ON THE UNCONDITIONAL CONVERGENCE IN SEMI-ORDERED LINEAR SPACES

By

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In continuous semi-ordered linear spaces, an unconditionally convergent[1] series is not, in general, absolutely convergent. In this paper we shall give on the one hand, a sufficient condition of the space for this with some examples of the spaces satisfying it; the space of M-type,[2] the space of all functions on a set, the sequence space c_0 [3] and etc.. On the other hand, we shall show in every infinite-dimensional space of L_p-type[4] there exists a series which is unconditionally and not absolutely convergent. On account of it, we shall give a necessary and sufficient condition for monotone complete[5] normed spaces under which two notions of convergence are equivalent.

Let R be a continuous semi-ordered linear space throughout the paper.

1. Let $a_\nu (\nu=1,2,\cdots)$ be a sequence of elements in R. For every finite set J of natural numbers, we write $\sigma(J)$ for $\sum_{\nu\in J} a_\nu$, then the absolute convergence of the series $\sum_{\nu=1}^{\infty} a_\nu$ is equivalent to the order-boundness of all $\sigma(J)$, because we have always

$$\{\bigcup_{J\subset J_0} \sigma(J)\}^+ = \sum_{\nu\in J_0} a_\nu^+ .$$

If $\sum a_\nu$ is unconditionally convergent, then for every mutually orthogonal J_ν, $\sigma(J_\nu)$ is order-bounded, since there exists a rearrangement of the series in which every J_ν consists of some succesive

1) Convergence means order-convergence. Cf. [1].

2) The space with the norm and an element $e\geqslant 0$ for which $\|x\|\leqslant 1$ is equivalent to $|x|\leqslant e$.

3) The space of all numerical sequences that converge to 0.

4) A normed space is said to be L_p-type if we have $\|x+y\|^p=\|x\|^p+\|y\|^p$ for every mutually orthogonal x, y.

5) A norm is said to be monotone complete if $0\leqslant a_\lambda \uparrow_{\lambda\in\Lambda}$, $\sup_{\lambda\in\Lambda}\|a_\lambda\|<+\infty$, imply the existence of $\bigcap_{\lambda\in\Lambda} a_\lambda$.

numbers.

Let $\mathfrak{B}$ be the collection of all order-bounded sets, then $\mathfrak{B}$ is an ideal in the lattice of all subsets in R. For an ideal $\mathfrak{A}$, we write $\mathfrak{A}^2$ for the ideal generated by all the sets

$$A+B=\{x+y;\ x\in A,\ y\in B\} \qquad \text{for } A, B\in\mathfrak{A}.$$

Now we shall prove that *if* $\mathfrak{B}$ *is the intersection of all the ideal* $\mathfrak{A}^2$, *such that* $\mathfrak{A}$ *has a countable basis and* $\mathfrak{B}\subset\mathfrak{A}$, *then every unconditionally convergent series converges also absolutely.*

In fact, suppose the set $\{\sigma(J)\}$ is not order-bounded, and an ideal $\mathfrak{A}$ generated by A_ν $(\nu=1,2,\cdots)$ is such that $\{\sigma(J)\}\notin\mathfrak{A}^2$, then for every J_0, the set $\{\sigma(J);\ J\frown J_0=\emptyset\}$ is not in $\mathfrak{A}$, and so we can find a mutually orthogonal J_ν such that $\sigma(J_\nu)\notin A_\nu$ $(\nu=1,2,\cdots)$, that is, a sequence $\{\sigma(J_\nu)\}$ is not order-bounded. Thus the series can not be unconditionally convergent.

If the order-boundness coincides with the topological boundness by some linear topology in R, then the condition above for $\mathfrak{B}$ is evidently satisfied. The space of M-type and the space of all functions on a set have this property.

If R is generated by countable order-bounded sets, then $\mathfrak{B}$ itself has a countable basis. The "Stufenraum" of G. KÖTHE [2] is an example of the space of this type (it also satisfies the topological condition above).

