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ON THE UNCONDITIONAL CONVERGENCE
IN SEMI-ORDERED LINEAR SPACES

By

Ichiro AMEMIYA

In continuous semi-ordered linear spaces, an unconditionally convergent series is not, in general, absolutely convergent. In this paper we shall give on the one hand, a sufficient condition of the space for this with some examples of the spaces satisfying it; the space of $M$-type, the space of all functions on a set, the sequence space $c_0$ and etc.. On the other hand, we shall show in every infinite-dimensional space of $L_p$-type there exists a series which is unconditionally and not absolutely convergent. On account of it, we shall give a necessary and sufficient condition for monotone complete normed spaces under which two notions of convergence are equivalent.

Let $R$ be a continuous semi-ordered linear space throughout the paper.

1. Let $a_\nu (\nu =1,2, \cdots )$ be a sequence of elements in $R$. For every finite set $J$ of natural numbers, we write $\sigma (J)$ for $\sum_{\nu \in J} a_\nu$, then the absolute convergence of the series $\sum_{\nu =1}^\infty a_\nu$ is equivalent to the order-boundness of all $\sigma (J)$, because we have always

$$\{ \bigcup_{J \subset J_0} \sigma (J) \}^+ = \sum_{\nu \in J_0} a_\nu^+ .$$

If $\sum a_\nu$ is unconditionally convergent, then for every mutually orthogonal $J_\nu$, $\sigma (J_\nu)$ is order-bounded, since there exists a rearrangement of the series in which every $J_\nu$ consists of some successive

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1) Convergence means order-convergence. Cf. [1].
2) The space with the norm and an element $e \geq 0$ for which $\| x \| \leq 1$ is equivalent to $| x | \leq e$.
3) The space of all numerical sequences that converge to 0.
4) A normed space is said to be $L_p$-type if we have $\| x+y \|_p = | x |^p + | y |^p$ for every mutually orthogonal $x, y$.
5) A norm is said to be monotone complete if $0 \leq a_\lambda$ for all $\lambda$, $\sup_{\lambda \in A} \| a_\lambda \| < +\infty$, imply the existence of $\bigcap_{\lambda \in A} a_\lambda$.
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numbers.

Let $\mathcal{B}$ be the collection of all order-bounded sets, then $\mathcal{B}$ is an ideal in the lattice of all subsets in $\mathcal{L}$. For an ideal $\mathcal{I}$, we write $\mathcal{I}^\perp$ for the ideal generated by all the sets

$$A + B = \{x + y; \ x \in A, \ y \in B\} \quad \text{for } A, B \in \mathcal{I}.$$ 

Now we shall prove that if $\mathcal{B}$ is the intersection of all the ideal $\mathcal{I}$, such that $\mathcal{I}$ has a countable basis and $\mathcal{B} \subset \mathcal{I}$, then every unconditionally convergent series converges also absolutely.

In fact, suppose the set $\{\sigma(J)\}$ is not order-bounded, and an ideal $\mathcal{I}$ generated by all the sets $A + B = \{x + y; x \in A, y \in B\}$ for $A, B \in \mathcal{B}$. Then for every $\mathcal{I}$, the set $\{\sigma(J); J \sim J_0 = 0\}$ is not in $\mathcal{I}$, and so we can find a mutually orthogonal $J_\nu$ such that $\sigma(J_\nu) \in A_\nu (\nu = 1, 2, \cdots)$, that is, a sequence $\{\sigma(J_\nu)\}$ is not order-bounded. Thus the series can not be unconditionally convergent.

If the order-boundness coincides with the topological boundness by some linear topology in $\mathcal{L}$, then the condition above for $\mathcal{B}$ is evidently satisfied. The space of $M$-type and the space of all functions on a set have this property.

If $\mathcal{L}$ is generated by countable order-bounded sets, then $\mathcal{B}$ itself has a countable basis. The "Stufenraum" of G. Köthe [2] is an example of the space of this type (it also satisfies the topological condition above).

The sequence space $c_0$ is also in our case. In fact, for every $\epsilon > 0$, let $\mathcal{I}_\epsilon$ be the totality of the set $A$ for which there exists a fixed natural number $\nu_0$ such that $\{\xi_\nu\} \in A$ implies $|\xi_\nu| < \epsilon$ for every $\nu \geq \nu_0$, then $\mathcal{I}_\epsilon$ is an ideal with countable basis and we have $\mathcal{B} = \bigcap_{\epsilon < 0} \mathcal{I}_\epsilon$ since $\mathcal{I}_{2\epsilon} \subset \mathcal{I}_{\epsilon}$.

