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ON THE UNIFORMLY BOUNDED COMMUTATIVE GROUP OF LINEAR TRANSFORMATIONS IN THE HILBERT SPACE

By

Takasi ITO

F. Riesz and B. Sz-Nagy have remarked in [2], p. 393 that an uniformly bounded real parameter group \( \{T_t\} \) of linear transformations in Hilbert space is similar to a real parameter group \( \{U_t\} \) of unitary transformations ([1]), that is, we can find a bicontinuous transformation \( A \) such that \( T_t = A^{-1}U_tA \)(\(-\infty < t < +\infty\)). In this paper we prove that this theorem can be generalized for any uniformly bounded commutative group. The proof is based on Markoff-Kakutani's fixed point theorem ([3], appendix).

Let \( R \) be a Hilbert space where an inner product \((x, y)\) is defined. A complex valued functional \( \varphi (x, y) \) on \( R \times R \) is called a symmetric bilinear functional on \( R \) if 1) \( \varphi_y(x) = \varphi(x, y) \) is a linear functional on \( R \), 2) \( \varphi_x(y) = \overline{\varphi(x, y)} \) is a conjugate linear functional, 3) \( \varphi(x, y) = \overline{\varphi(y, x)} \), where \( \varphi(y, x) \) is a conjugate complex number of \( \varphi(x, y) \). Let \( \bar{R} \) be the total of symmetric bilinear functionals on \( R \). It is easy to see that \( \bar{R} \) is not empty and a real vector space if the addition and the scalar multiplication are defined as usual: \( (a\varphi)(x, y) = a\varphi(x, y), (\varphi + \psi)(x, y) = \varphi(x, y) + \psi(x, y) \) \((\varphi, \psi \in \bar{R}, a \text{ is a real number})\). The following lemma is evident.

Lemma 1. For any positive number \( \epsilon > 0 \), and finite points \( x_i \in R, y_j \in R, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m \), if we put

\[
V(\epsilon, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = \{ \varphi ; \varphi \in \bar{R}, | \varphi(x_i, y_j)| < \epsilon, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m \}
\]

then a separated linear topology on \( \bar{R} \) is given uniquely such that \( \{V(\epsilon, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)\} \) is a fundamental system of neighbourhoods at zero.

We can prove the next lemma in the same way as Schwarz's inequality.

Lemma 2. If \( \varphi \in \bar{R} \) is positive: \( \varphi(x, x) \geq 0 \), then we have
Lemma 3. For any two positive numbers $c_1$, $c_2$ as $0 < c_1 < c_2$, if we put
\[ K_{c_1, c_2} = \{ \varphi; c_1 \|x\|^2 \leq \varphi(x, x) \leq c_2 \|x\|^2 (x \in \mathbb{R}), \varphi \in \overline{R} \}, \]
then $K_{c_1, c_2}$ is convex compact in $\overline{R}$.

Proof. Evidently $K_{c_1, c_2}$ is convex, and if $\varphi \in K_{c_1, c_2}$, then we have
\[ |\varphi(x, y)|^2 \leq \varphi(x, x) \varphi(y, y) \quad (x, y \in \mathbb{R}). \]
We put
\[ S_{(x,y)} = \{ a; |a| \leq c_2 \|x\|^2 \|y\|^2 \} \quad (x \neq y), \quad S_{(x,x)} = \{ a; c_1 \|x\|^2 \leq a \leq c_2 \|x\|^2 \}, \]
then by Tychonoff's theorem the direct product $S = X_{a,y \in \mathbb{R}} S_{(a,y)}$ is a compact set. It is easy to see that $K_{c_1, c_2}$ is embedded homeomorphic in $S$, and $K_{c_1, c_2}$ is closed in $S$, therefore $K_{c_1, c_2}$ is compact. q. e. d.

Lemma 4. For a linear transformation $T$ on $\mathbb{R}$ we have a continuous linear transformation $\overline{T}$ on $\overline{R}$ such that $(\overline{T} \varphi)(x, y) = \varphi(Tx, Ty) (\varphi \in \overline{R})$.

Proof. $\overline{T}$ is obviously linear and from
\[ \overline{T}(V(\epsilon, Tx_1, \cdots, Tx_n, Ty_1, \cdots, Ty_m)) \subseteq V(\epsilon, x_1, \cdots, x_n, y_1, \cdots, y_m) \]
$\overline{T}$ is continuous on $\overline{R}$. q. e. d.

