda} m(x_\lambda).
\]

**Theorem 2.1.** If a modular \( m \) on \( R \) is upper semi-continuous, then the second norm of \( m \) is semi-continuous, that is, \( 0 \leq x_\lambda \downarrow \lambda \in \Lambda \) \( x \) implies \( \sup_{\lambda \in \Lambda} \|x_\lambda\| = \|x\| \).

**Proof.** If \( 0 \leq x_\lambda \uparrow \lambda \in \Lambda \) \( x \) and \( \sup_{\lambda \in \Lambda} \|x_\lambda\| < \|x\| \), then we can find a positive number \( \alpha \) such that
\[
\sup_{\lambda \in \Lambda} \|ax_\lambda\| < 1 < \|ax\|.
\]

Thus we have for such \( \alpha \)
\[
\sup_{\lambda \in \Lambda} m(ax_\lambda) \leq 1 < m(ax), \quad 0 \leq ax_\lambda \downarrow \lambda \in \Lambda \ ax.
\]

Therefore we obtain our assertion.

A modular \( m \) on \( R \) is said to be lower semi-continuous, if \( m \) is monotone and \( x_\lambda \downarrow \lambda \in \Lambda \) \( 0 \), \( m(x_\lambda) < +\infty \) for every \( \lambda \in \Lambda \) implies \( \inf_{\lambda \in \Lambda} m(x_\lambda) = 0 \). If a
modular $m$ on $R$ is upper and lower semi-continuous simultaneously, then $m$ is said to be semi-continuous.

A modular $m$ on $R$ is said to be continuous, if $m$ is monotone and $x_{\lambda} \downarrow_{\lambda \in \Lambda} 0$ implies always $\inf_{\lambda \in \Lambda} m(x_{\lambda}) = 0$.

**Theorem 2.2.** Every continuous modular is semi-continuous.

*Proof.* If a modular $m$ on $R$ is continuous, then $m$ is obviously lower semi-continuous by definition. Since $m$ is monotone, we have for $0 \leq x_{\lambda} \downarrow_{\lambda \in \Lambda} x \in R$.

\[ \sup_{\lambda \in \Lambda} m(x_{\lambda}) \leq m(x) \]

On the other hand we have by the formula (3) for $0 < \varepsilon \leq 1$

\[ m\left(\frac{1}{1+\varepsilon} x\right) \leq m\left(\frac{1}{1+\varepsilon} x_{\lambda}\right) + \frac{\varepsilon}{1+\varepsilon} m(x_{\lambda}) + \frac{\varepsilon^{2}}{1+\varepsilon} m\left(\frac{1}{\varepsilon^{2}} (x-x_{\lambda})\right) \]

\[ \leq \frac{1+2\varepsilon}{1+\varepsilon} \sup_{\lambda \in \Lambda} m(x_{\lambda}) + \frac{\varepsilon}{1+\varepsilon} m\left(\frac{1}{\varepsilon^{2}} (x-x_{\lambda})\right). \]

Since $\frac{1}{\varepsilon^{2}} (x-x_{\lambda}) \downarrow_{\lambda \in \Lambda} 0$, we obtain by assumption

\[ m\left(\frac{1}{1+\varepsilon} x\right) \leq \frac{1+2\varepsilon}{1+\varepsilon} \sup_{\lambda \in \Lambda} m(x_{\lambda}). \]

This relation yields $m(x) \leq \sup_{\lambda \in \Lambda} m(x)$, because $\sup_{\lambda \in \Lambda} \frac{1}{1+\varepsilon} x = m(x)$ by the postulate 6). Therefore $m$ is upper semi-continuous too.

