FOURIER SERIES. VII. UNIFORM CONVERGENCE FACTORS OF FOURIER SERIES

By

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Let $f(x)$ be an integrable and periodic function with period 2π, and its Fourier series be

$$f(x) \sim \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

Let $\{\lambda_n\}$ be an infinite sequence of real numbers. If the series

$$\frac{1}{2} a_0 \lambda_0 + \sum_{\nu=1}^{\infty} \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

converges uniformly, then the sequence $\{\lambda_n\}$ is called a sequence of uniform convergence factors of Fourier series of $f(x)$. For the case of Fourier-Stieltjes series also we may define sequence of their uniform convergence factors.

Recently J. KARAMATA [1] has made use of F. RIESZ' theorem concerning the convergence of sequence of linear functionals, to characterize sequence of uniform convergence factors of Fourier series of all continuous functions, and proved the following

Theorem. *A necessary and sufficient condition that $\{\lambda_n\}$ be a sequence of uniform convergence factors of Fourier series of all continuous functions, is that*

$$\int_0^{\pi} |K_n(t)|\, dt = O(1) \qquad (n \to \infty),$$

where

$$K_n(t) = \frac{1}{2} \lambda_0 + \sum_{\nu=1}^{n} \lambda_\nu \cos \nu t .$$

In the present note, we shall find along his idea necessary and sufficient conditions for $\{\lambda_n\}$ to be a sequence of uniform convergence factors of Fourier series in various spaces of functions. We denote by $\mathfrak{C}$ the space of all continuous and periodic functions with period

2π, by $\mathfrak{M}$ that of all measurable, bounded and periodic functions, by $\mathfrak{L}_p\,(p>1)$ that of all $\mathfrak{L}_p$-integrable and periodic functions, and by $\mathfrak{L}$ that of all integrable and periodic functions.

1. Theorems we are now going to establish are as follows:

Theorem 1. *A necessary and sufficient condition that $\{\lambda_n\}$ be a sequence of uniform convergence factors of Fourier series of all functions belonging to $\mathfrak{M}$, is that the following two conditions be satisfied:*

1) $$\int_{-\pi}^{\pi} |K_n(t)|\, dt = O(1) \qquad (n \to \infty)\,,$$

2) *for any $\varepsilon>0$, there exists an $\eta>0$ such that for any subset $H\subset[-\pi,\pi]$ of measure $m(H)<\eta$,*

$$\left| \int_H (K_m(t)-K_n(t))\, dt \right| < \varepsilon \qquad (m,n=1,2,\cdots).^{*)}$$

Theorem 2. *A necessary and sufficient condition that $\{\lambda_n\}$ be a sequence of uniform convergence factors of Fourier series of all functions belonging to $\mathfrak{L}_p(p>1)$, is that*

$$\int_{-\pi}^{\pi} |K_n(t)|^q dt = O(1) \qquad (n \to \infty)\,,$$

where

$$\frac{1}{p}+\frac{1}{q}=1\,.$$

Theorem 3. *A necessary and sufficient condition that $\{\lambda_n\}$ be a sequence of uniform convergence factors of Fourier series of all functions belonging to $\mathfrak{L}$, is that*

$$K_n(t) = O(1)$$

uniformly both in n and in t.

Theorem 4. *A necessary and sufficient condition that $\{\lambda_n\}$ be a sequence of uniform convergence factors of all Fourier-Stieltjes series, is that $K_n(t)$ converge uniformly.*

2. For the proof of these theorems, we need some lemmas. Lemma 1 is a slightly modified form of the following

Theorem of LEBESGUE [2]. *Let $\{\varphi_n(t)\}$ be a sequence of integrable functions, and (a,b) a finite interval. In order that for every $f\in\mathfrak{M}$,*

$$\lim_{n\to\infty} \int_a^b \varphi_n(t) f(t)\, dt = 0\,,$$

*) We note the condition 1) is implied by the condition 2).

it is necessary and sufficient that the following three conditions be satisfied:

1) $$\int_a^b |\varphi_n(t)|\,dt = O(1) \qquad (n \to \infty),$$

2) *for any* $\varepsilon > 0$, *there is an* $\eta > 0$ *such that for any subset* $H \subset [a, b]$ *of measure* $m(H) < \eta$,

$$\left| \int_H \varphi_n(t)\,dt \right| < \varepsilon \qquad (n = 1, 2, \cdots),$$

3) $$\lim_{n\to\infty} \int_a^\tau \varphi_n(t)\,dt = 0 \qquad (a \leqslant \tau \leqslant b).$$

