ON THE UNIFORMLY BOUNDED COMMUTATIVE GROUP OF LINEAR TRANSFORMATIONS IN THE HILBERT SPACE

By

Takasi ITŌ

F. RIESZ and B. SZ-NAGY have remarked in [2], p. 393 that an uniformly bounded real parameter group $\{T_t\}$ of linear transformations in Hilbert space is *similar* to a real parameter group $\{U_t\}$ of unitary transformations ([1]), that is, we can find a bicontinuous transformation A such that $T_t = A^{-1}U_tA\ (-\infty < t < +\infty)$. In this paper we prove that this theorem can be generalized for any uniformly bounded commutative group. The proof is based on MARKOFF-KAKUTANI's *fixed point theorem* ([3], appendix).

Let R be a Hilbert space where an inner product (x, y) is defined. A complex valued functional $\varphi(x, y)$ on $R \times R$ is called a symmetric bilinear functional on R if 1) $\varphi_y(x) = \varphi(x, y)$ is a linear functional on R, 2) $\varphi_x(y) = \varphi(x, y)$ is a conjugate linear functional, 3) $\varphi(x, y) = \overline{\varphi(y, x)}$, where $\overline{\varphi(y, x)}$ is a conjugate complex number of $\varphi(y, x)$. Let $\widetilde{R}$ be the total of symmetric bilinear functionals on R. It is easy to see that $\widetilde{R}$ is not empty and a real vector space if the addition and the scalar multiplication are defined as usual: $(\alpha\varphi)(x, y) = \alpha\varphi(x, y)$, $(\varphi + \psi)(x, y) = \varphi(x, y) + \psi(x, y)$ ($\varphi, \psi \in \widetilde{R}$, α is a real number). The following lemma is evident.

Lemma 1. *For any positive number $\varepsilon > 0$, and finite points $x_i \in R$, $y_j \in R$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$, if we put*

$$V(\varepsilon, x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m)$$
$$= \{\varphi\ ;\ \varphi \in \widetilde{R}, |\varphi(x_i, y_j)| < \varepsilon, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m\}$$

then a separated linear topology on $\widetilde{R}$ is given uniquely such that $\{V(\varepsilon, x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m)\}$ is a fundamental system of neighbourhoods at zero.

We can prove the next lemma in the same way as SCHWARZ's inequality.

Lemma 2. *If $\varphi \in \widetilde{R}$ is positive: $\varphi(x, x) \geqq 0$, then we have*

$$|\varphi(x, y)|^2 \leqq \varphi(x, x)\,\varphi(y, y) \qquad (x, y \in R).$$

Lemma 3. *For any two positive numbers c_1, c_2 as $0 < c_1 < c_2$, if we put*

$$K_{c_1, c_2} = \{\varphi;\ c_1 \|x\|^2 \leqq \varphi(x, x) \leqq c_2 \|x\|^2 (x \in R),\ \varphi \in \widetilde{R}\},$$

then K_{c_1, c_2} is convex compact in $\widetilde{R}$.

Proof. Evidently K_{c_1, c_2} is convex, and if $\varphi \in K_{c_1, c_2}$, then we have from lemma 2 $|\varphi(x, y)| \leqq c_2 \|x\|\,\|y\| (x, y \in R)$. We put

$$S_{(x,y)} = \{\alpha;\ |\alpha| \leqq c_2 \|x\|\,\|y\|\}\ (x \neq y),\quad S_{(x,x)} = \{\alpha;\ c_1 \|x\|^2 \leqq \alpha \leqq c_2 \|x\|^2\},$$

then by TYCHONOFF's theorem the direct product $S = \underset{x, y \in R}{X} S_{(x,y)}$ is a compact set. It is easy to see that K_{c_1, c_2} is embedded homeomorphic in S, and K_{c_1, c_2} is closed in S, therefore K_{c_1, c_2} is compact. q. e. d.

Lemma 4. *For a linear transformation T on R we have a continuous linear transformation $\widetilde{T}$ on $\widetilde{R}$ such that $(\widetilde{T}\varphi)(x, y) = \varphi(Tx, Ty)\ (\varphi \in \widetilde{R})$.*

Proof. $\widetilde{T}$ is obviously linear and from

$$\widetilde{T}(V(\varepsilon, Tx_1, \cdots, Tx_n, Ty_1, \cdots, Ty_m)) \subseteqq V(\varepsilon, x_1, \cdots, x_n, y_1, \cdots, y_m)$$

$\widetilde{T}$ is continuous on $\widetilde{R}$. q. e. d.

