ON THE UNIFORMLY BOUNDED COMMUTATIVE GROUP OF LINEAR TRANSFORMATIONS IN THE HILBERT SPACE

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F. RIESZ and B. Sz-NAGY have remarked in [2], p. 393 that an uniformly bounded real parameter group \( \{T_t\} \) of linear transformations in Hilbert space is similar to a real parameter group \( \{U_t\} \) of unitary transformations ([1]), that is, we can find a bicontinuous transformation \( A \) such that \( T_t = A^{-1} U_t A (-\infty < t < +\infty) \). In this paper we prove that this theorem can be generalized for any uniformly bounded commutative group. The proof is based on Markoff-Kakutani's fixed point theorem ([3], appendix).

Let \( R \) be a Hilbert space where an inner product \((x, y)\) is defined. A complex valued functional \( \varphi(x, y) \) on \( R \times R \) is called a symmetric bilinear functional on \( R \) if 1) \( \varphi_y(x) = \varphi(x, y) \) is a linear functional on \( R \), 2) \( \varphi_x(y) = \varphi(x, y) \) is a conjugate linear functional, 3) \( \varphi(x, y) = \overline{\varphi(y, x)} \), where \( \overline{\varphi(y, x)} \) is a conjugate complex number of \( \varphi(y, x) \). Let \( \overline{R} \) be the total of symmetric bilinear functionals on \( R \). It is easy to see that \( \overline{R} \) is not empty and a real vector space if the addition and the scalar multiplication are defined as usual: \( (a \varphi)(x, y) = a \varphi(x, y) \), \( (\varphi + \psi)(x, y) = \varphi(x, y) + \psi(x, y) \) (\( \varphi, \psi \in \overline{R}, a \) is a real number). The following lemma is evident.

**Lemma 1.** For any positive number \( \epsilon > 0 \), and finite points \( x_i \in R, y_j \in R, i=1, 2, \ldots, n, j=1, 2, \ldots, m \), if we put

\[
V(\epsilon, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = \{ \varphi \in \overline{R} \mid \varphi(x_i, y_j) < \epsilon, i=1, 2, \ldots, n, j=1, 2, \ldots, m \}
\]

then a separated linear topology on \( \overline{R} \) is given uniquely such that \( \{ V(\epsilon, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) \} \) is a fundamental system of neighbourhoods at zero.

We can prove the next lemma in the same way as Schwarz's inequality.

**Lemma 2.** If \( \varphi \in \overline{R} \) is positive: \( \varphi(x, x) \geq 0 \), then we have
Lemma 3. For any two positive numbers $c_1$, $c_2$ as $0 < c_1 < c_2$, if we put
\[ K_{c_1, c_2} = \{ \varphi ; c_1 \| x \|^2 \leq \varphi(x, x) \leq c_2 \| x \|^2 (x \in R), \varphi \in \bar{R} \}, \]
then $K_{c_1, c_2}$ is convex compact in $\bar{R}$.

Proof. Evidently $K_{c_1, c_2}$ is convex, and if $\varphi \in K_{c_1, c_2}$, then we have from lemma 2 \[ | \varphi(x, y) | \leq c_2 \| x \| \| y \| \] (x, y ∈ R). We put
\[ S_{(x,y)} = \{ a ; | a | \leq c_2 \| x \| \| y \| \} (x \equiv y), \ S_{(x,x)} = \{ a ; c_1 \| x \|^2 \leq a \leq c_2 \| x \|^2 \}, \]
then by Tychonoff’s theorem the direct product $S = \prod_{(x,y) \in R} S_{(x,y)}$ is a compact set. It is easy to see that $K_{c_1, c_2}$ is embedded homeomorphic in $S$, and $K_{c_1, c_2}$ is closed in $S$, therefore $K_{c_1, c_2}$ is compact. q. e. d.

Lemma 4. For a linear transformation $T$ on $R$ we have a continuous linear transformation $\overline{T}$ on $\bar{R}$ such that $(\overline{T} \varphi)(x, y) = \varphi(Tx, Ty) (\varphi \in \bar{R})$.

