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ON THE UNIFORMLY BOUNDED COMMUTATIVE GROUP OF LINEAR TRANSFORMATIONS IN THE HILBERT SPACE

By

Takasi ITÔ

F. RIESZ and B. Sz-NAGY have remarked in [2], p. 393 that an uniformly bounded real parameter group \( \{T_t\} \) of linear transformations in Hilbert space is similar to a real parameter group \( \{U_t\} \) of unitary transformations ([1]), that is, we can find a bicontinuous transformation \( A \) such that \( T_t = A^{-1}U_tA (-\infty < t < +\infty) \). In this paper we prove that this theorem can be generalized for any uniformly bounded commutative group. The proof is based on Markoff-Kakutani's fixed point theorem ([3], appendix).

Let \( R \) be a Hilbert space where an inner product \( (x, y) \) is defined. A complex valued functional \( \varphi(x, y) \) on \( R \times R \) is called a symmetric bilinear functional on \( R \) if 1) \( \varphi_y(x) = \varphi(x, y) \) is a linear functional on \( R \), 2) \( \varphi_x(y) = \overline{\varphi(x, y)} \) is a conjugate linear functional, 3) \( \varphi(x, y) = \overline{\varphi(y, x)} \), where \( \overline{\varphi(y, x)} \) is a conjugate complex number of \( \varphi(y, x) \). Let \( \overline{R} \) be the total of symmetric bilinear functionals on \( R \). It is easy to see that \( \overline{R} \) is not empty and a real vector space if the addition and the scalar multiplication are defined as usual: \( (a\varphi)(x, y) = a\varphi(x, y) \), \( (\varphi + \psi)(x, y) = \varphi(x, y) + \varphi(x, y) \) (\( \varphi, \psi \in \overline{R}, a \) is a real number). The following lemma is evident.

Lemma 1. For any positive number \( \varepsilon > 0 \), and finite points \( x_i \in R, y_j \in R, i=1,2,\cdots,n, j=1,2,\cdots,m \), if we put

\[
V(\varepsilon, x_1, x_2,\cdots,x_n, y_1, y_2,\cdots,y_m) = \{ \varphi; \varphi \in \overline{R}, |\varphi(x_i, y_j)| < \varepsilon, i=1,2,\cdots,n, j=1,2,\cdots,m \}
\]

then a separated linear topology on \( \overline{R} \) is given uniquely such that \( \{V(\varepsilon, x_1, x_2,\cdots,x_n, y_1, y_2,\cdots,y_m)\} \) is a fundamental system of neighbourhoods at zero.

We can prove the next lemma in the same way as Schwarz's inequality.

Lemma 2. If \( \varphi \in \overline{R} \) is positive: \( \varphi(x, x) \geq 0 \), then we have
Lemma 3. For any two positive numbers $c_1$, $c_2$ as $0 < c_1 < c_2$, if we put

$$K_{c_1, c_2} = \{ \varphi; c_1 \| x \|^2 \leq \varphi(x, x) \leq c_2 \| x \|^2 | x \in R, \varphi \in \bar{R} \},$$

then $K_{c_1, c_2}$ is convex compact in $\bar{R}$.

Proof. Evidently $K_{c_1, c_2}$ is convex, and if $\varphi \in K_{c_1, c_2}$, then we have from lemma 2 $|\varphi(x, y)| \leq c_2 \| x \| \| y \| (x, y \in R)$. We put

$$S_{(x, y)} = \{ a; a \leq c_2 \| x \| \| y \| \} (x \neq y), \quad S_{(x, x)} = \{ a; c_1 \| x \|^2 \leq a \leq c_2 \| x \|^2 \},$$

then by Tychonoff's theorem the direct product $S = \prod_{x, y \in R} S_{(x, y)}$ is a compact set. It is easy to see that $K_{c_1, c_2}$ is embedded homeomorphic in $S$, and $K_{c_1, c_2}$ is closed in $S$, therefore $K_{c_1, c_2}$ is compact. q.e.d.

Lemma 4. For a linear transformation $T$ on $R$ we have a continuous linear transformation $\tilde{T}$ on $\bar{R}$ such that $(\tilde{T} \varphi)(x, y) = \varphi(Tx, Ty) (\varphi \in \bar{R})$.

