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ON THE UNIFORMLY BOUNDED COMMUTATIVE GROUP OF LINEAR TRANSFORMATIONS IN THE HILBERT SPACE

By

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F. RIESZ and B. Sz-NAGY have remarked in [2], p. 393 that an uniformly bounded real parameter group \( \{T_t\} \) of linear transformations in Hilbert space is similar to a real parameter group \( \{U_t\} \) of unitary transformations ([1]), that is, we can find a bicontinuous transformation \( A \) such that \( T_t = A^{-1}U_tA \) \((-\infty < t < +\infty)\). In this paper we prove that this theorem can be generalized for any uniformly bounded commutative group. The proof is based on Markoff-Kakutani's fixed point theorem ([3], appendix).

Let \( R \) be a Hilbert space where an inner product \((x, y)\) is defined. A complex valued functional \( \varphi(x, y) \) on \( R \times R \) is called a symmetric bilinear functional on \( R \) if 1) \( \varphi_{x}(x) = \varphi(x, y) \) is a linear functional on \( R \), 2) \( \varphi_{y}(y) = \varphi(x, y) \) is a conjugate linear functional, 3) \( \varphi(x, y) = \overline{\varphi(y, x)} \), where \( \overline{\varphi(y, x)} \) is a conjugate complex number of \( \varphi(y, x) \). Let \( \overline{R} \) be the total of symmetric bilinear functionals on \( R \). It is easy to see that \( \overline{R} \) is not empty and a real vector space if the addition and the scalar multiplication are defined as usual: \( (a\varphi)(x, y) = a\varphi(x, y), \ (\varphi + \psi)(x, y) = \varphi(x, y) + \psi(x, y) \) (\( \varphi, \psi \in \overline{R}, a \) is a real number). The following lemma is evident.

Lemma 1. For any positive number \( \varepsilon > 0 \), and finite points \( x_i \in R, y_j \in R, i=1, 2, \cdots, n, j=1, 2, \cdots, m \), if we put

\[
V(\varepsilon, x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) = \{ \varphi; \varphi \in \overline{R}, |\varphi(x_i, y_j)| < \varepsilon, i=1, 2, \cdots, n, j=1, 2, \cdots, m \}
\]

then a separated linear topology on \( \overline{R} \) is given uniquely such that \( \{V(\varepsilon, x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m)\} \) is a fundamental system of neighbourhoods at zero.

We can prove the next lemma in the same way as Schwarz's inequality.

Lemma 2. If \( \varphi \in \overline{R} \) is positive: \( \varphi(x, x) \geq 0 \), then we have
Lemma 3. For any two positive numbers $c_1$, $c_2$ as $0 < c_1 < c_2$, if we put

$$K_{c_1, c_2} = \{ \varphi; c_1 |x|^2 \leq \varphi(x, x) \leq c_2 |x|^2 |x \in R, \varphi \in \bar{R} \},$$

then $K_{c_1, c_2}$ is convex compact in $\bar{R}$.

Proof. Evidently $K_{c_1, c_2}$ is convex, and if $\varphi \in K_{c_1, c_2}$, then we have from lemma 2 $|\varphi(x, y)| \leq c_2 |x| |y| (x, y \in R)$. We put

$$S_{(x, y)} = \{ a; a \leq c_2 |x| |y| \} \quad (x \neq y), \quad S_{(x, x)} = \{ a; c_1 |x|^2 \leq a \leq c_2 |x|^2 \},$$

then by Tychonoff's theorem the direct product $S = \prod_{x, y \in R} S_{(x, y)}$ is a compact set. It is easy to see that $K_{c_1, c_2}$ is embedded homeomorphic in $S$, and $K_{c_1, c_2}$ is closed in $S$, therefore $K_{c_1, c_2}$ is compact. q.e.d.

Lemma 4. For a linear transformation $T$ on $R$ we have a continuous linear transformation $\bar{T}$ on $\bar{R}$ such that $(\bar{T} \varphi)(x, y) = \varphi(Tx, Ty) (\varphi \in \bar{R})$.

Proof. $\bar{T}$ is obviously linear and from

$$\bar{T}(V(\varepsilon, Tx_1, \cdots, Tx_n, Ty_1, \cdots, Ty_m)) \subseteq V(\varepsilon, x_1, \cdots, x_n, y_1, \cdots, y_m)$$

$\bar{T}$ is continuous on $\bar{R}$. q.e.d.