**Theorem 2.3.** A monotone modular $m$ on $R$ is continuous, if and only if the first or the second norm of $m$ is continuous: $x_{\lambda} \downarrow_{\lambda \in \Lambda} 0$ implies

\[ \inf_{\lambda \in \Lambda} \|x\| = 0 \quad \text{or} \quad \inf_{\lambda \in \Lambda} \|x\| = 0. \]

*Proof.* It is obvious that $\inf_{\lambda \in \Lambda} \|x_{\lambda}\| = 0$ is equivalent to $\inf_{\lambda \in \Lambda} \|x_{\lambda}\| = 0$. If $m$ is continuous, then for $x_{\lambda} \downarrow_{\lambda \in \Lambda} 0$ we have $x_{\lambda} \downarrow_{\lambda \in \Lambda} 0$ for every $\nu = 1, 2, \cdots$, and hence we can find $\lambda_{\nu} \in \Lambda$ ($\nu = 1, 2, \cdots$) such that $m(x_{\lambda_{\nu}}) \leq 1$ ($\nu = 1, 2, \cdots$). Then we have $\|\nu x_{\lambda_{\nu}}\| \leq 1$, namely $\|x_{\lambda_{\nu}}\| \leq \frac{1}{\nu}$ for every $\nu = 1, 2, \cdots$, and this relation yields $\inf_{\lambda \in \Lambda} \|x_{\lambda}\| = 0$. Thus the second norm of $m$ is continuous.

Conversely, if the second norm of $m$ is continuous, then for $x_{\lambda} \downarrow_{\lambda \in \Lambda} 0$ we can find $\lambda_{\nu} \in \Lambda$ ($\nu = 1, 2, \cdots$) such that $\|x_{\lambda_{\nu}}\| \leq 1$, and hence

\[ m(x_{\lambda_{\nu}}) \leq \frac{1}{\nu} m(\nu x_{\lambda_{\nu}}) \leq \frac{1}{\nu}. \]
for every $\nu=1,2,\cdots$. This relation yields $\inf_{\lambda \in \Lambda} m(x_{\lambda}) = 0$. Thus $m$ is continuous by definition.

A monotone modular $m$ on $R$ is said to be monotone complete, if

$$0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda}, \quad \sup_{\lambda \in \Lambda} m(x_{\lambda}) < +\infty$$

implies the existence of $\bigcup_{\lambda \in \Lambda} x_{\lambda}$. If $m$ is monotone complete, then $R$ must be universally continuous, because $0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda}, x_{\lambda} \leq x (\lambda \in \Lambda)$ implies $\sup_{\lambda \in \Lambda} m(ax_{\lambda}) < +\infty$ for some positive number $a$ such that $m(ax) < +\infty$.

**Theorem 2.4.** A monotone modular $m$ on $R$ is monotone complete if and only if the first or the second norm of $m$ is monotone complete.

**Proof.** If $\sup_{\lambda \in \Lambda} m(x_{\lambda}) \leq a$ for some $a > 1$, then we have

$$m\left(\frac{1}{a} x_{\lambda}\right) \leq \frac{1}{a} m(x_{\lambda}) \leq 1$$

for every $\lambda \in \Lambda$

and hence $\sup_{\lambda \in \Lambda} \|x_{\lambda}\| \leq 1$, that is, $\sup_{\lambda \in \Lambda} \|x_{\lambda}\| \leq a$. Conversely if $\sup_{\lambda \in \Lambda} \|x_{\lambda}\| \leq a$ for some $a > 0$, then we have

$$\sup_{\lambda \in \Lambda} m\left(\frac{1}{a} x_{\lambda}\right) \leq 1.$$ 

Therefore we can conclude our assertion.

**Theorem 2.5.** For any monotone modular $m$ on $R$, its conjugate modular $\overline{m}$ is upper semi-continuous and monotone complete.

**Proof.** The modular associated space $\overline{R}$ of $R$ is always universally continuous. (cf. [2]) The conjugate modular $\overline{m}$ is obviously monotone by definition. If $0 \leq x_{\lambda} \uparrow_{\lambda \in \Lambda} x$, then we have

$$\overline{m}(x) = \sup_{x \in R} \{x - m(x)\} = \sup_{0 \leq x \in R} \{\sup_{\lambda \in \Lambda} x_{\lambda}(x) - m(x)\}$$

$$= \sup_{0 \leq x \in R} \{\sup_{\lambda \in \Lambda} (x_{\lambda}(x) - m(x))\} = \sup_{\lambda \in \Lambda} \overline{m}(x_{\lambda}).$$

Thus $\overline{m}$ is upper semi-continuous. The first norm of $\overline{m}$ is the conjugate norm of the second norm of $m$, and hence monotone complete. (cf. [2]) Thus $\overline{m}$ is monotone complete by Theorem 2.4.