Lemma 1. *Let* $\{\varphi_n(t)\}$ *be a sequence of integrable functions, and* (a, b) *a finite interval. In order that for every function* $f \in \mathfrak{M}$, *there exists the limit*

$$\lim_{n\to\infty} \int_a^b \varphi_n(t) f(t)\,dt,$$

it is necessary and sufficient that the following three conditions be satisfied:

1) $$\int_a^b |\varphi_n(t)|\,dt = O(1) \qquad (n \to \infty),$$

2) *for any* $\varepsilon > 0$, *there is an* $\eta > 0$ *such that for any subset* $H \subset [a, b]$ *of measure* $m(H) < \eta$,

$$\left| \int_H (\varphi_m(t) - \varphi_n(t))\,dt \right| < \varepsilon \qquad (m, n = 1, 2, \cdots),$$

3) $$\lim_{n\to\infty} \int_a^\tau \varphi_n(t)\,dt \quad \textit{exists} \qquad (a \leqslant \tau \leqslant b).$$

In fact, condition 1) is an immediate consequence of condition 2) and that condition 3) is necessary is trivial. As to the necessity of condition 2), we note that, if we suppose the contrary, there would exist for some $\varepsilon > 0$ two infinite increasing sequences of positive integers $\{m_k\}$, $\{n_k\}$ and a sequence of sets $\{H_k\}$ of measure tending to zero, such that

$$\left| \int_{H_k} (\varphi_{m_k}(t) - \varphi_{n_k}(t))\,dt \right| \geqq \varepsilon.$$

However, this is imposible by LEBESGUE's theorem quoted above since

$$\lim_{k\to\infty} \int_a^b (\varphi_{m_k}(t) - \varphi_{n_k}(t)) f(t)\,dt = 0.$$

Proof of the sufficiency of the conditions is clear by LEBESGUE's theorem.

Lemma 2. *Let $\{A_n(f)\}$ be a sequence of linear functionals defined on a Banach space $\mathfrak{F}$, and $\mathfrak{B}$ a base of the space. In order that $\lim_{n\to\infty} A_n(f)$ exists for every $f\in\mathfrak{F}$, it is necessary and sufficient that*

$$1) \qquad \|A_n\| = O(1) \qquad (n\to\infty),$$

and

$$2) \qquad \lim_{n\to\infty} A_n(g) \text{ exists for every } g\in\mathfrak{B}.$$

This is well known as Riesz' theorem.

Lemma 3. *In order that a trigonometrical series should be a Fourier-Stieltjes series, it is necessary and sufficient that*

$$\int_{-\pi}^{\pi} |\sigma_n(t)|\, dt = O(1) \qquad (n\to\infty),$$

where $\sigma_n(t)$ is the n-th Cesàro mean of order 1 of the series [3, p. 79].

3. Now we shall prove Theorems 1–4.

Proof of Theorem 1. Let $\sigma_n^*(t)$ be the n-th Cesàro mean of the series

$$(1) \qquad \frac{1}{2}\lambda_0 + \sum_{\nu=1}^{\infty} \lambda_\nu \cos \nu t .$$

If condition 1) be satisfied, then

$$\int_{-\pi}^{\pi} |\sigma_n^*(t)|\, dt = O(1),$$

and hence, by Lemma 3, series (1) is a Fourier-Stieltjes series. Since the series obtained by termwise integration from (1), is the Fourier series of a function of bounded variation, $\int_{-\pi}^{\tau} K_n(t)dt$ converges for every $\tau\in[-\pi,\pi]$, the integral being a partial sum of that series.

Let $f(x)$ belong to $\mathfrak{M}$, its Fourier series be

$$f(x) \sim \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

and

$$S_n^*(f;x) = \frac{1}{2} a_0\lambda_0 + \sum_{\nu=1}^{n} \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x) .$$

Then

$$S_n^*(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t) f_x(t)\, dt \qquad (f_x(t)=f(x+t)).$$

Putting $[a, b]=[-\pi, \pi]$ and $\varphi_n(t)=K_n(t)$ in Lemma 1, we can obtain the theorem at once. It is easy to see the uniformity of convergence of $S_n^*(f; x)$. Given an $\varepsilon>0$, let η be a number >0 so small that condition 2) is satisfied. Then by LUSIN's theorem there exists a set H of measure $m(H)<\eta/2$ such that $f(t)=f(c+\tau)$ is continuous at every $t\in H^*=[-\pi, \pi]-H$, where c is a fixed point. For every x from a neighborhood N_c of c, put

$$H_1 = \{t; t\in H^*,\ t+x-c\in H^*\},\quad H_2 = H^*-H_1,$$

then $m(H_2)<\eta$. Let A be a bound of $\int_{-\pi}^{\pi}|K_n(t)|\,dt$ and B a bound of $f(x)$. From the continuity of $f(x)$ there exists a neighborhood N_c such that for every $x\in N_c$

$$|f_x(t)-f_c(t)|<\varepsilon,$$

from the condition 2)

$$\int_{H_2}|K_m(t)-K_n(t)|\,dt<\varepsilon,\qquad \int_{H}|K_m(t)-K_n(t)|\,dt<\varepsilon,$$

and from the convergency of $\int_{-\pi}^{\pi}K_n(t)f_c(t)\,dt$ there exists a number N such that

$$\left|\int_{-\pi}^{\pi}(K_m(t)-K_n(t))f_c(t)\,dt\right|<\varepsilon\qquad \text{for all } m, n>N.$$