If $\mathcal{G}$ is a group of linear transformations on $R$, then $\overline{\mathcal{G}} = \{ \overline{T}; T \in \mathcal{G} \}$ is a group of continuous linear transformations on $\bar{R}$. If $\mathcal{G}$ is commutative, then $\overline{\mathcal{G}}$ is commutative, too.

Lemma 5. Let $\mathcal{G}$ be a group of linear transformations and uniformly bounded: $||T|| \leq r(T \in \mathcal{G})$. If we put $K = \{ (Tx, Ty); T \in \mathcal{G} \} \subseteq \bar{R}$ and $\bar{K}$ is the least convex closed set that contains $K$, then $\bar{K}$ is compact and invariant under $\overline{\mathcal{G}}$, that is, if $\varphi \in \bar{K}$, then $\bar{T} \varphi \in \bar{K} (\bar{T} \in \overline{\mathcal{G}})$.

Proof. As $\overline{\mathcal{G}}$ is a group, $K$ is invariant under $\overline{\mathcal{G}}$. Let $K_1$ be the least convex extension of $K$, then for any $\varphi \in K_1$ we have $\varphi(x, y) = \sum_{i=1}^{n} \lambda_i (T_i x, T_i y)$ for some $\lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1, T_i \in \mathcal{G} (i = 1, 2, \cdots, n)$, so $K_1$ is invariant under $\overline{\mathcal{G}}$. From lemma 4 $\overline{T}(T \in \mathcal{G})$ is continuous on $\bar{R}$, hence

$$\overline{T}(\bar{K}) = \overline{T}(\bar{K}_1) \subseteq \overline{T}(K_1) = K_1 = \bar{K} \quad (\bar{T} \in \overline{\mathcal{G}}),$$

here a bar shows closure operation, therefore $\bar{K}$ is invariant under $\overline{\mathcal{G}}$. On the other hand since $\mathcal{G}$ is uniformly bounded, we have

$$\frac{1}{r^2} |x|^2 \leq (Tx, Ty) \leq r^2 |x|^2 \quad (x \in R)$$

hence $K \subseteq K_{(\frac{1}{r})^2, r^2}$. From lemma 3 $K_{(\frac{1}{r})^2, r^2}$ is convex and compact, therefore $\bar{K} \subseteq K_{(\frac{1}{r})^2, r^2}$, and hence $\bar{K}$ is compact. q.e.d.
Especially if \( \mathfrak{G} \) is commutative, by Markoff-Kakutani's fixed point theorem we obtain invariant elements in \( \overline{K} \) under \( \mathfrak{G} \). ([3], appendix). Let one of them be \( \langle x, y \rangle \), as \( \overline{K} \subseteq K_{e^{(1)/r}}^{(1)} \) from lemma 5, we have \( \frac{1}{r^{2}} \|x\|^{2} \leq \langle x, x \rangle \leq r^{2} \|x\|^{2} (x \in R) \), hence \( \langle x, y \rangle \) is an inner product on \( R \) equivalent to \( (x, y) \). From Riesz's theorem we can find a self-adjoint operator \( B \) such that \( \frac{1}{r^{2}} \leq B \leq r^{2} \) and \( \langle x, y \rangle = (Bx, y) \). If we put \( A = \sqrt{B} \), then \( \frac{1}{r} \leq A \leq r \) and \( \langle x, y \rangle = (Ax, Ay) \). On the other hand as \( \langle x, y \rangle \) is invariant under \( \mathfrak{G} \), we have \( \langle x, y \rangle = \langle Tx, Ty \rangle \). Therefore \( (Ax, Ay) = (ATx, ATy) \), hence \( T^{*}AT = A^{*} \). If we put \( U_{r} = ATA^{-1} \), we have \( U_{r}^{T}U_{r} = A^{-1}T^{*}AT = I \), so \( U_{r} \) is an unitary transformation, and \( \mathfrak{G} = \{U_{r}; T \in \mathfrak{G}\} \) is a group of unitary transformations. Thus we obtain the following.

**Theorem.** If \( \mathfrak{G} \) is an uniformly bounded commutative group of linear transformations in a Hilbert space \( R \), then \( \mathfrak{G} \) is similar to an unitary transformation group \( \mathfrak{G} \) in \( R \).

**Bibliography**