§ 3. Reflexivity of upper semi-continuous modulars

Now we suppose that $R$ is a universally continuous linear space and $m$ is a monotone modular on $R$. The totality of universally continuous linear functionals on $R$, which are modular bounded, is called
the modular conjugate space of $R$ and denoted by $\bar{R}$. $\bar{R}$ is a normal manifold of the modular associated space $\bar{R}$ of $R$. If $m$ is continuous, then the second norm of $m$ also is continuous by Theorem 2.3, and hence $\bar{R} = R$.

**Theorem 3.1.** If $R$ is semi-regular and $m$ is upper semi-continuous, then $m$ is reflexive, that is, we have for every $x \in R$

$$m(x) = \sup_{\bar{x} \in \bar{R}} \{ \bar{x}(x) - \overline{m(\bar{x})} \}$$

**Proof.** For any $0 \neq \bar{a} \in \bar{R}$ and $\nu=1,2,\ldots$, putting

$$m_{\nu}(x) = \inf_{|x|=|y|+|z|} \max \{ m(y), 2^\nu |\bar{a}|(|z|) \}$$

for $x \in [\bar{a}]R$, we obtain a monotone modular $m_{\nu}$ on $[\bar{a}]R$. Indeed we see easily that $m_{\nu}$ satisfies the all postulates except for 4). If $m_{\nu}(x)=0$ and $x \in [\bar{a}]R$, then we can find $0 \leq y_\mu, z_\mu \in R (\mu=1,2,\ldots)$ such that

$$|x| = y_\mu + z_\mu.$$ 

Max $\{ m(y_\mu), 2^\nu |\bar{a}|(z_\mu) \} \leq \frac{1}{2^{\nu\mu}}$

and putting $u_\mu = \cup z_\mu (\mu=1,2,\ldots)$, we have

$$2^\nu |\bar{a}|(u_\mu) \leq \frac{1}{2^{\nu\mu}} (\mu=1,2,\ldots)$$

and hence $2^\nu |\bar{a}|(\bigcap_{\mu=1}^\infty u_\mu) = 0$. This relation yields $\bigcap_{\mu=1}^\infty u_\mu = 0$, that is, $u_\mu \downarrow_{\mu=1}^\infty 0$. Thus we have $|x| - u_\mu \uparrow_{\mu=1}^\infty |x|$ and

$$m(|x| - u_\mu) \leq m(y_\mu) \leq \frac{1}{2^{\nu\mu}} (\mu=1,2,\ldots).$$

Therefore we obtain $m(x)=0$, because $m$ is upper semi-continuous by assumption, and we conclude that $m_{\nu}(x)=0$ and $x \in [\bar{a}]R$ implies $m(x)=0$. Consequently the postulate 4) also is satisfied.

The modular $m_{\nu}$ on $[\bar{a}]R$ is continuous for every $\nu=1,2,\ldots$, because we have obviously

$$m_{\nu}(x) \leq 2^\nu |\bar{a}|(|x|)$$

for every $x \in [\bar{a}]R$.

Thus the modular associated space $\bar{R}_\nu$ of $[\bar{a}]R$ by $m_{\nu}$ coincides with the modular conjugate space of $[\bar{a}]R$ by $m_{\nu}$ and hence $\bar{R}_\nu$ is included in the modular conjugate space $\bar{R}$ of $R$ by $m$, because we have obviously

$$m_{\nu}(x) \leq m(x)$$

for every $x \in [\bar{a}]R$.

Therefore we have for every $x \in [\bar{a}]R$
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\[ m_\nu(x) = \sup_{\bar{x} \in \bar{R}_\nu} \{ \bar{x}(x) - m_\nu(\bar{x}) \} \leq \sup_{x \in [a]R} \{ \bar{x}(x) - m_\nu(x) \} = m_\nu(\bar{x}). \]

because we have for \( \bar{x} \in \bar{R}_\nu \)

\[ m(\bar{x}) = \sup_{x \in [a]R} \{ \bar{x}(x) - m(x) \} \leq \sup_{x \in [a]R} \{ \bar{x}(x) - m_\nu(x) \} = m_\nu(\bar{x}). \]