Hence

$$\begin{aligned}\left|\int_{-\pi}^{\pi}(K_m(t)-K_n(t))f_x(t)\,dt\right| &\leqq \left|\int_{H_1}(K_m(t)-K_n(t))(f_x(t)-f_c(t))\,dt\right|\\ &+\left|\int_{H_2}(K_m(t)-K_n(t))(f_x(t)-f_c(t))\,dt\right|\\ &+\left|\int_{H}(K_m(t)-K_n(t))(f_x(t)-f_c(t))\,dt\right|\\ &+\left|\int_{-\pi}^{\pi}(K_m(t)-K_n(t))f_c(t)\,dt\right|\\ &<2A\cdot\varepsilon+2B\cdot\varepsilon+2B\cdot\varepsilon+\varepsilon\end{aligned}$$

for all $x\in N_c$ and all $n, m>N$. Covering $[-\pi, \pi]$ with a finite number of N_c, we can now conclude the proof of the theorem.

Proof of Theorem 2. Let $f(x)$ belong to $\mathfrak{L}_p$,

$$f(x)\sim\frac{1}{2}a_0+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu x+b_\nu\sin\nu x)$$

and

$$S_n^*(f;x) = \frac{1}{2} a_0 \lambda_0 + \sum_{\nu=1}^{n} \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x)$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t) f_x(t)\, dt .$$

Since $K_n(t) \in \mathfrak{L}_q$, we may consider linear functionals

$$A_n(f) = \int_{-\pi}^{\pi} K_n(t) f(t)\, dt$$

on $\mathfrak{L}_p$. It follows from Lemma 2 that for $S_n^*(f;x)$ to converge it is necessary and sufficient that

1) $$\| A_n \| = \left(\int_{-\pi}^{\pi} |K_n(t)|^q dt \right)^{1/q} = O(1) \qquad (n \to \infty) ,$$

and

2) $$\lim_{n\to\infty} A_n(\cos kt) \text{ exists for all } k.$$

On the other hand, since

$$\int_{-\pi}^{\pi} \cos kt\, K_n(t)\, dt = \begin{cases} 0 & n \neq k , \\ \lambda_k \pi & n = k , \end{cases}$$

condition 2) is satisfied always. Given $\varepsilon > 0$, since $f(x) \in \mathfrak{L}_p$, there exists a neighborhood N_c of c such that

$$\left(\int_{-\pi}^{\pi} |f(x+t) - f(x+c)|^p\, dt \right)^{1/p} < \varepsilon \qquad \text{for all } x \in N_c ,$$

and there exists a number N such that

$$| A_n(f_c) - A(f_c) | < \varepsilon \qquad \text{for all } n > N ,$$

since $A_n(f_c) \to A(f_c)$. Therefore

$$| A_n(f_x) - A(f_x) | \leqq | (A_n - A)(f_x - f_c) | + | (A_n - A) f_c |$$
$$< \| A_n - A \| \cdot \varepsilon + \varepsilon$$

for all $x \in N_c$ and all $n > N$. Covering $[-\pi, \pi]$ with a finite number of N_c, the uniformity of convergence of $A_n(f_x)$ is proved.

The proof of Theorem 3 is quite similar to that of Theorem 2, if we consider linear functionals

$$A_n(f) = \int_{-\pi}^{\pi} K_n(t) f(t)\, dt$$

on $\mathfrak{L}$, where $K_n(t) \in \mathfrak{M}$, and

$$\| A_n \| = \sup_t | K_n(t) | .$$

Proof of Theorem 4. Let $g(x)$ be a periodic function of bounded variation,

$$dg(x) \sim \frac{1}{2} a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

and

$$S_n^*(f; x) = \frac{1}{2} a_0 \lambda_0 + \sum_{\nu=1}^{n} \lambda_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x) .$$

Then

$$S_n^*(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t-x)\, dg(t) .$$

As is well known, to any function $g(x)$ of bounded variation there corresponds a linear functional $A(f)$ on $\mathfrak{C}$ such that

$$\int_{-\pi}^{\pi} f(x)\, dg(x) = A(f) ,$$

and this correspondence is one to one.