Proof. $\overline{T}$ is obviously linear and from
\[ \overline{T}(V(\epsilon, T_x_1, \cdots, T_x_n, T_y_1, \cdots, T_y_m)) \subseteq V(\epsilon, x_1, \cdots, x_n, y_1, \cdots, y_m) \]
$\overline{T}$ is continuous on $\bar{R}$. q. e. d.

If $\mathfrak{G}$ is a group of linear transformations on $R$, then $\mathfrak{G} = \{ \overline{T}; T \in \mathfrak{G} \}$ is a group of continuous linear transformations on $\bar{R}$. If $\mathfrak{G}$ is commutative, then $\mathfrak{G}$ is commutative, too.

Lemma 5. Let $\mathfrak{G}$ be a group of linear transformations and uniformly bounded: $\| T \| \leq r (T \in \mathfrak{G})$. If we put $K = \{ (Tx, Ty); T \in \mathfrak{G} \} \subseteq \bar{R}$ and $\bar{K}$ is the least convex closed set that contains $K$, then $\bar{K}$ is compact and invariant under $\mathfrak{G}$, that is, if $\varphi \in \bar{K}$, then $\overline{T} \varphi \in \bar{K}$ ($\overline{T} \in \mathfrak{G}$).

Proof. As $\mathfrak{G}$ is a group, $K$ is invariant under $\mathfrak{G}$. Let $K_1$ be the least convex extension of $K$, then for any $\varphi \in K_1$ we have $\varphi(x, y) = \sum_{i=1}^{n} \lambda_i (T_i x, T_i y)$ for some $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $T_i \in \mathfrak{G}$ (i = 1, 2, ..., n), so $K_1$ is invariant under $\mathfrak{G}$. From lemma 4 $\overline{T} (T \in \mathfrak{G})$ is continuous on $\bar{R}$, hence
\[ \overline{T}(\bar{K}) = \overline{T}(K_1) \subseteq \overline{T}(K_1) = K_1 = \bar{K} \quad (\overline{T} \in \mathfrak{G}) \]
here a bar shows closure operation, therefore $\bar{K}$ is invariant under $\mathfrak{G}$. On the other hand since $\mathfrak{G}$ is uniformly bounded, we have
\[ \frac{1}{r^2} \| x \|^2 \leq (Tx, Ty) \leq r^2 \| x \|^2 \quad (x \in R) \]
hence $K \subseteq K_{r^2, r^2}$. From lemma 3 $K_{r^2, r^2}$ is convex and compact, therefore $\bar{K} \subseteq K_{r^2, r^2}$, and hence $\bar{K}$ is compact. q. e. d.
Especially if $\mathcal{G}$ is commutative, by Markoff-Kakutani's fixed point theorem we obtain invariant elements in $\overline{K}$ under $\mathcal{G}$. ([3], appendix). Let one of them be $\langle x, y \rangle$, as $\overline{K} \subseteq K(\frac{1}{r})^2, \gamma^2$ from lemma 5, we have $\frac{1}{\gamma^2} ||x||^2 \leq \langle x, x \rangle \leq \gamma^2 ||x||^2 (x \in R)$, hence $\langle x, y \rangle$ is an inner product on $R$ equivalent to $(x, y)$. From Riesz's theorem we can find a self-adjoint operator $B$ such that $\frac{1}{\gamma^2} \leq B \leq \gamma^2$ and $\langle x, y \rangle = (Bx, y)$. If we put $A = \sqrt{B}$, then $\frac{1}{\gamma} \leq A \leq \gamma$ and $\langle x, y \rangle = (Ax, Ay)$. On the other hand as $\langle x, y \rangle$ is invariant under $\mathcal{G}$, we have $\langle x, y \rangle = \langle Tx, Ty \rangle$. Therefore $(Ax, Ay) = (ATx, ATy)$, hence $T^* A^* T = A$. If we put $U_T = A T A^{-1}$, we have $U_T^* U_T = A^{-1} T^* A^* T A^{-1} = I$, so $U_T$ is an unitary transformation, and $\mathfrak{U} = \{U_T; T \in \mathcal{G}\}$ is a group of unitary transformations. Thus we obtain the following.

**Theorem.** If $\mathcal{G}$ is an uniformly bounded commutative group of linear transformations in a Hilbert space $R$, then $\mathcal{G}$ is similar to an unitary transformation group $\mathfrak{U}$ in $R$.

**Bibliography**