On the other hand we have

\[ \lim_{\nu \to \infty} m_\nu(x) = m(x) \quad \text{for every } x \in [\bar{a}]R. \]

Because, for any \( x \in [\bar{a}]R \) we can find \( 0 \leq y_\nu, z_\nu \in R (\nu = 1, 2, \ldots) \) such that

\[ |x| = y_\nu + z_\nu, \quad m(y_\nu) \leq m_\nu(x) + \frac{1}{2^\nu}, \quad 2^\nu |a| (z_\nu) \leq m_\nu(x) + \frac{1}{2^\nu}. \]

Then putting \( u_\nu = \bigcup_{\rho \geq \nu} z_\rho (\nu = 1, 2, \ldots) \), we conclude \( u_\nu \downarrow \nu = 1 \infty 0 \) and

\[ m(|x| - u_\nu) \leq m(y_\nu) \leq m_\nu(x) + \frac{1}{2^\nu} \leq m(x) + \frac{1}{2^\nu}, \]

as obtained above. This relation yields \( m(x) = \lim_{\nu \to \infty} m_\nu(x) \). Therefore we conclude

\[ m(x) \leq \sup_{x \in R} \{ \bar{x}(x) - m(\bar{x}) \} \quad \text{for every } x \in [\bar{a}]R. \]

Since \( R \) is semi-regular by assumption, we have \( [\bar{a}]x \uparrow [\bar{a}] \in R \), and hence we obtain furthermore

\[ m(x) \leq \sup_{x \in R} \{ \bar{x}(x) - m(\bar{x}) \} \quad \text{for every } x \in R. \]

On the other hand it is obvious by definition

\[ m(x) \geq \sup_{x \in R} \{ \bar{x}(x) - m(\bar{x}) \} \quad \text{for every } x \in R. \]

Thus we conclude our assertion.

Recalling Theorem 2.4, we obtain immediately

Theorem 3.2. If \( R \) is semi-regular and \( m \) is upper semi-continuous and monotone complete, then \( R \) is reflexive and the modular conjugate space of \( R \) by \( m \) coincides with the conjugate space of \( R \).

\[ \S \text{4. Semi-additive modulars} \]

A modular \( m \) on a lattice ordered linear space \( R \) is said to be upper semi-additive, if \( m \) is monotone and

\[ m(a + b) \geq m(a) + m(b) \quad \text{for } 0 \leq a, b \in R. \]

Theorem 4.1. If an upper semi-additive modular \( m \) is upper semi-
continuous, then \( m \) is lower semi-continuous, and hence semi-continuous.

**Proof.** For \( x_{\lambda} \downarrow \lambda \in A, m(x_{\lambda}) < +\infty (\lambda \in A) \) we have

\[
m(x_{\lambda}) \leq m(x_{\lambda}) - m(x_{\lambda} - x_{\lambda}),
\]

for \( x_{\lambda} \leq x_{\lambda} \), because \( m \) is upper semi-additive by assumption. Since

\[
x_{\lambda} - x_{\lambda} \uparrow x_{\lambda} \leftarrow x_{\lambda} \\
\]

and \( m \) is upper semi-continuous by assumption, we have

\[
\sup_{x \leq x_{\lambda}} m(x_{\lambda} - x_{\lambda}) = m(x_{\lambda}).
\]

Thus we obtain \( m(x_{\lambda}) \downarrow \lambda \in A, 0 \).

A modular \( m \) on \( R \) is said to be lower semi-additive, if \( m \) is monotone and

\[
m(a \cdot b) \leq m(a) + m(b) \quad \text{for} \quad 0 \leq a, b \in R.
\]

A modular \( m \) on \( R \) is said to be additive, if \( m \) is upper and lower semi-additive simultaneously. Additive modulars are discussed in detail already in [2]. When \( R \) is universally continuous, if a modular \( m \) on \( R \) is upper semi-continuous and

\[
m(a + b) = m(a) + m(b) \quad \text{for} \quad a \leftrightarrow b = 0,
\]

then \( m \) is additive. (cf. [2])

**Theorem 4.2.** The conjugate modulars of the upper semi-additive modulars are lower semi-additive, and the conjugate modulars of the lower semi-additive modulars are upper semi-additive.