The sequence space c_0 is also in our case. In fact, for every $\varepsilon>0$, let $\mathfrak{A}_\varepsilon$ be the totality of the set A for which there exists a fixed natural number ν_0 such that $\{\xi_\nu\}\in A$ implies $|\xi_\nu|\leqq\varepsilon$ for every $\nu\geqq\nu_0$, then $\mathfrak{A}_\varepsilon$ is an ideal with countable basis and we have $\mathfrak{B}=\bigcap_{\varepsilon<0}\mathfrak{A}_\varepsilon^2$ since $\mathfrak{A}_\varepsilon^2\subset\mathfrak{A}_{2\varepsilon}$.

2. In the sequel we shall restrict ourselves to the case R is a normed semi-ordered linear space where the norm is monotone complete.

We put

$$\alpha=\alpha(R)=\inf_{a_1,a_2,\cdots a_n}\ \sup_{i_1,i_2,\cdots i_n}\ \frac{\|a(i_1,i_2,\cdots i_n)\|}{\|\,|a_1|+|a_2|+\cdots+|a_n|\,\|},$$

where the infimum is taken for every finite set $a_1, a_2,\cdots a_n$ in R, the supremum for every permutation $i_1, i_2,\cdots i_n$ of $1,2,\cdots n$ and $a(i_1,i_2,\cdots i_n)$ $=|a_{i_1}|\smile|a_{i_1}+a_{i_2}|\smile\cdots|a_{i_1}+a_{i_2}+\cdots+a_{i_n}|$. We have always $0\leqq\alpha\leqq\frac{1}{2}$ and for every space of M-type, as easily be seen, $\alpha=\frac{1}{2}$.

$\alpha(N)$ is also defined for every normal manifold N of R and we can prove easily the following relations:

(1) $\alpha(N)$, $\alpha(M)\geqslant\varepsilon$ and $N\perp M$ imply $\alpha(N+M)\geqslant\frac{\varepsilon}{2}$,

(2) $N_\lambda\uparrow_{\lambda\in\Lambda}N$ and $\alpha(N)=0$ imply $\inf_{\lambda\in\Lambda}\alpha(N_\lambda)=0$.[6]

Now we shall prove that *the two notions of convergence are equivalent if and only if* $\alpha>0$.

If $\alpha>0$ and $\sum a_\nu$ is not absolutely convergent, then by the monotone completeness of the norm, we have

$$\| |a_{\mu+1}|+|a_{\mu+2}|+\cdots+|a_{\mu+\nu}| \| \uparrow_\nu +\infty \qquad \text{for every } \mu=1,2,\cdots.$$

Since for some permutation $i_1, i_2, \cdots i_\nu$ of $1,2,\cdots\nu$ we have

$$\|a(i_1, i_2, \cdots i_\nu)\| \geqslant \frac{\alpha}{2} \| |a_{\mu+1}|+|a_{\mu+2}|+\cdots+|a_{\mu+\nu}| \|,$$

where $a(i_1, i_2, \cdots i_\nu)$ is as defined above for $a_{\mu+1}, a_{\mu+2}, \cdots a_{\mu+\nu}$, we can find a rearrangement of $a_\nu(\nu=1,2,\cdots)$ such that all the sum of successive elements are not order-bounded, and hence, $\sum a_\nu$ is not unconditionally convergent.

Next we suppose $\alpha=0$. If for every normal manifold N, either $\alpha(N)$ or $\alpha(N^\perp)$ is not 0 (anyway one of them is 0 by (1)), then $\{N; \alpha(N)=0\}$ is an maximal dual ideal of the Boolean algebra of all normal manifolds and this ideal is not atomic since $\alpha=\frac{1}{2}$ for every one dimensional space, and hence we have by (2) $\inf_{\alpha(N)>0}\alpha(N)=0$. Therefore for every N such that $\alpha(N)=0$, we can devide N into two orthogonal normal manifolds N', N'' where one of $\alpha(N')$ and $\alpha(N'')$ is 0 and the other is either 0 or arbitrarily small, and hence we can find mutually orthogonal normal manifolds $N_\nu\neq 0$ $(\nu=1,2,\cdots)$ such that $\alpha(N_\nu)<\frac{1}{\nu 2^\nu}$.