First, we shall prove the sufficiency of the condition. Since the functions $K_n(t-x)$ are continuous and converge to $K(t-x)$ uniformly, $A(K_n(t-x))$ converges to $A(K(t-x))$ by the continuity of linear functionals, that is,

$$\begin{aligned} \lim_{n\to\infty} S_n^*(f; x) &= \lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t-x)\, dg(t) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} K(t-x)\, dg(t) . \end{aligned}$$

Moreover, by the inequality

$$\begin{aligned} | A(K_n(t-x)) - A(K(t-x)) | &\leqq \| A \| \cdot \| K_n(t-x) - K(t-x) \| \\ &= \| A \| \cdot \| K_n(t) - K(t) \| , \end{aligned}$$

we find that the convergence is uniform. As to the necessity, let

$$g(t) = \begin{cases} 0 & -\pi \leqq t < 0 , \\ 1 & 0 \leqq t < \pi , \end{cases}$$

and $\{x_n\}$ be a sequence converging to a point x, arbitrary but fixed in $[-\pi, \pi[$. If $S_n^*(f; x)$ converges uniformly, then

$$\left| \int_{-\pi}^{\pi} (K_m(t+x_m) - K_n(t+x_n))\, dg(t) \right| = | K_m(x_m) - K_n(x_n) | \to 0$$
$$(m, n \to \infty),$$

and $K_n(t)$ is uniformly convergent.

4. Finally, using those theorems we shall give some sufficient conditions for $\{\lambda_n\}$ to be a sequence of uniform convergence factors of Fourier series in each case treated above.

Corollary 1. *If $\{\lambda_n\}$ is a convex sequence converging to 0, and*

$$\lambda_n = o\left(\frac{1}{\log n}\right),$$

then $\{\lambda_n\}$ is a sequence of uniform convergence factors of Fourier series of all functions belonging to $\mathfrak{M}$.

For if $\{\lambda_n\}$ satisfy the conditions, the series

$$\frac{1}{2}\lambda_0 + \sum_{\nu=1}^{\infty} \lambda_\nu \cos \nu t$$

is a Fourier series of an integrable function $f(t)$ and

$$\int_{-\pi}^{\pi} |f(t) - K_n(t)|\, dt \to 0 \qquad (n \to \infty) \qquad ([3, \text{p. } 110]).$$

Therefore

$$\int_{-\pi}^{\pi} | K_m(t) - K_n(t) |\, dt \to 0 \qquad (m, n \to \infty),$$

and the $K_n(t)$ satisfy the conditions of Theorem 1.

Corollary 2. *If $\{\lambda_n\}$ is a decreasing sequence converging to 0 and*

$$\sum_{n=1}^{\infty} \lambda_n^q n^{q-2} < \infty,$$

then $\{\lambda_n\}$ is a sequence of uniform convergence factors of Fourier series of all functions belonging to $\mathfrak{L}_p$.

For, if the condition is satisfied, the function

$$g(t) = \frac{1}{2}\lambda_0 + \sum_{\nu=1}^{\infty} \lambda_\nu \cos \nu t$$

belongs to $\mathfrak{L}_p$ and

$$\left(\int_{-\pi}^{\pi} |g(t) - K_n(t)|^q dt \right)^{1/q} \to 0 \qquad (n \to \infty) \qquad ([3, \text{p. } 212]).$$

Accordingly

$$\int_{-\pi}^{\pi} |K_n(t)|^q dt = O(1) \qquad (n \to \infty).$$

Then the result follows from Theorem 2.

Corollary 3. *If* $\{\lambda_n\}$ *is a decreasing sequence converging to* 0 *and satisfies the following conditions:*

1) $$\sum_{\nu=n}^{2n} |\lambda_\nu - \lambda_{\nu+1}| = O\left(\frac{1}{n}\right)$$

2) *the Abel means of* $\sum_{n=1}^{\infty} \lambda_n$ *are uniformly bounded,*

then $\{\lambda_n\}$ *is a sequence of uniform convergence factors of Fourier series of all functions belonging to* $\mathfrak{L}$.

Corollary 4. *If* $\{\lambda_n\}$ *is a decreasing sequence converging to* 0 *and satisfies the following conditions:*

1) $$\sum_{\nu=n}^{2n} |\lambda_\nu - \lambda_{\nu+1}| = O\left(\frac{1}{n}\right)$$

2) $$\sum_{n=1}^{\infty} \lambda_n \text{ is Abel summable},$$

then $\{\lambda_n\}$ *is a sequence of uniform convergence factors of all Fourier-Stieltjes series.*

Corollary 4 is an immediate consequence of a theorem due to O. Szász [4] and corollary 3 is obtained by a slight change of the same theorem.

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