**Proof.** If a modular \( m \) on \( R \) is upper semi-additive, then for the conjugate modular \( \overline{m} \) of \( m \) and the modular associated space \( \overline{R} \) of \( R \) we have by definition for \( 0 \leq x, y \in \overline{R} \)

\[
\overline{m}(x) + \overline{m}(y) = \sup_{x, y \in \overline{R}} \{ x(x) + y(y) - m(x) - m(y) \}
\]

\[
\geq \sup_{0 \leq x, y \in R} \sup_{a, y \geq 0} \{ x(a) + y(y) - m(x) - m(y) \}
\]

\[
= \sup_{a \in R} \{ x(a) - m(x) \} = \overline{m}(x),
\]

and hence \( \overline{m} \) is lower semi-additive by definition. If \( m \) is lower semi-additive, then we have by definition for \( 0 \leq x, y \in \overline{R} \)

\[
\overline{m}(x) + \overline{m}(y) = \sup_{0 \leq x, y \in R} \{ x(x) + y(y) - m(x) - m(y) \}
\]

\[
\leq \sup_{0 \leq x, y \in R} \{ x(x) + y(y) - m(x) - m(y) \}
\]
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\[ \sup_{x \in R} \{ \bar{x}(z) + \bar{y}(z) - m(z) \} = \bar{m}(\bar{x} + \bar{y}) , \]

and hence \( \bar{m} \) is upper semi-additive by definition.

§ 5. Bimodulars

Let \( R, S \) be two lattice ordered linear spaces. A functional \( M(x, y) \) \((x \in R, y \in S)\) is said to be a bimodular, if \( M(x, y) \) is an additive upper semi-continuous modular on \( R \) for every fixed \( 0 \leq y \in S \),

\[
M(x, |y_1| + |y_2|) = M(x, y_1) + M(x, y_2),
\]

and for any \( x \in R \) we can find a positive number \( \alpha \) such that

\[
M(\alpha x, y) < +\infty \quad \text{for every } y \in S.
\]

A bimodular \( M(x, y) (x \in R, y \in S) \) is said to be finite, if

\[
M(x, y) < +\infty \quad \text{for every } x \in R, y \in S.
\]

If \( S \) is a normed space and complete, then putting

\[
m(x) = \sup_{\|y\| \leq 1} M(x, y) \quad (x \in R, y \in S),
\]

we obtain a modular \( m \) on \( R \). This modular \( m \) is said to be a norm-modular of \( M \) by the norm of \( S \).

Theorem 5.1. Every norm-modular of a bimodular \( M(x, y) (x \in R, y \in S) \) is lower semi-additive and upper semi-continuous.

Proof. For \( 0 \leq x_1, x_2 \in R \) we have by definition

\[
m(x_1 - x_2) = \sup_{\|y\| \leq 1} M(x_1 - x_2, y)
\]

\[
\leq \sup_{\|y\| \leq 1} M(x_1, y) + \sup_{\|y\| \leq 1} M(x_2, y) = m(x_1) + m(x_2),
\]

because \( M(x_1 - x_2, y) \leq M(x_1, y) + M(x_2, y) \). Thus the norm-modular \( m \) is lower semi-additive. For \( 0 \leq x_{\lambda} \rfloor_{\lambda \in \Lambda} x \) we have by definition

\[
m(x) = \sup_{\|y\| \leq 1} M(x, y) = \sup \{ \sup_{\|y\| \leq 1} M(x_{\lambda}, y) \}
\]

\[
= \sup_{\lambda \in \Lambda} \{ \sup_{\|y\| \leq 1} M(x_{\lambda}, y) \} = \sup_{\lambda \in \Lambda} m(x_{\lambda}).
\]

Thus the norm-modular \( m \) is upper semi-continuous by definition.

Theorem 5.2. If a bimodular \( M(x, y) (x \in R, y \in S) \) is finite, then the norm-modular of \( M \) is finite.