Proof. $\tilde{T}$ is obviously linear and from

$$\tilde{T}(V(\varepsilon, Tx_1, \cdots, Tx_n, Ty_1, \cdots, Ty_m)) \subseteq V(\varepsilon, x_1, \cdots, x_n, y_1, \cdots, y_m)$$

$\tilde{T}$ is continuous on $\bar{R}$. q.e.d.

If $\mathfrak{G}$ is a group of linear transformations on $R$, then $\overline{\mathfrak{G}} = \{ \tilde{T}; T \in \mathfrak{G} \}$ is a group of continuous linear transformations on $\bar{R}$. If $\mathfrak{G}$ is commutative, then $\overline{\mathfrak{G}}$ is commutative, too.

Lemma 5. Let $\mathfrak{G}$ be a group of linear transformations and uniformly bounded: $\|T\| \leq r(T \in \mathfrak{G})$. If we put $K = \{ (Tx, Ty); T \in \mathfrak{G} \} \subseteq \bar{R}$ and $\bar{K}$ is the least convex closed set that contains $K$, then $\bar{K}$ is compact and invariant under $\mathfrak{G}$, that is, if $\varphi \in \bar{K}$, then $\tilde{T} \varphi \in \bar{K} (\tilde{T} \in \mathfrak{G})$.

Proof. As $\mathfrak{G}$ is a group, $K$ is invariant under $\mathfrak{G}$. Let $K_i$ be the least convex extension of $K$, then for any $\varphi \in K_i$ we have $\varphi(x, y) = \sum_{i=1}^{n} \lambda_i (T_i x, T_i y)$ for some $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $T_i \in \mathfrak{G} (i = 1, 2, \cdots, n)$, so $K_i$ is invariant under $\mathfrak{G}$. From lemma 4 $\overline{\tilde{T}}(T \in \mathfrak{G})$ is continuous on $\bar{R}$, hence

$$\overline{\tilde{T}}(\bar{K}) = \overline{\tilde{T}}(K_i) \subseteq \bar{K} = \bar{K} \quad (\tilde{T} \in \mathfrak{G})$$

here a bar shows closure operation, therefore $\bar{K}$ is invariant under $\mathfrak{G}$. On the other hand since $\mathfrak{G}$ is uniformly bounded, we have

$$\frac{1}{r^2} \| x \|^2 \leq (Tx, Ty) \leq r^2 \| x \|^2 \quad (x \in R)$$

hence $K \subseteq K(\frac{1}{r^2}, r^2)$. From lemma 3 $K(\frac{1}{r^2}, r^2)$ is convex and compact, therefore $\bar{K} \subseteq K(\frac{1}{r^2}, r^2)$, and hence $\bar{K}$ is compact. q.e.d.
Especially if $\mathfrak{G}$ is commutative, by Markoff-Kakutani’s fixed point theorem we obtain invariant elements in $\overline{K}$ under $\mathfrak{G}$. ([3], appendix). Let one of them be $\langle x, y \rangle$, as $\overline{K} \subseteq K(\frac{1}{r})^{2}$ from lemma 5, we have $\frac{1}{r^{2}} \|x\|^{2} \leq \langle x, x \rangle \leq r^{2} \|x\|^{2} (x \in R)$, hence $\langle x, y \rangle$ is an inner product on $R$ equivalent to $(x, y)$. From Riesz’s theorem we can find a self-adjoint operator $B$ such that $\frac{1}{r^{2}} \leq B \leq r^{2}$ and $\langle x, y \rangle = \langle Bx, y \rangle$. If we put $A = \sqrt{B}$, then $\frac{1}{r} \leq A \leq r$ and $\langle x, y \rangle = \langle Ax, Ay \rangle$. On the other hand as $\langle x, y \rangle$ is invariant under $\mathfrak{G}$, we have $\langle x, y \rangle = \langle Tx, Ty \rangle$. Therefore $(Ax, Ay) = (ATx, ATy)$, hence $T^{\ast}A^{\ast}T = A^{\ast}$. If we put $U_{r} = ATA^{-1}$, we have $U_{r}^{\ast}U_{r} = A^{-1}T^{\ast}A^{\ast}T = I$, so $U_{r}$ is an unitary transformation, and $\mathfrak{U} = \{U_{r} ; T \in \mathfrak{G}\}$ is a group of unitary transformations. Thus we obtain the following.

**Theorem.** If $\mathfrak{G}$ is an uniformly bounded commutative group of linear transformations in a Hilbert space $R$, then $\mathfrak{G}$ is similar to an unitary transformation group $\mathfrak{U}$ in $R$.

**Bibliography**