Now from every N_ν we take a finite sequence of elements $a_1, a_2, \cdots a_n$ such that $\| |a_1|+|a_2|+\cdots+|a_n| \|=\nu$ and for every permutation $i_1, i_2, \cdots i_n$, $\|a(i_1, i_2, \cdots i_n)\|\leqslant\frac{1}{2^{\nu-1}}$ then putting such a sequence $a_1, a_2, \cdots a_n$ one after another for $\nu=1,2\cdots$, we obtain an infinite sequence of which the series is obviously unconditionally and not absolutely convergent.

6) To prove this we may make use of the following property of the monotone complete norm, that is, there exists $\varepsilon>3$ such that

$$0\leqslant a_\lambda\uparrow_{\lambda\in\Lambda} a \text{ implies always } \sup_{\lambda\in\Lambda}\|a_\lambda\|\geqslant\varepsilon\|a\|. \qquad \text{Cf. [3].}$$

Thus the proof was completed.

3. Here we shall show $\alpha=0$ for every infinite-dimensional space of L_p-type $(1\leqslant p<\infty)$.

For this purpose we shall compute $\alpha=\alpha_n$ for the 2^n-dimensional space where the norm of an element $x\equiv\{\xi_m\}$ is defined as

$$\|x\| = \sum_{m=1}^{2^n} |\xi_m|^p .$$

Let M_n be the $(n, 2^n)$-matrix below

$$\begin{pmatrix} +1 & +1 & \cdots & +1 & -1 & -1 & \cdots & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & +1 & -1 & -1 & +1 & +1 & & \cdots\cdots \\ +1 & -1 & +1 & -1 & +1 & -1 & & \cdots\cdots \end{pmatrix}$$

that is, for the i^{th} row, the first 2^{i-1} are all $+1$ and the next 2^{i-1} are -1 and so on alternatively. Let a_i be the i^{th} row-vector, then for every permutation $i_1, i_2, \cdots i_n$ of $1, 2, \cdots n$, there exists a permutation of the co-ordinate axes by which the sequence $a_1, a_2, \cdots a_n$ is transformed to $a_{i_1}, a_{i_2}, \cdots a_{i_n}$, since any of 2^n possible sequences of ± 1 emerges as a column of M_n.

Consequently we have

$$\|a(i_1, i_2, \cdots i_n)\| = \|a(1, 2, \cdots n)\| ,$$

making use of the same notation as in 2.

Putting

$$a(+) = a_1^+ \smile (a_1+a_2)^+ \smile \cdots \smile (a_1+a_2+\cdots+a_n)^+ ,$$

and $$a(-) = a_1^- \smile (a_1+a_2)^- \smile \cdots \smile (a_1+a_2+\cdots+a_n)^- ,$$

we have $$a(1, 2, \cdots n) = a(+) \smile a(-) , \quad \|a(+)\| = \|a(-)\| ,$$

and hence $\|a(i_1, i_2, \cdots i_n)\| \leqslant 2\|a(+)\|$, or by the definition of α_n,

(1) $2^{n-1} n^p \alpha_n \leqslant \|a(+)\|$.

In the sequel we write $a^{(n)}$ for $a(+)$ and β_n for $\|a^{(n)}\|$.

The matrix M_{n+1} has the form

$$\begin{bmatrix} +1 & +1 & \cdots & +1 & -1 & -1 & \cdots & -1 \\ & & M_n & & & & M_n & \end{bmatrix}$$

so if $a^{(n)} \equiv \{\xi_1, \xi_2, \cdots \xi_{2^n}\}$, then we can see easily that

$$a^{(n+1)} \equiv \{\xi_1+1, \xi_2+1, \cdots \xi_{2^n}+1, \xi_1+\varepsilon_1, \xi_2+\varepsilon_2, \cdots\cdots\}$$

where

$$\varepsilon_m = \begin{cases} -1 & \text{for every } m \text{ such that } \xi_m > 0 \\ 0 & \text{elsewhere}. \end{cases}$$

Since $\beta_n = \sum_{m=1}^{2^n} \xi_m^p$, we have

(2) $\beta_{n+1} = 2\beta_n + \gamma_n + \sum_{\xi_m>0} \{(\xi_m+1)^p + (\xi_m-1)^p - 2\xi_m^p\}$

where γ_n is the number of those m's for which $\xi_m = 0$.