Proof. For each \( x \in R \), since \( M(x, y) < +\infty \) by assumption, putting
$x(y) = M(x, y^+) - M(x, y^-) \quad (y \in S)$

we obtain a positive linear functional $x$ on $S$. Since the norm of $S$ is complete by assumption, this linear functional $x$ on $S$ is norm bounded, and hence

$$\sup_{\|y\| \leq 1} M(x, y) < +\infty \quad \text{for every } x \in R.$$ 

Thus the norm-modular of $M$ is finite by definition.

For an additive complete modular $m_s$ on $S$, putting

$$m(x) = \sup_{y \in S} \{M(x, y) - m_s(y)\} \quad (x \in R)$$

we obtain a monotone modular $m$ on $R$. This modular $m$ on $R$ is said to be the double-modular of $M$ by $m_s$.

**Theorem 5.3.** Every double-modular of a bimodular $M(x, y) (x \in R, y \in S)$ is upper semi-additive and semi-continuous.

**Proof.** For $0 \leq x_1, x_2 \in R$ we have by definition

$$m(x_1 + x_2) = \sup_{y \in S} \{M(x_1 + x_2, y) - m_s(y)\}$$

\[ \geq \sup_{y \in S} \{M(x_1, y) + M(x_2, y) - m_s(y)\} \]

\[ \geq \sup_{0 \leq y_1, y_2 \in S} \{M(x_1, y_1) + M(x_2, y_2) - m_s(y_1 \cup y_2)\} \]

\[ \geq \sup_{0 \leq y_1, y_2 \in S} \{M(x_1, y_1) + M(x_2, y_2) - m_s(y_1) - m_s(y_2)\} \]

because

$$M(x_1 + x_2, y) \geq M(x_1, y) + M(x_2, y),$$

$$m_s(y_1 \cup y_2) \leq m_s(y_1) + m_s(y_2).$$

Thus the double-modular $m$ is upper semi-additive. For $0 \leq x_\lambda \in \Lambda \in A \times x$ we have by definition

$$m(x) = \sup_{y \in S} \left\{ \sup_{\lambda \in \Lambda} \{M(x_\lambda, y) - m_s(y)\} \right\} = \sup_{\lambda \in \Lambda} m(x_\lambda).$$

Thus $m$ is upper semi-continuous. Recalling Theorem 4.1, we conclude therefore that $m$ is semi-continuous.

**Theorem 5.4.** Let $m_s$ be a complete, additive modular on $S$. For a bimodular $M(x, y) (x \in R, y \in S)$, denoting by $m_d$ the double-modular of $M$ by $m_s$ and by $m_n$ the norm-modular of $M$ by the first norm of $m_s$, then we have

$$m_d(x) \leq m_n(x) \quad \text{for} \quad m_n(x) \leq 1,$$

$$m_d(x) \geq m_n(x) \quad \text{for} \quad m_n(x) \geq 1,$$
and the second norm of $m_d$ coincides with that of $m_n$.

**Proof.** If $M(x, y) < +\infty$ for every $y \in S$, then putting

$$x(y) = M(x, y^+) - M(x, y^-)$$

($y \in S$)

we obtain a positive linear functional $x$ on $S$. Since the modular $m_S$ is complete by assumption, this linear functional $x$ is modular bounded. Thus we have by definition

$$m_d(x) = \overline{m}_S(x), \quad m_n(x) = \|x\|$$

for the conjugate modular $\overline{m}_S$ of $m_S$ and the second norm $\|x\|$ of $\overline{m}_S$. If $M(x, y) = +\infty$ for some $y \in S$, then we have obviously by definition

$$m_d(x) = m_n(x) = +\infty$$

Therefore we conclude that $m_n(x) \leq 1$ implies $m_d(x) \leq m_n(x)$, and that $m_n(x) \geq 1$ implies $m_d(x) \geq m_n(x)$. Consequently the second norm of $m_d$ coincides with that of $m_n$.

§ 6. Proper bimodular

Let $m$ be an additive upper semi-continuous modular on a universally continuous semi-ordered linear space $R$, and $\mathcal{S}$ the proper space of $R$. We denote by $D_m$ the totality of such dilatators $T$ in $R$ that for any $x \in R$ we can find a positive number $a$ for which

$$\int_{\mathcal{S}} \left(\frac{|T|}{1}, p\right) m(dp_X) < +\infty.$$

Then, putting

$$M_m(x, T) = \int_{\mathcal{S}} \left(\frac{|T|}{1}, p\right) m(dp_X)$$

($x \in R$, $T \in D_m$)

we obtain a bimodular $M_m$. Here we see easily that $D_m$ is a semi-normal manifold of the dilatator space and $1 \in D_m$, because $M_m(x, 1) = m(x)$. This bimodular $M_m$ is said to be the proper bimodular of $m$.