2. In the sequel we shall restrict ourselves to the case $\mathcal{L}$ is a normed semi-ordered linear space where the norm is monotone complete. We put

$$\alpha = \alpha(R) = \inf_{a_1, a_2, \cdots, a_n} \sup_{i_1, i_2, \cdots, i_n} \frac{||a(\hat{i}_1, \hat{i}_2, \cdots, \hat{i}_n)||}{|||a_1| + |a_2| + \cdots + |a_n|||},$$

where the infimum is taken for every finite set $a_1, a_2, \cdots, a_n$ in $\mathcal{L}$, the supremum for every permutation $i_1, i_2, \cdots, i_n$ of 1, 2, \cdots, $n$ and $a(\hat{i}_1, \hat{i}_2, \cdots, \hat{i}_n) = |a_{i_1}| |a_{i_2} + a_{i_1}| \cdots |a_{i_2} + a_{i_1} + \cdots + a_{i_n}|$. We have always $0 \leq \alpha \leq \frac{1}{2}$ and for every space of $M$-type, as easily be seen, $\alpha = \frac{1}{2}$. 

\[ a(N) \] is also defined for every normal manifold \( N \) of \( R \) and we can prove easily the following relations:

1. \( a(N), a(M) \geq \varepsilon \) and \( N \perp M \) imply \( a(N + M) \geq \frac{\varepsilon}{2} \),

2. \( N \uparrow \lambda \in A N \) and \( a(N) = 0 \) imply \( \inf_{\lambda} a(N_{\lambda}) = 0 \).

Now we shall prove that the two notions of convergence are equivalent if and only if \( a > 0 \).

If \( a > 0 \) and \( \sum a_{\nu} \) is not absolutely convergent, then by the monotone completeness of the norm, we have
\[
\| a_{\mu+1} + |a_{\mu+2}| + \cdots |a_{\mu+n}| \uparrow \nu + \infty \quad \text{for every } \mu = 1, 2, \ldots .
\]
Since for some permutation \( i_{1}, i_{2}, \ldots i_{\nu} \) of \( 1, 2, \ldots \nu \) we have
\[
\| a(i_{1}, i_{2}, \ldots i_{\nu}) \| \geq \frac{a}{2} \| a_{\mu+1} + |a_{\mu+2}| + \cdots |a_{\mu+n}| \|
\]
where \( a(i_{1}, i_{2}, \ldots i_{\nu}) \) is as defined above for \( a_{\nu+1}, a_{\mu+2}, \ldots a_{\mu+n} \), we can find a rearrangement of \( a_{\nu}(\nu = 1, 2, \ldots) \) such that all the sum of successive elements are not order-bounded, and hence, \( \sum a_{\nu} \) is not unconditionally convergent.

Next we suppose \( a = 0 \). If for every normal manifold \( N \), either \( a(N) \) or \( a(N^\perp) \) is not 0 (anyway one of them is 0 by (1)), then \( \{N; a(N) = 0\} \) is a maximal dual ideal of the Boolean algebra of all normal manifolds and this ideal is not atomic since \( a = \frac{1}{2} \) for every one dimensional space, and hence we have by (2) \( \inf_{a(N) > 0} a(N) = 0 \). Therefore for every \( N \) such that \( a(N) = 0 \), we can devide \( N \) into two orthogonal normal manifolds \( N', N'' \) where one of \( a(N') \) and \( a(N'') \) is 0 and the other is either 0 or arbitrarily small, and hence we can find mutually orthogonal normal manifolds \( N_{\nu} \neq 0 (\nu = 1, 2, \ldots) \) such that \( a(N_{\nu}) < \frac{1}{\nu 2^{\nu}} \).