Now we devide the three cases.

(i) $1<p<2$: Then $(x+1)^p+(x-1)^p-2x^p$ is a decreasing function of $x \geqslant 1$, and hence the third term in the right side of (2) is smaller than $2^n(2^p-2)$. Therefore we have the inequality

$$\beta_{n+1} \leqslant 2\beta_n + 2^n(2^p-1),$$

or since $\beta_1 = 1$,

$$\beta_{n+1} \leqslant 2^n + n2^n(2^p-1) = 2^n O(n),$$

and hence by (1)

$$\alpha_n = O(n^{1-p}).$$

(ii) $2 \leqslant p$: Then $(x+1)^p+(x-1)^p-2x^p$ is increasing and so we have, since $(n+1)^p+(n-1)^p-2n^p = O(n^{p-2})$,

$$\beta_{n+1} \leqslant 2\beta_n + 2^n O(n^{p-2})$$

or

$$\beta_{n+1} \leqslant 2^n + n2^n O(n^{p-2}) = 2^n O(n^{p-1})$$

that is,

$$\alpha_n = O(n^{-1}).$$

(iii) $p=1$: Here the third term in the right side of (2) vanishes and we have

$$\beta_{n+1} = 2^n + \sum_{r=1}^{n} 2^{n-r}\gamma_r .$$

If we had proved that the second term of the right side does not exceed $2^{n+1}\sqrt{n}$, then we can conclude from that

$$\alpha_n = O(n^{-\frac{1}{2}}).$$

Let $\varepsilon_1, \varepsilon_2, \cdots \varepsilon_n$ ($\varepsilon_i = \pm 1$) be the m^{th} column of M_n for an arbitrary $m=1, 2, \cdots 2^n$, then $\xi_m = 0$ if and only if

$$\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i \leqslant 0 \qquad \text{for every } i = 1, 2, \cdots$$

If r is the smallest of those numbers for which $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r$ has the least value of all $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$, then $\varepsilon_r = -1$ and both the sequence $-\varepsilon_{r+1}, -\varepsilon_{r+2}, \cdots -\varepsilon_n$ and $\varepsilon_{r-1}, \varepsilon_{r-2}, \cdots \varepsilon_1$ has the same property as mentioned above, that is, every partial sum of them does not exceed 0, and these conditions determine r. If we put $\varphi(m) = r$ where r is defined above, then since there exist γ_{n-r} number of possible case for $\varepsilon_{r+1}, \varepsilon_{r+2}, \cdots \varepsilon_n$, and γ_{r-1} for $\varepsilon_{r-1}, \varepsilon_{r-2}, \cdots \varepsilon_1$, the number of those m's for which $\varphi(m) = r$ is $\gamma_{n-r}\gamma_{r-1} (\gamma_0 = 1)$. Since then $\sum_{r=1}^{n} \gamma_{n-r}\gamma_{r-1} = 2^n$ we have, putting $\gamma = \sum_{i=1}^{n} 2^{n-i}\gamma_i$,

$$\gamma^2 = \sum_{r,s=1}^{n} 2^{2n-r-s}\gamma_r\gamma_s \leqslant \sum_{t=1}^{2n} \sum_{r=1}^{t} 2^{2n-t+1}\gamma_{t-r}\gamma_{r-1} = 2n2^{2n+1},$$

that is, $\gamma \leqslant 2^{n+1}\sqrt{n}$ as required.

Therefore we have always

$$\lim_{n\to\infty} \alpha_n = 0,$$

and since in our space we can find a subspace which is isomorphic to the 2^n-dimensional space considered above for every n, the proof was completed.

To construct an example of the series which is unconditionally and not absolutely convergent in the sequence space l_p, let us decompose the space into mutually orthogonal normal manifolds $N_n (n = 1, 2, \cdots)$ where N_n is of 2^n-dimension, and in every N_n take a sequence $a_1, a_2, \cdots a_n$ as constructed above but multiplying them by $2^{-\frac{n}{p}} n^{-(1+\frac{1}{p})}$, then the union of all such sequences for $n = 1, 2, \cdots$ provides our example.

(September, 1954)

References

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