If $\mathfrak{G}$ is a group of linear transformations on $\mathbb{R}$, then $\overline{\mathfrak{G}} = \{ \overline{T}; T \in \mathfrak{G} \}$ is a group of continuous linear transformations on $\overline{R}$. If $\mathfrak{G}$ is commutative, then $\overline{\mathfrak{G}}$ is commutative, too.

Lemma 5. Let $\mathfrak{G}$ be a group of linear transformations and uniformly bounded: $\|T\| \leq r(T \in \mathfrak{G})$. If we put $K = \{ (Tx, Ty); T \in \mathfrak{G} \} \subseteq \overline{R}$ and $\overline{K}$ is the least convex closed set that contains $K$, then $\overline{K}$ is compact and invariant under $\overline{\mathfrak{G}}$, that is, if $\varphi \in \overline{K}$, then $\overline{T} \varphi \in \overline{K} (\overline{T} \in \overline{\mathfrak{G}})$. 

Proof. As $\overline{\mathfrak{G}}$ is a group, $K$ is invariant under $\overline{\mathfrak{G}}$. Let $K_i$ be the least convex extension of $K$, then for any $\varphi \in K_i$ we have $\varphi(x, y) = \sum_{i=1}^{n} \lambda_i (T_i x, T_i y)$ for some $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $T_i \in \mathfrak{G} (i = 1, 2, \cdots, n)$, so $K_i$ is invariant under $\overline{\mathfrak{G}}$. From lemma 4 $\overline{T}(T \in \mathfrak{G})$ is continuous on $\overline{R}$, hence
\[ \overline{T}(\overline{K}) = \overline{T}(K_i) \subseteq \overline{T}(K_i) = K_i = \overline{K} \quad (\overline{T} \in \overline{\mathfrak{G}}) \]
here a bar shows closure operation, therefore $\overline{K}$ is invariant under $\overline{\mathfrak{G}}$. On the other hand since $\mathfrak{G}$ is uniformly bounded, we have
\[ \frac{1}{r^2} \|x\|^2 \leq (Tx, Ty) \leq r^2 \|x\|^2 \quad (x \in \mathbb{R}) \]
hence $K \subseteq K^{\frac{r^2}{r^2}}$. From lemma 3 $K^{\frac{r^2}{r^2}}$ is convex and compact, therefore $\overline{K} \subseteq K^{\overline{\frac{r^2}{r^2}}}$, and hence $\overline{K}$ is compact. q. e. d.
Especially if $\mathfrak{G}$ is commutative, by Markoff-Kakutani's fixed point theorem we obtain invariant elements in $\overline{K}$ under $\mathfrak{G}$. ([3], appendix). Let one of them be $\langle x, y \rangle$, as $\overline{K} \subseteq K_{(\frac{1}{r} \gamma^{2})}$ from lemma 5, we have $\frac{1}{\gamma^{2}} ||x||^{2} \leq \langle x, x \rangle \leq r^{2} ||x||^{2} (x \in R)$, hence $\langle x, y \rangle$ is an inner product on $R$ equivalent to $(x, y)$. From Riesz's theorem we can find a self-adjoint operator $B$ such that $\frac{1}{\gamma^{2}} \leq B \leq r^{2}$ and $\langle x, y \rangle = (Bx, y)$. If we put $A = \sqrt{B}$, then $\frac{1}{r} \leq A \leq r$ and $\langle x, y \rangle = (Ax, Ay)$. On the other hand as $\langle x, y \rangle$ is invariant under $\mathfrak{G}$, we have $\langle x, x \rangle = \langle Tx, Ty \rangle$. Therefore $(Ax, Ay) = (ATx, ATy)$, hence $T^* A^* T = A^*$. If we put $U_T = A T A^{-1}$, we have $U_T^* U_T = A^{-1} T^* A^* T A^{-1} = I$, so $U_T$ is an unitary transformation, and $\mathfrak{U} = \{U_T; T \in \mathfrak{G}\}$ is a group of unitary transformations. Thus we obtain the following.

**Theorem.** If $\mathfrak{G}$ is an uniformly bounded commutative group of linear transformations in a Hilbert space $R$, then $\mathfrak{G}$ is similar to an unitary transformation group $\mathfrak{U}$ in $R$.

**Bibliography**