If $\mathfrak{G}$ is a group of linear transformations on R, then $\widetilde{\mathfrak{G}} = \{\widetilde{T};\ T \in \mathfrak{G}\}$ is a group of continuous linear transformations on $\widetilde{R}$. If $\mathfrak{G}$ is commutative, then $\widetilde{\mathfrak{G}}$ is commutative, too.

Lemma 5. *Let $\mathfrak{G}$ be a group of linear transformations and uniformly boundeed: $\|T\| \leqq \gamma (T \in \mathfrak{G})$. If we put $K = \{(Tx, Ty);\ T \in \mathfrak{G}\} \subseteqq \widetilde{R}$ and $\widetilde{K}$ is the least convex closed set that contains K, then $\widetilde{K}$ is compact and invariant under $\widetilde{\mathfrak{G}}$, that is, if $\varphi \in \widetilde{K}$, then $\widetilde{T}\varphi \in \widetilde{K}\ (\widetilde{T} \in \widetilde{\mathfrak{G}})$.*

Proof. As $\widetilde{\mathfrak{G}}$ is a group, K is invariant under $\widetilde{\mathfrak{G}}$. Let K_1 be the least convex extension of K, then for any $\varphi \in K_1$ we have $\varphi(x, y) = \sum_{i=1}^{n} \lambda_i (T_i x, T_i y)$ for some $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $T_i \in \mathfrak{G}\ (i = 1, 2, \cdots, n)$, so K_1 is invariant under $\widetilde{\mathfrak{G}}$. From lemma 4 $\widetilde{T}\ (T \in \mathfrak{G})$ is continuous on $\widetilde{R}$, hence

$$\widetilde{T}(\widetilde{K}) = \widetilde{T}(\overline{K_1}) \subseteqq \overline{\widetilde{T}(K_1)} = \overline{K_1} = \widetilde{K} \qquad (\widetilde{T} \in \widetilde{\mathfrak{G}})$$

here a bar shows closure operation, therefore $\widetilde{K}$ is invariant under $\widetilde{\mathfrak{G}}$. On the other hand since $\mathfrak{G}$ is uniformly bounded, we have

$$\frac{1}{\gamma^2} \|x\|^2 \leqq (Tx, Ty) \leqq \gamma^2 \|x\|^2 \qquad (x \in R)$$

hence $K \subseteqq K_{(\frac{1}{\gamma})^2, \gamma^2}$. From lemma 3 $K_{(\frac{1}{\gamma})^2, \gamma^2}$ is convex and compact, therefore $\widetilde{K} \subseteqq K_{(\frac{1}{\gamma})^2, \gamma^2}$, and hence $\widetilde{K}$ is compact. q. e. d.

Especially if $\mathfrak{G}$ is commutative, by MARKOFF-KAKUTANI's fixed point theorem we obtain invariant elements in $\widetilde{K}$ under $\widetilde{\mathfrak{G}}$. ([3], appendix). Let one of them be $\langle x, y\rangle$, as $\widetilde{K} \subseteqq K_{(\frac{1}{\gamma})^2, \gamma^2}$ from lemma 5, we have $\frac{1}{\gamma^2}\|x\|^2 \leqq \langle x, x\rangle \leqq \gamma^2 \|x\|^2$ $(x \in R)$, hence $\langle x, y\rangle$ is an inner product on R equivalent to (x, y). From RIESZ's theorem we can find a self-adjoint operator B such that $\frac{1}{\gamma^2} \leqq B \leqq \gamma^2$ and $\langle x, y\rangle = (Bx, y)$. If we put $A = \sqrt{B}$, then $\frac{1}{\gamma} \leqq A \leqq \gamma$ and $\langle x, y\rangle = (Ax, Ay)$. On the other hand as $\langle x, y\rangle$ is invariant under $\widetilde{\mathfrak{G}}$, we have $\langle x, y\rangle = \langle Tx, Ty\rangle$. Therefore $(Ax, Ay) = (ATx, ATy)$, hence $T^*A^2T = A^2$. If we put $U_T = ATA^{-1}$, we have $U_T^* U_T = A^{-1}T^*A^2TA^{-1} = I$, so U_T is an unitary transformation, and $\mathfrak{U} = \{U_T; T \in \mathfrak{G}\}$ is a group of unitary transformations. Thus we obtain the following.

Theorem. *If $\mathfrak{G}$ is an uniformly bounded commutative group of linear transformations in a Hilbert space R, then $\mathfrak{G}$ is similar to an unitary transformation group $\mathfrak{U}$ in R.*

Bibliography

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