For a semi-normal manifold $D$ of $D_m$ containing $1$, and for a complete norm $\|T\|$ ($T \in D$) on $D$, putting

$$m_n(x) = \sup_{\|T\| \leq 1, T \in D} M_m(x, T)$$

($x \in R$), we obtain a norm-modular $m_n$ of $M_m$.

**Theorem 6.1.** If the modular $m$ on $R$ is monotone complete, then every norm-modular of the proper bimodular $M_m$ of $m$ also is monotone complete.
Proof. If $0 \leq x_\lambda \uparrow_{\lambda \in \Lambda}$, $\sup_{\lambda \in \Lambda} m_n(x_\lambda) < +\infty$, then we have by definition
\[ \sup_{\lambda \in \Lambda} m(x_\lambda) = \sup_{\lambda \in \Lambda} M_m(x_\lambda, 1) < +\infty \]
and hence $x_\lambda$ ($\lambda \in \Lambda$) is bounded, because $m$ is monotone complete by assumption. Therefore the norm-modular $m_n$ also is monotone complete.

For a complete, additive modular $m_D(T)$ ($T \in D$) on $D$, putting
\[ m_d(x) = \sup_{x \in D} \{ M_m(x, T) - m_D(T) \} \quad (x \in R), \]
we obtain a double-modular $m_d$ of $M_m$.

Theorem 6.2. Every double-modular of the proper bimodular $M_m$ of $m$ also is additive.
Proof. If $M_m(x, T) < +\infty$ for every $T \in D$, then, putting
\[ x(T) = \int_{\mathfrak{E}} \left( \frac{T}{1}, y \right) m(d\mathfrak{p}x) \quad (T \in D), \]
we obtain a positive linear functional $x(T)$ ($T \in D$) on $D$. Furthermore if $x \sim y = 0$, $M_m(x, T) < +\infty$, $M_m(y, T) < +\infty$ for every $T \in D$, then we also have $x \sim y = 0$ considering both $x$ and $y$ linear functionals on $D$, and hence
\[ \overline{m}_D(x + y) = \overline{m}_D(x) + \overline{m}_D(y) \]
for the conjugate modular $\overline{m}_D$ of $m_D$, because $m_D$ is additive by assumption. On the other hand we have by definition
\[ m_d(x) = \begin{cases} \overline{m}_D(x) & \text{if } M_m(x, T) < +\infty \quad \text{for every } T \in D, \\ +\infty & \text{if } M_m(x, T) = +\infty \quad \text{for some } T \in D. \end{cases} \]
Thus we conclude that $x \sim y = 0$ implies $m_d(x + y) = m_d(x) + m_d(y)$. Therefore the double-modular $m_d$ is additive. (cf. [2])

Theorem 6.3. If the modular $m$ on $R$ is monotone complete, then every double-modular of the proper bimodular $M_m$ of $m$ also is monotone complete.
Proof. For an additive complete modular $m_D$ on $D$, we can find a positive number $\alpha$ such that $m_D(\alpha) < +\infty$ considering $\alpha$ a dilatator in $R$. If $0 \leq x_\lambda \uparrow_{\lambda \in \Lambda}$ and $\sup_{\lambda \in \Lambda} m_d(x_\lambda) < +\infty$, then we have
\[ \sup_{\lambda \in \Lambda} m(x_\lambda) = \frac{1}{\alpha} \sup_{\lambda \in \Lambda} M_m(x_\lambda, \alpha) \leq \frac{1}{\alpha} \left\{ \sup_{\lambda \in \Lambda} m_d(x_\lambda) + m_D(\alpha) \right\} < +\infty, \]
and hence $x_\lambda$ ($\lambda \in \Lambda$) is bounded, because $m$ is monotone complete by
assumption. Therefore the double-modular $m_d$ also is monotone complete by definition.

References