Now from every \( N_{\nu} \) we take a finite sequence of elements \( a_{1}, a_{2}, \ldots a_{n} \) such that \( \| a_{1} + a_{2} + \cdots + a_{n} \| = \nu \) and for every permutation \( i_{1}, i_{2}, \ldots i_{n} \), \( \| a(i_{1}, i_{2}, \ldots i_{n}) \| \leq \frac{1}{2^{\nu-1}} \) then putting such a sequence \( a_{1}, a_{2}, \ldots a_{n} \) one after another for \( \nu = 1, 2, \ldots \), we obtain an infinite sequence of which the series is obviously unconditionally and not absolutely convergent.

\[ \alpha \] To prove this we may make use of the following property of the monotone complete norm, that is, there exists \( \varepsilon > 3 \) such that
\[
0 \leq a_{\lambda} \uparrow A a \quad \text{implies always } \sup_{\lambda} a_{\lambda} \geq \varepsilon \| a \|. \quad \text{Cf. [3].} \]
Thus the proof was completed.

3. Here we shall show \( a = 0 \) for every infinite-dimensional space of \( L_p \)-type \( (1 \leq p < \infty) \).

For this purpose we shall compute \( a = a_n \) for the \( 2^n \)-dimensional space where the norm of an element \( x \equiv \{ \xi_m \} \) is defined as

\[
||x|| = \sum_{m=1}^{2^n} |\xi_m|^p .
\]

Let \( M_n \) be the \((n, 2^n)\)-matrix below

\[
\begin{pmatrix}
+1 & +1 & \cdots & +1 & -1 & -1 & \cdots & -1 \\
+1 & +1 & \cdots & +1 & -1 & -1 & \cdots & -1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
+1 & -1 & -1 & +1 & +1 & \cdots & \cdots & \cdots \\
+1 & 1 & -1 & 1 & +1 & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

that is, for the \( i^{th} \) row, the first \( 2^{i-1} \) are all \( +1 \) and the next \( 2^{i-1} \) are \( -1 \) and so on alternatively. Let \( a_i \) be the \( i^{th} \) row-vector, then for every permutation \( i_1, i_2, \cdots i_n \) of \( 1, 2, \cdots n \), there exists a permutation of the co-ordinate axes by which the sequence \( a_1, a_2, \cdots a_n \) is transformed to \( a_{i_1}, a_{i_2}, \cdots a_{i_n} \), since any of \( 2^n \) possible sequences of \( \pm 1 \) emerges as a column of \( M_n \).

Consequently we have

\[
||a(i_1, i_2, \cdots i_n)|| = ||a(1, 2, \cdots n)|| ,
\]

making use of the same notation as in 2.

Putting

\[
a(+) = a_{1}^{+} \cup (a_{1} + a_{2})^{+} \cup \cdots (a_{1} + a_{2} + \cdots a_{n})^{+} ,
\]

and

\[
a(-) = a_{1}^{-} \cup (a_{1} + a_{2})^{-} \cup \cdots (a_{1} + a_{2} + \cdots a_{n})^{-} ,
\]

we have

\[
a(1, 2, \cdots n) = a(+) \cup a(-) , \quad ||a(+)|| = ||a(-)|| ,
\]

and hence

\[
||a(i_1, i_2, \cdots i_n)|| \leq 2||a(+)|| , \quad \text{or by the definition of} \ a_n ,
\]

(1) \( 2^{n-1}n^p a_n \leq ||a(+)|| .

In the sequel we write \( a^{(n)} \) for \( a(+) \) and \( \beta_n \) for \( ||a^{(n)}|| \).

The matrix \( M_{n+1} \) has the form

\[
\begin{pmatrix}
+1 & +1 & \cdots & +1 & -1 & -1 & \cdots & -1 \\
M_n & M_n \\
\end{pmatrix}
\]

so if \( a^{(n)} \equiv \{ \xi_1, \xi_2, \cdots \xi_{2^n} \} \), then we can see easily that
\[ a^{(n+1)} \equiv \{ \xi_1 + 1, \xi_2 + 1, \cdots \xi_{2^n} + 1, \xi_1 + \epsilon_1, \xi_2 + \epsilon_2, \cdots \} \]

where \[ \epsilon_m = \begin{cases} -1 & \text{for every } m \text{ such that } \xi_m > 0 \\ 0 & \text{elsewhere} \end{cases} \]

Since \[ \beta_n = \sum_{m=1}^{2^n} \xi_m \]
we have

\[ \beta_{n+1} = 2\beta_n + \tau_n + \sum_{\xi_m > 0} \{(\xi_m + 1)^p + (\xi_m - 1)^p - 2\xi_m^p\} \]

where \( \tau_n \) is the number of those \( m \)'s for which \( \xi_m = 0 \).

Now we divide the three cases.

(i) \( 1 < p < 2 \): Then \((x + 1)^p + (x - 1)^p - 2x^p\) is a decreasing function of \( x \geq 1 \), and hence the third term in the right side of (2) is smaller than \( 2^n(2^p - 2) \). Therefore we have the inequality

\[ \beta_{n+1} \leq 2\beta_n + 2^n(2^p - 1), \]

or since \( \beta_1 = 1 \),

\[ \beta_{n+1} \leq 2^n + n2^n(2^p - 1) = 2^nO(n), \]

and hence by (1)

\[ a_n = O(n^{1-p}). \]

(ii) \( 2 \leq p \): Then \((x + 1)^p + (x - 1)^p - 2x^p\) is increasing and so we have, since \((n + 1)^p + (n - 1)^p - 2n^p = O(n^{p-2})\),

\[ \beta_{n+1} \leq 2\beta_n + 2^nO(n^{p-2}) \]

or

\[ \beta_{n+1} \leq 2^n + n2^nO(n^{p-2}) = 2^nO(n^{p-1}) \]

that is,

\[ a_n = O(n^{-1}). \]

(iii) \( p = 1 \): Here the third term in the right side of (2) vanishes and we have

\[ \beta_{n+1} = 2^n + \sum_{r=1}^{n} 2^{n-r}\tau_r. \]

If we had proved that the second term of the right side does not exceed \( 2^{n+1}\sqrt{n} \), then we can conclude from that

\[ a_n = O(n^{-\frac{1}{2}}). \]

Let \( \epsilon_1, \epsilon_2, \cdots \epsilon_n (\epsilon_i = \pm 1) \) be the \( m^{th} \) column of \( M_n \) for an arbitrary \( m = 1, 2, \cdots 2^n \), then \( \xi_m = 0 \) if and only if
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\[ \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i \leq 0 \quad \text{for every } i = 1, 2, \ldots \]

If \( r \) is the smallest of those numbers for which \( \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r \) has the least value of all \( \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i \), then \( \varepsilon_r = -1 \) and both the sequence \(-\varepsilon_{r+1}, -\varepsilon_{r+2}, \ldots -\varepsilon_n \) and \( \varepsilon_{r-1}, \varepsilon_{r-2}, \ldots \varepsilon_1 \) has the same property as mentioned above, that is, every partial sum of them does not exceed 0, and these conditions determine \( r \). If we put \( \varphi(m) = r \) where \( r \) is defined above, then since there exist \( r_{n-r} \) number of possible case for \( \varepsilon_{r+1}, \varepsilon_{r+2}, \ldots \varepsilon_n \), and \( r_{r-1} \) for \( \varepsilon_{r-1}, \varepsilon_{r-2}, \ldots \varepsilon_1 \), the number of those \( m \)'s for which \( \varphi(m) = r \) is \( r_{n-r} r_{r-1} (r_0 = 1) \). Since then \( \sum_{r=1}^{n} r_{n-r} r_{r-1} = 2^n \) we have, putting \( r = \sum_{r=1}^{n} 2^{n-r} \gamma_r \),

\[ \gamma^2 = \sum_{r,s=1}^{n} 2^{2n-r-s} \gamma_r \gamma_s \leq \sum_{r=1}^{2n} \sum_{s=1}^{r} 2^{2n-r+1} \gamma_r \gamma_{r-1} = 2n 2^{2n+1}, \]

that is, \( \gamma \leq 2^{n+1} \sqrt{n} \) as required.

Therefore we have always

\[ \lim_{n \to \infty} a_n = 0, \]

and since in our space we can find a subspace which is isomorphic to the \( 2^n \)-dimensional space considered above for every \( n \), the proof was completed.

To construct an example of the series which is unconditionally and not absolutely convergent in the sequence space \( l_p \), let us decompose the space into mutually orthogonal normal manifolds \( N_n (n = 1, 2, \ldots) \) where \( N_n \) is of \( 2^n \)-dimension, and in every \( N_n \) take a sequence \( a_1, a_2, \ldots a_n \) as constructed above but multiplying them by \( 2^{-\frac{n}{p}} n^{-\frac{1}{p} + \frac{1}{p}} \), then the union of all such sequences for \( n = 1, 2, \ldots \) provides our example.

(September, 1954)

References